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Orbital stability of solitary wave solutions of Zakharov–Rubenchik equation*

XiaoHua Liu

Abstract: We study orbital stability of bell-shaped solitary wave solutions of the Zakharov–Rubenchik equation for the interaction of high–frequency and low–frequency waves in an arbitrary medium. Our approach is based on the theories of orbital stability presented by Grillakis, Shatah and Strauss, and relies on a reformulation of the coupled equations in Hamiltonian form. We investigate stability of solitary wave solutions by ascertaining the number of negative eigenvalues of the linear operator and the number of positive eigenvalues of its Hessian of the scalar function.

Keywords: Solitary wave solution, undeterminded coefficient method, orbital stability, the Zakharov–Rubenchik equation.

1. Introduction

The Zakharov equation [1] in a commonly form

$$
(1.1)\qquad \begin{cases} iB_t + \Delta B = B\rho, \qquad x \in R^n,\ t \in R,\ n \geq 1 \\ \rho_{tt} - \Delta\rho = \Delta(|B|^2) \end{cases}
$$

deduced by Zakharov, model the nonlinear interaction of high–frequency and low–frequency waves in an arbitrary medium, where B denotes the slowly varying envelope of high–frequency electic field, ρ denotes the low-frequency variation of density of ions. Tsutsumi and Ozawa [2] investigated the local well-posedness of equation (1.1). The more research about Zakharov equation and their multiple variants were presented in the literatures [3, 4, 5, 6, 7, 8, 9].

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By a quantum fluid approach Garcia, Haas, and Oliveira et al. [8] obtained the following modified Zakharov equation:

$$\begin{cases} iB_t + B_{xx} - H^2 B_{xxxx} = B\rho, \\ \rho_{tt} - \rho_{xx} + H^2 \rho_{xxxx} = (|B|^2)_{xx}. \end{cases} \tag{1.2}$$

You, Guo and Ning [3] proved the existence and uniqueness of solutions to the initial boundary value problem for equation (1.2).

The one–dimension version of equation (1.1) is the following Zakharov–Rubenchik equation [10]:

$$\begin{cases} iB_t + wB_{xx} = \gamma(u - \frac{v}{2}\rho + q|B|^2)B, \\ \theta\rho_t + (u - v\rho)_x = -\gamma(|B|^2)_x, \\ \theta u_t + (\beta\rho - vu)_x = \frac{\gamma}{2}(|B|^2)_x, \end{cases} \tag{1.3}$$

where $x,\ t \in R$, B is a complex–valued function, ρ and u are real-valued functions, $\gamma,\ \beta,\ q,\ w,\ v$ and θ are real parameters and satisfy the conditions that $w > 0,\ \gamma q < 0,\ q = \gamma + \frac{v(\gamma v - 1)}{2(\beta - v^2)},\ \beta > 0$ and $\beta \neq v^2$.

Setting $u = 0$, equation (1.3) become the Benny equation

$$\begin{cases} iB_t + B_{xx} + ef\psi B + c|B|^2 B = 0, \\ \varepsilon\psi_t + e\psi_x = f|B|^2_x. \end{cases} \tag{1.4}$$

The well-posedness of equation (1.4) has been studied in [11], and the study of stability of solitary wave type solution of equation (1.4) in the case where $c = 0$ has been found in [12].

Oliveira [13] studied stability and existence of solitary wave solutions of equation (1.3) for small enough positive θ, and he also proved the local and global well–posedness for data in $H^2(R) \times H^1(R) \times H^1(R)$ by the ideas of Tsutsumi and Ozawa [2]. Oliveira [14] investigated that the magnetic field B of equation (1.3) converge point by point to a solution of the nonlinear Schrödinger equation $iB_t + wB_{xx} + \frac{kv}{4(\beta - v^2)}|B|^2 B = 0$, in the case of adiabatic limit $\theta \to 0$. Ponce and Saut [15] studied the local well-posedness of multi–dimensional version of equation (1.3). The numerical solution of equation (1.3) has been considered in [16].

Setting

$$\rho = \psi_1 + \psi_2,\ u = \sqrt{\beta}(\psi_1 - \psi_2),$$

equation (1.3) become the following form:

$$\begin{cases} iB_t + wB_{xx} = \gamma(\sqrt{\beta} - \frac{v}{2})B\psi_1 - \gamma(\sqrt{\beta} + \frac{v}{2})B\psi_2 + \gamma q|B|^2 B, \\ \theta\psi_{1t} + (\sqrt{\beta} - v)\psi_{1x} = \frac{\gamma}{2}(-1 + \frac{v}{2\sqrt{\beta}})(|B|^2)_x, \\ \theta\psi_{2t} - (\sqrt{\beta} + v)\psi_{2x} = \frac{\gamma}{2}(-1 - \frac{v}{2\sqrt{\beta}})(|B|^2)_x. \end{cases} \tag{1.5}$$

Linares and Matheus [17] proved that equation (1.5) was globally well-posedness for data in $H^k(R) \times H^l(R) \times H^l(R)(0 \leq k \leq l + \frac{1}{2})$ with the help of the methods presented by Colliander, Holmer and Tzirakis in [9]. To the best of our knowledge, there is no references in the literature of the stability of solitary wave solutions of equation (1.5) for any $\theta > 0$.

Our purpose is to investigate the orbital stability of solitary wave solutions for equation (1.5). The rest of content is structured as follows, in Section 2, we state the main result of the orbital stability of solitary wave solutions of equation (1.5) for any positive θ, using the abstract theories of orbital stability proposed by Grillakis, Shatah and Strauss [18, 19]. In Section 3, we prove the stability. The main intermediate step is to determine the number of negative eigenvalues of the linear operator $H_{\lambda,c}$ and the number of positive eigenvalues of its Hessian of the scalar function $d(\lambda, c)$ at c, which $H_{\lambda,c}$ and $d(\lambda, c)$ is denoted by (2.33) and (2.40), respectively.

