Pure and Applied Mathematics Quarterly
Volume 13, Number 4, 639–692, 2017

Bloch's conjecture for certain hyperkähler fourfolds

Robert Laterveer

Abstract: On a hyperkähler fourfold X, Bloch's conjecture predicts that any involution acts trivially on the deepest level of the Bloch–Beilinson filtration on the Chow group of 0–cycles. We prove a version of Bloch's conjecture when X is the Hilbert scheme of 2 points on a generic quartic in $\mathbb{P}^3$, and the involution is the non–natural, non–symplectic involution on X constructed by Beauville. This has interesting consequences for the Chow groups of the quotient.
Keywords: Algebraic cycles, Chow groups, motives, Bloch's conjecture, Bloch–Beilinson filtration, hyperkähler varieties, K3 surfaces, Hilbert schemes, non–symplectic involution, multiplicative Chow–Künneth decomposition, "spread" of algebraic cycles in a family.

1. Introduction

For a smooth projective variety X over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow group of codimension i algebraic cycles modulo rational equivalence with $\mathbb{Q}$–coefficients. Let $A^i_{hom}(X)$ and $A^i_{AJ}(X) \subset A^i(X)$ denote the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles. Notoriously, Chow groups of codimension $i > 1$ cycles are still but poorly understood. To cite one prominent example, there is Bloch's conjecture (which even for surfaces of geometric genus 0 remains stubbornly conjectural, reminiscent of a castle lying under siege for many years but showing no intention of being ready to hoist the white flag of complete surrender):

Conjecture 1.1 (Bloch [6]). *Let X be a smooth projective variety of dimension n. Let $\Gamma \in A^n(X \times X)$ be a correspondence such that*

$$\Gamma^* = 0\colon \quad H^p(X, \mathcal{O}_X) \ \to\ H^p(X, \mathcal{O}_X) \quad \textit{for all } p > 0\ .$$

Then

$$\Gamma^* = 0\colon \quad A^n_{hom}(X) \ \to\ A^n_{hom}(X)\ .$$

Received September 29, 2017.
2010 Mathematics Subject Classification: 14C15, 14C25, 14C30.

One could also state a variant of Bloch's conjecture for codimension 2 cycles:

Conjecture 1.2. *Let X be a smooth projective variety of dimension n. Let $\Gamma \in A^n(X \times X)$ be a correspondence such that*

$$\Gamma^* = 0\colon \quad H^2(X, \mathcal{O}_X) \ \to \ H^2(X, \mathcal{O}_X)\ .$$

Then

$$\Gamma^* = 0\colon \quad A^2_{AJ}(X) \ \to \ A^2_{AJ}(X)\ .$$

Now, let us restrict focus to the realm of hyperkähler varieties (by which we mean: projective irreducible holomorphic symplectic manifolds [2]). In this case, $H^*(X, \mathcal{O}_X)$ is generated by $H^2(X, \mathcal{O}_X)$ which is of dimension 1, and so Conjecture 1.1 takes on a particularly appealing form:

Conjecture 1.3. *Let X be a hyperkähler variety of dimension n. Let $\omega \in H^{2,0}(X)$ be a holomorphic 2–form. Let $\Gamma \in A^n(X \times X)$ be a correspondence such that*

$$\Gamma^*(\omega^r) = 0 \quad \textit{for all } r > 0\ .$$

Then

$$\Gamma^* = 0\colon \quad A^n_{hom}(X) \ \to \ A^n_{hom}(X)\ .$$

We also get the following particular case:

Conjecture 1.4. *Let X be a hyperkähler variety of dimension $n = 4m$ (where $m \in \mathbb{N}$). Let $\iota \in \mathrm{Aut}(X)$ be an involution. Then*

$$\iota^* = \mathrm{id}\colon \quad F^nA^n(X) \ \to \ F^nA^n(X)\ .$$

Here $F^nA^n(X)$ denotes the "deepest level" of the Bloch–Beilinson filtration, which conjecturally exists for all smooth projective varieties [18], [19], [20], [25], [26], and for which good candidates are known to exist unconditionally for certain hyperkähler varieties [35], [46]. (The point of Conjecture 1.4 is that the action of ι on $F^nA^n(X)$ is conjecturally determined by the action of ι on $H^{n,0}(X) = H^{4m,0}(X)$, which is the identity.)

In dimension $n = 2$, certain cases of Conjecture 1.3 have been proven:

Theorem 1.5 (Huybrechts [17], Voisin [41])**.** *Let X be a $K3$ surface. Let $f \in \mathrm{Aut}(X)$ be a finite order automorphism that is symplectic. Then*

$$f^* = \mathrm{id}\colon \quad A^2(X) \ \to \ A^2(X)\ .$$

That is, Conjecture 1.3 is true when X is a $K3$ surface and $\Gamma = \Gamma_f - \Delta_X$ (where Γ_f denotes the graph of f, and f is as in Theorem 1.5).

In dimension $n > 2$, certain cases of Conjecture 1.3 have been proven for the Fano variety of lines on a cubic fourfold [15]. There is also a result for what is perhaps the prime series of examples of hyperkähler fourfolds: the Hilbert scheme $S^{[2]}$ of 2 points on a $K3$ surface S [35, Proposition 5.2]:

Theorem 1.6 (Shen–Vial [35])**.** *Let S be a $K3$ surface, and let $X = S^{[2]}$. Let $f \in \mathrm{Aut}(X)$ be a natural automorphism of finite order that is symplectic. Then*

$$\begin{aligned} f^* = \mathrm{id}\colon \quad & A^4(X) \ \to \ A^4(X) \ , \\ f^* = \mathrm{id}\colon \quad & A^2_{hom}(X) \ \to \ A^2_{hom}(X) \ . \end{aligned}$$

That is, Conjecture 1.3 is true for $X = S^{[2]}$ and $\Gamma = \Gamma_f - \Delta_X$.

Here, a *natural automorphism* is by definition an automorphism of X that is induced by an automorphism of S. Theorem 1.6 is proven by reducing to Theorem 1.5. The goal of this article is to go beyond Theorem 1.6, by also considering *non–natural* and *non–symplectic* automorphisms of $S^{[2]}$.

Let $X = S^{[2]}$ be a Hilbert scheme with S a $K3$ surface, and assume that X has an anti–symplectic involution ι. The involution ι, being anti–symplectic, has the property that

$$\iota^* = -\,\mathrm{id}\colon \quad H^2(X, \mathcal{O}_X) \ \to \ H^2(X, \mathcal{O}_X) \ .$$

Conjecture 1.2 thus predicts that

$$\iota^* \overset{??}{=} -\,\mathrm{id}\colon \quad A^2_{hom}(X) \ \to \ A^2_{hom}(X) \ . \tag{1}$$

Can one prove this conjectural equality?

One classical case where an anti–symplectic involution exists is that of the Hilbert scheme $X = S^{[2]}$, where $S \subset \mathbb{P}^3$ is a smooth quartic with Picard number $\rho(S) = 1$. In this case, it is known ([9], cf. Theorem 2.21 below) that the only non–trivial automorphism of X is the non–symplectic, non–natural involution

$$\iota\colon X \ \to \ X$$

which was first studied by Beauville [1]. Our main result implies that in this case, Conjecture 1.4 and a weak version of the conjectural equality (1) are true:

Theorem (=Theorem 4.1)**.** *Let $S \subset \mathbb{P}^3$ be a smooth quartic with Picard number $\rho(S) = 1$, and let $X = S^{[2]}$. Let $\iota \in \mathrm{Aut}(X)$ be the non–symplectic involution of [1]. Then*

$$\begin{aligned} \iota^* = -\,\mathrm{id}\colon \quad & A^i_{(2)}(X) \ \to\ A^i_{(2)}(X) \quad \textit{for } i = 2, 4\ ; \\ \iota^* = \mathrm{id}\colon \quad & A^4_{(j)}(X) \ \to\ A^4_{(j)}(X) \quad \textit{for } j = 0, 4\ . \end{aligned}$$

Here, $A^*_{(*)}(X)$ denotes the bigraded ring structure constructed by Shen–Vial [35] using (their version of) the Fourier transform. To establish equality (1) for X as in Theorem 4.1, it remains to prove the conjectural equality

$$A^2_{(2)}(X) \stackrel{??}{=} A^2_{hom}(X)\ . \tag{2}$$

Unfortunately, equality (2) does not seem to be known for any Hilbert square $X = S^{[2]}$. Some evidence for equality (2) is that it is true if there exists a Bloch–Beilinson filtration on $A^*(X)$ of which the Fourier decomposition $A^*_{(*)}(X)$ is a splitting; more concretely, equality (2) is equivalent to Murre's conjecture D for X [35, Theorem 3.3].

Theorem 4.1 has a nice implication for the quotient (this quotient is a slightly singular Calabi–Yau variety):

Corollary (=Corollary 5.4)**.** *Let X and ι be as in Theorem 4.1, and let $Y := X/\iota$ be the quotient. For any $r \in \mathbb{N}$, let*

$$E^*(Y^r)\ \subset\ A^*(Y^r)$$

be the subring generated by (pullbacks of) $A^1(Y)$ and $A^2(Y)$. The cycle class map

$$E^k(Y^r)\ \to\ H^{2k}(Y^r)$$

is injective for $k \geq 4r - 1$.

In particular, taking $r = 1$, we find that the subspaces

$$\begin{aligned} &\mathrm{Im}\Bigl(A^2(Y) \otimes A^1(Y)\ \to\ A^3(Y)\Bigr)\ , \\ &\mathrm{Im}\Bigl(A^2(Y) \otimes A^2(Y)\ \to\ A^4(Y)\Bigr) \end{aligned}$$

are of dimension 1 (Corollary 5.6). This is analogous to known results for 0–cycles on $K3$ surfaces [5] and on complete intersection Calabi–Yau varieties [40], [14] (cf. Remark 5.7 below).

To prove Theorem 4.1, we employ the technique of "spread" of algebraic cycles in a family, as developed by Voisin in her work on the Bloch/Hodge equivalence for complete intersections [42], [43], [44], [45]. At the heart of our proof is a result of Voisin about the triviality of certain Chow groups of the relative fourfold fibre product of the family of all smooth quartics, provided the (Lefschetz or Voisin) standard conjecture is true ([42, Proposition 4.11], cf. also Theorem 4.5 below). The most delicate part of the proof is to circumvent recourse to the standard conjectures in Voisin's result; in this case, this works because we can reduce the problem to a certain relative correspondence of codimension 2 (rather than 4).

Another case where an anti–symplectic involution exists on $X = S^{[2]}$ is when S is a degree 2 $K3$ surface (i.e., a double cover of the plane ramified along a smooth sextic). In this case, the anti–symplectic involution is natural (induced by the covering involution of S), and the statement of Theorem 4.1 can be easily proven for this case (cf. Proposition 3.1). Other cases where an anti–symplectic involution exists on $X = S^{[2]}$ are when S is a generic $K3$ of degree $20, 26$ or 34 (Theorem 2.21). Proving the statement of Theorem 4.1 for these cases would be interesting, but appears to be difficult (cf. Question 6.3).

Conventions. *In this article, the word variety refers to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients*: we denote by $A_j(X)$ the Chow group of j–dimensional cycles on X with $\mathbb{Q}$–coefficients; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ will be used interchangeably.*

The notations $A^j_{hom}(X)$, $A^j_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f\colon X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [34], [26]) will be denoted $\mathcal{M}_{\mathrm{rat}}$.

We will write $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

Given an involution ι on X, we will write $A^j(X)^\iota$ (and $H^j(X)^\iota$) for the subgroup of elements fixed by ι.

2. Preliminary

2.1. Quotient varieties

Definition 2.1. *A* projective quotient variety *is a variety*

$$X = Y/G \ ,$$

where Y is a smooth projective variety and $G \subset Aut(Y)$ is a finite group.

Proposition 2.2 (Fulton [16])**.** *Let X be a projective quotient variety of dimension n. Let $A^*(X)$ denote the operational Chow cohomology ring. The natural map*

$$A^i(X) \ \to \ A_{n-i}(X)$$

is an isomorphism for all i.

Proof. This is [16, Example 17.4.10]. □

Remark 2.3. *It follows from Proposition 2.2 that the formalism of correspondences goes through unchanged for projective quotient varieties (this is also noted in [16, Example 16.1.13]). We can thus consider motives $(X, p, 0) \in \mathcal{M}_{\mathrm{rat}}$, where X is a projective quotient variety and $p \in A^n(X \times X)$ is a projector. For a projective quotient variety $X = Y/G$, one readily proves (using Manin's identity principle) that there is an isomorphism*

$$h(X) \cong h(Y)^G := (Y, \Delta_Y^G, 0) \quad \text{in } \mathcal{M}_{\mathrm{rat}} \ ,$$

where Δ_Y^G denotes the idempotent $\frac{1}{|G|}\sum_{g\in G}\Gamma_g$.

2.2. MCK decomposition

Definition 2.4 (Murre [25])**.** *Let X be a smooth projective variety of dimension n. We say that X has a* CK decomposition *if there exists a decomposition of the diagonal*

$$\Delta_X = \Pi_0^X + \Pi_1^X + \cdots + \Pi_{2n}^X \quad \text{in } A^n(X \times X) \ ,$$

such that the Π_i^X are mutually orthogonal idempotents and $(\Pi_i^X)_* H^*(X) = H^i(X)$.

(NB: "CK decomposition" is shorthand for "Chow–Künneth decomposition".)

Remark 2.5. *The existence of a CK decomposition for any smooth projective variety is part of Murre's conjectures [25], [18].*

Definition 2.6 (Shen–Vial [35]). *Let X be a smooth projective variety of dimension n. Let $\Delta^X_{sm} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal*

$$\Delta^X_{sm} := \{(x,x,x) \mid x \in X\} \subset X \times X \times X .$$

An MCK decomposition is a CK decomposition $\{\Pi^X_i\}$ of X that is multiplicative*, i.e. it satisfies*

$$\Pi^X_k \circ \Delta^X_{sm} \circ (\Pi^X_i \times \Pi^X_j) = 0 \quad \textit{in } A^{2n}(X \times X \times X) \quad \textit{for all } i+j \neq k .$$

(NB: "MCK decomposition" is shorthand for "multiplicative Chow–Künneth decomposition".)

Remark 2.7. *The small diagonal (seen as a correspondence from $X \times X$ to X) induces the* multiplication morphism

$$\Delta^X_{sm} \colon \quad h(X) \otimes h(X) \ \to \ h(X) \quad \textit{in } \mathcal{M}_{\mathrm{rat}} .$$

Suppose X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \textit{in } \mathcal{M}_{\mathrm{rat}} .$$

By definition, this decomposition is multiplicative if for any i, j the composition

$$h^i(X) \otimes h^j(X) \ \to \ h(X) \otimes h(X) \ \xrightarrow{\Delta^X_{sm}} \ h(X) \quad \textit{in } \mathcal{M}_{\mathrm{rat}}$$

factors through $h^{i+j}(X)$. It follows that if X has an MCK decomposition, then setting

$$A^i_{(j)}(X) := (\Pi^X_{2i-j})_* A^i(X) ,$$

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A^i_{(j)}(X) \otimes A^{i'}_{(j')}(X)$ to $A^{i+i'}_{(j+j')}(X)$.

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville's "weak splitting property" [3]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [35, Section 8] and [39] and [36].

Lemma 2.8 (Vial [39]). *Let X, X' be birational hyperkähler varieties. Then X has an MCK decomposition if and only if X' has one.*

Proof. This is noted in [39, Introduction]; the idea (as indicated in loc. cit.) is that Rieß's result [33] implies that X and X' have isomorphic Chow motives and the isomorphism is compatible with the multiplicative structure. (For more details, cf. [22, Lemma 2.13].) $\square$

2.3. MCK for $K3^{[2]}$

Theorem 2.9 (Shen–Vial [35])**.** *Let S be a $K3$ surface, and $X = S^{[2]}$. There exists an MCK decomposition $\{\Pi_j^X\}$ for X. In particular, setting*

$$A^i_{(j)}(X) := (\Pi^X_{2i-j})_* A^i(X)$$

defines a bigraded ring structure $A^*_{(*)}(X)$ on $A^*(X)$. Moreover, $A^*_{(*)}(X)$ coincides with the bigrading defined by the Fourier transform.

Proof. The existence of $\{\Pi_j^X\}$ is a special case of [35, Theorem 13.4]. The "moreover" part is [35, Theorem 15.8]. $\square$

Remark 2.10. *The first statement of Theorem 2.9 actually holds for $X = S^{[r]}$ for any $r \in \mathbb{N}$ [39].*

Any $K3$ surface S has an MCK decomposition [35]. Since this property is stable under products, S^2 has an MCK decomposition. The following lemma records a basic compatibility between the bigradings on $A^*(S^{[2]})$ and on $A^*(S^2)$:

Lemma 2.11. *Let S be a $K3$ surface, and $X = S^{[2]}$. Let $\Psi \in A^4(X \times S^2)$ be the correspondence coming from the diagram*

$$\begin{array}{ccc} S^{[2]} & \leftarrow & \widetilde{S^2} \\ {\scriptstyle h}\downarrow & & \downarrow \\ S^{(2)} & \xleftarrow{g} & S^2 \end{array}$$

(the arrow labelled h is the Hilbert–Chow morphism; the right vertical arrow is the blow–up of the diagonal). Then

$$\begin{aligned} (\Psi)_* R(X) &\subset R(S^2) , \\ ({}^t\Psi)_* R(S^2) &\subset R(X) , \end{aligned}$$

where $R = A^4_{(4)}$ or $A^4_{(2)}$ or $A^2_{(2)}$ or $A^2_{(0)} \cap A^2_{hom}$.

