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Upper k-tuple total domination in graphs

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Abstract: Let $G = (V, E)$ be a simple graph. For any integer $k \geq 1$, a subset of V is called a k-tuple total dominating set of G if every vertex in V has at least k neighbors in the set. The minimum cardinality of a minimal k-tuple total dominating set of G is called the k-tuple total domination number of G. In this paper, we introduce the concept of upper k-tuple total domination number of G as the maximum cardinality of a minimal k-tuple total dominating set of G, and study the problem of finding a minimal k-tuple total dominating set of maximum cardinality on several classes of graphs, as well as finding general bounds and characterizations. Also, we find some results on the upper k-tuple total domination number of the Cartesian and cross product graphs.

Keywords: k-Tuple total domination number, upper k-tuple total domination number, Cartesian and cross product graphs, hypergraph, (upper) k-transversal number.

1. Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [27]. Let $G = (V, E)$ be a graph with the *vertex set* V of *order* $n(G)$ and the *edge set* E of *size* $m(G)$. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$, while its cardinality is the *degree* of v and denoted by $deg_G(v)$. The *closed neighborhood* of a vertex $v \in V$ is also $N[v] = N_G(v) \cup \{v\}$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write K_n, C_n and P_n for a *complete graph*, a *cycle*, and a *path* of order n, respectively, while $K_{n_1,\ldots,n_p}$ denotes a *complete p-partite graph*. Also for a subset $S \subseteq V$, $G[S]$ denotes the *induced subgraph* of G by S in which $V(G[S]) = S$ and for any two vertices $x, y \in S$, $xy \in E(G[S])$ if and only if $xy \in E(G)$.

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Definition 1. Let $k \geq 1$ be an integer and let $v \in S \subseteq V$. A vertex v' is called a *k-open private neighbor* of v with respect to S, or simply a (S,k)*-opn* of v if $v \in N_G(v')$ and $|N_G(v') \cap S| = k$, in other words, there exists a k-subset $S_v \subseteq S$ containing v such that $N_G(v') \cap S = S_v$. The set

$$opn_k(v; S) = \{v' \in V | v' \text{ is a } (S,k)\text{-opn of } v\}$$

is called the *k-open private neighborhood set* of v with respect to S. Also, a k-open private neighbor of v with respect to S is called *external* or *inner* if the vertex is in $V - S$ or S, respectively.

Hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a set V of elements, called *vertices*, together with a multiset E of arbitrary subsets of V, called *edges*. For integer $k \geq 1$, a *k-uniform hypergraph* is a hypergraph in which every edge has size k. Every simple graph is a 2-uniform hypergraph. For a graph $G = (V, E)$, $H_G = (V, C)$ denotes the *open neighborhood hypergraph* of G with the vertex set V and edge set C consisting of the open neighborhoods of vertices of V in G.

A *transversal* in a hypergraph $H = (V, E)$ is a subset $S \subseteq V$ such that $|S \cap e| \geq 1$ for every edge $e \in E$; that is, the set S meets every edge in H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. In a natural way, Wanless et al. generalized the concept of transversal in a Latin square to k-transversal [26]. We recall that for any integere $n \geq 1$, a *Latin square* of order n is a n-by-n grid, each entry of which is a number from the set $[n] = \{1, 2, \cdots, n\}$ such that no number appears twice in any row or column.

Definition 2. [26] For any positive integer k, a *k-transversal* or a *k-plex* in a Latin square of order n is a set of nk cells, k from each row, k from each column, in which every symbol occurs exactly k times. The maximum number of disjoint k-transversals in a Latin square L is called its *k-transversal number* and denoted by $\tau_k(L)$. Obviously $\tau_k(L) \leq n/k$. A Latin squre L has *a decomposition into disjoint k-transversals* means $\tau_k(L) = n/k$.

In a similar way, we generalize the concept of transversal in a hypergraph to k-transversal as following:

Definition 3. For any integer $k \geq 1$, a *k-transversal* in a hypergraph $H = (V, E)$ is a subset $S \subseteq V$ such that $|S \cap e| \geq k$ for every edge $e \in E$; that is, every edge in H contains at least k vertices from the set S. The *k-transversal number* $\tau_k(H)$ of H is the minimum cardinality of a minimal k-transversal in H, while the *upper k-transversal number* $\Upsilon_k(H)$ of H is defined as the maximum cardinality of a minimal k-transversal in H.