2. The main result

In order to state the main result, we firstly study the exact solitary wave solutions of equation (1.5).

2.1. The exact solitary wave solutions

By substituting $k_1 = \sqrt{\beta} - \frac{v}{2},\ k_2 = \sqrt{\beta} + \frac{v}{2},\ k_3 = \sqrt{\beta} - v,\ k_4 = \sqrt{\beta} + v,\ k_5 = -1 + \frac{v}{2\sqrt{\beta}}$ and $k_6 = 1 + \frac{v}{2\sqrt{\beta}}$ into equation (1.5), it can be rewrited as

$$\begin{cases} iB_t = -wB_{xx} + \gamma k_1 B\psi_1 - \gamma k_2 B\psi_2 + \gamma q|B|^2 B, \\ \theta\psi_{1t} = -k_3\psi_{1x} + \frac{\gamma}{2}k_5(|B|^2)_x, \\ \theta\psi_{2t} = k_4\psi_{2x} - \frac{\gamma}{2}k_6(|B|^2)_x. \end{cases} \tag{2.1}$$

Let

$$\begin{aligned} B(x,t) &= e^{-i\lambda t}e^{if(x-ct)}\varphi(x-ct), \\ \psi_1(x,t) &= \phi_1(x-ct),\ \psi_2(x,t) = \phi_2(x-ct) \end{aligned} \tag{2.2}$$

be the solitary wave solutions of equation (2.1), where $\lambda,\ f,\ c$ are real numbers, $\varphi,\ \phi_1,\ \phi_2$ are real functions. Substituting (2.2) into (2.1), we have the following equations:

$$\begin{cases} cf\varphi + \lambda\varphi - ic\varphi = -w\varphi'' - 2iwf\varphi' + wf^2\varphi + \gamma k_1\varphi\phi_1 - \gamma k_2\varphi\phi_2 + \gamma q\varphi^3, \\ -c\theta\phi_1' = -k_3\phi_1' + \gamma k_5\varphi\varphi', \\ -c\theta\phi_2' = k_4\phi_2' - \gamma k_6\varphi\varphi', \end{cases} \tag{2.3}$$

where the prime denotes the derivative with respect to x. In order to study the exact solitary wave solutions of equation (2.1), we consider the exact solutions of ordinary differential equation (2.3).

Firstly, by integrating the second and third equation of (2.3) with respect to x from $-\infty$ to x, we obtain that

$$\phi_1 = \frac{\gamma k_5}{2(k_3 - c\theta)}\varphi^2, \quad \phi_2 = \frac{\gamma k_6}{2(k_4 + c\theta)}\varphi^2, \tag{2.4}$$

in the above process of calculation, we use the boundary conditions: $\varphi,\ \phi_1$ and $\phi_2 \to 0$ as $|x| \to \infty$.

By calculating and simplifying the first equation of (2.3) with the help of expressions (2.4), we have

$$f = \frac{c}{2w} \tag{2.5}$$

and

$$\varphi'' + \frac{c^2 + 4\lambda w}{4w^2}\varphi - \frac{m}{4w^2}\varphi^3 = 0, \tag{2.6}$$

where

$$m = \frac{2\gamma^2 w k_1 k_5}{k_3 - \theta c} - \frac{2\gamma^2 w k_2 k_6}{k_4 + \theta c} + 4w\gamma q. \tag{2.7}$$

Equation (2.6) is a second order ordinary differential equation, we can convert it into the equivalent of planar dynamical system, by taking advantage of the theories of qualitative analysis and method provided by Zhang [20, 21], we can investigate many exact traveling wave solutions. In here, we mainly consider the orbital stability of exact bell solitary wave solutions. By using the undeterminded coefficient method which has been presented by Zhang [22], we may assume that equation (2.6) have solutions in the following form

$$\varphi(x) = \frac{A\mathrm{sech}^2(\frac{r}{2}x)}{4 + B\mathrm{sech}^2(\frac{r}{2}x)} + D, \tag{2.8}$$

where A, B, D and r are undeterminded real parameters.

Substituting (2.8) and $\varphi''(x)$ into (2.6), we obtain the following algebraic equations with respect to A, B, D and r:

$$\begin{cases} -\frac{m}{4w^2}D^3 + \frac{c^2+4w\lambda}{4w^2}D = 0, \\ r^2 - \frac{3m}{4w^2}D^2 + \frac{c^2+4w\lambda}{4w^2} = 0, \\ -\frac{3m}{4w^2}DA - 3r^2(2+B) = 0, \\ -\frac{m}{4w^2}A^2 - 8r^2 - DA(2+B)\frac{3m}{4w^2} - r^2(2+B)^2 = 0. \end{cases} \tag{2.9}$$

We solve equations (2.9) by using Maple 10.0 and obtain that its solutions are

$$r = \pm\sqrt{-\frac{c^2+4w\lambda}{4w^2}}, \quad A = \pm\sqrt{\frac{8(c^2+4w\lambda)}{m}}, \quad B = -2, \quad D = 0, \tag{2.10}$$

the solutions (2.10) of A, B, D and r are meaning in the case of $m < 0,\ c^2 + 4w\lambda < 0$.