Proof. We prove the statement for ${}^t\Psi$ and $R = A^2_{(2)}$ or $A^2_{(0)} \cap A^2_{hom}$, which are the only cases we'll be using (the other statements can be proven similarly). By construction of the MCK decomposition for X (cf. [35, Theorem 13.4]), there is a relation

$$(3) \qquad \Pi^X_k = \frac{1}{2}\, {}^t\Psi \circ \Pi^{S^2}_k \circ \Psi + \text{Rest} \quad \text{in } A^4(X \times X)\ , \quad (k = 0, 2, 4, 6, 8)\ ,$$

where $\{\Pi^{S^2}_k\}$ is a product MCK decomposition for S^2, and "Rest" is a term coming from $\Delta_S \subset S \times S$ which does not act on $A^4(X)$ and on $A^2_{AJ}(X)$. Since $\frac{1}{2}\, {}^t\Psi \circ \Psi$ is the identity on $A^2_{hom}(X) = A^2_{AJ}(X)$, we can write

$$({}^t\Psi)_*(\Pi^{S^2}_k)_* = ({}^t\Psi \circ \Pi^{S^2}_k)_* = (\frac{1}{2}\, {}^t\Psi \circ \Psi \circ {}^t\Psi \circ \Pi^{S^2}_k)_* \colon \quad A^2_{hom}(S^2) \ \to\ A^2_{hom}(X)\ .$$

In view of Sublemma 2.12 below, this implies

$$({}^t\Psi)_*(\Pi^{S^2}_k)_* = (\frac{1}{2}\, {}^t\Psi \circ \Pi^{S^2}_k \circ \Psi \circ {}^t\Psi)_* \colon \quad A^2_{hom}(S^2) \ \to\ A^2_{hom}(X)\ .$$

But then, plugging in relation (3), we find

$$({}^t\Psi)_*(\Pi^{S^2}_k)_* A^2_{hom}(S^2) \ \subset\ (\Pi^X_k)_* A^2_{hom}(X)\ .$$

Taking $k = 2$, this proves

$$({}^t\Psi)_* A^2_{(2)}(S^2) \ \subset\ A^2_{(2)}(X)\ .$$

Taking $k = 4$, this proves

$$({}^t\Psi)_*\Big(A^2_{(0)}(S^2) \cap A^2_{hom}(S^2)\Big) \ \subset\ A^2_{(0)}(X) \cap A^2_{hom}(X)\ .$$

Sublemma 2.12. *There is commutativity*

$$\Psi \circ {}^t\Psi \circ \Pi^{S^2}_k = \Pi^{S^2}_k \circ \Psi \circ {}^t\Psi \quad \textit{in } A^4(S^4)\ .$$

To prove the sublemma, we remark that $h_* h^* = 2\,\mathrm{id} \colon A^i(S^{(2)}) \to A^i(S^{(2)})$, and so

$$(\Psi \circ {}^t\Psi)_* = 2\ g^* g_* = 2(\Delta_{S^2} + \Gamma_\iota)_* \colon \quad A^i(S^2) \to A^i(S^2)\ ,$$

where ι denotes the involution switching the two factors. But $\{\Pi^{S^2}_k\}$, being a product decomposition, is symmetric and hence

$$\Gamma_\iota \circ \Pi^{S^2}_k \circ \Gamma_\iota = (\iota \times \iota)^* \Pi^{S^2}_k = \Pi^{S^2}_k \quad \text{in } A^4(S^4)\ .$$

This implies commutativity

$$\Gamma_\iota \circ \Pi_k^{S^2} = \Pi_k^{S^2} \circ \Gamma_\iota \quad \text{in } A^4(S^4) \ ,$$

which proves the sublemma. $\square$

Remark 2.13. *Lemma 2.11 is probably true for any (i,j) (i.e., Ψ should be "of pure grade 0" in the language of [36, Definition 1.1]). I have not been able to prove this.*

2.4. MCK for $S \times S$

Notation 2.14. *Let $\mathcal{S} \to B$ be a family (i.e., a smooth projective morphism). For $r \in \mathbb{N}$, we write $\mathcal{S}^{r/B}$ for the relative r–fold fibre product*

$$\mathcal{S}^{r/B} := \mathcal{S} \times_B \mathcal{S} \times_B \cdots \times_B \mathcal{S}$$

(r copies of $\mathcal{S}$).

Proposition 2.15. *Let $\mathcal{S} \to B$ be a family of $K3$ surfaces. There exist relative correspondences*

$$\Pi_j^{\mathcal{S}^{2/B}} \quad \in A^4(\mathcal{S}^{4/B}) \quad (j = 0, 2, 4, 6, 8) \ ,$$

such that
(i) for each $b \in B$, the restriction

$$\Pi_j^{(S_b)^2} := \Pi_j^{\mathcal{S}^{2/B}}|_{(S_b)^4} \quad \in A^4((S_b)^4)$$

defines a self–dual MCK decomposition for $(S_b)^2$;
(ii) there is a decomposition

$$\Pi_2^{\mathcal{S}^{2/B}} = P_1 \circ Q_1 + P_2 \circ Q_2 \quad \textit{in } A^4(\mathcal{S}^{4/B}) \ ,$$

where $P_i \in A^2(\mathcal{S}^{4/B})$ and $Q_i \in A^6(\mathcal{S}^{4/B})$ for $i = 1, 2$.

Proof. (i) On any $K3$ surface S_b, there is the distinguished 0–cycle $\mathfrak{o}_{S_b}$ such that $c_2(S_b) = 24\mathfrak{o}_{S_b}$ [5]. Let $p_i \colon \mathcal{S} \times_B \mathcal{S} \to \mathcal{S}$, $i = 1, 2$, denote the projections to the two factors. Let $T_{\mathcal{S}/B}$ denote the relative tangent bundle. The assignment

$$\begin{aligned}
\Pi_0^{\mathcal{S}} &:= (p_1)^*\big(\frac{1}{24} c_2(T_{\mathcal{S}/B})\big) \quad \in \ A^2(\mathcal{S} \times_B \mathcal{S}) \ , \\
\Pi_4^{\mathcal{S}} &:= (p_2)^*\big(\frac{1}{24} c_2(T_{\mathcal{S}/B})\big) \quad \in \ A^2(\mathcal{S} \times_B \mathcal{S}) \ , \\
\Pi_2^{\mathcal{S}} &:= \Delta_{\mathcal{S}} - \Pi_0^{\mathcal{S}} - \Pi_4^{\mathcal{S}}
\end{aligned}$$

defines (by restriction) an MCK decomposition for each fibre:

$$\Pi_j^{S_b} := \Pi_j^{\mathcal{S}}|_{S_b\times S_b} \quad \in A^2(S_b\times S_b) \quad (j=0,2,4)$$

is an MCK decomposition [35, Example 8.17].

Next, we consider the fourfold relative fibre product $\mathcal{S}^{4/B}$. Let

$$p_{ij}\colon \mathcal{S}^{4/B} \ \to\ \mathcal{S}^{2/B} \quad (1\le i<j\le 4)$$

denote projection to the i-th and j-th factor. We set

$$\Pi_j^{\mathcal{S}^{2/B}} := \sum_{k+\ell=j} (p_{13})^*(\Pi_k^{\mathcal{S}})\cdot (p_{24})^*(\Pi_\ell^{\mathcal{S}}) \quad \in A^4(\mathcal{S}^{4/B})\ , \quad (j=0,2,4,6,8)\ .$$

By construction, the restriction to each fibre induces an MCK decomposition (the "product MCK decomposition")

$$\Pi_j^{(S_b)^2} := \Pi_j^{\mathcal{S}^{2/B}}|_{(S_b)^4} = \sum_{k+\ell=j} \Pi_k^{S_b}\times \Pi_\ell^{S_b} \quad \in A^4((S_b)^4)\ , \quad (j=0,2,4,6,8)\ .$$

Since the $\Pi_j^{S_b}$ are self–dual, so are the $\Pi_j^{(S_b)^2}$. (ii) Define

$$\begin{aligned} P_1 &:= (p_{13})^*(\Pi_2^{\mathcal{S}}) \quad \in A^2(\mathcal{S}^{4/B})\ ,\\ Q_1 &:= (p_{134})^*(\Delta_{\mathcal{S},sm})\cdot (p_2)^*\big(\frac{1}{24}c_2(T_{\mathcal{S}/B})\big) \quad \in A^6(\mathcal{S}^{4/B})\ ,\\ P_2 &:= (p_{24})^*(\Pi_2^{\mathcal{S}}) \quad \in A^2(\mathcal{S}^{4/B})\ ,\\ Q_2 &:= (p_{234})^*(\Delta_{\mathcal{S},sm})\cdot (p_1)^*\big(\frac{1}{24}c_2(T_{\mathcal{S}/B})\big) \quad \in A^6(\mathcal{S}^{4/B})\ . \end{aligned}$$

Here $p_{134}\colon \mathcal{S}^{4/B} \to \mathcal{S}^{3/B}$ is projection on the first, third and fourth factor (and similarly for p_2, etc.), and $\Delta_{\mathcal{S},sm}$ is the "relative small diagonal" (i.e., the image of the natural morphism $\mathcal{S}\to\mathcal{S}^{3/B}$.

We will now show that for each $b\in B$, there is equality

$$(4)\qquad \begin{aligned} (P_1\circ Q_1)|_{(S_b)^4} &= \Pi_2^{S_b}\times \Pi_0^{S_b} \quad \in A^4((S_b)^4)\ ,\\ (P_2\circ Q_2)|_{(S_b)^4} &= \Pi_0^{S_b}\times \Pi_2^{S_b} \quad \in A^4((S_b)^4)\ . \end{aligned}$$

This suffices to prove the proposition, because it implies that $P_1\circ Q_1+P_2\circ Q_2$ restricts to

$$\Pi_2^{(S_b)^2} = \Pi_2^{S_b}\times \Pi_0^{S_b} + \Pi_0^{S_b}\times \Pi_2^{S_b} \quad \in A^4((S_b)^4)\ ,$$

which is part of a product MCK decomposition on each fibre.

For a given S_b let $x = \mathfrak{o}_{S_b} \in A^2(S_b)$ denote the distinguished 0–cycle of [5]. We note that

$$\begin{aligned}(P_1 \circ Q_1)|_{(S_b)^4} &= (P_1|_{(S_b)^4}) \circ (Q_1|_{(S_b)^4})\\ &= ((p_{13})^*(\Pi_2^{S_b})) \circ (\{(s,x,s,s) \in (S_b)^4\})\\ &= (p_{1256})_*\Big((p_{35})^*(\Pi_2^{S_b}) \cdot (\{(s,x,s,s)\} \times S_b \times S_b)\Big)\\ &= (p_{1256})_*\Big((p_{15})^*(\Pi_2^{S_b}) \cdot (\{(s,x,s,s)\} \times S_b \times S_b)\Big)\\ &= (p_{13})^*(\Pi_2^{S_b}) \cdot (\{(s,x)\} \times S_b \times S_b) \quad \text{in } A^4((S_b)^4)\\ &= \Pi_2^{S_b} \times x \times S_b \quad \text{in } A^4((S_b)^4)\ .\end{aligned}$$

Likewise,

$$\begin{aligned}(P_2 \circ Q_2)|_{(S_b)^4} &= (P_2|_{(S_b)^4}) \circ (Q_2|_{(S_b)^4})\\ &= ((p_{24})^*(\Pi_2^{S_b})) \circ (\{(x,s,s,s) \in (S_b)^4\})\\ &= (p_{1256})_*\Big((p_{46})^*(\Pi_2^{S_b}) \cdot (\{(x,s,s,s)\} \times S_b \times S_b)\Big)\\ &= (p_{1256})_*\Big((p_{26})^*(\Pi_2^{S_b}) \cdot (\{(x,s,s,s)\} \times S_b \times S_b)\Big)\\ &= (p_{24})^*(\Pi_2^{S_b}) \cdot (\{(x,s)\} \times S_b \times S_b) \quad \text{in } A^4((S_b)^4)\\ &= x \times S_b \times \Pi_2^{S_b} \quad \text{in } A^4((S_b)^4)\ .\end{aligned}$$

This proves the equalities (4), and so the proposition is proven. ☐

2.5. Relative MCK for $K3^{[2]}$

Proposition 2.16. *Let $\mathcal{S} \to B$ be a family of $K3$ surfaces (i.e. each fibre S_b is a $K3$ surface), and let $\mathcal{X} \to B$ be the family of associated Hilbert schemes (i.e., a fibre X_b is $(S_b)^{[2]}$). There exist relative correspondences*

$$\Pi_j^{\mathcal{X}} \quad \in A^4(\mathcal{X} \times_B \mathcal{X}) \quad (j = 0,2,4,6,8)\ ,$$

such that for each $b \in B$, the restrictions

$$\Pi_j^{X_b} := \Pi_j^{\mathcal{X}}|_{X_b \times X_b} \quad \in A^4(X_b \times X_b) \quad (j = 0,2,4,6,8)$$

define an MCK decomposition for X_b.

Proof. The construction of an MCK decomposition for X_b given in [35, Theorem 13.4] can be done in a relative setting. That is, let $\{\Pi_j^{\mathcal{S}}\}$ be a relative MCK decomposition for $\mathcal{S}$ as in Proposition 2.15, and let $\{\Pi_j^{\mathcal{S}^{2/B}}\}$ be the induced relative MCK decomposition for $\mathcal{S}^{2/B}$ as in Proposition 2.15. Let

$$\mathcal{Z} \to B$$

be the family obtained by blowing–up $\mathcal{S}\times_B\mathcal{S}$ along the relative diagonal $\Delta_{\mathcal{S}}$. As in the proof of [35, Propositions 13.2 and 13.3][1], one can use $\{\Pi_j^{\mathcal{S}^{2/B}}\}$ and $\{\Pi_j^{\mathcal{S}}\}$ to define relative correspondences

$$\Pi_j^{\mathcal{Z}} \quad \in A^4(\mathcal{Z}\times_B\mathcal{Z}) \quad (j=0,2,4,6,8)\ ,$$

which restrict to an MCK decomposition of each fibre Z_b. Let

$$p\colon\quad \mathcal{Z} \to \mathcal{X}$$

denote the morphism of B-schemes induced by the action of the symmetric group $\mathfrak{S}_2$, and let $\Gamma_p\in A^4(\mathcal{Z}\times_B\mathcal{X})$ be the graph of p. We define

$$\Pi_j^{\mathcal{X}} := \frac{1}{2}\Gamma_p\circ\Pi_j^{\mathcal{Z}}\circ {}^t\Gamma_p \quad \in A^4(\mathcal{X}\times_B\mathcal{X}) \quad (j=0,2,4,6,8)\ .$$

The restrictions $\Pi_j^{X_b} := \Pi_j^{\mathcal{X}}|_{X_b\times X_b}$ define an MCK decomposition for each fibre by [35, Theorem 13.4]. ☐

2.6. Multiplicative structure of Chow ring of $K3^{[2]}$

Theorem 2.17 (Shen–Vial [35])**.** *Let S be a $K3$ surface, and $X=S^{[2]}$.*
(i) Intersection product induces a surjection

$$A^2_{(2)}(X)\otimes A^2_{(2)}(X) \twoheadrightarrow A^4_{(4)}(X)\ .$$

(ii) There is a distinguished class $l\in A^2_{(0)}(X)$ such that intersection induces an isomorphism

$$\cdot l\colon\quad A^2_{(2)}(X) \xrightarrow{\cong} A^4_{(2)}(X)\ .$$

Proof. This is [35, Theorem 3]. ☐

[1]The statement and proof of [35, Proposition 13.2] should be slightly modified, as noted in [36, Remark 2.8].

2.7. Refined CK decomposition

Theorem 2.18 (Vial [38]). *Let X be a smooth projective variety of dimension $n \le 5$. Assume the Lefschetz standard conjecture $B(X)$ holds (in particular, the Künneth components $\pi_i \in H^{2n}(X \times X)$ are algebraic). Then there is a splitting into mutually orthogonal idempotents*

$$\pi_i = \sum_j \pi_{i,j} \quad \in H^{2n}(X \times X) \ ,$$

such that

$$(\pi_{i,j})_* H^*(X) = gr^j_{\widetilde{N}} H^i(X) \ ,$$

where $\widetilde{N}^*$ denotes Vial's niveau filtration *[38].*

In particular,

$$\begin{aligned} (\pi_{2,1})_* H^j(X) &= H^2(X) \cap F^1 \ , \\ (\pi_{2,0})_* H^j(X) &= H^2_{tr}(X) \ . \end{aligned}$$

(Here F^* denotes the Hodge filtration, and $H^2_{tr}(X)$ is the orthogonal complement to $H^2(X) \cap F^1$ under the pairing

$$\begin{aligned} H^2(X) \otimes H^2(X) \ &\to \ \mathbb{Q} \ , \\ a \otimes b \ &\mapsto \ a \cup h^{n-2} \cup b \ .) \end{aligned}$$

The projector $\pi_{2,1}$ is supported on $C \times D$, where $C \subset X$ is a curve and $D \subset X$ is a divisor.

