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Jacobian-squared function-germs

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Abstract: In this paper, it is shown that, for any equidimensional C^∞ map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, the map-germ $F : (\mathbb{R}^n, 0) \to \mathbb{R}^n \times \mathbb{R}^\ell$ defined by $F(x) = \big(f(x), \mu_1(x)|Jf|^2(x), \cdots, \mu_\ell(x)|Jf|^2(x)\big)$ is always a frontal, where μ_i is a C^∞ function-germ and $|Jf|$ is the Jacobian-determinant of f. Moreover, it is also shown that when the multiplicity of f is less than or equal to 3, any frontal constructed from f must be $\mathcal{A}$-equivalent to a frontal F of the above form.
Keywords: Jacobian-squared, Frontal, Opening, Ramification module, Equidimensional map-germ.

1. Introduction

Throughout this paper, let n, ℓ be positive integers; and all map-germs, vector fields and differential forms are of class C^∞ unless otherwise stated.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a map-germ and let $|Jf|$ denote the Jacobian-determinant of f. The square of $|Jf|$ is called the *Jacobian-squared* function-germ of f. In the theory of singularities of mappings, it is well-known that the function-germ $|Jf|$ plays an essential role to investigate the behavior of f (for instance, see [2, 17]). However, to the best of author's knowledge, so far there have been no literatures to emphasize the importance of the Jacobian-squared $|Jf|^2$. In this paper, it is explained that $|Jf|^2$ is very significant to construct non-trivial frontals from a given equidimensional map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.

Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$ be a map-germ and let $T\mathbb{R}^{n+\ell}$ be the tangent bundle of $\mathbb{R}^{n+\ell}$. A map-germ $\Phi : (\mathbb{R}^n, 0) \to T\mathbb{R}^{n+\ell}$ is called a *vector field along* F if the equality $\pi \circ \Phi(x) = F(x)$ holds, where $\pi : T\mathbb{R}^{n+\ell} \to \mathbb{R}^{n+\ell}$ is the canonical projection. Namely, Φ has the form $\Phi(x) = (F(x), \phi(x))$ where $\phi(x) \in T_{F(x)}\mathbb{R}^{n+\ell}$. A map-germ $F : (\mathbb{R}^n, 0) \to \mathbb{R}^{n+\ell}$ is called a *frontal* if there exist vector fields $\Phi_1, \ldots, \Phi_\ell : (\mathbb{R}^n, 0) \to T\mathbb{R}^{n+\ell}$ along F such that the following two conditions are satisfied:

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(1) $\phi_i(x) \cdot tF(\xi)(x) = 0$ for any i $(1 \le i \le \ell)$ and any $\xi \in \theta(n)$, where $\Phi_i(x) = (F(x), \phi_i(x))$ and the dot in the center stands for the scalar product of two vectors in $T_{F(x)}\mathbb{R}^{n+\ell}$.
(2) $\phi_1(0), \ldots, \phi_\ell(0)$ are linearly independent.

Here, tF and $\theta(n)$ are the notations defined by J. Mather in [12] (For details, see § 2). Given an equidimensional map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, the *pull-back* $f^* : \mathcal{E}_n \to \mathcal{E}_n$ is defined by $f^*(\eta) = \eta \circ f$ (for the definition of $\mathcal{E}_n$, see Subsection 2.1). The pull-back f^* gives an $\mathcal{E}_n$-module structure via f^*. In this paper, an $\mathcal{E}_n$-module via f^* is called and denoted by an $f^*(\mathcal{E}_n)$-module. For example, $f^*(\mathcal{E}_n)$ itself is an $f^*(\mathcal{E}_n)$-module.

Definition 1 ([9, 10])**.** Let $f = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ.

(1) Let Ω^1_n denote the $\mathcal{E}_n$-module of 1-forms on $(\mathbb{R}^n, 0)$. Then, the $\mathcal{E}_n$-module generated by df_i $(i = 1, \ldots, n)$ in Ω^1_n is called the *Jacobi module* of f and is denoted by $\mathcal{J}_f$, where dh for a function-germ $h : (\mathbb{R}^n, 0) \to \mathbb{R}$ stands for the exterior differential of h.
(2) The *ramification module* of f (denoted by $\mathcal{R}_f$) is defined as the $f^*(\mathcal{E}_n)$-module consisting of all function-germs φ such that $d\varphi$ belongs to $\mathcal{J}_f$.

Though frontals have been already well-investigated (for instance see [10]), it seems to be desired to obtain how to construct non-trivial frontals easily and systematically from a given equidimensional map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ which is not necessarily finite (for the definition of finite map-germ, see Subsection 2.1).

Theorem 1. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. Then, the following inclusion holds:*

$$|Jf|\Omega^1_n \subset \mathcal{J}_f.$$

Since $d(\mu|Jf|^2) = |Jf|(|Jf|d\mu + 2\mu\, d|Jf|) \in |Jf|\Omega^1_n$ for any $\mu \in \mathcal{E}_n$, the following corollary can be obtained from Theorem 1.

Corollary 1. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. For any i $(1 \le i \le \ell)$, let $\mu_i : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function-germ. Then, the map-germ $F : (\mathbb{R}^n, 0) \to \mathbb{R}^{n+\ell}$ defined by*

$$F = \left(f, \mu_1|Jf|^2, \ldots, \mu_\ell|Jf|^2\right)$$

is always a frontal.