We simplify expression (2.8) by using the solutions (2.10), and obtain the following solutions of equation (2.6):

$$\varphi(x) = \pm\sqrt{\frac{c^2+4w\lambda}{m}}\mathrm{sech}\sqrt{-\frac{c^2+4w\lambda}{4w^2}}(x), \tag{2.11}$$

In the above process of calculation, we use the property of $\mathrm{sech}(x) = \mathrm{sech}(-x)$. On the basis of expressions (2.4) and (2.5), along with solutions (2.11) of equation (2.6), the exact solutions of ordinary differential equation (2.3) are:

$$\begin{cases} \varphi(x) = \pm\sqrt{\frac{c^2+4w\lambda}{m}}\mathrm{sech}\sqrt{-\frac{c^2+4w\lambda}{4w^2}}(x), \\ \phi_1(x) = \frac{\gamma k_5(c^2+4w\lambda)}{(k_3-c\theta)m}\mathrm{sech}^2\sqrt{-\frac{c^2+4w\lambda}{4w^2}}(x), \\ \phi_2(x) = \frac{\gamma k_6(c^2+4w\lambda)}{(k_4+c\theta)m}\mathrm{sech}^2\sqrt{-\frac{c^2+4w\lambda}{4w^2}}(x), \end{cases} \tag{2.12}$$

where $m<0,\ c^2+4w\lambda<0$.

In according to the relationship between the exact solitary wave solutions of equation (2.1) and exact solutions of ordinary differential equation (2.3), We have result:

Theorem 2.1. *If parameters m, c, λ, w satisfy $m<0$, $c^2+4w\lambda<0$, then equation (2.1) has the solitary wave solution with the form (2.2), where f and φ, ϕ_1, ϕ_2 denoted by (2.5) and (2.12) respectively.*

In the following content, we study the orbital stability of solitary wave solutions (2.2) of equation (2.1) for any $\theta>0$ by employing the method represented by Guo and Chen [23].

2.2. The orbital stability of solitary wave solutions

Setting $B=B_1+iB_2$ and rewriting equation (2.1), we have

$$\begin{cases} B_{1t}=-wB_{2xx}+\gamma k_1\psi_1B_2-\gamma k_2\psi_2B_2+\gamma q(B_1^2+B_2^2)B_2,\\ B_{2t}=wB_{1xx}-\gamma k_1\psi_1B_1+\gamma k_2\psi_2B_1-\gamma q(B_1^2+B_2^2)B_1,\\ \psi_{1t}=\frac{1}{\theta}[-k_3\psi_1+\frac{\gamma}{2}k_5(B_1^2+B_2^2)]_x,\\ \psi_{2t}=\frac{1}{\theta}[k_4\psi_2-\frac{\gamma}{2}k_6(B_1^2+B_2^2)]_x. \end{cases} \tag{2.13}$$

Assuming that

$$\vec{u}=(B_1,\,B_2,\,\psi_1,\,\psi_2)^T, \tag{2.14}$$

where $.^T$ denotes the column vector. Let $X=H^1(R)\times H^1(R)\times L^2(R)\times L^2(R)$ with the inner product

$$(\vec{f},\vec{g})=\int_R(f_1g_1+f_{1x}g_{1x}+f_2g_2+f_{2x}g_{2x}+f_3g_3+f_4g_4)dx, \tag{2.15}$$

where $\vec{f}=(f_1,\,f_2,\,f_3,\,f_4)^T$, $\vec{g}=(g_1,\,g_2,\,g_3,\,g_4)^T$ in X. The dual space $X^*=H^{-1}(R)\times H^{-1}(R)\times L^2(R)\times L^2(R)$, a natural isomorphism $I:X\to X^*$ denoted by

$$\left\langle I\vec{f},\vec{g}\right\rangle=\left(\vec{f},\vec{g}\right), \tag{2.16}$$

with $\left\langle \vec{f},\vec{g}\right\rangle=\int_R(\sum_{i=1}^4 f_ig_i)dx$.

According to (2.15) and (2.16), we have

$$I=\begin{pmatrix} 1-\partial_{xx} & 0 & 0 & 0\\ 0 & 1-\partial_{xx} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{2.17}$$

Let T_1, T_2 one-parameter group of unitary operators on X, and

$$T_1(s_1)\vec{u}(\cdot)=\vec{u}(\cdot - s_1),\qquad for\quad \vec{u}(\cdot)\in X,\ s_1\in R, \tag{2.18}$$

$$T_2(s_2)\vec{u}(\cdot)=\begin{pmatrix} \cos s_2 & \sin s_2 & 0 & 0\\ -\sin s_2 & \cos s_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} B_1\\ B_2\\ \psi_1\\ \psi_2\end{pmatrix}, \tag{2.19}$$

$$for\quad \vec{u}(\cdot)\in X,\ s_2\in R$$

We derivative (2.18) and (2.19) with respect to s_1, s_2, respectively, and have

(2.20)

$$T_1'(0)=\begin{pmatrix} -\partial_x & 0 & 0 & 0\\ 0 & -\partial_x & 0 & 0\\ 0 & 0 & -\partial_x & 0\\ 0 & 0 & 0 & -\partial_x\end{pmatrix},\qquad T_2'(0)=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Letting

(2.21)

$$\vec{\Phi}_{\lambda,c}(x)=\left(\varphi_{\lambda,c}(x)\cos\frac{c}{2w}x,\quad \varphi_{\lambda,c}(x)\sin\frac{c}{2w}x,\quad \phi_{1\lambda,c}(x),\quad \phi_{2\lambda,c}(x)\right).$$

In the light of definitions (2.18) and (2.19) of T_1, T_2, comparing (2.21) with (2.12), we find that $T_1(ct)T_2(\lambda t)\vec{\Phi}_{\lambda,c}(x)$ is the solitary wave solution of equation (2.13). Next we need the following definition of orbital stability of solitary wave solutions:

Definition 2.2. [17] The solitary wave solution $T_1(ct)T_2(\lambda t)\vec{\Phi}_{\lambda,c}(x)$ is orbitally stable: if for all $\varepsilon>0$, there exists $\delta>0$, with the following property:

if $\| \vec{u}_0 - \vec{\Phi} \|_X < \delta$ and $\vec{u}(t)$ is a solution of (2.13) in some interval $[0, \quad t_0)$ with $\vec{u}(0) = \vec{u}_0$, then $\vec{u}(t)$ can be continued to a solution in $0 \le t < +\infty$, and

$$\sup_{0<t<+\infty} \inf_{s_1 \in R} \inf_{s_2 \in R} \| \vec{u}(t) - T_1(s_1)T_2(s_2)\vec{\Phi} \|_X < \varepsilon.$$

Otherwise, $T_1(ct)T_2(\lambda t)\vec{\Phi}_{\lambda,c}(x)$ is called orbitally unstable.