Proof. This is [38, Theorem 1], plus the fact that $gr^1_{\widetilde{N}} H^2(X) = H^2(X) \cap F^1$ is the Néron–Severi group of X (cf. loc. cit.). ☐

Remark 2.19. *Vial's* niveau filtration *[38] is conjecturally (but not provably) the same as the coniveau filtration. The construction of Theorem 2.18 is inspired by [21], where for any surface S, the "transcendental part of the Chow motive of S" is constructed.*

2.8. The automorphism group of $K3^{[2]}$

Proposition 2.20 (Boissière et al. [9]). *Let S be a projective $K3$ surface of Picard number $\rho(S) = 1$, and let $X = S^{[2]}$. Suppose* Pic(S) *is generated by a divisor H with $H^2 = 2$. Then* Aut$(X) = \mathbb{Z}/2\mathbb{Z}$*, and the non–trivial involution ι of X is anti–symplectic, induced by the covering involution of S.*

Proof. This is [9, Proposition 5.1]. □

Theorem 2.21 (Boissière et al. [9]). *Let S be a projective $K3$ surface of Picard number $\rho(S) = 1$. Suppose* $\mathrm{Pic}(S)$ *is generated by a divisor H with $H^2 = 2t$, $t \geq 2$.*
(i) The Hilbert scheme $X = S^{[2]}$ has a non–trivial automorphism if and only if there exists an ample divisor $D \in \mathrm{Pic}(X)$ of square 2 (with respect to the Beauville–Bogomolov quadratic form). This is the case for $t = 1, 10, 13, 17, \ldots$
(ii) If $\mathrm{Aut}(X) \neq 0$ *then* $\mathrm{Aut}(X) = \mathbb{Z}/2\mathbb{Z}$ *and the only non–trivial automorphism is an anti–symplectic involution ι leaving the divisor D invariant (i.e. $\iota^*(D) = D$ in $NS(X)$).*

Proof. Statement (i) is [9, Theorem 5.5] (combined with results concerning solutions of Pell's equation to compute the first values of t; these values are stated in [9, Introduction]). Statement (ii) is [9, Lemma 5.3]. □

Proposition 2.22 (Beauville [1]). *Let $S \subset \mathbb{P}^3$ be a smooth quartic with Picard number $\rho(S) = 1$, and let $X = S^{[2]}$. Let G denote the Grassmannian of lines in $\mathbb{P}^3$, and let $\phi\colon X \to G$ be the morphism sending a length–two subscheme Z to its one–dimensional span $< Z > \subset \mathbb{P}^3$.*
(i) There exists an anti–symplectic involution

$$\iota\colon \quad X \ \to \ X \ ,$$

defined by sending $Z \in X$ to the residual subscheme of $< Z > \cap S$, i.e.

$$< Z > \cap S = Z \amalg \iota(Z) \ .$$

(ii) There exists an ample divisor $D \in A^1(X)$ of square 2 (with respect to the Beauville–Bogomolov form), and such that the linear system $|D|$ is base–point–free. Define the morphism f as the composition

$$f\colon X \ \xrightarrow{\phi}\ G \ \xrightarrow{\psi}\ \mathbb{P}^5 \ ,$$

where ψ is the Plücker embedding. Then f is the same as the morphism defined by $|D|$.
(iii) The involution ι acts on $NS(X)$ as reflection in the span of D.
(iv) The involution ι is non–natural (i.e., there exists no pair (S', τ) with S' a $K3$ surface and $\tau \in Aut(S')$ and such that $(X, \iota) = ((S')^{[2]}, \tau^{[2]})$).

Proof. Statement (i) is [1, Section 6]. Statements (ii) and (iii) are contained in [28, Section 4.1.2], or [9, Section 6.1]. Finally, point (iv) is proven in [10, page 6] by computing the *index* $\lambda(\iota)$ (as defined in loc. cit.) of ι. □

Remark 2.23. *Let X be the Hilbert scheme $X = S^{[2]}$ of a generic quartic $S \subset \mathbb{P}^3$. Combining Proposition 2.22 and Theorem 2.21, it follows that Beauville's involution ι is the unique non–trivial automorphism of X.*

Remark 2.24. *Let $S \subset \mathbb{P}^3$ be a smooth quartic of any Picard number, and let $X = S^{[2]}$. Clearly, the above construction gives a rational map*

$$\iota\colon \quad X \dashrightarrow X \ ,$$

which is well–defined outside of the locus of zero–dimensional subschemes contained in a line on S.

Remark 2.25. *Oguiso [32, Section 4 Example 2] has used Beauville's involution to construct interesting automorphisms of $S^{[2]}$ where $S \subset \mathbb{P}^3$ is a certain quartic of Picard number 2. These automorphisms are non–natural and of positive entropy.*

Remark 2.26. *The (X, ι) as in Proposition 2.22 form a 19–dimensional family. It is known that this family is a degeneration of the 20–dimensional family (Z, i), where Z is a double EPW sextic (in the sense of [29], [30], [31]) and i is the anti–symplectic involution such that the quotient Z/i is an EPW sextic. This degeneration is explained in [13, Section 4], and also noted in [4, Section 3.5]. In the notation of Proposition 2.22, the morphism $f = \phi_{|D|}$ (given by the linear system of D) admits a factorization*

$$f = \psi \circ \phi\colon \qquad \begin{array}{ccccc} X & \xrightarrow{\phi} & G & \xrightarrow{\psi} & \mathbb{P}^5 \\ \downarrow & \nearrow & & & \\ Y := X/\iota & & & & \end{array}$$

where ϕ is generically 6 to 1. It follows that the image $\phi_{|D|}(X) \subset \mathbb{P}^5$ is $3Q$ (the Plücker quadric Q with multiplicity 3), which can be seen as a degenerate EPW sextic. It also follows that the quotient Y is the triple cover of the quadric $Q \subset \mathbb{P}^5$.

3. Degree 2 $K3$ surfaces

As a warm–up before proving the main result (which is about the Hilbert square of $K3$ surfaces of degree 4), we consider the Hilbert square of $K3$ surfaces of degree 2. This case is easy, because the anti–symplectic involution is natural.

Proposition 3.1. *Let S be a $K3$ surface of degree 2 (i.e. there exists an ample divisor $H \in \mathrm{Pic}(S)$ with $H^2 = 2$) and with $\rho(S) = 1$. Let $X = S^{[2]}$, and let $\iota \in \mathrm{Aut}(X)$ be the unique non–trivial automorphism of Proposition 2.20. Then*

$$\begin{aligned} \iota^* = -\,\mathrm{id}\colon \quad & A^i_{(2)}(X) \ \to \ A^i_{(2)}(X) \quad (i = 2, 4)\ , \\ \iota^* = \mathrm{id}\colon \quad & A^4_{(4)}(X) \ \to \ A^4_{(4)}(X)\ . \end{aligned}$$

Proof. The natural correspondence $\Psi \in A^4(X \times S^2)$ induces a a split injection

$$\Psi_*\colon \quad A^4(X) \ \to \ A^4(S^2)\ ,$$

which is compatible with the bigrading $A^4_{(j)}$ for $j = 2, 4$ (Lemma 2.11). The involution ι being natural (i.e. induced by the covering involution i of S), there is a commutative diagram

$$\begin{array}{ccc} A^4_{(j)}(X) & \xrightarrow{\Psi_*} & A^4_{(j)}(S^2) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle (i\times i)^*} \\ A^4(X) & \xrightarrow{\Psi_*} & A^4(S^2) \end{array} \tag{5}$$

for $j = 2, 4$. We are thus reduced to proving a statement for S^2.

Lemma 3.2. *Set–up as in Proposition 3.1. Then*

$$A^2_{hom}(S)^i = 0\ .$$

Proof. The quotient variety S/i has geometric genus 0. Since quotient singularities are rational singularities, there exists a resolution $Y \to S/i$ with $p_g(Y) = 0$. Since Y is not of general type, Bloch's conjecture is known to hold for Y [7], i.e. $A^2_{hom}(Y) = 0$. This implies that also $A^2_{hom}(S/i) = A^2_{hom}(S)^i = 0$. □

Lemma 3.2 implies that i acts as $-\,\mathrm{id}$ on $A^2_{hom}(S)$. Looking at the action of i on $H^{2,2}(S) \cong \mathbb{C}$, one finds that i acts as id on $A^2_{(0)}(S)$. This implies that

$$\begin{aligned} (i \times i)^* = -\,\mathrm{id}\colon \quad & A^4_{(2)}(S^2) \ \to \ A^4(S^2)\ , \\ (i \times i)^* = \mathrm{id}\colon \quad & A^4_{(2)}(S^2) \ \to \ A^4(S^2)\ . \end{aligned}$$

Using diagram (5), this proves Proposition 3.1 for $A^4_{(2)}$ and for $A^4_{(4)}$.

It remains to prove the statement for $A^2_{(2)}(X)$. This can be done as follows: the above implies there is a decomposition

$$A^4(X) = A^4(X)^\iota \oplus A^4_{(2)}(X) \ ,$$

and so the correspondence

$$\Gamma := \Pi_6^X \circ (\Gamma_\iota + \Delta_X) \quad \in A^4(X \times X)$$

acts trivially on 0–cycles:

$$\Gamma_\ast A^4(X) = 0 \quad \text{in } A^4(X) \ .$$

Using the Bloch–Srinivas argument [8], this implies Γ is supported on $D \times X$, where $D \subset X$ is a divisor. This holds for any MCK decomposition $\{\Pi_i^X\}$ for X. Let us now take an MCK decomposition of X that is self–dual (this exists: [36, Remark 2.8]). The transpose

$${}^t\Gamma = (\Gamma_\iota + \Delta_X) \circ \Pi_2^X \quad \in A^4(X \times X)$$

is supported on $X \times D$. As such, it does not act on $A^2_{hom}(X) = A^2_{AJ}(X)$:

$$({}^t\Gamma)_\ast = 0\colon \quad A^2_{hom}(X) \ \to \ A^2_{hom}(X) \ .$$

Since $A^2_{(2)}(X) = (\Pi_2^X)_\ast A^2_{hom}(X)$, this implies

$$(\Gamma_\iota + \Delta_X)_\ast = 0\colon \quad A^2_{(2)}(X) \ \to \ A^2(X) \ ,$$

proving the statement for $A^2_{(2)}(X)$. □

4. Main result

This section contains the proof of the main result of this note (Theorem 4.1). The global strategy is as follows: we start by proving (Theorem 4.2) that the involution ι has the expected action on $A^2_{(2)}(X)$. As will be apparent to the well–informed reader, the proof of Theorem 4.2 is directly inspired by Voisin's seminal work on the Bloch/Hodge equivalence for complete intersections [42], [43], [45], reasoning family–wise and spreading out correspondences to the family. At the heart of our proof is a result of Voisin [42] concerning the triviality of certain Chow groups of the fourfold relative fibre product $\mathcal{S}^{4/B}$ of the family of smooth quartic surfaces (Theorem 4.5). Voisin's result is conditional to the standard conjectures; however, we manage to bypass the need for the standard conjectures by only using Voisin's result in codimension 2, where it is unconditional.

Next, we consider the action of the involution ι on 0-cycles (Theorem 4.15). Here, we rely on the result for $A^2_{(2)}(X)$, plus the relations in $A^*(X)$ discovered by Shen–Vial (Theorem 2.17). In order to be able to use these relations, we apply once again (Proposition 4.16) Voisin's method of "spread". This second application of the method of "spread" is easier than the first, as everything happens on $\mathcal{S}\times_B\mathcal{S}$, rather than on the fourfold relative fibre product $\mathcal{S}^{4/B}$.

Here is the main result of this note:

Theorem 4.1. *Let $S\subset\mathbb{P}^3$ be a smooth quartic with Picard number $\rho(S)=1$, and let $X=S^{[2]}$. Let $\iota\in\mathrm{Aut}(X)$ be the non–symplectic involution of Beauville (cf. Proposition 2.22). Then*

$$\begin{aligned} \iota^*=-\,\mathrm{id}\colon\quad & A^i_{(2)}(X)\ \to\ A^i_{(2)}(X)\quad \textit{for } i=2,4\ ;\\ \iota^*=\mathrm{id}\colon\quad & A^4_{(j)}(X)\ \to\ A^4_{(j)}(X)\quad \textit{for } j=0,4\ . \end{aligned}$$

Theorem 4.1 is a combination of Theorems 4.2 and 4.15.

4.1. Action on $A^2_{(2)}$

Theorem 4.2. *Let X and ι be as in Theorem 4.1. Then*

$$\iota^*=-\,\mathrm{id}\colon\quad A^2_{(2)}(X)\ \to\ A^2_{(2)}(X)\ .$$

Proof. We consider the family

$$\mathcal{S}\ \to\ B$$

of all smooth quartics S_b with Picard number $\rho(S_b)=1$. Here the base B is a Zariski–open in a projective space $B\subset\bar{B}:=\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(4))$. We will denote

$$\mathcal{X}\ \to\ B$$

the family of Hilbert schemes, and we write $X_b=(S_b)^{[2]}$ for a fibre of $\mathcal{X}\to B$. It will be convenient to also consider the family $\mathcal{S}\times_B\mathcal{S}$ (whose fibres are products $S_b\times S_b$). This family is related to the family $\mathcal{X}\to B$ by a "hat" of morphisms over B

$$\begin{array}{ccccc} & & \widetilde{\mathcal{S}\times_B\mathcal{S}} & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & & & \mathcal{S}\times_B\mathcal{S} \end{array}\tag{6}$$

where $\widetilde{\mathcal{S}\times_B\mathcal{S}}$ is the blow–up of $\mathcal{S}\times_B\mathcal{S}$ with centre the relative diagonal. This diagram (6) gives rise to relative correspondences

$$\Psi\in A^4(\mathcal{X}\times_B\mathcal{S}\times_B\mathcal{S})\ ,\quad {}^t\Psi\in A^4(\mathcal{S}\times_B\mathcal{S}\times_B\mathcal{X})\ .$$

(For details on relative correspondences, cf. [26], and also [12], [11], [27].) Restricting to a fibre over $b\in B$, diagram (6) induces the familiar diagram

$$\begin{array}{ccc} & \widetilde{S_b\times S_b} & \\ \swarrow & & \searrow \\ X_b=(S_b)^{[2]} & & S_b\times S_b \end{array}$$

(where $\widetilde{S_b\times S_b}$ is the blow–up of $S_b\times S_b$ along the diagonal), and the (absolute) correspondences

$$\Psi_b\in A^4(X_b\times S_b\times S_b)\ ,\quad {}^t\Psi_b\in A^4(S_b\times S_b\times X_b)\ .$$

The morphism $\phi_b\colon X_b\to G$ (where G is the Grassmannian of lines in $\mathbb{P}^3$) extends to the family, in the sense that there is a morphism of B–schemes

$$\phi\colon\ \ \mathcal{X}\ \to\ \mathcal{G}:=G\times B$$

such that the restriction to a fibre gives ϕ_b. This implies that the Beauville involution also extends to the family: there exists an involution of B–schemes

$$\iota\colon\ \ \mathcal{X}\ \to\ \mathcal{X}\ ,$$

such that restriction to a fibre gives the involution $\iota_b\colon X_b\to X_b$ of Proposition 2.22.

Let $\Gamma_\iota\in A^4(\mathcal{X}\times_B\mathcal{X})$ denote the graph of ι. The fact that ι_b acts as -1 on $H^{2,0}(X_b)$ for all $b\in B$ implies that

$$(\Gamma_{\iota_b}+\Delta_{X_b})_\ast H^2_{tr}(X_b)=0\quad\forall b\in B\ .$$

In view of the refined Chow–Künneth decomposition (Theorem 2.18, which applies since X_b verifies the standard conjectures), this implies that

$$(7)\qquad (\Gamma_{\iota_b}+\Delta_{X_b})\circ(\pi_2^{X_b})=\gamma_b\quad\text{in } H^8(X_b\times X_b)\ ,\quad\forall b\in B\ ,$$

where γ_b is some cycle supported on $Y_b\times Y_b$, for $Y_b\subset X_b$ a divisor.