Domination. Domination in graphs is now well-studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [11, 12]. A subset $S \subseteq V$ is a *dominating set* (resp. *total dominating set*) of G if each vertex in $V \setminus S$ (resp. V) is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ (resp. *total domination number* $\gamma_t(G)$) of G is the minimum cardinality of a dominating set (resp. total dominating set) of G. The following extension of total domination number introduced by Henning and Kazemi in [13] (see [14, 19, 20, 21, 22] for more information).

Definition 4. [13] Let $k \geq 1$ be an integer and let G be a graph with $\delta(G) \geq k$. A subset $S \subseteq V$ is called a *k-tuple total dominating set,* briefly kTDS, of G if for each $x \in V$, $|N(x) \cap S| \geq k$. The minimum number of vertices of a kTDS of G is called the *k-tuple total domination number* of G and denoted by $\gamma_{\times k,t}(G)$. A kTDS with cardinality $\gamma_{\times k,t}(G)$ is called a *min-kTDS* of G.

Finding the maximum cardinality of the set of minimal subsets of the vertices (or edges or both) of a graph with a property is one of the important problems in graph theory. According to this fact, in this paper, we initiate to study the problem of finding a minimal kTDS of maximum cardinality in a graph. This leads to our next definition.

Definition 5. The *upper k-tuple total domination number* $\Gamma_{\times k,t}(G)$ of a graph G is the maximum cardinality of a minimal kTDS of G, and a minimal kTDS with cardinality $\Gamma_{\times k,t}(G)$ is called a *$\Gamma_{\times k,t}(G)$-set,* or a *$\Gamma_{\times k,t}$-set of G*. Also, we say that a graph G is a *$\Gamma_{\times k,t}$-external graph* if it has a $\Gamma_{\times k,t}$-set S such that every vertex in it has an external k-open private neighbor with respect to S.

Obviously, for every graph G and every positive integer k, $\gamma_{\times k,t}(G) \leq \Gamma_{\times k,t}(G)$, and this bound is sharp for any complete graph of order $n > k$. We remark that the upper 1-tuple total domination number $\Gamma_{\times 1,t}(G)$ is the well-studied *upper total domination number* $\Gamma_t(G)$, while the upper 2-tuple total domination number is known as the *upper double total domination number.* The redundancy involved in upper k-tuple total domination makes it useful in many applications.

In this paper, as we mentioned, we initiate to study the problem of finding a minimal kTDS of maximum cardinality on several classes of graphs, as well as finding general bounds and characterizations. Also we present a Vizing-like conjecture on the upper k-tuple total domination number, and prove it for a family of graphs. Proving

$$\Gamma_{\times k\ell,t}(G \times H) \geq \Gamma_{\times k,t}(G) \cdot \Gamma_{\times \ell,t}(H) \quad \text{(for any } k, \ell \geq 1)$$

is our next work in which $G \times H$ denotes the cross product of two graphs G and H. Then we characterize graphs G satisfying $\Gamma_{\times k,t}(G) = \gamma_{\times k,t}(G)$, and show that for any graph G with minimum degree at least k,

1. $\Gamma_{\times k,t}(G) = \Upsilon_k(H_G)$, and
2. $\Gamma_{\times k,t}(G) = \gamma_{\times k,t}(G)$ if and only if $\Upsilon_k(H_G) = \tau_k(H_G)$.

We begin our discussion with the following useful observation.

Observation 1. *Let $k \geq 1$ be an integer and let G be a graph of order n with $\delta(G) \geq k$. Then*

i. $\gamma_{\times k,t}(G) \leq \Gamma_{\times k,t}(G) \leq n$,

ii. *every kTDS S of G is minimal if and only if $opn_k(v; S) \neq \emptyset$ for every vertex $v \in S$,*

iii. *all neighbors of every vertex of degree k in G belong to every kTDS of G, and*

iv. *if H is a spanning subgraph of G which has a $\Gamma_{\times k,t}$-set that is also a minimal kTDS of G, then $\Gamma_{\times k,t}(H) \leq \Gamma_{\times k,t}(G)$.*

Observation 1 (iii) implies the next proposition.