The above definition can also be found in [22]. We can rewrite equation (2.13) as the following Hamiltonian system with the form

$$\frac{d\vec{u}}{dt} = J'E(\vec{u}), \tag{2.22}$$

where $\vec{u}$ denoted by (2.14),

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_5}{k_1\theta}\partial_x & 0 \\ 0 & 0 & 0 & \frac{k_6}{k_2\theta}\partial_x \end{pmatrix}, \tag{2.23}$$

and

$$\begin{aligned} E(\vec{u}) \quad &= \int_R \left[\tfrac{w}{2}(B_{1x}^2 + B_{2x}^2) + \tfrac{\gamma}{2}k_1\psi_1(B_1^2 + B_2^2) - \tfrac{\gamma}{2}k_2\psi_2(B_1^2 + B_2^2) \right. \\ &\left. + \tfrac{q\gamma}{4}(B_1^2 + B_2^2)^2 - \tfrac{k_1k_3}{2k_5}\psi_1^2 + \tfrac{k_2k_4}{2k_6}\psi_2^2 \right] dx, \end{aligned} \tag{2.24}$$

$E'(\vec{u})$ is the Fréchet derivative of $E(\vec{u})$.

E is invariant under T_1, T_2, that is

$$E(T_1(s_1)T_2(s_2)\vec{u}) = E(\vec{u}), \qquad \text{for} \quad s_1, \; s_2 \in R, \vec{u} \in X \tag{2.25}$$

and for $t \in R$, $E(\vec{u}(t)) = E(\vec{u}(0))$ under the flow $\vec{u}$ of equation (2.13).

Let be

$$F_1 = \begin{pmatrix} 0 & \partial_x & 0 & 0 \\ -\partial_x & 0 & 0 & 0 \\ 0 & 0 & -\frac{\theta k_1}{k_5} & 0 \\ 0 & 0 & 0 & -\frac{\theta k_2}{k_6} \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

By (2.20) and (2.23), we have

$$T_1'(0) = JF_1, \qquad T_2'(0) = JF_2.$$

Introducing the functional

$$Q_1(\vec{u}) = \frac{1}{2}\langle F_1\vec{u},\, \vec{u}\rangle = \frac{1}{2}\int_R \left[B_1B_{2x} - B_2B_{1x} - \frac{\theta k_1}{k_5}\psi_1^2 - \frac{\theta k_2}{k_6}\psi_2^2\right] dx \tag{2.26}$$

and

$$Q_2(\vec{u}) = \frac{1}{2}\langle F_2\vec{u},\, \vec{u}\rangle = \frac{1}{2}\int_R \left(B_1^2 + B_2^2\right) dx. \tag{2.27}$$

From (2.18) and (2.19), we obtain that Q_1, Q_2 are also invariant under T_1, T_2,

$$\begin{array}{l} Q_1(T_1(s_1)T_2(s_2)\vec{u}) = Q_1(\vec{u}), \\ Q_2(T_1(s_1)T_2(s_2)\vec{u}) = Q_2(\vec{u}), \end{array} \text{ for } \quad s_1,\; s_2 \in R,\, \vec{u} \in X \tag{2.28}$$

Q_1, Q_2 are conserved under the flow $\vec{u}$ of equation (2.13), namely

$$Q_1(\vec{u}(t)) = Q_1(\vec{u}(0)), \quad Q_2(\vec{u}(t)) = Q_2(\vec{u}(0)),\, t \in R \tag{2.29}$$

By (2.24), we obtain the Fréchet derivatives $E'(\vec{u})$ of $E(\vec{u})$, that is

$$E'(\vec{u}) = \begin{pmatrix} -wB_{1xx} + \gamma k_1\psi_1 B_1 - \gamma k_2\psi_2 B_1 + \gamma q B_1(B_1^2 + B_2^2) \\ -wB_{2xx} + \gamma k_1\psi_1 B_2 - \gamma k_2\psi_2 B_2 + \gamma q B_2(B_1^2 + B_2^2) \\ \frac{\gamma}{2}k_1(B_1^2 + B_2^2) - \frac{k_1k_3}{k_5}\psi_1 \\ -\frac{\gamma}{2}k_2(B_1^2 + B_2^2) + \frac{k_2k_4}{k_6}\psi_2 \end{pmatrix}, \tag{2.30}$$

According to (2.26) and (2.27), we have the Fréchet derivatives $Q'_{1,2}(\vec{u})$ of $Q_{1,2}(\vec{u})$:

$$Q'_1(\vec{u}) = \begin{pmatrix} B_{2x} \\ -B_{1x} \\ -\frac{\theta k_1}{k_5}\psi_1 \\ -\frac{\theta k_2}{k_6}\psi_2 \end{pmatrix}, \qquad Q'_2(\vec{u}) = \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix}. \tag{2.31}$$

By (2.13), (2.21), (2.30) and (2.31), after simply calculation, we get

$$E'(\vec{\Phi}_{\lambda,c}) - cQ'_1(\vec{\Phi}_{\lambda,c}) - \lambda Q'_2(\vec{\Phi}_{\lambda,c}) = 0. \tag{2.32}$$