Let $\{\Pi_j^{\mathcal{X}}\}$ be a relative MCK decomposition as in Proposition 2.16. The relation (7) implies the following: the relative correspondence

$$\Gamma_0 := (\Gamma_\iota + \Delta_{\mathcal{X}}) \circ \Pi_2^{\mathcal{X}} \quad \in A^4(\mathcal{X} \times_B \mathcal{X})$$

has the property that for each $b \in B$, there exists a divisor $Y_b \subset X_b$ and a cycle γ_b supported on $Y_b \times Y_b$ such that

$$(\Gamma_0)|_{X_b \times X_b} = \gamma_b \quad \text{in } H^8(X_b \times X_b) \ .$$

At this point, we recall Voisin's "spread–out" result:

Proposition 4.3 (Voisin [42])**.** *Let $\mathcal{X} \to B$ be a smooth projective morphism of relative dimension n. Let $\Gamma \in A^n(\mathcal{X} \times_B \mathcal{X})$ be a cycle such that for all $b \in B$, there exists a closed algebraic subset $Y_b \subset X_b$ of codimension c, and a cycle $\gamma_b \in A_n(Y_b \times Y_b)$ such that*

$$\Gamma|_{X_b \times X_b} = \gamma_b \quad \textit{in } H^{2n}(X_b \times X_b) \ .$$

Then there exists a closed algebraic subset $\mathcal{Y} \subset \mathcal{X}$ of codimension c, and a cycle $\gamma \in A_*(\mathcal{Y} \times_B \mathcal{Y})$ such that

$$\Gamma|_{X_b \times X_b} = \gamma|_{X_b \times X_b} \quad \textit{in } H^{2n}(X_b \times X_b) \quad \forall b \in B.$$

Proof. This is a Hilbert schemes argument [42, Proposition 3.7]. ☐

Applying Proposition 4.3 to Γ_0, it follows there exists a divisor $\mathcal{Y} \subset \mathcal{X}$ and a cycle $\gamma \in A_*(\mathcal{Y} \times_B \mathcal{Y})$ such that

$$(\Gamma_0 - \gamma)|_{X_b \times X_b} = 0 \quad \text{in } H^8(X_b \times X_b) \ , \quad \forall b \in B \ .$$

That is, the relative correspondence

$$\Gamma_1 := \Gamma_0 - \gamma \quad \in A^4(\mathcal{X} \times_B \mathcal{X})$$

has the property of being homologically trivial on every fibre:

$$(\Gamma_1)|_{X_b \times X_b} = 0 \quad \text{in } H^8(X_b \times X_b) \ , \quad \forall b \in B.$$

At this point, it is convenient to consider the family $\mathcal{S} \times_B \mathcal{S}$ (of products of surfaces $S_b \times S_b$), rather than the family $\mathcal{X}$ (of Hilbert schemes $(S_b)^{[2]}$). That is, we consider the relative correspondence

$$\Gamma_2 := \Psi \circ \Gamma_1 \circ {}^t\Psi \quad \in A^4(\mathcal{S}^{4/B}) \ ,$$

where

$$\mathcal{S}^{4/B} := \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S} \ .$$

Since

$$(\Gamma_2)|_{(S_b)^4} = (\Psi_b) \circ ((\Gamma_1)|_{X_b \times X_b}) \circ {}^t\Psi_b \quad \text{in } A^4((S_b)^4)$$

(restriction and composition of relative correspondences commute), the relative correspondence Γ_2 has the property of being homologically trivial on every fibre:

$$(\Gamma_2)|_{(S_b)^4} = 0 \quad \text{in } H^8((S_b)^4) \ , \quad \forall b \in B.$$

Thanks to the following result, we can improve this fibre–wise homological vanishing to a global homological vanishing:

Proposition 4.4 (Voisin [42])**.** *Let $\Gamma \in A^4(\mathcal{S}^{4/B})$ be such that*

$$\Gamma|_{(S_b)^4} = 0 \quad \textit{in } H^8((S_b)^4) \quad \forall b \in B \ .$$

Then, after shrinking B to a non–empty Zariski–open subset, we have

$$\Gamma + \sum_{j=1}^{6} \psi_j = 0 \quad \textit{in } H^8(\mathcal{S}^{4/B}) \ ,$$

where ψ_1 (resp. $\psi_2, \ldots, \psi_6$) is the restriction of a cycle on $B \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathcal{S} \times_B \mathcal{S}$ (resp. on a copy of $B \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathcal{S} \times_B \mathcal{S}$, where the factors are permuted).

Proof. This is an extension of the Leray spectral sequence argument [42, Lemmas 3.11 and 3.12] to the fourfold relative fibre product $\mathcal{S}^{4/B}$. The fact that such an extension is true is stated in [42, Proof of Theorem 4.10], which also uses the fourfold relative fibre product $\mathcal{S}^{4/B}$. □

Applying Proposition 4.4 to Γ_2, we obtain a relative correspondence

$$\Gamma_3 := \Gamma_2 + \psi \quad \in A^4(\mathcal{S}^{4/B})$$

that is homologically trivial (i.e. $\Gamma_3 \in A^4_{hom}(\mathcal{S}^{4/B})$. Here ψ is a cycle of the form

$$\psi = \psi_1 + \cdots + \psi_6 \quad \in A^4(\mathcal{S}^{4/B}) \ ,$$

where $\psi_1, \ldots, \psi_6$ are restrictions of cycles coming from larger varieties as in Proposition 4.4.

We now come to the "trivial Chow groups" statement which is at the heart of our proof:

Theorem 4.5 (Voisin [42])**.** *Let $\mathcal{S} \to B$ denote the family of all smooth hypersurfaces $S_b \subset \mathbb{P}^3$ of a given degree d (where $d \geq 3$). Let*

$$i\colon \quad \mathcal{S}_0^{4/B} \ \subset \ \mathcal{S}^{4/B}$$

denote the complement of the small relative diagonal $\mathcal{S} \subset \mathcal{S}^{4/B}$. There exists a smooth proper surjective morphism

$$f\colon \widetilde{\mathcal{S}_0^{4/B}} \ \to \ \mathcal{S}_0^{4/B} \ ,$$

and a smooth quasi–projective variety M containing $\widetilde{\mathcal{S}_0^{4/B}}$ as a Zariski–open and such that

$$A^i_{hom}(M) = 0 \quad \forall i \leq 4 \ .$$

Proof. This is (contained in the proof of) [42, Proposition 4.11]. The variety M is constructed as a projective bundle over the variety $\widetilde{(\mathbb{P}^3)^4_0}$ of [42, Lemma 4.12]. $\square$

The relative correspondence Γ_3 being homologically trivial, we also have that

$$\Gamma_4 := f^\ast i^\ast(\Gamma_3) \quad \in A^4(\widetilde{\mathcal{S}_0^{4/B}})$$

is homologically trivial. Now, *if* we assume the Lefschetz standard conjecture (or the Voisin standard conjecture [42, Conjecture 1.6], [45, Conjecture 2.29]) is true, we can find a cycle

$$\bar{\Gamma}_4 \in A^4(M)$$

which restricts to Γ_4 and is homologically trivial. In view of Theorem 4.5, we then obtain a rational equivalence

$$\bar{\Gamma}_4 = 0 \quad \text{in } A^4(M) \ ,$$

and we can conclude the argument. However, as we do not want to end up with a conditional statement we need to avoid recourse to the Voisin (or Lefschetz) standard conjecture. To this end, we slightly modify the relative correspondence Γ_3, by defining

$$\Gamma_5 := \Gamma_3 \circ \Pi_2^{\mathcal{S}^{2/B}} \quad \in A^4(\mathcal{S}^{4/B}) \ ,$$

where $\Pi_2^{\mathcal{S}^{2/B}}$ is part of a "relative MCK decomposition" for $\mathcal{S}^{2/B}$ as in Proposition 2.15. Since Γ_3 is homologically trivial, Γ_5 is so as well:

$$\Gamma_5 \quad \in A^4_{hom}(\mathcal{S}^{4/B}) \ .$$

Now, using the factorization $\Pi_2^{\mathcal{S}^{2/B}} = P_1 \circ Q_1 + P_2 \circ Q_2$ of Proposition 2.15, we obtain a factorization

$$\Gamma_5 = \Gamma_3 \circ P_1 \circ Q_1 + \Gamma_3 \circ P_2 \circ Q_2 \quad \text{in } A^4(\mathcal{S}^{4/B}) \ ,$$

where $\Gamma_3 \circ P_j \in A^2(\mathcal{S}^{4/B})$ and $Q_j \in A^6(\mathcal{S}^{4/B})$ for $j = 1, 2$. Moreover,

$$\Gamma_3 \circ P_1 \ , \quad \Gamma_3 \circ P_2 \quad \in A^2_{hom}(\mathcal{S}^{4/B})$$

(since Γ_3 is homologically trivial). It follows that the pullbacks

$$\Gamma_{6,j} := f^* i^* (\Gamma_3 \circ P_j) \quad \in A^2(\widetilde{\mathcal{S}_0^{4/B}}) \quad (j = 1, 2)$$

are also homologically trivial. But the cycles $\Gamma_{6,j}$ can be extended to M, i.e. there exist

$$\bar{\Gamma}_{6,j} \in A^2(M) \quad (j = 1, 2)$$

which restrict to $\Gamma_{6,j}$ and are homologically trivial (indeed, the Voisin standard conjecture is true in codimension 2, essentially because the Hodge conjecture is true in codimension 1 [42, Lemma 2.1]). But then, using Theorem 4.5, we find that

$$\bar{\Gamma}_{6,j} = 0 \quad \text{in } A^2(M) \ ,$$

and so

$$\Gamma_{6,j} = 0 \quad \text{in } A^2(\widetilde{\mathcal{S}_0^{4/B}}) \quad (j = 1, 2) \ .$$

It follows that also

$$\Gamma_3 \circ P_j = i_* f_* f^* i^* (\Gamma_3 \circ P_j) = 0 \quad \text{in } A^2(\mathcal{S}^{4/B}) \quad (j = 1, 2)$$

(note that $i_* i^* = \text{id}$ on codimension 2 cycles, for dimension reasons), and so

$$\Gamma_5 = \Gamma_3 \circ P_1 \circ Q_1 + \Gamma_3 \circ P_2 \circ Q_2 = 0 \quad \text{in } A^4(\mathcal{S}^{4/B}) \ .$$

In particular, restricting to a fibre, we obtain

$$(\Gamma_5)|_{(S_b)^4} = 0 \quad \text{in } A^4((S_b)^4) \quad \forall b \in B \ . \tag{8}$$

We now make the connection with the relative correspondence Γ_0 that we started out with.

Claim 4.6. *We have*

$$((\Psi\circ\Gamma_0\circ{}^t\Psi+\gamma'+\psi')|_{(S_b)^4})_*=0\colon\quad A^2_{AJ}(S_b\times S_b)\ \to\ A^2_{AJ}(S_b\times S_b)\quad \forall b\in B\ ,$$

where γ' is a cycle supported on $\mathcal{D}\times_B\mathcal{D}$ for some divisor $\mathcal{D}\subset\mathcal{S}\times_B\mathcal{S}$, and ψ' is a sum of restrictions of cycles coming from larger varieties as in Proposition 4.4.

Proof. Recall that Γ_0 was defined as

$$\Gamma_0:=(\Gamma_\iota+\Delta_{\mathcal{X}})\circ\Pi_2^{\mathcal{X}}\quad\in A^4(\mathcal{X}\times_B\mathcal{X})\ ,$$

and Γ_1 was defined as the difference

$$\Gamma_1:=\Gamma_0-\gamma\quad\in A^4(\mathcal{X}\times_B\mathcal{X})\ ,$$

where γ is a cycle supported on $\mathcal{Y}\times_B\mathcal{Y}$ for some divisor $\mathcal{Y}\subset\mathcal{X}$. The next step was to define

$$\Gamma_2:=\Psi\circ\Gamma_1\circ{}^t\Psi\quad\in A^4(\mathcal{S}^{4/B})\ ,$$

and then

$$\Gamma_3:=\Gamma_2+\psi\quad\in A^4(\mathcal{S}^{4/B})\ ,$$

where ψ is a sum of restrictions of cycles coming from larger varieties as in Proposition 4.4. This implies that

$$\Gamma_3=\Psi\circ\Gamma_0\circ{}^t\Psi+\gamma_1+\psi\quad\in A^4(\mathcal{S}^{4/B})\ ,\tag{9}$$

where $\gamma_1=\Psi\circ\gamma\circ{}^t\Psi$ is supported on $\mathcal{D}\times_B\mathcal{D}$ for some divisor $\mathcal{D}\subset\mathcal{S}\times_B\mathcal{S}$. The relative correspondence Γ_5 was defined as $\Gamma_5:=\Gamma_3\circ\Pi_2^{\mathcal{S}^{2/B}}$, and so (by substituting using equality (9)) we find an equality

$$\Gamma_5=\Psi\circ\Gamma_0\circ{}^t\Psi\circ\Pi_2^{\mathcal{S}^{2/B}}+\gamma'+\psi'\quad\in A^4(\mathcal{S}^{4/B})\ ,\tag{10}$$

where $\gamma':=\gamma_1\circ\Pi_2^{\mathcal{S}^{2/B}}$ is supported on $\mathcal{D}\times_B\mathcal{D}$, and $\psi':=\psi\circ\Pi_2^{\mathcal{S}^{2/B}}$ is a sum of restrictions of cycles coming from larger varieties as in Proposition 4.4. But we know that Γ_5 is rationally trivial on each fibre (equality (8)), and so equality (10) implies

$$(\Psi\circ\Gamma_0\circ{}^t\Psi\circ\Pi_2^{\mathcal{S}^{2/B}})|_{(S_b)^4}+(\gamma'+\psi')|_{(S_b)^4}=0\quad\text{in }A^4((S_b)^4)\quad\forall b\in B\ .\tag{11}$$

Applying both sides of the equality of correspondences (11) to codimension 2 cycles implies Claim 4.6, in view of the following lemma:

Lemma 4.7. *For any $b\in B$, there is equality*

$$\begin{aligned}&\bigl((\Psi\circ\Gamma_0\circ{}^t\Psi\circ\Pi_2^{\mathcal{S}^{2/B}})|_{(S_b)^4}\bigr)_\ast\\ &\quad=\bigl((\Psi\circ\Gamma_0\circ{}^t\Psi)|_{(S_b)^4}\bigr)_\ast\colon\quad A^2_{AJ}(S_b\times S_b)\ \to\ A^2_{AJ}(S_b\times S_b)\ .\end{aligned}$$

Proof. We start by observing there is a commutativity relation

$$(\Pi_2^{X_b}\circ{}^t\Psi_b)_\ast=(\Pi_2^{X_b}\circ{}^t\Psi_b\circ\Pi_2^{(S_b)^2})_\ast\colon\quad A^2_{hom}(S_b\times S_b)\ \to\ A^2_{hom}(X_b)\ . \tag{12}$$

Indeed, we have seen (Lemma 2.11) that

$$\begin{aligned}&({}^t\Psi)_\ast A^2_{(2)}(S_b\times S_b)\ \subset\ A^2_{(2)}(X_b)\ ,\\ &({}^t\Psi_b)_\ast\Bigl(A^2_{(0)}(S_b\times S_b)\cap A^2_{hom}(S_b\times S_b)\Bigr)\ \subset\ A^2_{(0)}(X_b)\cap A^2_{hom}(X_b)\ ,\end{aligned}$$

and so

$$(\Pi_2^{X_b}\circ{}^t\Psi_b)_\ast=0\colon\quad A^2_{(0)}(S_b\times S_b)\cap A^2_{hom}(S_b\times S_b)\ \to\ A^2(X_b)\ .$$

Since

$$A^2_{hom}(S_b\times S_b)=A^2_{(2)}(S_b\times S_b)\oplus A^2_{(0)}(S_b\times S_b)\cap A^2_{hom}(S_b\times S_b)\ ,$$

it follows that

$$\begin{aligned}&\mathrm{Im}\bigl(A^2_{hom}(S_b\times S_b)\ \xrightarrow{({}^t\Psi_b)_\ast}\ A^2(X_b)\ \xrightarrow{(\Pi_2^{X_b})_\ast}\ A^2_{(2)}(X_b)\bigr)=\\ &\mathrm{Im}\bigl(A^2_{(2)}(S_b\times S_b)\ \xrightarrow{({}^t\Psi_b)_\ast}\ A^2(X_b)\ \xrightarrow{(\Pi_2^{X_b})_\ast}\ A^2_{(2)}(X_b)\bigr)=\\ &\mathrm{Im}\bigl(A^2(S_b\times S_b)\ \xrightarrow{(\Pi_2^{(S_b)^2})_\ast}\ A^2_{(2)}(S_b\times S_b)\ \xrightarrow{({}^t\Psi_b)_\ast}\ A^2(X_b)\ \xrightarrow{(\Pi_2^{X_b})_\ast}\ A^2_{(2)}(X_b)\bigr)\ .\end{aligned}$$

This proves equality (12).

To prove Lemma 4.7, one notes that the left–hand–side of Lemma 4.7 is

$$(\Psi_b\circ(\Gamma_0)|_{(S_b)^4}\circ{}^t\Psi_b\circ\Pi_2^{(S_b)^2})_\ast\colon\quad A^2_{hom}(S_b\times S_b)\ \to\ A^2_{hom}(S_b\times S_b)\ .$$

Plugging in the definition of Γ_0, we obtain

$$\begin{aligned}&(\Psi_b\circ(\Gamma_0)|_{(S_b)^4}\circ{}^t\Psi_b\circ\Pi_2^{(S_b)^2})_\ast\\ &\quad=(\Psi_b\circ(\Gamma_{\iota_b}+\Delta_{X_b})\circ\Pi_2^{X_b}\circ{}^t\Psi_b\circ\Pi_2^{(S_b)^2})_\ast\\ &\quad=(\Psi_b\circ(\Gamma_{\iota_b}+\Delta_{X_b})\circ\Pi_2^{X_b}\circ{}^t\Psi_b)_\ast\\ &\quad=(\Psi_b\circ(\Gamma_0)|_{X_b\times X_b}\circ{}^t\Psi_b)_\ast\colon\quad A^2_{hom}((S_b)^2)\ \to\ A^2_{hom}((S_b)^2)\ ,\end{aligned}$$

where the second equality is thanks to the relation (12). ☐

☐

This ends the proof of Claim 4.6. The next step is the following:

Claim 4.8. *For all $b \in B$, we have*

$$((\Psi \circ \Gamma_0 \circ {}^t\Psi + \psi')|_{(S_b)^4})_* = 0\colon \quad A^2_{AJ}(S_b \times S_b) \ \to \ A^2_{AJ}(S_b \times S_b) \ ,$$

where ψ' is a sum of restrictions of cycles coming from larger varieties as in Proposition 4.4.