It is clear that $f^*(\mathcal{E}_n)$ itself is contained in $\mathcal{R}_f$. By using the notion of the ramification module $\mathcal{R}_f$, Corollary 1 is equivalent to assert that

$$\langle |Jf|^2 \rangle_{\mathcal{E}_n} + f^*(\mathcal{E}_n) \subset \mathcal{R}_f,$$

where $\langle |Jf|^2 \rangle_{\mathcal{E}_n}$ is the $\mathcal{E}_n$-module generated by $|Jf|^2$.

Definition 2 ([9, 10])**.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ and let $\psi_1, \ldots, \psi_\ell$ be elements of $\mathcal{R}_f$. Then, the map-germ

$$(f, \psi_1, \ldots, \psi_\ell) : (\mathbb{R}^n, 0) \to \mathbb{R}^n \times \mathbb{R}^\ell$$

is called an *opening* of f.

By definition, for any equidimensional map-germ, its opening is always a frontal. Corollary 1 guarantees that even for non-finite equidimensional map-germs, it is constructed automatically infinitely many non-trivial openings. This is one advantage of Theorem 1. In Subsections 5.1-5.7, by direct elementary calculations, it is easily shown that the normal forms of known frontals are actually $\mathcal{A}$-equivalent to the frontals constructed by Corollary 1. In other words, in Subsections 5.1-5.7, frontals constructed by using Jacobian-squared function-germs are already very near (with respect to $\mathcal{A}$-equivalence) to the normal forms of celebrated frontals, and therefore criteria for these noticeable frontals are not needed. Since the normal forms of these frontals seem to be not easy to memorize, this may be another advantage of Theorem 1.

Proposition 1 ([10])**.** *For any frontal germ* $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$*, there exist germs of diffeomorphism* $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *and* $H : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$*, an equidimensional map-germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *and elements* $\psi_1, \ldots, \psi_\ell$ *of* $\mathcal{R}_f$ *such that the following equality holds:*

$$H \circ F \circ h = (f, \psi_1, \ldots, \psi_\ell).$$

Based on Proposition 1, it is natural to ask the converse of Corollary 1. Subsections 5.1-5.7 suggest that if f satisfies $\dim_{\mathbb{R}} Q(f) \leq 3$, then asking the converse of Corollary 1 is reasonable (for the definition of $Q(f)$, see Section 2). This question can be answered affirmatively as follows.

Theorem 2. *Let* $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$ *be a frontal germ. Suppose that there exist germs of diffeomorphism* $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *and* $H : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$*, an equidimensional map-germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *with* $\dim_{\mathbb{R}} Q(f) \leq 3$ *and elements* $\psi_1, \ldots, \psi_\ell$ *of* $\mathcal{R}_f$ *such that the following equality holds:*

$$H \circ F \circ h = (f, \psi_1, \ldots, \psi_\ell).$$

Then, the following holds:

$$\left\langle |Jf|^2 \right\rangle_{\mathcal{E}_n} + f^*\left(\mathcal{E}_n\right) = \mathcal{R}_f.$$

Corollary 2. *Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$ be a frontal germ. Suppose that there exist germs of diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$, an equidimensional map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with $\dim_{\mathbb{R}} Q(f) \leq 3$ and elements $\psi_1, \ldots, \psi_\ell$ of $\mathcal{R}_f$ such that the following equality holds:*

$$H \circ F \circ h = (f, \psi_1, \ldots, \psi_\ell).$$

Then, there exist a germ of diffeomorphism $\widetilde{H} : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$ and function-germs $\mu_i : (\mathbb{R}^n, 0) \to \mathbb{R}$ $(1 \leq i \leq \ell)$ such that

$$\widetilde{H} \circ H \circ F \circ h = (f, \mu_1 |Jf|^2, \ldots, \mu_\ell |Jf|^2).$$

On the other hand, if $\dim_{\mathbb{R}} Q(f) > 3$, it turns out that there exist counterexamples against the converse of Corollary 1 (see Subsection 5.9). Thus, in general, it is natural to ask the following:

Question 1. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. Then, does the following equality hold?*

$$\langle |Jf|^2 \rangle_{\mathcal{E}_n} = \left\{ h \in \mathcal{E}_n \mid dh \in |Jf| \Omega^1_n \right\}.$$

Question 2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. Then, does there exist a finitely generated $\mathcal{E}_n$-module A such that the following holds?*

$$A + f^*\left(\mathcal{E}_n\right) = \mathcal{R}_f.$$

Notice that by Ishikawa ([8, 9], see also [10]), it is known if "f is finite and of corank one" or "it is $\mathcal{A}$-equivalent to a finite analytic map-germ", then there exists a finitely generated $f^*(\mathcal{E}_n)$-module B satisfying the equality:

$$B + f^*\left(\mathcal{E}_n\right) = \mathcal{R}_f.$$

Notice also that in the case of Mather's $\mathcal{A}_e$ tangent space for a map-germ $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, the corresponding $\mathcal{E}_n$-module is nothing but $tg(\theta(n))$ (for the definition of Mather's $\mathcal{A}_e$ tangent space, see Section 2). Thus, Question 2 asks whether or not the ramification module $\mathcal{R}_f$ has a similar structure as $T\mathcal{A}_e(g)$.

This paper is organized as follows. In Section 2, preliminaries are given. Theorem 1 (resp., Theorem 2) is proved in Section 3 (resp., Section 4). Finally, in Section 5, examples concerning Theorem 1 and Theorem 2 are given.