Proposition 1. *For any k-regular graph G, $\Gamma_{\times k,t}(G) = n$.*

The converse of Proposition 1 does not hold. For example, if G is the graph obtained by the union of two disjoint complete graphs of order $k+1 \geq 3$, with an edge between them, then G is not regular but $\Gamma_{\times k,t}(G) = 2k+2$. The next two propositions are useful for our investigations. First we recall that for any positive integer k, the *k-join* $G \circ_k H$ of a graph G to a graph H with $\delta(H) \geq k$ is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H.

Proposition 2. [9] *For any path P_n of order $n \geq 2$, $\Gamma_t(P_n) = 2\lfloor (n+1)/3 \rfloor$.*

Proposition 3. [13] *Let G be a graph with $\delta(G) \geq k$. Then $\gamma_{\times k,t}(G) = k+1$ if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$ for some graph F.*

2. Cycles and complete mutipartite graphs

In this section, we calculate the upper k-tuple total domination number of a cycle and a complete multipartite graph. Proposition 1 implies $\Gamma_{\times 2,t}(C_n) = n$. The next proposition calculates $\Gamma_t(C_n)$.

Proposition 4. *For any cycle C_n of order $n \geq 3$, $\Gamma_t(C_n) = \lfloor \frac{2n}{3} \rfloor$.*

Proof. Let $V(C_n) = \{1, 2, \cdots, n\}$, and let $ij \in E(C_n)$ if and only if $j \equiv i+1 \pmod{n}$. Let S be a $\Gamma_t(C_n)$-set. If at least one vertex of any two consecutive vertices belongs to S, then $n \equiv 0 \pmod 3$. Since, otherwise, S will contain at least three consecutive vertices of $V(C_n)$, which contracts the minimality of S. Hence $|S| = \lfloor \frac{2n}{3} \rfloor$, when $n \equiv 0 \pmod 3$. Now, assume there exist two consecutive vertices, say 1 and n, out of S. Then S is also a minimal TDS in the path $P_n = C_n - \{e\}$ in which $e = 1n \in E(C_n)$. This implies

$$\begin{array}{rcl} |S| & = & \Gamma_t(C_n) \\ & \leq & \Gamma_t(P_n) \\ & = & 2\lfloor (n+1)/3 \rfloor \quad \text{(by Proposition 2)}. \end{array}$$

Now since $\{3i+1, 3i+2 | 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\}$ is a minimal TDS in C_n with cardinality $\Gamma_t(P_n)$ when $n \not\equiv 2 \pmod 3$, we obtain $\Gamma_t(C_n) = 2\lfloor \frac{n+1}{3} \rfloor = \lfloor \frac{2n}{3} \rfloor$. Now let $n \equiv 2 \pmod 3$ and let S be a minimal TDS of C_n with cardinality $\Gamma_t(P_n)$. Then there exist seven consecutive vertices, say $1, 2, \cdots, 7$, such that $S \cap \{1, 2, \cdots, 7\} = \{i\}$ in which $i = 2$ or 4. Since $S - \{i+1\}$ is a TDS of C_n, we obtain $|S| < \Gamma_t(P_n)$, and so $\Gamma_t(C_n) \leq \Gamma_t(P_n) - 1$. Now since $\{3i+1, 3i+2 | 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\} \cup \{n\}$ is a minimal TDS of C_n with cardinality $\Gamma_t(P_n) - 1 = \lfloor \frac{2n}{3} \rfloor$, we obtain $\Gamma_t(C_n) = \lfloor \frac{2n}{3} \rfloor$. $\square$

Theorem 1. *Let $G = K_{n_1, n_2, \cdots, n_p}$ be a complete p-partite graph with $\delta(G) \geq k \geq 1$ in which $n_1 \leq n_2 \leq \cdots \leq n_p$. Then*

$$\Gamma_{\times k,t}(G) = k + \max\{x \mid (\ell - 1)x = k \text{ and } x \leq \min\{k, n_{p-\ell+1}, \cdots, n_p\}\}.$$