In order to discuss the stability, we need the linear operator $H_{\lambda,c}$ from X to X^*, denoted by

$$H_{\lambda,c} = E''(\vec{\Phi}_{\lambda,c}) - cQ_1''(\vec{\Phi}_{\lambda,c}) - \lambda Q_2''(\vec{\Phi}_{\lambda,c}). \tag{2.33}$$

Noting that $\varphi = \varphi_{\lambda,c}$, $\phi_1 = \phi_{1\lambda,c}$, $\phi_2 = \phi_{2\lambda,c}$, then

(2.34)

$$H_{\lambda,c} = \begin{pmatrix} L + \gamma q\varphi^2 \cos\frac{c}{w}x & \gamma q\varphi^2 \sin\frac{c}{w}x - c\partial_x & \gamma k_1\varphi\cos\frac{c}{2w}x & -\gamma k_2\varphi\cos\frac{c}{2w}x \\ \gamma q\varphi^2 \sin\frac{c}{w}x + c\partial_x & L - \gamma q\varphi^2\cos\frac{c}{w}x & \gamma k_1\varphi\sin\frac{c}{2w}x & -\gamma k_2\varphi\sin\frac{c}{2w}x \\ \gamma k_1\varphi\cos\frac{c}{2w}x & \gamma k_1\varphi\sin\frac{c}{2w}x & \frac{k_1}{k_5}(c\theta - k_3) & 0 \\ -\gamma k_2\varphi\cos\frac{c}{2w}x & -\gamma k_2\varphi\sin\frac{c}{2w}x & 0 & \frac{k_2}{k_6}(c\theta + k_4) \end{pmatrix},$$

where $L = -w\partial_{xx} + \gamma k_1\phi_1 - \gamma k_2\phi_2 - \lambda + 2\gamma q\varphi^2$.

It is easy to test that $H_{\lambda,c}$ is a self-adjoint operator in the sense that $H^*_{\lambda,c} = H_{\lambda,c}$. This means that $I^{-1}H_{\lambda,c}$ is a bounded operator. The spectrum of $H_{\lambda,c}$ consists of the real numbers μ such that $H_{\lambda,c} - \mu I$ is irreversible.

By (2.20), (2.21), (2.34) and (2.32), we have

$$H_{\lambda,c}(T_1'(0)\vec{\Phi}_{\lambda,c}(x)) = 0, \qquad H_{\lambda,c}(T_2'(0)\vec{\Phi}_{\lambda,c}(x)) = 0, \tag{2.35}$$

so we claim that $\lambda = 0$ belongs to the spectrum of $H_{\lambda,c}$ and $T_1'(0)\vec{\Phi}_{\lambda,c}(x)$, $T_2'(0)\vec{\Phi}_{\lambda,c}(x)$ are the corresponding eigenvectors with eigenvalue zero.

Denoted by

$$Z = \left\{ h_1T_1'(0)\vec{\Phi}_{\lambda,c}(x) + h_2T_2'(0)\vec{\Phi}_{\lambda,c}(x) \mid h_1, h_2 \in R \right\}. \tag{2.36}$$

It follows (2.35) that Z is included in the kernel of $H_{\lambda,c}$.

Lemma 2.3. *(Spectral decomposition of $H_{\lambda,c}$) For any $\theta > 0$, if the wave velocity c satisfies $\frac{-\sqrt{\beta - v}}{\theta} < c < \frac{\sqrt{\beta-v}}{\theta}$, then the space X is decomposed as a direct sum*

$$X = N + Z + P, \tag{2.37}$$

where Z is defined by (2.36), N is a finite-dimensional subspace such that

$$\langle H_{\lambda,c}\vec{u}, \vec{u}\rangle < 0, \qquad \text{for} \quad 0 \neq \vec{u} \in N, \tag{2.38}$$

and P is a closed subspace such that

$$\langle H_{\lambda,c}\vec{u}, \vec{u}\rangle \geq \delta \parallel \vec{u} \parallel_X^2, \qquad \text{for} \quad \vec{u} \in P, \tag{2.39}$$

with some constant $\delta > 0$ which is independent of $\vec{u}$.

In Section 3, we prove Lemma 2.3. Let $d(\lambda, c) : R \times R \to R$ be

$$d(\lambda, c) = E(\vec{\Phi}_{\lambda,c}) - cQ_1(\vec{\Phi}_{\lambda,c}) - \lambda Q_2(\vec{\Phi}_{\lambda,c}) \tag{2.40}$$

and $d''(\lambda, c)$ be the Hessian matrix of function $d(\lambda, c)$.

Lemma 2.4. *[15] For any $(B_{10}, B_{20}, \phi_{10}, \phi_{20},) \in H^{l+\frac{1}{2}}(R) \times H^{l+\frac{1}{2}}(R) \times H^l(R) \times H^l(R)$, $l > \frac{1}{2}$, there exists a unique function $B : [0, T] \times R \to C$ satisfying (2.13) such that $\vec{u} \in X$ for all $T > 0$.*

Lemma 2.5. *For $w > 0$, $m < 0$, if parameters c, λ, w satisfy $c^2 + 4w\lambda < 0$, then $p(d'') = n(H_{\lambda,c})$, where $n(H_{\lambda,c})$ denotes the number of negative eigenvalues of $H_{\lambda,c}$, $p(d'')$ denotes the number of positive eigenvalues for Hessian matrix d'' at (λ, c).*

The proof of $p(d'') = n(H_{\lambda,c})$ has been given in Section 3.

We give the main results in the following theorem.