2. Preliminaries

2.1. Theory of singularities of mappings

In this subsection, it is partially reviewed several well-known notions/terminologies in the theory of singularities of mappings which are mainly developed by J. Mather in [12, 13]. [17] is an excellent survey article on notions/terminologies reviewed in this subsection, which is recommended to readers.

Two map-germ $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}^p$ are said to be $\mathcal{A}$-equivalent if there exist germs of diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^p, f(0)) \to (\mathbb{R}^p, g(0))$ such that $H \circ f \circ h = g$.

Given a map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, the $\mathcal{A}$-equivalence class of f is denoted by $\mathcal{A}(f)$. Let $\mathcal{E}_n$ be the $\mathbb{R}$-algebra consisting of function-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ and let m_n be the unique maximal ideal of $\mathcal{E}_n$. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a map-germ. The set consisting of all vector fields along f is denoted by $\theta(f)$. Notice that $\theta(f)$ is a finitely generated $\mathcal{E}_n$-module and it is a $\mathcal{E}_p$-module via f. For the identity map-germ $id_n : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $\theta(id_n)$ is denoted by $\theta(n)$. The mapping $tf : \theta(n) \to \theta(f)$ (resp., $\omega f : \theta(p) \to \theta(f)$) is defined by $tf(\xi) = df \circ \xi$ (resp., $\omega f(\eta) = \eta \circ f$). The set $\mathcal{A}(f)$ may be regarded as an orbit of f by the direct product of the following two groups:

$$\{\text{a germ of diffeomorphism } h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\},$$
$$\{\text{a germ of diffeomorphism } H : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)\}.$$

Thus, the tangent space of $\mathcal{A}(f)$ at f is naturally defined as follows:

$$T\mathcal{A}(f) = tf(m_n\theta(n)) + \omega f(m_p\theta(p)).$$

It is meaningful if {diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$} (resp., {diffeomorphism $H : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$}) is replaced with {diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, h(0))$} (resp., {diffeomorphism $H : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, H(0))$}). In this case, $tf(\theta(n)) + \omega f(\theta(p))$ is denoted by $T\mathcal{A}_e(f)$.

$$T\mathcal{A}_e(f) = tf(\theta(n)) + \omega f(\theta(p)).$$

Two map-germ $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be $\mathcal{R}$-equivalent if there exists a germ of diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f \circ h = g$. Given a map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, similarly as in the case of $\mathcal{A}$-equivalence class of f, $T\mathcal{R}(f)$ and $T\mathcal{R}_e(f)$ can be naturally defined as follows:

$$T\mathcal{R}(f) = tf(m_n\theta(n))$$

$$T\mathcal{R}_e(f) \quad = \quad tf(\theta(n)).$$

Therefore, the condition (1) of the definition of frontal may be regarded as the condition that $\phi \in \theta(F)$ is perpendicular to $T\mathcal{R}_e(F)$ in $\theta(F)$.

Given a map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, the $\mathbb{R}$-algebra $\mathcal{E}_n/f^*m_p\mathcal{E}_n$ is denoted by $Q(f)$. The $\mathbb{R}$-algebra $Q(f)$ is called the *local algebra* of the map-germ f. Since the local algebra $Q(f)$ is an $\mathbb{R}$-algebra, it is a vector space. A map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is said to be *finite* if the vector space $Q(f)$ is of finite dimension. The dimension of $Q(f)$ as $\mathbb{R}$ vector space is called the *multiplicity* of f.

2.2. Openings

In this subsection, it is partially reviewed several well-known notions/terminologies in the theory of openings which are mainly developed by G. Ishikawa. [9, 10] are excellent survey articles on notions/terminologies reviewed in this subsection, which are recommended to readers. There is one remark. In [10], openings are defined for a map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ $(n \leq m)$; and in [9] they are defined even for any multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^m, 0)$ $(n \leq m)$. However, in this paper, it is needed only openings of an equidimensional mono-germ. Thus, for the sake of clearness, in this subsection, we concentrate on reviewing notions/terminologies of opening only for an equidimensional mono-germ.

Definition 3. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. An opening $F = (f, \psi_1, \ldots, \psi_\ell)$ of f is called a *versal opening* (resp., *mini-versal opening*) of f if $1, \psi_1, \ldots, \psi_\ell$ form a system (resp., minimal system) of generators of $\mathcal{R}_f$ as an $f^*(\mathcal{E}_n)$-module.

Lemma 1 ([9]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. Then, the following hold.*

(1) $f^*(\mathcal{E}_n) \subset \mathcal{R}_f \subset \mathcal{E}_n$.
(2) *$\mathcal{R}_f$ is a $f^*(\mathcal{E}_n)$-module.*
(3) *$\mathcal{R}_f$ is a C^∞ subring of $\mathcal{E}_n$.*
(4) *For a germ of diffeomorphism on the target space $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\mathcal{R}_{H\circ f} = \mathcal{R}_f$ holds. For a germ of diffeomorphism on the source space $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\mathcal{R}_{f\circ h} = h^*(\mathcal{R}_f)$ holds.*

Proposition 2 ([8]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a finite map-germ of corank one. Then, the following hold.*