Proof. This follows from Claim 4.6, provided we manage to convince ourselves that

$$(13) \qquad (\gamma'|_{(S_b)^4})_* = 0\colon \quad A^4(S_b \times S_b) \ \to \ A^4(S_b \times S_b) \quad \forall b \in B \ .$$

For general $b \in B$, (13) is clearly true: by construction, γ' is supported on $\mathcal{D} \times_B \mathcal{D}$, and for general b the fibre $(S_b)^4$ will meet the divisor $\mathcal{D}$ in a divisor $D_b \subset (S_b)^2$; since a 0–cycle on $(S_b)^2$ can avoid the divisor D_b, the restriction $(\gamma')|_{(S_b)^4}$ does not act on 0–cycles.

Now let $b_0 \in B$ be any given point. The Hilbert schemes argument (Proposition 4.3) can be made relative to b_0, to the effect that one obtains a divisor $\mathcal{D}$ in general position with respect to the fibre $(S_{b_0})^4$. As above, one then obtains the vanishing (13) for the fibre over b_0. ☐

The next step is a further improvement on Claim 4.8:

Claim 4.9. *For all $b \in B$, we have*

$$((\Psi \circ \Gamma_0 \circ {}^t\Psi)|_{(S_b)^4})_* = 0\colon \quad A^2_{AJ}(S_b \times S_b) \ \to \ A^2_{AJ}(S_b \times S_b) \ .$$

Proof. This follows from Claim 4.8, provided we manage to convince ourselves that

$$(14) \qquad (\psi|_{(S_b)^4})_* = 0\colon \quad A^2_{AJ}(S_b \times S_b) \ \to \ A^2_{AJ}(S_b \times S_b) \quad \forall b \in B \ ,$$

where $\psi \in A^4(\mathcal{S}^{4/B})$ is a cycle which is coming from larger varieties as in Proposition 4.4.

For a given $b \in B$, let us write

$$\begin{aligned} \psi_b &:= \psi|_{(S_b)^4} \quad && \in A^4((S_b)^4) \ , \\ \psi_{b,j} &:= (\psi_j)|_{(S_b)^4} \quad && \in A^4((S_b)^4) \quad (j = 1, \ldots, 6) \ , \end{aligned}$$

where

$$\psi_b = \psi_{b,1} + \cdots + \psi_{b,6} \quad \text{in } A^4((S_b)^4) ,$$

and $\psi_{b,1}$ (resp. $\psi_{b,2}, \ldots, \psi_{b,6}$) is the restriction of a cycle on $\mathbb{P}^3 \times \mathbb{P}^3 \times S_b \times S_b$ (resp. on $\mathbb{P}^3 \times S_b \times \mathbb{P}^3 \times S_b$, …, resp. on $S_b \times S_b \times \mathbb{P}^3 \times \mathbb{P}^3$).

Obviously,

$$(\psi_{b,j})_\ast = 0\colon \quad A^2_{AJ}(S_b \times S_b) \ \to \ A^2_{AJ}(S_b \times S_b) \quad \text{for } j = 1, 6$$

(Indeed, the action of $\psi_{b,1}$ factors over $A^4_{AJ}(\mathbb{P}^3 \times \mathbb{P}^3) = 0$, and the action of $\psi_{b,6}$ factors over $A^2_{AJ}(\mathbb{P}^3 \times \mathbb{P}^3) = 0$).

For the $\psi_{b,j}$ with $j = 2, \ldots, 5$, some more work is needed. We will treat the case of $\psi_{b,2}$ in detail (the argument for the cases $j = 3, 4, 5$ is the same, up to permutation of the factors). Since

$$A^4(\mathbb{P}^3 \times S_b \times \mathbb{P}^3 \times S_b) = \bigoplus_{k+\ell+m=4} A^k(S_b \times S_b) \otimes A^\ell(\mathbb{P}^3) \otimes A^m(\mathbb{P}^3) ,$$

we can write $\psi_{b,2}$ uniquely as a sum

$$\psi_{b,2} = \sum_{k+\ell+m=4} (h_b)^\ell \times a_{b,k,\ell,m} \times (h_b)^m \quad \text{in } A^4((S_b)^4) , \tag{15}$$

where $h_b \in A^1(S_b)$ is an ample class with $(h_b)^2 = 16$ in $H^4(S_b)$, and $a_{b,k,\ell,m} \in A^k(S_b \times S_b)$ is understood to be in the 2nd and 4th factor. (More precisely, expression (15) should be taken to mean that

$$\psi_{b,2} = \sum_{k+\ell+m=4} (p_1)^\ast(h_b)^\ell \cdot (p_3)^\ast(h_b)^m \cdot (p_{24})^\ast(a_{b,k,\ell,m}) \quad \text{in } A^4((S_b)^4) ,$$

where the p_i and p_{24} denote the obvious projections.)

Likewise, the other $\psi_{b,j}$ decompose as sums in $A^4((S_b)^4)$:

$$\begin{aligned}
\psi_{b,1} &= \sum_{k+\ell+m=4} (h_b)^\ell \times (h_b)^m \times a^1_{b,k,\ell,m} ,\\
\psi_{b,3} &= \sum_{k+\ell+m=4} (p_{23})^\ast(a^3_{b,k,\ell,m}) \cdot (p_1)^\ast(h_b)^\ell \cdot (p_4)^\ast(h_b)^m ,\\
\psi_{b,4} &= \sum_{k+\ell+m=4} (p_{14})^\ast(a^4_{b,k,\ell,m}) \cdot (p_2)^\ast(h_b)^\ell \cdot (p_3)^\ast(h_b)^m ,\\
\psi_{b,5} &= \sum_{k+\ell+m=4} (p_{13})^\ast(a^5_{b,k,\ell,m}) \cdot (p_2)^\ast(h_b)^\ell \cdot (p_4)^\ast(h_b)^m ,\\
\psi_{b,6} &= \sum_{k+\ell+m=4} a^6_{b,k,\ell,m} \times (h_b)^\ell \times (h_b)^m ,
\end{aligned} \tag{16}$$

where $a^j_{b,k,\ell,m} \in A^k(S_b \times S_b)$.

Lemma 4.10. *Let $a_{b,k,\ell,m} \in A^k(S_b \times S_b)$ be as in expression (15). We have*

$$(\psi_{b,2})_* = ((h_b)^2 \times a_{b,2,2,0} \times (h_b)^0)_* \colon \quad A^2_{AJ}(S_b \times S_b) \ \to\ A^2_{AJ}(S_b \times S_b)\ .$$

Proof. Suppose $(\ell, m) \neq (2,0)$. Thanks to Lieberman's lemma, there is a factorization

$$\begin{array}{ccc} A^3_{AJ}(\mathbb{P}^3 \times S_b) & \xrightarrow{(h^\ell \times a_{b,k,\ell,m} \times h^m)_*} & A^2_{AJ}(\mathbb{P}^3 \times S_b) \\ \uparrow & & \downarrow \\ A^2_{AJ}(S_b \times S_b) & \xrightarrow{((h_b)^\ell \times a_{b,k,\ell,m} \times (h_b)^m)_*} & A^2_{AJ}(S_b \times S_b) \end{array}$$

(where $h \in A^1(\mathbb{P}^3)$ denotes an ample class restricting to $h_b \in A^1(S_b)$). But

$$A^3_{AJ}(\mathbb{P}^3 \times S_b) = A^1(\mathbb{P}^3) \otimes A^2_{AJ}(S_b)\ ,$$

i.e. any $c \in A^3_{AJ}(\mathbb{P}^3 \times S_b)$ can be written $c = h \times d$ with $d \in A^2_{AJ}(S_b)$. It follows that

$$(h^\ell \times a_{b,k,\ell,m} \times h^m)_*(c) = (h^\ell \times h^m)_*(h) \times (a_{b,k,\ell,m})_*(d) = 0 \quad \in A^2(\mathbb{P}^3 \times S_b)$$
$$\text{for } \ell \neq 2$$

(since clearly $(h^\ell \times h^m)_*(h) = 0$ in $A^*(\mathbb{P}^3)$ for all $\ell \neq 2$). Suppose now $\ell = 2$, and so (by hypothesis) $m = 1$ or 2. Then

$$(h^2 \times a_{b,k,2,m} \times h^m)_*(c) \ \in A^m(\mathbb{P}^3) \otimes A^{2-m}(S_b) \ \subset\ A^2(\mathbb{P}^3 \times S_b)\ .$$

But $(h^2 \times a_{b,k,2,m} \times h^m)_*(c)$ is also Abel–Jacobi trivial, and so

$$(h^2 \times a_{b,k,2,m} \times h^m)_*(c) \ \in \Big(A^m(\mathbb{P}^3) \otimes A^{2-m}(S_b)\Big) \cap A^2_{AJ}(\mathbb{P}^3 \times S_b) = 0$$
$$\text{in } A^2(\mathbb{P}^3 \times S_b)\ .$$

∎

Lemma 4.11. *Let $a_{b,k,\ell,m} \in A^k(S_b \times S_b)$ be as in expression (15). Then $a_{b,k,\ell,m} \in H^{2k}(S_b \times S_b)$ is in the image of the natural map*

$$\Big(A^k(\mathbb{P}^3 \times S_b) \oplus A^k(S_b \times \mathbb{P}^3)\Big) \ \to\ A^k(S_b \times S_b) \ \to\ H^{2k}(S_b \times S_b)\ ,$$

i.e. there exist $\alpha_{b,k,\ell,m,p} \in A^p(S_b)$ *and* $\alpha_{b,k,\ell,m,p'} \in A^p(S_b)$ *such that there is equality of cycles modulo homological equivalence*

$$a_{b,k,\ell,m} = \sum_{p=0}^{k} \alpha_{b,k,\ell,m,p} \times (h_b)^{k-p} + \sum_{p'=0}^{k} (h_b)^{k-p'} \times \alpha_{b,k,\ell,m,p'} \quad \textit{in } H^{2k}(S_b \times S_b)\ .$$

Proof. By construction, $\psi_b \in A^4((S_b)^4)$ is homologically trivial:

$$\psi_b = \psi_{b,1} + \psi_{b,2} + \cdots + \psi_{b,6} = 0 \quad \text{in } H^8((S_b)^4)\ .$$

In particular, intersecting and pushing forward we find a vanishing

$$(17) \qquad \frac{1}{16}(p_{24})_*\Big(\psi_b \cdot (p_1)^*(h_b)^{2-\ell} \cdot (p_3)^*(h_b)^{2-m}\Big) = 0 \quad \text{in } H^4(S_b \times S_b)\ .$$

On the other hand,

$$\begin{aligned} \frac{1}{16}(p_{24})_*\Big(\psi_b \cdot (p_1)^*(h_b)^{2-\ell} \cdot (p_3)^*(h_b)^{2-m}\Big) = a_{b,k,\ell,m} + \\ \frac{1}{16}\sum_{j\neq 2}(p_{24})_*\Big(\psi_{b,j} \cdot (p_1)^*(h_b)^{2-\ell} \cdot (p_3)^*(h_b)^{2-m}\Big) \quad \text{in } A^2(S_b \times S_b)\ . \end{aligned}$$

Combined with the vanishing (17), we obtain

$$a_{b,k,\ell,m} = -\frac{1}{16}\sum_{j\neq 2}(p_{24})_*\Big(\psi_{b,j} \cdot (p_1)^*(h_b)^{2-\ell} \cdot (p_3)^*(h_b)^{2-m}\Big) \quad \text{in } H^4(S_b \times S_b)\ .$$

But we have seen (expressions (16)) that the $\psi_{b,j}$ for $j \neq 2$ contain an element $(h_b)^i$ in either the 2nd or 4th factor, and so this proves Lemma 4.11. More in detail: let us consider $j = 1$. Using (16), we find

$$\begin{aligned} &(p_{24})_*\Big(\psi_{b,1} \cdot (p_1)^*(h_b)^{2-\ell} \cdot (p_3)^*(h_b)^{2-m}\Big) \\ &\quad = (p_{24})_*\Big(\sum_{k+\ell'+m'=4} ((h_b)^{\ell'} \times (h_b)^{m'} \times a^1_{b,k,\ell',m'}) \cdot (p_1)^*(h_b)^{2-\ell} \\ &\qquad \cdot (p_3)^*(h_b)^{2-m}\Big) \\ &\quad = \sum_{k+\ell'+m'=4} (p_{24})_*\Big(((h_b)^{2-\ell+\ell'} \times (h_b)^{m'} \times a^1_{b,k,\ell',m'}) \cdot (p_3)^*(h_b)^{2-m}\Big) \\ &\quad = \sum_{\substack{k+\ell'+m'=4 \\ \ell'=\ell}} (p_{24})_*\Big(((h_b)^2 \times [S_b] \times a^1_{b,k,\ell,m'}) \cdot (p_3)^*(h_b)^{2-m} \end{aligned}$$

$$
\begin{aligned}
&\quad \cdot (p_{24})^*\big((h_b)^{m'}\times [S_b]\big)\Big)\\
&= \sum_{k+\ell+m'=4} (p_{24})_*\Big((h_b)^2\times [S_b]\times a^1_{b,k,\ell,m'}\big)\cdot (p_3)^*(h_b)^{2-m}\Big)\\
&\quad \cdot \Big((h_b)^{m'}\times [S_b]\Big)\\
&= \sum_{k+\ell+m'=4} (p_{24})_*\Big((h_b)^2\times [S_b]\times (\text{something})\Big)\cdot \Big((h_b)^{m'}\times [S_b]\Big)\\
&= \sum_{k+\ell+m'=4} \Big([S_b]\times (\text{something})\Big)\cdot \Big((h_b)^{m'}\times [S_b]\Big)\\
&= \sum_{k+\ell+m'=4} (h_b)^{m'}\times (\text{something}) \quad \text{in } A^2(S_b\times S_b)\ .
\end{aligned}
$$

This shows that

$$(p_{24})_*\Big(\psi_{b,1}\cdot (p_1)^*(h_b)^{2-\ell}\cdot (p_3)^*(h_b)^{2-m}\Big)$$

can be written in the form of the right–hand–side of Lemma 4.11. The proof for the other $\psi_{b,j}$ is similar. □

We now upgrade (a weak version of the $k=2$ part of) the equality of Lemma 4.11 to rational equivalence:

Lemma 4.12. *Let $a_{b,k,\ell,m}\in A^k(S_b\times S_b)$ be as in expression (15). Then $a_{b,2,\ell,m}\in A^2(S_b\times S_b)$ can be written*

$$a_{b,2,\ell,m} = \gamma_{b,2,\ell,m,0}+\gamma_{b,2,\ell,m,1}+\gamma_{b,2,\ell,m,2}\quad \textit{in } A^2(S_b\times S_b)\ ,$$

where $\gamma_{b,2,\ell,m,j}$ is supported on $V_{b,2,\ell,m,j}\times W_{b,2,\ell,m,j}$ for $j=0,1,2$, and $V_{b,2,\ell,m,j}\subset S_b$ is closed of codimension j and $W_{b,2,\ell,m,j}\subset S_b$ is closed of codimension $2-j$.

Proof. This is another application of the technique of "spread" developed in [42], [43]. The application in this instance is easier than the above, for we only need to reason on the fibre product $\mathcal{S}\times_B\mathcal{S}$, and *not* on the fourfold relative fibre product $\mathcal{S}^{4/B}$.

The first thing to do is to find a relative cycle inducing the $a_{b,2,\ell,m}$ for the various b. This can be done as follows: let us define

$$a_{2,\ell,m} := \frac{1}{16}(p_{24})_*\Big(\psi_2\cdot (p_1)^*(H^{2-\ell})\cdot (p_3)^*(H^{2-m})\Big)\quad \in A^2(\mathcal{S}\times_B\mathcal{S})\ .$$

(Here $H \in A^1(\mathcal{S})$ denotes a relatively ample class with $(H^2)|_{S_b} = 16$ in $H^4(S_b)$, and p_i, p_{24} denote the obvious projections.) The relative cycle $a_{2,\ell,m}$ has the property that the restriction to a fibre is

$$\begin{aligned} a_{2,\ell,m}|_{(S_b)^2} &= \frac{1}{16}(p_{24})_*\Big(\psi_{b,2}\cdot (p_1)^*((h_b)^{2-\ell})\cdot (p_3)^*((h_b)^{2-m})\Big) \\ &= a_{b,2,\ell,m} \quad \in A^2(S_b\times S_b) \quad \forall b\in B\ , \end{aligned}$$

in view of expression (15).