Theorem 2.6. *Under the condition of Theorem 2.1, if for any $\theta > 0$, the wave velocity c satisfying $\frac{-\sqrt{\beta - v}}{\theta} < c < \frac{\sqrt{\beta - v}}{\theta}$, and $w > 0$, then the solitary wave solutions $T_1(ct)T_2(\lambda t)\vec{\Phi}_{\lambda,c}(x)$ of (2.13) with the expression (2.2), (2.5) and (2.12) are orbitally stable.*

Proof. According to Lemma 2.3 and $p(d'') = n(H_{\lambda,c})$, along with theorem 4.1 of [18], we know that the result is right. □

3. The proof of lemma

In this section, we prove the stability of solitary wave solutions of equation (2.13). The main intermediate step is to determine the number of negative eigenvalues of the linear operator $H_{\lambda,c}$ and the number of positive eigenvalues of the Hessian of scalar function $d(\lambda, c)$.

3.1. The proof of Lemma 2.3

In this content, we consider the spectrum of the linear operator (2.33). Firstly, let

$$\vec{\psi} = \begin{pmatrix} \cos\frac{c}{2w}x & -\sin\frac{c}{2w}x & 0 & 0 \\ \sin\frac{c}{2w}x & \cos\frac{c}{2w}x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \tag{3.1}$$

and

$$\vec{y} = (y_1,\, y_2,\, y_3,\, y_4)^T \in X.$$

Obviously, $\left\langle \vec{\psi},\, \vec{\psi} \right\rangle = \langle \vec{y},\, \vec{y} \rangle$, by (2.34) we have

(3.2)

$$\left\langle H_{\lambda,c}\vec{\psi},\, \vec{\psi} \right\rangle = \langle L_1 y_1,\, y_1 \rangle + \langle L_2 y_2,\, y_2 \rangle \; + \int_R \tfrac{k_1}{k_5}(\theta c - k_3) \left[y_3 + \tfrac{\gamma k_5}{\theta c - k_3} \varphi y_1 \right]^2 dx + \int_R \tfrac{k_2}{k_6}(\theta c + k_4) \left[y_4 - \tfrac{\gamma k_6}{\theta c + k_4} \varphi y_1 \right]^2 dx,$$

where

$$L_1 = -w\partial_{xx} - \frac{c^2 + 4\lambda w}{4w} + \frac{3m}{4w}\varphi^2, \qquad L_2 = -w\partial_{xx} - \frac{c^2 + 4\lambda w}{4w} + \frac{m}{4w}\varphi^2. \tag{3.3}$$

By the Wely theorem and the solitary wave solutions (2.12), the essential spectrum of L_1, L_2, denoted by

$$\sigma_{\mathrm{ess}}(L_1) = \sigma_{\mathrm{ess}}(L_2) = \left[-\frac{c^2 + 4\lambda w}{4w},\, +\infty \right). \tag{3.4}$$

By (2.3), (2.5), (2.7) and (2.32),(3.3), we get

$$L_1 \varphi_x = 0, \qquad L_2 \varphi = 0. \tag{3.5}$$

By the Strum-Liouville theorem and the fact that φ_x has simple zero at $x = 0$, we can deduce that zero is the second eigenvalue of L_1 which only owns one negative eigenvalue denoted by $-\sigma^2$, the corresponding eigenvector denoted by χ_1, that is

$$L_1 \chi_1 = -\sigma^2 \chi_1, \qquad \langle \chi_1,\, \chi_1 \rangle = 1. \tag{3.6}$$

Zero is also the simple eigenvalue of L_2. From (3.3),(3.4) and (3.5), we obtain the following two lemmas.

Lemma 3.1. *For any real function $y_1 \in H^1(R)$, satisfying $\langle y_1,\, \chi_1 \rangle = \langle y_1,\, \varphi_x \rangle = 0$, then there exists a positive constant δ_1 such that*

$$\langle Ł_1 y_1,\, y_1 \rangle \geq \delta_1 \parallel y_1 \parallel_{H^1}^2 . \tag{3.7}$$

Lemma 3.2. *For any real function $y_2 \in H^1(R)$, satisfying $\langle y_2,\, \varphi \rangle = 0$, there exists a positive constant δ_2 such that*

$$\langle Ł_2 y_2,\, y_2 \rangle \geq \delta_2 \parallel y_2 \parallel_{H^1}^2 . \tag{3.8}$$

Let be

$$y_1^- = \chi_1, \quad y_2^- = 0, \quad y_3^- = -\tfrac{\gamma k_5}{\theta c - k_3}\varphi\chi_1, \quad y_4^- = \tfrac{\gamma k_6}{\theta c + k_4}\varphi\chi_1, \qquad (3.9)$$
$$\vec{\psi}^- = (y_1^-,\, y_2^-,\, y_3^-,\, y_4^-)^T.$$

By using (3.2), we have

$$\left\langle H_{\lambda,c}\vec{\psi}^-,\, \vec{\psi}^-\right\rangle = \langle L_1\chi_1,\, \chi_1\rangle = -\sigma^2\langle\chi_1,\, \chi_1\rangle < 0. \qquad (3.10)$$

Assuming that

(3.11)
$$Z = \left\{h_1\vec{\psi}_{01} + h_2\vec{\psi}_{02} \mid h_1, h_2 \in R\right\}, \quad N = \left\{h\vec{\psi} \mid h \in R\right\},$$
$$P = \left\{\vec{p} \in X \mid \vec{p} = (p_1,\, p_2,\, p_3,\, p_4)^T\right\},\ \langle p_1,\, \chi_1\rangle = \langle p_1,\, \varphi_x\rangle = \langle p_1,\, \varphi\rangle = 0.$$

On the base of (3.10), (3.11), (2.38) is correct.