(1) *$\mathcal{R}_f$ is a finitely generated $f^*(\mathcal{E}_n)$-module. Therefore, there is a versal opening of f.*
(2) *Function-germs $1, \psi_1, \ldots, \psi_\ell \in \mathcal{R}_f$ generate $\mathcal{R}_f$ as $f(\mathcal{E}_n)$-module if and only if $[1], [\psi_1], \ldots, [\psi_\ell]$ generate the vector space $\mathcal{R}_f / (f^* m_n) \mathcal{R}_f$, where $[a]$ stands for $a + (f^* m_n) \mathcal{R}_f$.*

Proposition 3. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a map-germ $\mathcal{A}$-equivalent to a finite analytic map-germ. Then, the following hold.*

(1) *$\mathcal{R}_f$ is a finitely generated $f^*(\mathcal{E}_n)$-module. Therefore, there is a versal opening of f.*
(2) *Function-germs $1, \psi_1, \ldots, \psi_\ell \in \mathcal{R}_f$ generate $\mathcal{R}_f$ as $f(\mathcal{E}_n)$-module if and only if $[1], [\psi_1], \ldots, [\psi_\ell]$ generate the vector space $\mathcal{R}_f / (f^* m_n) \mathcal{R}_f$.*

For the proof of Proposition 3, see [7, 9].

Proposition 4 ([9]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a finite equidimensional map-germ. Then, every versal opening $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$ is injective.*

Here, a map-germ is said to be *injective* if it has an injective representative.

3. Proof of Theorem 1

Let $\widetilde{Jf}$ be the cofactor matrix of the Jacobian matrix Jf. Then, notice that $\widetilde{Jf} Jf = |Jf| E_n$ where E_n is the $n \times n$ unit matrix. For any 1-form $\alpha = \sum_{i=1}^n a_i dx_i$, we have the following:

$$|Jf|\alpha = (a_1, \ldots, a_n) \widetilde{Jf} Jf \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = (a_1, \ldots, a_n) \widetilde{Jf} \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} \in \mathcal{J}_f.$$

This completes the proof. $\square$

4. Proof of Theorem 2

4.1. The case $n = 1$

Set $\delta = \dim_{\mathbb{R}} Q(f)$. In the case of $n = 1$, there exists a germ of diffeomorphism $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $g(x) = f \circ h(x) = \frac{1}{\delta} x^\delta$. By [6], the ramification module of g can be described as follows.

$$\mathcal{R}_g = \begin{cases} \langle 1 \rangle_{g^*(\mathcal{E}_1)} = \langle 1 \rangle_{\mathcal{E}_1} & (\delta = 1), \\ \langle 1, x^3 \rangle_{g^*(\mathcal{E}_1)} & (\delta = 2), \\ \langle 1, x^4, x^5 \rangle_{g^*(\mathcal{E}_1)} & (\delta = 3). \end{cases}$$

On the other hand, since $|Jg|^2(x) = x^{2(\delta-1)}$, the following holds.

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_1} = \begin{cases} \langle 1 \rangle_{\mathcal{E}_1} & (\delta = 1), \\ \langle x^2, x^3 \rangle_{\mathcal{E}_1} & (\delta = 2), \\ \langle x^4, x^5 \rangle_{\mathcal{E}_1} & (\delta = 3). \end{cases}$$

Thus, we have the following inclusion.

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_1} + g^* \left(\mathcal{E}_1\right) \supset \mathcal{R}_g.$$

Combining this inclusion with Theorem 1, the following equality holds.

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_1} + g^* \left(\mathcal{E}_1\right) = \mathcal{R}_g.$$

Since h is a germ of diffeomorphism, we have the following by the chain rule.

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_1} = \left\langle |J(f \circ h)|^2 \right\rangle_{\mathcal{E}_1} = h^* \left(\left\langle |Jf|^2 \right\rangle_{\mathcal{E}_1} \right).$$

It follows

$$h^* \left(\left\langle |Jf|^2 \right\rangle_{\mathcal{E}_1} + f^* \left(\mathcal{E}_1\right) \right) = h^* \left(\mathcal{R}_f\right).$$

Hence, by pulling both sides back by $\left(h^{-1}\right)^*$, the desired equality can be obtained as follows.

$$\left\langle |Jf|^2 \right\rangle_{\mathcal{E}_1} + f^* \left(\mathcal{E}_1\right) = \mathcal{R}_f.$$

□

4.2. The case $n \geq 2$

Again in this subsection, $\dim_{\mathbb{R}} Q(f)$ is denoted by δ. The assumption $\delta \leq 3$ implies that f is of corank one. Since $\frac{1}{2}x^2 + tx$ (resp., $\frac{1}{3}x^3 + tx$) is an $\mathcal{R}$-versal unfolding of $\frac{1}{2}x^2$ (resp., $\frac{1}{3}x^3$), it is deduced that there exist germs of diffeomorphism $h, H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a function-germ $\alpha : (\mathbb{R}^{n-1}, 0) \to \mathbb{R}$ such that

$$H \circ f \circ h(x, \lambda_1, \ldots, \lambda_{n-1}) = \begin{cases} (x, \lambda_1, \ldots, \lambda_{n-1}) & (\delta = 1), \\ (\frac{1}{2}x^2 + \alpha(\lambda_1, \ldots, \lambda_{n-1})x, \lambda_1, \ldots, \lambda_{n-1}) & (\delta = 2), \\ (\frac{1}{3}x^3 + \alpha(\lambda_1, \ldots, \lambda_{n-1})x, \lambda_1, \ldots, \lambda_{n-1}) & (\delta = 3), \end{cases}$$

where $(x, \lambda_1, \ldots, \lambda_{n-1})$ is an element of $\mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$. Set $g = H \circ f \circ h$ and $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$. By [6], the ramification module of g can be described as follows.