The next thing to do is to find a fibrewise homological property of the relative cycle. Lemma 4.11 implies (after regrouping of the summands) that for each $b \in B$, there exist closed subvarieties $V_{b,2,\ell,m,j}$ and $W_{b,2,\ell,m,j} \subset S_b$ of codimension j resp. $2-j$ ($j = 0, 1, 2$), and cycles

$$\gamma_{b,2,\ell,m,j} \in A_2(V_{b,2,\ell,m,j} \times W_{b,2,\ell,m,j})\ ,$$

such that

$$a_{2,\ell,m}|_{(S_b)^2} = \gamma_{b,2,\ell,m,0} + \gamma_{b,2,\ell,m,1} + \gamma_{b,2,\ell,m,2} \quad \text{in } H^4(S_b\times S_b)\ .$$

Applying Proposition 4.13 below, these fibrewise cycles can be spread out to the family: there exist subvarieties

$$\mathcal{V}_{2,\ell,m,j} \subset \mathcal{S}\ ,\quad \mathcal{W}_{2,\ell,m,j} \subset \mathcal{S}$$

of codimension j resp. $2-j$, and relative cycles

$$\gamma_{2,\ell,m,j} \in A_*(\mathcal{V}_{2,\ell,m,j} \times_B \mathcal{W}_{2,\ell,m,j}) \quad (j = 0, 1, 2)\ ,$$

such that for each $b \in B$, there is a homological equivalence

$$a_{2,\ell,m}|_{(S_b)^2} = \Big(\gamma_{2,\ell,m,0} + \gamma_{2,\ell,m,1} + \gamma_{2,\ell,m,2}\Big)|_{(S_b)^2} \quad \text{in } H^4(S_b\times S_b)\ .$$

In other words, the relative cycle

$$C_0 := a_{2,\ell,m} - \gamma_{2,\ell,m,0} + \gamma_{2,\ell,m,1} + \gamma_{2,\ell,m,2} \quad \in A^2(\mathcal{S}\times_B \mathcal{S})$$

has the property of being homologically trivial on every fibre:

$$C_0|_{S_b\times S_b} = 0 \quad \text{in } H^4(S_b\times S_b)\ ,\quad \forall\, b\in B\ . \tag{18}$$

Applying the Leray spectral sequence argument [42, Lemmas 3.11 and 3.12], up to shrinking the base B one can render C_0 globally homologically trivial, i.e.

$$C_1 := C_0 + \theta = 0 \quad \text{in } H^4(\mathcal{S} \times_B \mathcal{S}) \ ,$$

where $\theta \in A^2(\mathcal{S} \times_B \mathcal{S})$ is the restriction of a cycle in $A^2(B \times \mathbb{P}^3 \times \mathbb{P}^3)$. But

$$A^2_{hom}(\mathcal{S} \times_B \mathcal{S}) = 0$$

([42, Proposition 3.13], combined with the fact that the Voisin standard conjecture [42, Conjecture 1.6] is true in codimension 2), and so C_1 is rationally trivial. In particular, restricting to a fibre one obtains

$$(C_0 + \theta)|_{S_b \times S_b} = 0 \quad \text{in } A^2(S_b \times S_b) \ , \quad \forall \, b \in B \ . \tag{19}$$

The restriction $\theta|_{S_b \times S_b}$ (coming from $A^2(\mathbb{P}^3 \times \mathbb{P}^3)$) is of the form $\sum_j (h_b)^j \times (h_b)^{2-j}$. Thus (after modifying the $V_{b,2,\ell,m,j}$ and $W_{b,2,\ell,m,j}$), we find that

$$C_0|_{S_b \times S_b} = \Big(a_{2,\ell,m} - \gamma_{2,\ell,m,0} - \gamma_{2,\ell,m,1} - \gamma_{2,\ell,m,2} \Big)|_{S_b \times S_b} = 0 \quad \text{in } A^2(S_b \times S_b) \ , \quad \forall b \in B \ ,$$

proving Lemma 4.12 for the smaller base B. To extend to the original B, one invokes [45, Lemma 3.2].

(Alternatively, using the approach of [43], one could forsake the Leray spectral sequence argument in the above proof, and skip directly from equality (18) to equality (19) by invoking [43, Proposition 1.6].)

Proposition 4.13. *Let $\mathcal{X} \to B$ be a smooth projective morphism of relative dimension n, and let $\Gamma \in A^n(\mathcal{X} \times_B \mathcal{X})$. Assume that for the very general $b \in B$, there exist closed subvarieties $V_{b,j} \subset X_b$, $W_{b,j} \subset X_b$ of codimension j resp. $n-j$, and cycles $\gamma_{b,j} \in A_n(V_{b,j} \times W_{b,j})$ such that*

$$\Gamma|_{X_b \times X_b} = \gamma_{b,0} + \cdots + \gamma_{b,n} \quad \text{in } H^{2n}(X_b \times X_b) \ .$$

Then there exist closed subvarieties $\mathcal{V}_j \subset \mathcal{X}$, $\mathcal{W}_j \subset \mathcal{X}$ of codimension j resp. $n-j$, and cycles $\gamma_j \in A_*(\mathcal{V}_j \times_B \mathcal{W}_j)$, such that

$$\Gamma|_{X_b \times X_b} = (\gamma_0 + \cdots + \gamma_n)|_{X_b \times X_b} \quad \text{in } H^{2n}(X_b \times X_b) \ ,$$

for all $b \in B$.

Proof. This is the same Hilbert schemes argument as [42, Proposition 3.7] (i.e., Proposition 4.3 above). The point is that the data of all the $V_{b,j}, W_{b,j}, \gamma_{b,j}$ can be encoded by a countably infinite set of varieties. Since by assumption, this countably infinite set dominates B, one of the varieties must dominate B. □

□

We are now in position to wrap up the proof of Claim 4.9. We have

$$\begin{aligned}(\psi_{b,2})_* &= ((h_b)^2 \times a_{b,2,2,0} \times [S_b])_* \\ &= \Big((h_b)^2 \times (\sum_{j=0}^{2} \gamma_{b,2,2,0,j}) \times [S_b]\Big)_* \\ &\overset{(*)}{=} 0 \quad : A^2_{AJ}((S_b)^2) \ \to \ A^2_{AJ}((S_b)^2)\ .\end{aligned}$$

Here, the first equality is Lemma 4.10, and the second equality is Lemma 4.12. As for the equality labelled $(*)$, this is true for dimension reasons: indeed, there is a factorization

$$\begin{array}{ccc} A^2_{AJ}((h_b)^2 \times \widetilde{V_{b,2,\ell,m,j}}) & \to & A^j_{AJ}(\widetilde{W_{b,2,\ell,m,j}} \times S_b) \\ \uparrow & & \downarrow \\ A^2_{AJ}(S_b \times S_b) & \xrightarrow{((h_b)^2 \times \gamma_{b,2,2,0,j} \times [S_b])_*} & A^2_{AJ}(S_b \times S_b)\ . \end{array}$$

(Here the $\widetilde{V}$ and $\widetilde{W}$ denote resolutions of singularities.) The upper–right corner $A^j_{AJ}()$ is 0 unless $j = 2$. However, for $j = 2$ the dimension of $(h_b)^2 \times \widetilde{V_{b,2,\ell,m,j}}$ is 0 and so in this case the upper–left corner is 0. This proves equality $(*)$ for general $b \in B$. For any given $b_0 \in B$, the cycles $\gamma_{2,2,0,j}$ can be moved in general position with respect to the fibre $S_{b_0} \times S_{b_0}$, and then the above argument applies to prove $(*)$ for S_{b_0}.

We have now proven that the correspondence $\psi_{b,2}$ does not act on $A^2_{AJ}((S_b)^2)$ for all $b \in B$. The same argument also proves that the correspondences $\psi_{b,j}$, $j = 3, 4, 5$ do not act on $A^2_{AJ}((S_b)^2)$ (the argument is only notationally different), and so

$$(\psi_b)_* = 0\colon \quad A^2_{AJ}((S_b)^2) \ \to \ A^2_{AJ}((S_b)^2) \quad \forall b \in B\ .$$

This proves equality (14), and hence also Claim 4.9. □

The last step is to return from the product $S_b \times S_b$ to the Hilbert scheme X_b:

Claim 4.14. *For all $b \in B$, we have*

$$((\Gamma_0)|_{X_b\times X_b}))_* = 0\colon\quad A^2_{AJ}(X_b) \ \to\ A^2(X_b)\ .$$

Proof. This is immediate from Claim 4.9, since

$$({}^t\Psi_b)_*(\Psi_b)_* = \mathrm{id}\colon\quad A^2_{AJ}(X_b) \ \to\ A^2_{AJ}(X_b)\ .$$

□

By definition of Γ_0, Claim 4.14 implies that

$$(\Gamma_{\iota_b} + \Delta_{X_b})_*(\Pi_2^{X_b})_* = 0\colon\quad A^2_{AJ}(X_b) \ \to\ A^2(X_b) \quad \forall b \in B\ .$$

But $\Pi_2^{X_b}$ is a projector on $A^2_{(2)}(X_b) \subset A^2_{AJ}(X_b)$ and so

$$(\Gamma_{\iota_b} + \Delta_{X_b})_* = 0\colon\quad A^2_{(2)}(X_b) \ \to\ A^2(X_b) \quad \forall b \in B\ ,$$

which concludes the proof of Theorem 4.2. □

4.2. Action on A^4

In this subsection, we finish the proof of our main result (Theorem 4.1), by checking that the involution ι has the expected action on A^4:

Theorem 4.15. *Let X and ι be as in Theorem 4.1. Then*

$$\iota^*\colon\quad A^4_{(j)}(X) \ \to\ A^4(X) = \begin{cases} \mathrm{id} & \text{if } j = 0,4\ ;\\ -\mathrm{id} & \text{if } j = 2\ .\end{cases}$$

Proof. The case $j = 0$ is easy: there is a ι–invariant ample divisor D (Proposition 2.22). As D is ample, the intersection D^4 is non–zero and so (since $D^4 \in A^4_{(0)}(X)$, and $A^4_{(0)}(X)$ is one–dimensional)

$$A^4_{(0)}(X) = \mathbb{Q}[D^4]$$

is ι–invariant.

Next, let us consider the case $j = 4$. As we have seen (Theorem 2.17), Shen–Vial have proven the multiplication map

$$A^2_{(2)}(X) \otimes A^2_{(2)}(X) \ \to\ A^4_{(4)}(X)$$

is surjective. Given $b \in A^4_{(4)}(X)$, we can thus write

$$b = a_1^1 \cdot a_2^1 + \cdots + a_1^r \cdot a_2^r \quad \text{in } A^4(X) \ ,$$

where $a_1^k, a_2^k \in A^2_{(2)}(X)$. But then, using Theorem 4.2 we find

$$\iota^*(b) = \sum_{k=1}^r \iota^*(a_1^k) \cdot \iota^*(a_2^k) = \sum_{k=1}^r (-a_1^k) \cdot (-a_2^k) = \sum_{k=1}^r a_1^k \cdot a_2^k = b \quad \text{in } A^4(X) \ .$$

It remains to prove Theorem 4.15 for $j = 2$. As we have seen (Theorem 2.17), Shen–Vial have established an isomorphism

$$\cdot\, l\colon \quad A^2_{(2)}(X) \ \xrightarrow{\cong}\ A^4_{(2)}(X) \ . \tag{20}$$

Theorem 4.15 now follows, provided we understand the action of ι on the class $l \in A^2(X)$. To this end, we will prove the following:

Proposition 4.16. *Let X and ι be as in Theorem 4.1. Let $l \in A^2_{(0)}(X)$ be the class as in Theorem 2.17. Then*

$$\iota^*(l) = \pm l \quad \textit{in } A^2(X) \ .$$

Proposition 4.16 suffices to prove Theorem 4.1. Indeed, let us suppose for a moment that

$$\iota^*(l) = -l \quad \text{in } A^2(X) \ .$$

Using the isomorphism (20) and Theorem 4.2, this would imply

$$\iota^* = \mathrm{id}\colon \quad A^4_{(2)}(X) \ \to\ A^4(X) \ .$$

Since ι acts as the identity on $A^4_{(j)}(X)$ for $j = 0, 4$, this would imply

$$\iota^* = \mathrm{id}\colon \quad A^4(X) \ \to\ A^4(X) \ .$$

Using the Bloch–Srinivas argument [8] applied to $\Gamma_\iota - \Delta_X$, this would imply that

$$\Gamma_\iota - \Delta_X = \gamma \quad \text{in } A^4(X \times X) \ ,$$

where γ is a cycle supported on $X \times D$ for $D \subset X$ a divisor. In particular, this would imply

$$\iota^* = \mathrm{id}\colon \quad H^{2,0}(X) \ \to\ H^{2,0}(X) \ ,$$

which is absurd since we know that ι is non–symplectic. The minus sign in Proposition 4.16 can thus be excluded; assuming Proposition 4.16 is true, we must have $\iota^*(l) = l$.

Now let $c \in A^4_{(2)}(X)$. Using the isomorphism (20), we can find $a \in A^2_{(2)}(X)$ such that

$$c = l \cdot a \quad \text{in } A^4(X) \ .$$

But then

$$\iota^*(c) = \iota^*(l) \cdot \iota^*(a) = l \cdot (-a) = -l \cdot a = -c \quad \text{in } A^4(X) \ .$$

Here, the second equality comes from Proposition 4.16 and Theorem 4.2. This proves Theorem 4.1, assuming Proposition 4.16.

We now proceed with the proof of Proposition 4.16. The first step is to prove the statement in homology:

Lemma 4.17. *Let S be any $K3$ surface and let $X = S^{[2]}$. Let $l \in A^2(X)$ be the class of Theorem 2.17, and let $\iota \in \mathrm{Aut}(X)$ be an involution. We have*

$$\iota^*(l) = \pm l \quad \textit{in } H^4(X) \ .$$

Proof. Shen and Vial have constructed a distinguished cycle $L \in A^2(X \times X)$ (whose cohomology class is the Beauville–Bogomolov class denoted $\mathfrak{B}$ in loc. cit.), and an eigenspace decomposition

$$A^2(X) = \Lambda^2_{25} \oplus \Lambda^2_2 \oplus \Lambda^2_0 \ , \tag{21}$$

where

$$\Lambda^i_\lambda := \{a \in A^i(X) \mid (L^2)_*(a) = \lambda a\} \ ,$$

and

$$\Lambda^2_{25} = \mathbb{Q}[l]$$

(This is [35, Theorem 14.5, Propositions 14.6 and 14.8], combined with [35, Theorem 2.2]).

We now observe the following commutativity relation in cohomology:

Lemma 4.18. *Set–up as in Lemma 4.17. Then*

$$(L^2)_* \iota^* = \iota^* (L^2)_* \colon \quad H^i(X) \ \to \ H^i(X) \ .$$

Proof. Let $L\in A^2(X\times X)$ be the Shen–Vial cycle as above. As proven in [35, Proposition 1.3(i)], the Shen–Vial cycle satisfies a quadratic relation

$$L^2=2\Delta_X-\frac{2}{25}(l_1+l_2)L-\frac{1}{23\cdot 25}(2l_1^2-23l_1l_2+2l_2^2)\quad\text{in } H^8(X\times X)\ , \tag{22}$$

where $l:=(i_\Delta)^*(L)$ (and $i_\Delta\colon X\to X\times X$ is the diagonal embedding) and $l_i:=(p_i)^*(l)$ (and p_i are the obvious projections).

Let us define a modified cycle

$$L':=\Gamma_\iota\circ L\circ\Gamma_\iota\quad\in A^2(X\times X)\ .$$

Using Lieberman's lemma [37, Lemma 3.3] plus the fact that ${}^t\Gamma_\iota=\Gamma_\iota$, we see that

$$L'=(\iota\times\iota)^*(L)\quad\text{in } A^2(X\times X)\ .$$

Define also $l':=(i_\Delta)^*(L')\in A^2(X)$ and $l_i':=(p_i)^*(l')\in A^2(X\times X)$, $i=1,2$. Since the diagram

$$\begin{array}{ccccc} X\times X & \xrightarrow{p_i} & X & \xrightarrow{i_\Delta} & X\times X\\ \downarrow \iota\times\iota & & \downarrow\iota & & \downarrow\iota\times\iota\\ X\times X & \xrightarrow{p_i} & X & \xrightarrow{i_\Delta} & X\times X \end{array}$$

commutes, we have the relations

$$l_i'=(\iota\times\iota)^*(l_i)\quad\text{in } A^2(X\times X)\ ,\quad i=1,2\ . \tag{23}$$

Let us apply $(\iota\times\iota)^*$ to the quadratic relation (22). The result is a relation

$$\begin{aligned}(\iota\times\iota)^*(L^2)=2\Delta_X&-\frac{2}{25}(\iota\times\iota)^*(l_1+l_2)L'\\ &-\frac{1}{23\cdot 25}(\iota\times\iota)^*(2l_1^2-23l_1l_2+2l_2^2)\quad\text{in } H^8(X\times X)\ .\end{aligned} \tag{24}$$

But

$$(\iota\times\iota)^*(L^2)=((\iota\times\iota)^*L)^2=(L')^2\quad\text{in } A^4(X\times X)\ . \tag{25}$$

Plugging this in equality (24), and also using the relations (23), we find that the cycle L' satisfies a quadratic relation

$$(L')^2=2\Delta_X-\frac{2}{25}(l_1'+l_2')L'-\frac{1}{23\cdot 25}(2(l_1')^2-23l_1'l_2'+2(l_2')^2)\quad\text{in } H^8(X\times X)\ . \tag{26}$$

But then, applying the unicity result [35, Proposition 1.3 (v)], we find there is equality

$$L' = \pm L \quad \text{in } H^4(X \times X) \ .$$

In particular, there is equality

$$(L')^2 = L^2 \quad \text{in } H^8(X \times X) \ .$$

In view of equality (25), this means

$$\Gamma_\iota \circ (L^2) \circ \Gamma_\iota = L^2 \quad \text{in } H^8(X \times X) \ ,$$

and so (by composing with Γ_ι)

$$\Gamma_\iota \circ (L^2) = (L^2) \circ \Gamma_\iota \quad \text{in } H^4(X \times X) \ .$$

This proves Lemma 4.18. □

The eigenspace decomposition (21) induces an eigenspace decomposition modulo homological equivalence:

$$\mathrm{Im}\big(A^2(X) \ \to \ H^4(X)\big) = \Lambda^2_{25} + \frac{\Lambda^2_2}{A^2_{(0)}(X) \cap A^2_{hom}(X)}$$

(this is the algebraic part of the eigenspace decomposition of $H^4(X)$ given in [35, Proposition 1.3(iii)]).