Proof. Let S be a minimal kTDS of $G = K_{n_1, n_2, \cdots, n_p}$ and let $V = X_1 \cup X_2 \cup \cdots \cup X_p$ be the partition of the vertex set of G to the p independent sets X_1, X_2, $\cdots$, X_p in which $|X_i| = n_i$ for each i and $n_1 \leq n_2 \leq \cdots \leq n_p$. Let $I = \{i_j \mid j = 1, \cdots, \ell\}$ be an index subset of $\{1, 2, \cdots, p\}$ for some $2 \leq \ell \leq p$ such that $S \cap X_i \neq \emptyset$ if and only if $i \in I$. Also assume $|S \cap X_{i_j}| = x_{i_j}$ for each $i_j \in I$, and $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_\ell}$. The minimality of S implies $x_{i_j} \leq k$ for each $i_j \in I$, and there exists a $(\ell - 1)$-subset $L \subseteq I$ such that $\sum_{i_j \in L} x_{i_j} = k$. Then, by the minimality of S, $\sum_{i_j \in L} x_{i_j} = k$ for every $(\ell - 1)$-subset $L \subseteq I$, which implies $x_{i_1} = x_{i_2} = \cdots = x_{i_\ell}$. Let $x := x_{i_1} = x_{i_2} = \cdots = x_{i_\ell}$. Then $x_{i_1} + x_{i_2} + \cdots + x_{i_\ell} = \ell x = k + x \leq \Gamma_{\times k,t}(G)$ where $x \leq \min\{k, n_{i_1}, ..., n_{i_\ell}\}$, and so

$$\begin{array}{rcl} \Gamma_{\times k,t}(G) & = & k + \max\{x \mid (\ell - 1)x = k \text{ and } x \leq \min\{k, n_{i_1}, \cdots, n_{i_\ell}\}\} \\ & = & k + \max\{x \mid (\ell - 1)x = k \text{ and } x \leq \min\{k, n_{p-\ell+1}, \cdots, n_p\}\}. \end{array}$$

□

Corollary 1. *Let $G = K_{n_1,n_2,\cdots,n_p}$ be a complete p-partite graph. For any integer $k \geq 1$ if $|\{ i \mid n_i \geq k\}| \geq 2$, then $\Gamma_{\times k,t}(G) = 2k$.*

In a similar way, the next theorem can be proved.

Theorem 2. *Let $G = K_{n_1,n_2,\cdots,n_p}$ be a complete p-partite graph with $\delta(G) \geq k \geq 1$ in which $n_1 \leq n_2 \leq \cdots \leq n_p$. Then*

$$\gamma_{\times k,t}(G) \leq k + \min\{x|(\ell-1)x = k \text{ and } x \leq \min\{k, n_1, \cdots, n_\ell\}\}.$$

3. Two upper bounds

In this section, we present two upper bounds for the upper k-tuple total domination number of a graph. The first one is in terms of k, n, δ, and the second is in terms of the upper ℓ-tuple total domination number of the graph for some $\ell < k$.

Theorem 3. *If G is a graph of order n with $\delta \geq k+1 \geq 2$, then $\Gamma_{\times k,t}(G) \leq n - \delta + k$, and this bound is sharp.*

Proof. Let G be a graph of order n with $\delta \geq k+1 \geq 2$ and let S be a $\Gamma_{\times k,t}(G)$-set. Then for every $v \in S$ there exist a k-subset $S_v \subseteq S$ and a vertex $v' \in V(G)$ such that $N_G(v') \cap S = S_v$, by Observation 1 (ii). If $v' \in S$, then $N_G(v') - S_v \subseteq V(G) - S$, and so

$$\delta - k \leq deg(v') - k \leq n - |S| = n - \Gamma_{\times k,t}(G),$$

which implies $\Gamma_{\times k,t}(G) \leq n - \delta + k$. If $v' \notin S$, then v' is not adjacent to at least $|S| - k$ vertices of $S - S_v$, and so

$$\delta \leq deg(v') \leq n - |S| + k - 1 = n - \Gamma_{\times k,t}(G) + k - 1,$$

which implies $\Gamma_{\times k,t}(G) < n - \delta + k$.