$$\mathcal{R}_g = \begin{cases} \langle 1 \rangle_{g^*(\mathcal{E}_n)} = \langle 1 \rangle_{\mathcal{E}_n} & (\delta = 1), \\ \langle 1, \frac{1}{3}x^3 + \frac{1}{2}\alpha(\lambda)x^2 \rangle_{g^*(\mathcal{E}_n)} & (\delta = 2), \\ \langle 1, \frac{1}{4}x^4 + \frac{1}{2}\alpha(\lambda)x^2, \frac{1}{5}x^5 + \frac{1}{3}\alpha(\lambda)x^3 \rangle_{g^*(\mathcal{E}_n)} & (\delta = 3). \end{cases}$$

On the other hand, we have

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_n} = \begin{cases} \langle 1 \rangle_{\mathcal{E}_n} & (\delta = 1), \\ \langle (x + \alpha(\lambda))^2 \rangle_{\mathcal{E}_n} & (\delta = 2), \\ \langle (x^2 + \alpha(\lambda))^2 \rangle_{\mathcal{E}_n} & (\delta = 3). \end{cases}$$

Since

$$\begin{aligned} (x + \alpha(\lambda))^2 &= 2\left(\frac{1}{2}x^2 + \alpha(\lambda)x\right) + (\alpha(\lambda))^2, \\ x\,(x + \alpha(\lambda))^2 &= 3\left(\frac{1}{3}x^3 + \frac{1}{2}\alpha(\lambda)x^2\right) + \alpha(\lambda)\left(\frac{1}{2}x^2 + \alpha(\lambda)x\right), \\ \left(x^2 + \alpha(\lambda)\right)^2 &= 4\left(\frac{1}{4}x^4 + \frac{1}{2}\alpha(\lambda)x^2\right) + (\alpha(\lambda))^2 \quad \text{and} \\ x\left(x^2 + \alpha(\lambda)\right)^2 &= 5\left(\frac{1}{5}x^5 + \frac{1}{3}\alpha(\lambda)x^3\right) + \alpha(\lambda)\left(\frac{1}{3}x^3 + \alpha(\lambda)x\right), \end{aligned}$$

the following holds.

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_n} + g^*\left(\mathcal{E}_n\right) \supset \mathcal{R}_g.$$

Combining this inclusion with Theorem 1, we have the following equality.

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_n} + g^*\left(\mathcal{E}_n\right) = \mathcal{R}_g.$$

Since h, H are germs of diffeomorphism, it follows the following.

$$\left\langle |Jg|^2 \right\rangle_{\mathcal{E}_n} = \left\langle |J(H \circ f \circ h)|^2 \right\rangle_{\mathcal{E}_n} = h^* \left(\left\langle |Jf|^2 \right\rangle_{\mathcal{E}_n} \right).$$

Combining this property with two facts $g^* (\mathcal{E}_n) = h^* ((H \circ f)^* (\mathcal{E}_n)) = h^* (f^* (\mathcal{E}_n))$ and $\mathcal{R}_g = \mathcal{R}_{H \circ f \circ h} = h^* (\mathcal{R}_{H \circ f}) = h^* (\mathcal{R}_f)$, we have

$$h^* \left(\langle |Jf|^2 \rangle_{\mathcal{E}_n} + f^* (\mathcal{E}_n) \right) = h^* (\mathcal{R}_f).$$

Therefore, also in this case, we have the following desired equality.

$$\left\langle |Jf|^2 \right\rangle_{\mathcal{E}_n} + f^* (\mathcal{E}_n) = \mathcal{R}_f.$$

□

5. Examples of Theorems 1 and 2

All map-germs $h_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H_j : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$ appearing in this section are germs of diffeomorphism.

5.1. Fold singularity

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the map-germ defined by $f(x, y) = (\frac{1}{2}x^2 + xy,\ y)$. The map-germ f is $\mathcal{A}$-equivalent to the map-germ named *Type 2* in the list of [14]. It is clear that $|Jf|(x, y) = x + y$ and $\dim_{\mathbb{R}} Q(f) = 2$. Set $\mu_1(x, y) = 1$. Then, by Theorem 1, the map-germ $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$\begin{aligned} F(x, y) &= \left(f(x, y), \mu_1(x, y)|Jf|^2(x, y) \right) \\ &= \left(\frac{1}{2}x^2 + xy,\ y,\ (x + y)^2 \right) = \left(\frac{1}{2} (x + y)^2 - \frac{1}{2}y^2,\ y,\ (x + y)^2 \right) \end{aligned}$$

is a frontal. Set $H_1(X, Y, Z) = (X - \frac{1}{2}Y^2,\ Y,\ Z)$. Then,

$$H_1 \circ F(x, y) = \left(\frac{1}{2}(x + y)^2,\ y,\ (x + y)^2 \right).$$

Secondly, set $h_1(x, y) = (x - y, y)$ and $H_2(X, Y, Z) = (2X,\ Y,\ Z - 2X)$. Then,

$$H_2 \circ H_1 \circ F \circ h_1(x, y) = \left(x^2,\ y,\ 0 \right).$$

In Differential Geometry, the map-germ $H_2 \circ H_1 \circ F \circ h_1$ is called the normal form of *fold singularity* ([4]).