For $\vec{u} = (y_1,\, y_2,\, y_3,\, y_4)^T \in X$, let be $a = \langle y_1,\, \chi_1\rangle$, $b_1 = \frac{\langle\varphi_x, y_1\rangle}{\langle\varphi_x, \varphi_x\rangle}$, $b_2 = \frac{\langle\varphi, y_2\rangle}{\langle\varphi, \varphi\rangle}$, then $\vec{u}$ can only expressed as

$$\vec{u} = a\vec{\psi}^- + b_1\vec{\psi}_{01} + b_2\vec{\psi}_{02} + \vec{p}, \qquad \vec{p} \in P. \qquad (3.12)$$

So the decompose expression (2.37) for direct sum of space X holds.

Lemma 3.3. *For any $\theta > 0$, the wave velocity c satisfying $\frac{-\sqrt{\beta - v}}{\theta} < c < \frac{\sqrt{\beta - v}}{\theta}$, there exists constant $\delta_3 > 0$ such that*

$$\langle H_{\lambda,c}\vec{p},\, \vec{p}\rangle \geq \delta_3 \parallel \vec{p} \parallel_X, \qquad (3.13)$$

where $\delta_3 > 0$ is independent of $\vec{p}$, $\vec{p} \in P$ denoted by (3.11).

Proof. For $\vec{p} \in P$, by (3.2) and (3.11), we have

(3.14)
$$\langle H_{\lambda,c}\vec{p},\, \vec{p}\rangle = \delta_1 \parallel p_1 \parallel_{H_1}^2 + \delta_2 \parallel p_2 \parallel_{H_1}^2 \quad + \tfrac{k_1}{k_5}(\theta c - k_3)\int_R \left[p_3 + \tfrac{\gamma k_5}{\theta c - k_3}\varphi p_1\right]^2 dx$$
$$+ \tfrac{k_2}{k_6}(\theta c + k_4)\int_R \left[p_4 - \tfrac{\gamma k_6}{\theta c + k_4}\varphi p_1\right]^2 dx,$$

By $\frac{k_1}{k_5} < 0$, $\frac{k_2}{k_6} < 0$ and $\frac{-\sqrt{\beta - v}}{\theta} < c < \frac{\sqrt{\beta - v}}{\theta}$, then $\frac{k_1}{k_5}(\theta c - k_3) > 0$, $\frac{k_2}{k_6}(\theta c + k_4) > 0$. We discuss that the sign of expression (3.14) in the following content:

Case 1 if $\parallel p_3 \parallel_{L^2}^2 \geq \frac{4\gamma^2 k_5^2 M}{(\theta c - k_3)^2 k_1^2} \parallel p_1 \parallel_{L^2}^2$, then

(3.15)
$$\frac{k_1}{k_5}(\theta c-k_3)\int_R\left[p_3+\frac{\gamma k_5}{\theta c-k_3}\varphi p_1\right]^2dx \geq \frac{k_1}{k_5}(\theta c-k_3)\parallel p_3\parallel_{L^2}^2-\frac{2\gamma^2k_5M}{(\theta c-k_3)k_1}\parallel p_1\parallel_{L^2}^2$$
$$\geq \frac{k_1}{k_5}(\theta c-k_3)\parallel p_3\parallel_{L^2}^2-\frac{2\gamma^2k_5M}{(\theta c-k_3)k_1}\frac{(\theta c-k_3)^2k_1^2}{4\gamma^2k_5^2M}\parallel p_3\parallel_{L^2}^2=\frac{k_1}{2k_5}(\theta c-k_3)\parallel p_3\parallel_{L^2}^2.$$

Case 2 if $\parallel p_3\parallel_{L^2}^2\leq \frac{4\gamma^2k_5^2M}{(\theta c-k_3)^2k_1^2}\parallel p_1\parallel_{L^2}^2$, then

$$\delta_1\parallel p_1\parallel_{H^1}^2\geq \frac{\delta_1}{2}\parallel p_1\parallel_{H^1}^2+\frac{\delta_1}{2}\frac{(\theta c-k_3)^2k_1^2}{4\gamma^2k_5^2M}\parallel p_3\parallel_{L^2}^2. \tag{3.16}$$

Case 3 if $\parallel p_4\parallel_{L^2}^2\geq \frac{4\gamma^2k_6^2M}{(\theta c+k_4)^2k_2^2}\parallel p_1\parallel_{L^2}^2$, then

(3.17)
$$\frac{k_2}{k_6}(\theta c+k_4)\int_R\left[p_4-\frac{\gamma k_6}{\theta c+k_4}\varphi p_1\right]^2dx \geq \frac{k_2}{k_6}(\theta c+k_4)\parallel p_4\parallel_{L^2}^2-\frac{2\gamma^2k_6M}{(\theta c+k_4)k_2}\parallel p_1\parallel_{L^2}^2$$
$$\geq \frac{k_2}{k_6}(\theta c+k_4)\parallel p_4\parallel_{L^2}^2-\frac{2\gamma^2k_6M}{(\theta c+k_4)k_2}\frac{(\theta c+k_4)^2k_2^2}{4\gamma^2k_6^2M}\parallel p_4\parallel_{L^2}^2=\frac{k_2}{2k_6}(\theta c+k_4)\parallel p_4\parallel_{L^2}^2.$$