Lemma (4.18) implies ι preserves this eigenspace decomposition modulo homological equivalence. In particular, $\iota^*\Lambda^2_{25} \subset \Lambda^2_{25}$ (modulo homologically trivial cycles), and so

$$\iota^*(l) = dl \quad \text{in } H^4(X) \ ,$$

for some $d \in \mathbb{Q}$. Since $A^4_{(0)}(X) = \mathbb{Q}[l^2]$ [35, Theorem 4.6], we have

$$\iota^*(l^2) = l^2 \quad \text{in } H^8(X) \ ,$$

and so $d = \pm 1$. This proves Lemma 4.17. □

The next step (in proving Proposition 4.16) is to upgrade to rational equivalence. Here, we use again the method of "spread" developed in [42], [43]. As in the proof of Theorem 4.2, let $\mathcal{S} \to B$ resp. $\mathcal{X} \to B$ denote the family of all smooth quartics $S_b \subset \mathbb{P}^3$ with Picard number 1, resp. of all Hilbert schemes $X_b = (S_b)^{[2]}$. We note that there exists a relative cycle

$$\mathcal{L} \in A^2(\mathcal{X})$$

such that restriction

$$\mathcal{L}|_{X_b} = l_b \quad \in A^2(X_b) \quad \forall b \in B \tag{27}$$

is the distinguished class (denoted l in Theorem 2.17) for the fibre X_b. Indeed, one defines $\mathcal{L}$ as

$$\mathcal{L} := \frac{5}{6} c_2(T_{\mathcal{X}/B}) \quad \in A^2(\mathcal{X}) \ ,$$

where $T_{\mathcal{X}/B}$ is the relative tangent bundle of the smooth morphism $\mathcal{X} \to B$. Since for any $b \in B$ there is a relation

$$l_b = \frac{5}{6} c_2(X_b) \quad \text{in } A^2(X_b)$$

[35, Equation (93)], this implies (27).

The relative cycle

$$\Gamma_0 := \mathcal{L} \pm \Gamma_\iota \circ \mathcal{L} \quad \in A^2(\mathcal{X})$$

is such that the restriction to each fibre is homologically trivial:

$$(\Gamma_0)|_{X_b} = 0 \quad \text{in } H^4(X_b) \ .$$

(Here, "$\pm$" is taken to mean $+$ (resp. $-$) if Lemma 4.17 is true with a $+$ (resp. a $-$).) Thus, the relative cycle

$$\Gamma_1 := \Psi_\ast(\Gamma_0) \quad \in A^2(\mathcal{S} \times_B \mathcal{S})$$

also is homologically trivial on each fibre. (Here, Ψ is the relative correspondence from $\mathcal{X}$ to $\mathcal{S} \times_B \mathcal{S}$ as in the proof of Theorem 4.2.)

Applying [42, Lemma 3.12], up to shrinking B we can make Γ_1 globally homologically trivial. That is, there exists

$$\psi \quad \in \mathrm{Im}\big(A^2(B \times \mathbb{P}^3 \times \mathbb{P}^3) \ \to \ A^2(\mathcal{S} \times_B \mathcal{S})\big)$$

such that (after replacing B by a non–empty open subset $B' \subset B$)

$$\Gamma_2 := \Gamma_1 + \psi \quad \in A^2(\mathcal{S} \times_{B'} \mathcal{S})$$

is actually in $A^2_{hom}(\mathcal{S} \times_{B'} \mathcal{S})$.

But $A^2_{hom}(\mathcal{S}\times_{B'}\mathcal{S})=0$ (this follows from [42, Proposition 3.13], combined with the fact that the "Voisin standard conjecture" [42, Conjecture 1.6] is known to hold in codimension 2), and so

$$\Gamma_2=0 \quad \in A^2(\mathcal{S}\times_{B'}\mathcal{S})\ .$$

Restricting to a fibre, we find

$$(\Gamma_1)|_{S_b\times S_b}+\psi|_{S_b\times S_b}=0 \quad \text{in } A^2(S_b\times S_b) \quad \forall b\in B'\ .$$

As Γ_1 is fibrewise homologically trivial, the same goes for ψ:

$$\psi|_{S_b\times S_b}=0 \quad \text{in } H^4(S_b\times S_b) \quad \forall b\in B'\ . \tag{28}$$

But $A^2(\mathbb{P}^3\times\mathbb{P}^3)=\oplus_i A^i(\mathbb{P}^3)\otimes A^{2-i}(\mathbb{P}^3)$ and so

$$\psi|_{S_b\times S_b}=\lambda_0[S_b]\times H_b^2+\lambda_1 H_b\times H_b+\lambda_2 H_b^2\times[S_b] \quad \text{in } A^2(S_b\times S_b)\ ,$$

where $\lambda_i\in\mathbb{Q}$ and $H_b\in A^1(S_b)$ is an ample class on S_b. It follows from the vanishing (28) that the λ_i must be 0, and so $\psi|_{S_b\times S_b}$ is rationally trivial, and hence also

$$(\Gamma_1)|_{S_b\times S_b}=0 \quad \text{in } A^2(S_b\times S_b)\ .$$

Composing with ${}^t\Psi_b$, it follows that also

$$({}^t\Psi_b)_*\big((\Gamma_1)|_{S_b\times S_b}\big)=({}^t\Psi_b)_*(\Psi_b)_*\big((\Gamma_0)|_{X_b}\big)=0 \quad \text{in } A^2(X_b) \quad \forall b\in B'\ .$$

On the other hand, as we have seen above $(\Gamma_0)|_{X_b}\in A^2_{hom}(X_b)$ and $({}^t\Psi_b)_*(\Psi_b)_*$ is the identity on $A^2_{hom}(X_b)$. It follows that

$$(\Gamma_0)|_{X_b}=\big(l_b\pm(\iota_b)^*(l_b)\big)|_{X_b}=0 \quad \text{in } A^2(X_b) \quad \forall b\in B'\ .$$

This proves Proposition 4.16 for general $b\in B$. To extend to *all* $b\in B$, one can invoke [45, Lemma 3.2]. Proposition 4.16 and Theorem 4.15 are now proven. □

For later use, we remark that the above argument also proves the following statement:

Corollary 4.19. *Let X and ι be as in Theorem 4.1. Then*

$$\iota^* A^2_{(0)}(X)\ \subset\ A^2_{(0)}(X)\ .$$

Proof. Let $b \in A^2_{(0)}(X)$, and suppose

$$\iota^*(b) = c_0 + c_2 \quad \text{in } A^2(X) \ ,$$

with $c_0 \in A^2_{(0)}(X)$ and $c_2 \in A^2_{(2)}(X)$.

Let $l \in A^2_{(0)}(X)$ be the distinguished class of Theorem 2.17. The 0–cycle $b \cdot l$ is in $A^4_{(0)}(X)$, and so

$$\iota^*(b \cdot l) = b \cdot l \quad \text{in } A^4_{(0)}(X)$$

On the other hand, we have

$$\iota^*(b \cdot l) = \iota^*(b) \cdot \iota^*(l) = (c_0 + c_2) \cdot l = c_0 \cdot l + c_2 \cdot l \quad \text{in } A^4(X) \ .$$

(Here we have used Proposition 4.16, which we have seen must be true with a + sign.) Since $c_0 \cdot l \in A^4_{(0)}(X)$ and $c_2 \cdot l \in A^4_{(2)}(X)$, we must have

$$c_0 \cdot l = b \cdot l \quad \text{in } A^4_{(0)}(X) \ , \quad c_2 \cdot l = 0 \quad \text{in } A^4_{(2)}(X) \ .$$

Using the injectivity part of Theorem 2.17, this implies that $c_2 = 0$. ☐

Remark 4.20. *Another way of proving the $j = 2$ case of Theorem 4.15 could be as follows: define a relative correspondence*

$$\Gamma_0' := \Pi_6^{\mathcal{X}} \circ (\Gamma_\iota + \Delta_{\mathcal{X}}) \quad \in A^4(\mathcal{X} \times_B \mathcal{X}) \ ,$$

and go through the proof of Theorem 4.2 with Γ_0' instead of Γ_0.

Remark 4.21. *Can one prove the commutativity of Lemma 4.18 also modulo rational equivalence, i.e. can one prove*

$$(L^2)_* \iota^* \overset{??}{=} \iota^* (L^2)_* : \quad A^i(X) \ \to \ A^i(X) \quad ? \tag{29}$$

This would imply that ι respects the eigenspace decomposition Λ^i_λ of [35] (and in particular, that ι respects the bigraded ring structure $A^*_{(*)}(X)$).

The proof of Lemma 4.18 given above does not extend to rational equivalence, for the following reason: The quadratic relation (22) still holds modulo rational equivalence [35, Theorem 14.5], and so L' satisfies the quadratic relation (26) modulo rational equivalence. However, the unicity result ([35, Proposition 1.3(v)]), that allowed us to conclude from this that $L = \pm L'$, is only known modulo homological equivalence.

(This unicity result modulo rational equivalence is conjecturally true, and would follow from the Bloch–Beilinson conjectures [35, Proposition 3.4].)

5. Complements

This section contains some corollaries and extensions of the main result.

Corollary 5.1. *Let $S\subset\mathbb{P}^3$ be any smooth quartic. Let $X=S^{[2]}$, and let*

$$\iota\colon X\ \dashrightarrow\ X$$

be the rational map defined in [1] (cf. Remark 2.24). Let X' be a hyperkähler fourfold birational to X, and let

$$\iota'\colon X'\ \dashrightarrow\ X'$$

be the rational map induced by ι. Then

$$\begin{aligned} (\iota')^\ast=-\operatorname{id}\colon\quad & A^i_{(2)}(X')\ \to\ A^i_{(2)}(X')\quad \textit{for } i=2,4\ ;\\ (\iota')^\ast=\operatorname{id}\colon\quad & A^4_{(j)}(X')\ \to\ A^4_{(j)}(X')\quad \textit{for } j=0,4\ . \end{aligned}$$

Proof. First, we note that X' has an MCK decomposition ([35] or Lemma 2.8 above), so the notation $A^\ast_{(\ast)}(X')$ makes sense. Since X and X' have isomorphic Chow rings [33], it suffices to prove the statement for X. Let

$$\mathcal{S}\ \to\ B_\rho\ ,\quad \mathcal{X}\ \to\ B_\rho$$

denote the families of all smooth quartics $S_b\subset\mathbb{P}^3$, resp. of all Hilbert schemes $X_b=(S_b)^{[2]}$. Note that there is an inclusion

$$B_\rho\ \supset\ B\ ,$$

where B is as before (parametrizing smooth quartics of Picard number 1), and the complement $B_\rho\setminus B$ is the union of countably many closed proper subsets (i.e., a very general point of B_ρ is in B). Let $\bar{\Gamma}_\iota\in A^4(\mathcal{X}\times_{B_\rho}\mathcal{X})$ denote the closure of the graph of the rational map

$$\iota\colon\quad \mathcal{X}\ \dashrightarrow\ \mathcal{X}\ .$$

One can define relative correspondences $\Gamma_0,\ldots,\Gamma_5$ for this larger family just as in the proof of Theorem 4.2. Since the restriction of Γ_5 to the fibre over a very general point of B_ρ is rationally trivial, it follows (using [45, Lemma 3.2]) the same is true over every point of B_ρ, i.e.

$$\Gamma_5|_{(S_b)^4}=0\quad \text{in } A^4((S_b)^4)\quad \forall\, b\in B_\rho\ .$$

Just as in the proof of Theorem 4.2, one deduces from this that

$$(\iota_b)^* = -\,\mathrm{id}\colon \quad A^2_{(2)}(X_b) \ \to\ A^2_{(2)}(X_b) \quad \forall\, b\in B_\rho\ .$$

To prove the result for A^4, one extends (again using [45, Lemma 3.2]) Proposition 4.16 to all of B_ρ, i.e.

$$(\iota_b)^*(l_b) = l_b \quad \text{in } A^2_{(0)}(X_b) \quad \forall\, b\in B_\rho\ .$$

Then, just as in the proof of Theorem 4.15, using the Shen–Vial isomorphism (Theorem 2.17), one finds that any $a\in A^4_{(2)}(X_b)$ can be written as $a=l_b\cdot d$ with $d\in A^2_{(2)}(X_b)$, and thus

$$(\iota_b)^*(a) = (\iota_b)^*(l_b\cdot d) \stackrel{!!}{=} (\iota_b)^*(l_b)\cdot(\iota_b)^*(d) = -l_b\cdot d \quad \text{in } A^4_{(2)}(X_b) \quad \forall\, b\in B_\rho\ .$$

(NB: On the boundary $b\in B_\rho\setminus B$, ι_b is not a morphism but only a rational map. Yet, the equality labelled "!!" is still valid since $d\in A^2_{AJ}(X_b)$; this is thanks to [35, Proposition B.6].)

Similarly, any $a\in A^4_{(4)}(X_b)$ can be written as $a=d_1\cdot d_2$ with $d_i\in A^2_{(2)}(X_b)$ (Theorem 2.17(i)). Again using [35, Proposition B.6], we find

$$(\iota_b)^*(a) = (\iota_b)^*(d_1\cdot d_2) = (\iota_b)^*(d_1)\cdot(\iota_b)^*(d_2) = d_1\cdot d_2 \quad \text{in } A^4_{(4)}(X_b) \quad \forall\, b\in B_\rho\ .$$

The case $A^4_{(0)}$ is easy: $A^4_{(0)}(X_b)$ is generated by $(l_b)^2$. Letting $\mathcal{L}\in A^2(\mathcal{X})$ be the relative cycle restricting to the distinguished class $l_b\in A^2(X_b)$ on each fibre (as in the proof of Theorem 4.15), we know from Theorem 4.15 that

$$\mathcal{L}^2 - \iota^*(\mathcal{L}^2) \quad \in A^4(\mathcal{X})$$

is rationally trivial on a very general fibre $b\in B_\rho$. Invoking [45, Lemma 3.2], this implies $\mathcal{L}^2-\iota^*(\mathcal{L}^2)$ must be rationally trivial on *every* fibre, i.e.

$$(l_b)^2 - (\iota_b)^*\big((l_b)^2\big) = 0 \quad \text{in } A^4(X_b) \quad \forall\, b\in B_\rho\ .$$

$\square$

Corollary 5.2. *Let $X=S^{[2]}$, where $S\subset\mathbb{P}^3$ is a quartic of Picard number* 2 *and not containing lines, as in [32, Section 4 Example 2]. Let $g_\ell\in\mathrm{Aut}(X)$ be the non–natural automorphism constructed in [32, Lemma 4.6]. Then*

$$\begin{aligned}(g_\ell)^* &= \mathrm{id}\colon \quad A^4(X)\ \to\ A^4(X)\ ,\\ (g_\ell)^* &= \mathrm{id}\colon \quad A^2_{hom}(X)\ \to\ A^2_{hom}(X)\ .\end{aligned}$$

Proof. The automorphism g_ℓ is defined as

$$g_\ell := (\iota_1 \circ \iota_2)^\ell \quad \in \mathrm{Aut}(X) \ ,$$

where ι_1, ι_2 are Beauville involutions corresponding to two different embeddings of S in $\mathbb{P}^3$. It follows from Corollary 5.1 that

$$(\iota_1 \circ \iota_2)^* = \mathrm{id}\colon \quad A^4(X) \ \to \ A^4(X) \ ,$$

hence in particular g_ℓ acts as the identity on $A^4(X)$.

The second assertion follows from the first by a Bloch–Srinivas argument [8]. ☐

Let X and ι be as in Theorem 4.1. As noted in the introduction, we are not able to prove the expected equality

$$\iota^*(a) \overset{??}{=} -a \quad \text{for all } a \in A^2_{hom}(X) \ .$$

This is because of the nuisance (already noted in [35]) of having the subgroup $A^2_{(0)}(X) \cap A^2_{hom}(X)$ which is conjecturally, but not provably, zero. As shown in the following corollary, at least this nuisance disappears when intersecting with a divisor:

Corollary 5.3. *Let X and ι be as in Theorem 4.1. Let $a \in A^2_{hom}(X)$ and $D \in A^1(X)$. Then*

$$\iota^*(a \cdot D) = -a \cdot \iota^*(D) \quad \textit{in } A^3(X) \ .$$

Proof. As shown by Shen–Vial [35, page 7],

$$\mathrm{Im}\bigl(A^2_{(0)}(X) \cap A^2_{hom}(X) \ \xrightarrow{\cdot D}\ A^3(X)\bigr) = 0 \ .$$

The result now follows from Theorem 4.1. ☐

The quotient of X under the anti–symplectic involution ι is a "singular Calabi–Yau variety" (cf. Remark 2.26 for an interpretation of this quotient as triple cover of a quadric). Since it is a quotient variety, the Chow groups with $\mathbb{Q}$–coefficients form a ring. The following result is about this ring structure:

Corollary 5.4. *Let X and ι be as in Theorem 4.1, and let $Y := X/\iota$ be the quotient. For any $r \in \mathbb{N}$, let*

$$E^*(Y^r) \ \subset \ A^*(Y^r)$$

be the subring generated by (pullbacks of) $A^1(Y)$ and $A^2(Y)$. The cycle class map

$$E^k(Y^r) \ \to\ H^{2k}(Y^r)$$

is injective for $k \geq 4r-1$.