5.2. Cuspidal edge

Not only in this subsection, but also until Subsection 5.4, we start from the same map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ as in Subsection 5.1. Namely, in this subsection, f is the map-germ defined by $f(x, y) = \left(\frac{1}{2}x^2 + xy,\ y\right)$. Set $\mu_1(x, y) = x$. Then, by Theorem 1, the map-germ $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$\begin{aligned} F(x,y) &= \left(f(x,y), \mu_1(x,y)|Jf|^2(x,y)\right) \\ &= \left(\frac{1}{2}x^2 + xy,\ y,\ x(x+y)^2\right) \\ &= \left(\frac{1}{2}(x+y)^2 - \frac{1}{2}y^2,\ y,\ (x+y)^3 - y(x+y)^2\right) \end{aligned}$$

is a frontal. Set $H_1(X, Y, Z) = (X - \frac{1}{2}Y^2,\ Y,\ Z)$. Then,

$$H_1 \circ F(x,y) = \left(\frac{1}{2}(x+y)^2,\ y,\ (x+y)^3 - y(x+y)^2\right).$$

Secondly, set $h_1(x, y) = (x - y, y)$ and $H_2(X, Y, Z) = (2X,\ Y,\ Z - 2XY)$. Then,

$$H_2 \circ H_1 \circ F \circ h_1(x,y) = \left(x^2,\ y,\ x^3\right),$$

well-known as the normal form of *cuspidal edge* (for instance, see [11]).

5.3. Folded umbrella (cuspidal crosscap)

As explained in the last subsection, in this subsection again, $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is the map-germ defined by $f(x, y) = \left(\frac{1}{2}x^2 + xy,\ y\right)$. Set $\mu_1(x, y) = x^2$. Then, by Theorem 1, the map-germ $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$\begin{aligned} F(x,y) &= \left(f(x,y), \mu_1(x,y)|Jf|^2(x,y)\right) \\ &= \left(\frac{1}{2}x^2 + xy,\ y,\ x^2(x+y)^2\right) \\ &= \left(\frac{1}{2}x^2 + xy,\ y,\ x^4 + 2x^3y + x^2y^2\right) \end{aligned}$$

is a frontal. Set $H_1(X, Y, Z) = (X,\ Y,\ Z - X^2)$. Then,

$$H_1 \circ F(x,y) = \left(\frac{1}{2}x^2 + xy,\ y,\ \frac{3}{4}x^4 + x^3y\right).$$

Secondly, set $h_1(x,y) = (x, \frac{1}{2}y)$. Then,

$$H_1 \circ F \circ h_1(x,y) = \left(\frac{1}{2}x^2 + \frac{1}{2}xy,\ \frac{1}{2}y,\ \frac{3}{4}x^4 + \frac{1}{2}x^3y\right).$$

Thirdly, set $H_2(X,Y,Z) = \left(2X, 2Y, \frac{4}{3}Z\right)$. Then, we have

$$H_2 \circ H_1 \circ F \circ h_1(x,y) = \left(x^2 + xy,\ y,\ x^4 + \frac{2}{3}x^3y\right),$$

called the normal form of *folded umbrella* (for instance, see [10]). A folded umbrella is also called a *cuspidal crosscap*. As the normal form of cuspidal crosscap, Some references adopt $(x,y) \mapsto (x^2,\ y,\ x^3y)$ (for instance, see [5]). Set $h_2(x,y) = \left(\frac{1}{2}(x-y), y\right)$ and

$$\begin{aligned} &H_3(X,Y,Z) \\ =& \left(4\left(X + \frac{1}{4}Y^2\right), Y, -6\left(Z - \left(X + \frac{1}{4}Y^2\right)^2 - \frac{1}{2}\left(X + \frac{1}{4}Y^2\right)Y^2 - \frac{1}{16}Y^4\right)\right). \end{aligned}$$

Then, it is easily confirmed

$$H_3 \circ H_2 \circ H_1 \circ F \circ h_1 \circ h_2(x,y) = \left(x^2,\ y,\ x^3y\right).$$

5.4. Open folded umbrella

As explained, we start from the same $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ as in Subsection 5.1. Thus, f is defined by $f(x,y) = \left(\frac{1}{2}x^2 + xy,\ y\right)$ in this subsection. Suppose that $\ell \geq 3$ and set $\mu_1(x,y) = x^2$, $\mu_2(x,y) = x^2|Jf|(x,y)$ and $\mu_i(x,y) = 0$ $(3 \leq i \leq \ell)$. Then,

$$\begin{aligned} F(x,y) &= \left(f(x,y), \mu_1|Jf|^2(x,y), \ldots, \mu_\ell(x,y)|Jf|^2(x,y)\right) \\ &= \left(\frac{1}{2}x^2 + xy,\ y,\ x^2(x+y)^2,\ x^2(x+y)^3, 0, \ldots, 0\right) \\ &= \left(\frac{1}{2}x^2 + xy,\ y,\ x^4 + 2x^3y + x^2y^2,\right. \\ &\qquad \left. x^5 + 3x^4y + 3x^3y^2 + x^2y^3, 0, \ldots, 0\right). \end{aligned}$$

Set

$$H_1(X, Y, U_1, U_2, U_3, \ldots, U_\ell) = (X,\ Y,\ U_1 - X^2,\ U_2 - YU_1,\ U_3, \ldots, U_\ell).$$