Case 4 if $\parallel p_4\parallel_{L^2}^2\leq \frac{4\gamma^2k_6^2M}{(\theta c+k_4)^2k_2^2}\parallel p_1\parallel_{L^2}^2$, then

$$\delta_1\parallel p_1\parallel_{H^1}^2\geq \frac{\delta_1}{2}\parallel p_1\parallel_{H^1}^2+\frac{\delta_1}{2}\frac{(\theta c+k_4)^2k_2^2}{4\gamma^2k_6^2M}\parallel p_4\parallel_{L^2}^2, \tag{3.18}$$

where $M=|\varphi|_\infty^2$. Let

$$\begin{aligned}\delta_1'&=\min\left\{\frac{k_1}{2k_5}(\theta c-k_3),\ \frac{\delta_1}{2}\frac{(\theta c-k_3)^2k_1^2}{4\gamma^2k_5^2M}\right\},\\ \delta_2'&=\min\left\{\frac{k_2}{2k_6}(\theta c+k_4),\ \frac{\delta_1}{2}\frac{(\theta c+k_4)^2k_2^2}{4\gamma^2k_6^2M}\right\}.\end{aligned} \tag{3.19}$$

It follows (3.14)–(3.19) that

$$\langle H_{\lambda,c}\vec{p},\vec{p}\rangle\geq\frac{\delta_1}{2}\parallel p_1\parallel_{H_1}^2+\delta_2\parallel p_2\parallel_{H_1}^2+\delta_1'\parallel p_3\parallel_{L^2}^2+\delta_2'\parallel p_4\parallel_{L^2}^2,$$

then

$$\langle H_{\lambda,c}\vec{p},\vec{p}\rangle\geq\delta_3\parallel\vec{p}\parallel_X, \tag{3.20}$$

where $\delta_3=\min\left\{\frac{\delta_1}{2},\ \delta_2,\ \delta_1',\ \delta_2'\right\}$ be independent of $\vec{p}$.

Using (3.10), (3.12) and (3.20), we establish the proof of Lemma 2.3 and $n(H_{\lambda,c})=1$. □

3.2. The proof of Lemma 2.5

By (2.40), we have

$$d_\lambda = -Q_2(\vec{\Phi}_{\lambda,c}), \quad d_c = -Q_1(\vec{\Phi}_{\lambda,c}). \tag{3.21}$$

According to (2.12), (2.21), (2.26), (2.27) and (3.21), after the simple calculation, we obtain

(3.22)

$$d_\lambda = -\frac{1}{2}\int_R \left(B_1^2 + B_2^2\right) dx = -\frac{c^2+4\lambda w}{2m}\int_R sech^2\sqrt{\frac{-c^2-4\lambda w}{4w^2}}x dx = \frac{y_0}{m} < 0$$

and

$$d_c = \frac{cy_0}{2wm} + \frac{y_0\gamma^2\theta k_1k_5(-c^2-4w\lambda)}{3m^2(k_3-\theta c)^2} + \frac{y_0\gamma^2\theta k_2k_6(-c^2-4w\lambda)}{3m^2(k_4+\theta c)^2}, \tag{3.23}$$

where $y_0 = \sqrt{-4w^2(c^2+4\lambda w)}$. Further more, by (3.22) and $w > 0$ we have

$$d_{\lambda\lambda} = -\frac{8w^3}{my_0} > 0 \tag{3.24}$$

and

$$d_{\lambda c} = -\frac{4cw^2}{my_0} - \frac{2\bar{y}y_0}{m^2}, \tag{3.25}$$

where $\bar{y} = \frac{\theta\gamma^2wk_1k_5}{(k_3-\theta c)^2} + \frac{\theta\gamma^2wk_2k_6}{(k_4+\theta c)^2}$.

By (3.23) we have

$$d_{c\lambda} = d_{\lambda c} \tag{3.26}$$

and

$$\begin{aligned} d_{cc} &= \frac{y_0}{2mw} - \frac{2wc^2}{my_0} - \frac{cy_0\bar{y}}{wm^2} - \frac{c\gamma^2\theta k_1k_5y_0}{m^2(k_3-c\theta)^2} + \frac{4\gamma^2\theta^2k_1k_5y_0(-c^2-4w\lambda)}{3m^2(k_3-c\theta)^2} \\ &\quad - \frac{4\gamma^2\theta k_1k_5y_0(-c^2-4w\lambda)}{3m^3(k_3-c\theta)^2}\bar{y} - \frac{c\gamma^2\theta k_2k_6y_0}{m^2(k_4+c\theta)^2} - \frac{4\gamma^2\theta^2k_2k_6y_0(-c^2-4w\lambda)}{3m^2(k_4+c\theta)^2} \\ &\quad - \frac{4\gamma^2\theta k_2k_6y_0(-c^2-4w\lambda)}{3m^3(k_4+c\theta)^2}\bar{y}. \end{aligned} \tag{3.27}$$

By (3.24)–(3.27), we get

$$
\begin{aligned}
\det(d'') &= d_{\lambda\lambda}d_{cc} - d_{\lambda c}d_{c\lambda} \\
&= -\frac{4w^2}{m^2} - \frac{16w^2(-c^2-4w\lambda)}{3m^4}\bar{y}^2 - \frac{32w^3\theta^2\gamma^2k_1k_5(-c^2-4w\lambda)}{3m^3(k_3-c\theta)^2} + \frac{32w^3\theta^2\gamma^2k_2k_6(-c^2-4w\lambda)}{3m^3(k_4+c\theta)^2},
\end{aligned} \tag{3.28}
$$

According to $m < 0$, $k_1k_5 < 0$, $k_2k_6 > 0$, $-c^2 - 4w\lambda > 0$ and (3.28), after simple analyzing, we have

$$\det(d'') < 0. \tag{3.29}$$

From (3.22), (3.24) and (3.29), we obtain that d'' only owns one positive eigenvalue and one negative eigenvalue. by using $n(H_{\lambda,c}) = 1$ which has been proved in Section 3.1, we get

$$p(d'') = n(H_{\lambda,c}) = 1. \tag{3.30}$$

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XiaoHua Liu
School of Data-Science and Information-Engineering
Guizhou Minzu University
Guizhou
Guiyang 550025
P.R. China
E-mail: lxhjkkl@yeah.net