Proof. The point is that X, and hence also X^r, has an MCK decomposition [35]. Let $p\colon X \to Y$ denote the quotient morphism.

Lemma 5.5. *We have*

$$p^*A^2(Y)\ \subset\ A^2_{(0)}(X)\ .$$

Proof. Clearly,

$$p^*A^2(Y)\ \subset\ A^2(X)^\iota\ .$$

Given $b \in A^2(Y)$, let us write

$$p^*(b) = c_0 + c_2 \quad \in A^2_{(0)}(X) \oplus A^2_{(2)}(X)\ .$$

Applying ι, we find

$$\iota^*p^*(b) = c_0 + c_2 \quad \in A^2_{(0)}(X) \oplus A^2_{(2)}(X)\ .$$

On the other hand,

$$\iota^*p^*(b) = \iota^*(c_0) + \iota^*(c_2) = \iota^*(c_0) - c_2 \quad \in A^2_{(0)}(X) \oplus A^2_{(2)}(X)$$

(where we have used Corollary 4.19 to obtain that $\iota^*(c_0) \in A^2_{(0)}(X)$, and Theorem 4.2 to obtain that $\iota^*(c_2) = -c_2$). Comparing these two expressions, we find

$$\iota^*(c_0) = c_0 \quad \text{in } A^2_{(0)}(X)\ , \quad -c_2 = c_2 \quad \text{in } A^2_{(2)}(X)\ ,$$

proving Lemma 5.5. □

Lemma 5.5, combined with the obvious fact that $A^1(X) = A^1_{(0)}(X)$, implies that

$$(p^r)^*E^*(Y^r)\ \subset\ A^*_{(0)}(X^r)\ .$$

Since there is a commutative diagram

$$\begin{array}{ccc} A^k_{(0)}(X^r) & \to & H^{2k}(X^r) \\ \uparrow {\scriptstyle (p^r)^*} & & \uparrow {\scriptstyle (p^r)^*} \\ E^k(Y^r) & \to & H^{2k}(Y^r)\ , \end{array}$$

and the cycle class map

$$A^k_{(0)}(X^r) \ \to \ H^{2k}(X^r)$$

is known to be injective for $k \geq 4r-1$ ([39, Introduction]; this follows for instance from [38, Section 4.3]), this establishes Corollary 5.4. □

We single out a particular case of Corollary 5.4:

Corollary 5.6. *Let X and ι be as in Theorem 4.1, and let $Y := X/\iota$ be the quotient. The subspaces*

$$\begin{aligned} &\mathrm{Im}\Bigl(A^2(Y)\otimes A^1(Y) \ \to \ A^3(Y)\Bigr)\ ,\\ &\mathrm{Im}\Bigl(A^2(Y)\otimes A^2(Y) \ \to \ A^4(Y)\Bigr) \end{aligned}$$

are of dimension 1.

Proof. This follows from Corollary 5.4, combined with the fact that

$$N^3(Y) := \mathrm{Im}\Bigl(A^3(Y) \ \to \ H^6(Y)\Bigr)$$

is of dimension 1. To see this, since the pairing

$$NS(X)^\iota \otimes N^3(X)^\iota \ \to \ N^4(X)^\iota \cong \mathbb{Q}$$

is non–degenerate, it suffices to prove that

$$\dim NS(Y) = \dim NS(X)^\iota = 1\ .$$

But ι acts on $NS(X)$ as reflection in the span of D (Proposition 2.22), and so $NS(X)^\iota = \mathbb{Q}[D]$ is of dimension 1. □

Remark 5.7. *It is instructive to compare Corollary 5.6 with known results concerning the Chow ring of $K3$ surfaces and of Calabi–Yau varieties. For any $K3$ surface S, it is known that*

$$\dim \mathrm{Im}\Bigl(A^1(S)\otimes A^1(S) \ \to \ A^2(S)\Bigr) = 1$$

[5]. For a generic Calabi–Yau complete intersection X of dimension n, it is known that

$$\dim \mathrm{Im}\Bigl(A^i(X)\otimes A^{n-i}(X) \ \to \ A^n(X)\Bigr) = 1\ , \quad \forall 0<i<n$$

[40], [14].

The new part of Corollary 5.6, with respect to these results, is the part about

$$\mathrm{Im}\Big(A^2(X)\otimes A^1(X)\to A^3(X)\Big)\ .$$

We also get the following corollary, providing an alternative description of the Fourier decomposition on $A^4(X)$:

Corollary 5.8. *Let X and ι be as in Theorem 4.1, and let $p\colon X\to Y:=X/\iota$ be the quotient morphism. Then*

$$\begin{aligned} A^4_{(4)}(X) &= p^\ast A^4_{hom}(Y)\ ,\\ A^4_{(2)}(X) &= \ker\Big(A^4(X)\ \xrightarrow{p_\ast}\ A^4(Y)\Big)\ .\end{aligned}$$

Proof. Theorem 4.1 implies that

$$A^4_{(4)}(X)=A^4(X)^\iota\cap A^4_{hom}(X)$$

(which proves the first statement of the corollary), and also that

$$A^4(X)=A^4(X)^\iota\oplus A^4_{(2)}(X)$$

(which proves the second statement of the corollary). ☐

Remark 5.9. *Let X and Y be as in Corollary 5.8. It seems likely that also*

$$A^2_{(2)}(X)\stackrel{??}{=}\ker\Big(A^2(X)\ \xrightarrow{p_\ast}\ A^2(Y)\Big)\ .$$

To prove this, it remains to establish that ι acts as the identity on $A^2_{(0)}(X)\cap A^2_{hom}(X)$ (which is conjecturally 0).

6. Open questions

Question 6.1. *Let X and ι be as in Theorem 4.1. Can one say anything about the action of ι on $A^3(X)$? This seems more difficult than Theorem 4.1. Indeed, the action of ι on A^2_{hom} and on A^4 is determined by "behaviour up to codimension 1 phenomena". The action of ι on $A^3_{(2)}$, on the other hand, should be determined by the action of ι on $H^{3,1}(X)$, which is not as neat as the action on $H^{2,0}(X)$ and $H^{4,0}(X)$. I am not even sure what the conjectural statement should be.*

Question 6.2. *Let X and ι be as in Theorem 4.1. Does the quotient $Y = X/\iota$ have a (self–dual) MCK decomposition? I have not been able to prove this (essentially, this reduces to the problem of showing that ι is of pure grade 0, in the sense of [36, Definition 1.1]).*

Question 6.3. *As we have seen (Theorem 2.21), other cases where a non–symplectic, non–natural involution exists on $S^{[2]}$ is when S is a generic $K3$ surface of degree $d = 20, 26, 34, ...$ (i.e., of genus $g = 11, 14, 18, \ldots$). It would be interesting to prove Bloch's conjecture for these cases as well.*

For the case $d = 34$ (i.e., $g = 18$), Mukai [24] has given a nice description of S in terms of sections of a vector bundle on an orthogonal Grassmannian, so there is at least some hope that the method of spread à la Voisin can be employed in this case as well. Let $\mathcal{S} \to B$ be the family of all smooth dimension 2 sections of this vector bundle. One major difficulty is in proving a version of Theorem 4.5 for the fourfold relative fibre product of this family $\mathcal{S} \to B$, i.e. one would need to prove

$$A^2_{hom}(\mathcal{S}^{4/B}) = 0 \ .$$

Is this feasible?

Question 6.4. *Let S be a generic $K3$ surface of degree $d = 10$ (i.e., of genus $g = 6$). The Hilbert scheme $X = S^{[2]}$ has no non–trivial automorphisms (Theorem 2.21), but there is a non–symplectic rational involution*

$$\iota\colon\ \ X \dashrightarrow X\ ,$$

constructed by O'Grady [28, Section 4.3]. Can one prove the statement of Theorem 4.1 in this set–up? Work of Mukai [23] realizes these $K3$ surfaces as complete intersections in a certain Grassmannian. Again, the main difficulty seems to consist in proving that

$$A^2_{hom}(\mathcal{S}^{4/B}) = 0$$

for this family. Is this feasible?

Question 6.5. *It would also be interesting to extend Theorem 4.1 to higher dimensional Hilbert schemes $S^{[r]}$, $r > 2$. Let $S \subset \mathbb{P}^{r+1}$ be a $K3$ surface of degree $2r$. The Hilbert scheme $S^{[r]}$ has an MCK decomposition [39], and so there is a bigraded ring structure $A^*_{(*)}(S^{[r]})$. As noted by Beauville [1], there is a non–trivial rational involution*

$$\iota\colon\ \ S^{[r]} \dashrightarrow S^{[r]}\ .$$

Can one prove something about the action of ι on $A^*_{(*)}(S^{[r]})$? Supposing one wants to follow the approach of the present article, the main difficulty consists in proving that

$$A^2_{hom}(\mathcal{S}^{2r/B}) = 0$$

(or even Griff$^2(\mathcal{S}^{2r/B}) = 0$*), where $\mathcal{S} \to B$ is the family of all smooth $K3$ surfaces of degree $2r$ in $\mathbb{P}^{r+1}$.*

Acknowledgements

Thanks to all participants of the Strasbourg 2014/2015 "groupe de travail" based on the monograph [45] for a stimulating atmosphere. Many thanks to Yasuyo, Kai and Len for efficiently managing the Schiltigheim Math Research Institute.

References

[1] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, in: Classification of algebraic and analytic manifolds (Katata, 1982), Birkhäuser Boston, Boston, 1983. MR0728605

[2] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differential Geom. **18**(4) (1983), 755–782. MR0730926

[3] A. Beauville, On the splitting of the Bloch–Beilinson filtration, in: Algebraic cycles and motives (J. Nagel and C. Peters, eds.). London Math. Soc. Lecture Notes, 344. Cambridge University Press, Cambridge, 2007. MR2187148

[4] A. Beauville, Antisymplectic involutions of holomorphic symplectic manifolds, Journal of Topology **4**(2) (2011), 300–304. MR2805992

[5] A. Beauville and C. Voisin, On the Chow ring of a $K3$ surface, J. Alg. Geom. **13** (2004), 417–426. MR2047674

[6] S. Bloch, Lectures on algebraic cycles, Duke Univ. Press, Durham, 1980. MR0558224

[7] S. Bloch, A. Kas and D. Lieberman, Zero cycles on surfaces with $p_g = 0$, Comp. Math. **33**(2) (1976), 135–145. MR0435073

[8] S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles, American Journal of Mathematics **105**(5) (1983), 1235–1253. MR0714776

[9] S. Boissière, A. Cattaneo, M. Nieper-Wisskirchen and A. Sarti, The automorphism group of the Hilbert scheme of two points on a generic projective $K3$ surface, arXiv:1410.8387v2, in: K3 Surfaces and Their Moduli, Proceedings of the Schiermonnikoog conference 2014 (C. Faber, G. Farkas and G. van der Geer, eds.). Progress in Maths, 315, Birkhäuser, Boston, 2016.

[10] S. Boissière and A. Sarti, A note on automorphisms and birational transformations of holomorphic symplectic manifolds, Proc. Amer. Math. Soc. **140** (2012), 4053–4062. MR2957195

[11] A. Corti and M. Hanamura, Motivic decomposition and intersection Chow groups, I, Duke Math. J. **103** (2000), 459–522. MR1763656

[12] C. Deninger and J. Murre, Motivic decomposition of abelian schemes and the Fourier transform. J. reine u. angew. Math. **422** (1991), 201–219. MR1133323

[13] A. Ferretti, The Chow ring of double EPW sextics, Algebra and Number Theory **6** no. 3 (2012), 539–560. MR2966709

[14] L. Fu, Decomposition of small diagonals and Chow rings of hypersurfaces and Calabi–Yau complete intersections, Advances in Mathematics (2013), 894—924. MR3077892

[15] L. Fu, On the action of symplectic automorphisms on the CH_0–groups of some hyper-Kähler fourfolds, Math. Z. **280** (2015), 307–334. MR3343908

[16] W. Fulton, Intersection theory, Springer–Verlag Ergebnisse der Mathematik, Berlin, Heidelberg, New York, Tokyo, 1984. MR0732620

[17] D. Huybrechts, Symplectic automorphisms of $K3$ surfaces of arbitrary order, Math. Res. Letters **19** (2012), 947–951. MR3008427

[18] U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: Motives (U. Jannsen et al., eds.). Proceedings of Symposia in Pure Mathematics, Part 1, 55, 1994. MR1265533

[19] U. Jannsen, Equivalence relations on algebraic cycles, in: The arithmetic and geometry of algebraic cycles, Banff Conference (B. Gordon et al., eds.). Kluwer, 1998. MR1744947

[20] U. Jannsen, On finite–dimensional motives and Murre's conjecture, in: Algebraic cycles and motives (J. Nagel and C. Peters, eds.). Cambridge University Press, Cambridge, 2007. MR2187152

[21] B. Kahn, J. Murre and C. Pedrini, On the transcendental part of the motive of a surface, in: Algebraic cycles and motives (J. Nagel and C. Peters, eds.). Cambridge University Press, Cambridge, 2007. MR2385298

[22] R. Laterveer, Algebraic cycles on a very special EPW sextic, Rend. Sem. Mat. Univ. Padova. MR3880213

[23] S. Mukai, Curves, K3 surfaces and Fano 3-folds of genus ≤ 10, in: Algebraic Geometry and Commutative Algebra I, Kinokuniya, Tokyo, 1988. MR0977768

[24] S. Mukai, Polarized K3 surfaces of genus 18 and 20, in: Complex projective geometry (Trieste, 1989/Bergen, 1989), London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992. MR1201388

[25] J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II, Indag. Math. **4** (1993), 177–201. MR1225267

[26] J. Murre, J. Nagel and C. Peters, Lectures on the theory of pure motives, Amer. Math. Soc. University Lecture Series, 61, Providence, 2013. MR3052734

[27] J. Nagel and M. Saito, Relative Chow-Künneth decompositions for conic bundles and Prym varieties, Int. Math. Res. Not. **2009**(16) (2009), 2978–3001. MR2533794

[28] K. O'Grady, Involutions and linear systems on holomorphic symplectic manifolds, Geom. Funct. Anal. **15**(6) (2005), 1223–1274. MR2221247

[29] K. O'Grady, Irreducible symplectic 4–folds and Eisenbud–Popescu–Walter sextics, Duke Math. J. **134**(1) (2006), 99–137. MR2239344

[30] K. O'Grady, EPW–sextics: taxonomy, Manuscripta Math. **138**(1) (2012), 221–272. MR2898755

[31] K. O'Grady, Double covers of EPW–sextics, Michigan Math. J. **62** (2013), 143–184. MR3049300

[32] K. Oguiso, A remark on dynamical degrees of automorphisms of hyperkähler manifolds, Manuscripta Math. **130** (2009), 101–111. MR2533769

[33] U. Riess, On the Chow ring of birational irreducible symplectic varieties, Manuscripta Math. **145** (2014), 473–501. MR3268859

[34] T. Scholl, Classical motives, in: Motives (U. Jannsen et al., eds.). Proceedings of Symposia in Pure Mathematics, Part 1, 55, 1994. MR1265529

[35] M. Shen and C. Vial, The Fourier transform for certain hyperKähler fourfolds, Memoirs of the AMS 240(1139) (2016) MR3460114

[36] M. Shen and C. Vial, The motive of the Hilbert cube $X^{[3]}$, arXiv:1503.00876.

[37] C. Vial, Remarks on motives of abelian type, to appear in Tohoku Math. J. MR3682163

[38] C. Vial, Niveau and coniveau filtrations on cohomology groups and Chow groups, Proceedings of the LMS **106**(2) (2013), 410–444. MR3021467

[39] C. Vial, On the motive of some hyperkähler varieties, to appear in J. für Reine u. Angew. Math. MR3630123

[40] C. Voisin, Chow rings and decomposition theorems for $K3$ surfaces and Calabi–Yau hypersurfaces, Geom. Topol. **16** (2012), 433–473. MR2916291

[41] C. Voisin, Symplectic involutions of $K3$ surfaces act trivially on CH_0, Documenta Math. **17** (2012), 851–860. MR3007678

[42] C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, Ann. Sci. Ecole Norm. Sup. **46**(3) (2013), 449–475. MR3099982

[43] C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, II, J. Math. Sci. Univ. Tokyo **22** (2015), 491–517. MR3329204

[44] C. Voisin, Bloch's conjecture for Catanese and Barlow surfaces, J. Differential Geometry **97** (2014), 149–175. MR3229054

[45] C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014. MR3186044

[46] C. Voisin, Remarks and questions on coisotropic subvarieties and 0–cycles of hyper–Kähler varieties, in: K3 Surfaces and Their Moduli, Proceedings of the Schiermonnikoog conference 2014 (C. Faber, G. Farkas and G. van der Geer, eds.). Progress in Maths, 315, Birkhäuser, Boston 2016. MR3524175

Robert Laterveer
Institut de Recherche Mathématique Avancée
CNRS – Université de Strasbourg
7 Rue René Descartes
67084 Strasbourg Cedex
France
E-mail: robert.laterveer@math.unistra.fr