Then,

$$H_1 \circ F(x,y) = \left(\frac{1}{2}x^2 + xy,\ y,\ \frac{3}{4}x^4 + x^3y,\ x^5 + 2x^4y + x^3y^2, 0, \ldots, 0\right).$$

Set $h_1(x,y) = (x, \frac{1}{2}y)$. Then,

$$\begin{aligned} &H_1 \circ F \circ h_1(x,y) \\ &= \left(\frac{1}{2}x^2 + \frac{1}{2}xy,\ \frac{1}{2}y,\ \frac{3}{4}x^4 + \frac{1}{2}x^3y,\ x^5 + x^4y + \frac{1}{4}x^3y^2, 0, \ldots, 0\right). \end{aligned}$$

Nextly, set

$$H_2(X, Y, U_1, U_2, U_3, \ldots, U_\ell) = (X,\ Y,\ U_1,\ U_2 - YU_1,\ U_3, \ldots, U_\ell).$$

Then, we have

$$\begin{aligned} &H_2 \circ H_1 \circ F \circ h_1(x,y) \\ = &\left(\frac{1}{2}x^2 + \frac{1}{2}xy,\ \frac{1}{2}y,\ \frac{3}{4}x^4 + \frac{1}{2}x^3y,\ x^5 + \frac{5}{8}x^4y, 0, \ldots, 0\right). \end{aligned}$$

Finally, set

$$H_3(X, Y, U_1, U_2, \ldots, U_\ell) = \left(2X,\ 2Y,\ \frac{4}{3}U_1,\ U_2, \ldots, U_\ell\right).$$

Then,

$$\begin{aligned} &H_3 \circ H_2 \circ H_1 \circ F \circ h_1(x,y) \\ = &\left(x^2 + xy,\ y,\ x^4 + \frac{2}{3}x^3y,\ x^5 + \frac{5}{8}x^4y,\ 0, \ldots, 0\right), \end{aligned}$$

called the normal form of *open folded umbrella* ([1, 10]).

5.5. Swallowtail

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the map-germ defined by $f(x,y) = (\frac{1}{3}x^3 + xy,\ y)$. The map-germ f is $\mathcal{A}$-equivalent to the map-germ named *Type 3* in the list of [14]. It is clear that $|Jf|(x,y) = x^2 + y$ and $\dim_{\mathbb{R}} Q(f) = 3$. Set $\mu_1(x,y) = 1$. We consider the map-germ $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$F(x,y) = \left(f(x,y), \mu_1(x,y)|Jf|^2(x,y)\right)$$

$$= \left(\frac{1}{3}x^3 + xy,\ y,\ (x^2+y)^2\right)$$
$$= \left(\frac{1}{3}x^3 + xy,\ y,\ x^4 + 2x^2y + y^2\right).$$

By Theorem 1, F is a frontal. Set $H_1(X,\ Y,\ Z) = (X,\ Y,\ Z - Y^2)$, $H_2(X,\ Y,\ Z) = (-12X,\ 6Y,\ 3Z)$. Then,

$$H_2 \circ H_1 \circ F(x,y) = \left(-4x^3 - 12xy, 6y, 3x^4 + 6x^2y\right).$$

Next, set $h_1(x,\ y) = (x,\ \frac{1}{6}y)$. Then,

$$H_2 \circ H_1 \circ \widetilde{F} \circ h_1(x,\ y) = \left(-4x^3 - 2xy,\ y,\ 3x^4 + x^2y\right),$$

well-known as the normal form of *swallowtail* (for instance see [3], page 129).

5.6. Open swallowtail

As in Subsection 5.5, let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the map-germ defined by $f(x,y) = (\frac{1}{3}x^3 + xy,\ y)$. Suppose that $\ell \geq 3$ and set $\mu_1(x,y) = 1, \mu_2(x,y) = x, \mu_i(x,y) = 0$ $(3 \leq i \leq \ell)$. Let $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^{2+\ell}, 0)$ be the map-germ defined by

$$\begin{aligned} F(x,y) &= \left(f(x,y), \mu_1|Jf|^2(x,y), \ldots, \mu_\ell|Jf|^2(x,y)\right) \\ &= \left(\frac{1}{3}x^3 + xy,\ y,\ (x^2+y)^2,\ x(x^2+y)^2,\ 0, \ldots, 0\right) \\ &= \left(\frac{1}{3}x^3 + xy,\ y,\ x^4 + 2x^2y + y^2, x^5 + 2x^3y + xy^2, 0, \ldots, 0\right). \end{aligned}$$

Set

$$H_1(X, Y, U_1, U_2, U_3, \ldots, U_\ell) = (X,\ Y,\ U_1 - Y^2,\ U_2 - XY,\ U_3, \ldots, U_\ell).$$

Then,

$$H_1 \circ F(x,y) = \left(\frac{1}{3}x^3 + xy,\ y,\ x^4 + 2x^2y,\ x^5 + \frac{5}{3}x^3y,\ 0, \ldots, 0\right).$$

Set $h_1(x,y) = (x,\ \frac{1}{3}y)$. Then,

$$H_1 \circ F \circ h_1(x,y) = \left(\frac{1}{3}x^3 + \frac{1}{3}xy,\ \frac{1}{3}y,\ x^4 + \frac{2}{3}x^2y\ x^5 + \frac{5}{9}x^3y,\ 0, \ldots, 0\right).$$

Set $H_2(X, Y, U_1, \dots, U_\ell) = (3X\ , 3Y,\ U_1, \dots, U_\ell)$. Then,

$$H_2 \circ H_1 \circ G \circ h_1(x, y) = \left(x^3 + xy,\ y,\ x^4 + \frac{2}{3}x^2y, x^5 + \frac{5}{9}x^3y,\ 0, \dots, 0\right),$$

known as the normal form of *open swallowtail* ([10]).

5.7. Opening of 4_k in the list of [14]

Let k be an integer greater than 1 and let $f_{k,\pm} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the map-germ defined by $f_{k,\pm}(x, y) = (\frac{1}{3}x^3 \pm xy^k,\ y)$. The map-germ $f_{k,\pm}$ is $\mathcal{A}$-equivalent to the map-germ named *Type* 4_k in the list of [14]. It is clear that $|Jf|(x, y) = x^2 \pm y^k$ and $\dim_{\mathbb{R}} Q(f) = 3$. Set $\mu_1(x, y) = 1$. We consider the map-germ $F_{k,\pm} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$\begin{aligned} F_{k,\pm}(x, y) &= \left(f_{k,\pm}(x, y), \mu_1(x, y)|Jf|^2(x, y)\right) \\ &= \left(\frac{1}{3}x^3 \pm xy^k,\ y,\ (x^2 \pm y^k)^2\right) \\ &= \left(\frac{1}{3}x^3 \pm xy^k,\ y,\ x^4 \pm 2x^2y^k + y^{2k}\right). \end{aligned}$$

Theorem 1 guarantees that $F_{k,\pm}$ is a frontal. Set $H_1(X,\ Y,\ Z) = (X,\ Y,\ Z - Y^{2k})$ and $H_2(X,\ Y,\ Z) = (6X,\ 6Y,\ 3Z)$. Then,

$$H_2 \circ H_1 \circ F_{k,\pm}(x, y) = \left(2x^3 \pm 6xy^k,\ 6y,\ 3x^4 \pm 6x^2y^k\right).$$

Next, set $h_1(x,\ y) = \left(x,\ \frac{1}{\sqrt[k]{6}}y\right)$ and $H_3(X,\ Y,\ Z) = (X,\ \frac{\sqrt[k]{6}}{6}Y,\ Z)$. Then,

$$H_3 \circ H_2 \circ H_1 \circ F_{k,\pm} \circ h_1(x,\ y) = \left(2x^3 \pm xy^k,\ y,\ 3x^4 \pm x^2y^k\right).$$

The form of $H_3 \circ H_2 \circ H_1 \circ F_{2,-} \circ h_1$ is exactly the same as the map-germ called a *double swallowtail* given in [15].

5.8. A non-analytic example

Let $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a flat function-germ (i.e. $j^\infty\psi(0) = 0$) and let $f_\infty : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the map-germ defined by $f_\infty(x, y) = (\frac{1}{3}x^3 \pm x\psi(y),\ y)$. It is clear that $\dim_{\mathbb{R}} Q(f_\infty) = 3$. Thus, by Theorem 2, the following equality holds even for the f_∞.

$$\left\langle |Jf_\infty|^2 \right\rangle_{\mathcal{E}_2} + f_\infty^* (\mathcal{E}_2) = \mathcal{R}_{f_\infty}.$$

Therefore, for any opening $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^{2+\ell}, 0)$ of f_∞ there exist germs of diffeomorphisms $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $H : (\mathbb{R}^{2+\ell}, 0) \to (\mathbb{R}^{2+\ell}, 0)$ and function-germs $\mu_1, \ldots, \mu_\ell : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that the following holds.

$$H \circ F \circ h = (f_\infty,\ \mu_1 |Jf_\infty|^2,\ \ldots,\ \mu_\ell |Jf_\infty|^2).$$

5.9. A_k-front singularity

Let k be an integer greater than 1 and let $f_k : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ be the map-germ defined as follows:

$$f_k(x_1, \ldots, x_k) = \left(\frac{1}{k+1} x_1^{k+1} + \frac{1}{k-1} x_1^{k-1} x_2 + \cdots + x_1 x_k, x_2, \ldots, x_k \right).$$

The map-germ f_k is a generalization of f given in Subsection 5.5 because f_2 is exactly the same as the f. The map-germ f_k is well-known as the normal form of corank-one isolated stable singularity (for instance, see [2, 13]). For the f_k, $\dim_{\mathbb{R}} Q(f_k) = k + 1$ and $|Jf_k|^2(x_1, 0, \ldots, 0) = x_1^k$.

From now on, suppose that k is greater than 2. Let $\mu : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function-germ. Since $2k - (k + 1) = k - 1 > 1$, for the normal vector field ν of the frontal $(f_k, \mu|Jf_k|^2)$, $\nu(x_1, 0, \ldots, 0)$ must be singular at the origin.

On the other hand, since f_k is a polynomial map-germ of corank one, $\mathcal{R}_{f_k}$ can be described explicitly by [6]. In particular, there exists an opening $\widetilde{F}_k$ of f_k whose normal vector field ν is non-singular with respect to x_1 (A concrete construction of ν can be found in [16]). Therefore, if $k \geq 3$, it is impossible to expect the converse of Theorem 1 for the frontal $\widetilde{F}_k$.

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