The sharpness of this bound can be checked as following: Let $\delta \geq k+1 \geq 2$. Consider b vertex-disjoint complete graphs K_{k+1} where $b \geq \lceil \frac{\delta}{k+1} \rceil$, and let $H_b = K_{k+1} + \cdots + K_{k+1}$ be the union of b vertex-disjoint complete graphs K_{k+1}. Also consider an empty graph T with $\delta - k$ vertices. Let $G_b = H_b \vee T$ be the *join* of H_b and T, which is the union of H_b and T such that every vertex of H_b is adjacent to all vertices in T. Then G_b is a connected graph of order $n = b(k+1) + \delta - k$ with minimum degree δ. Since $V(H_b)$ is a minimal kTDS of G_b, we obtain $\Gamma_{\times k,t}(G_b) \geq n - \delta + k$, and consequently $\Gamma_{\times k,t}(G_b) = n - \delta + k$. □

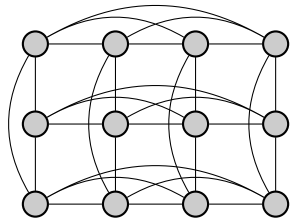

Figure 1: The 3×4 rook's graph, i.e., $K_3 \Box K_4$.

Theorem 4. *Let G be a graph with $\delta \geq k \geq 1$. Let $L = \cap_{v \in S} S_v$ be a set of cardinality ℓ in which S is a $\Gamma_{\times k,t}(G)$-set and S_v is the set given in Definition 1. If $\ell < k$, then $\Gamma_{\times k,t}(G) \leq \Gamma_{\times (k-\ell),t}(G) + \ell$.*

Proof. Let S be a $\Gamma_{\times k,t}(G)$-set and let $L = \cap_{v \in S} S_v$ be a set of cardinality ℓ in which S_v is the set given in Definition 1 and $\ell < k$. Since $S - L$ is a minimal $(k-\ell)$TDS of G, we obtain

$$\begin{array}{rcl} \Gamma_{\times (k-\ell),t}(G) & \geq & |S - L| \\ & = & |S| - \ell \\ & = & \Gamma_{\times k,t}(G) - \ell. \end{array}$$

□

4. The Cartesian product and a Vizing-like conjecture

The *Cartesian product* $G \Box H$ of two graphs G and H is a graph with the vertex set $V(G) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or $h_1 = h_2$ and $(g_1, g_2) \in E(G)$. For more information on product graphs see [24]. The Cartesian product $K_n \Box K_m$ is known as the $n \times m$ *rook's graph*, as edges represent possible moves by a rook on an $n \times m$ chess board. For example see Figure 1.

Now for integers $n \geq m \geq k+1 \geq 3$ we consider the $n \times m$ rook's graph $K_n \Box K_m$ with the vertex set $V = \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$. Since the set $\{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq k\}$ is a minimal kTDS of $K_n \Box K_m$, we have the following proposition.

Proposition 5. *For any integers $n \geq m \geq k+1 \geq 3$, $\Gamma_{\times k,t}(K_n \Box K_m) \geq kn$.*

As we will show in Proposition 6 for $n = m = k+1 \geq 3$, we guess equality holds in Proposition 5.

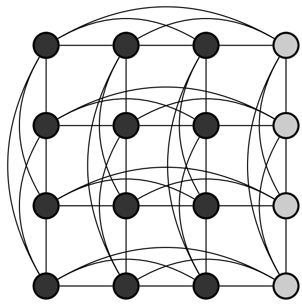

Figure 2: The dark vertices highlight a minimal 3TDS of $K_4 \Box K_4$ with maximum cardinality.

Proposition 6. *For any integer $k \geq 2$, $\Gamma_{\times k,t}(K_{k+1} \Box K_{k+1}) = k(k+1)$.*

Proof. Let $V(K_{k+1} \Box K_{k+1}) = \{(i,j) \mid 1 \leq i,j \leq k+1\}$ in which $k \geq 2$. We know $\Gamma_{\times k,t}(K_{k+1} \Box K_{k+1}) \geq k(k+1)$ by Proposition 5. Now let

$$S = \bigcup_{1 \leq i \leq k+1} S_i^r = \bigcup_{1 \leq j \leq k+1} S_j^c$$

be a minimal kTDS of $K_{k+1} \Box K_{k+1}$ with cardinality more than $k(k+1)$ in which

$$S_i^r = S \cap \{(i,j) \mid 1 \leq j \leq k+1\},$$

$$S_j^c = S \cap \{(i,j) \mid 1 \leq i \leq k+1\}.$$

Then $|S_i^r| \geq k$ and $|S_j^c| \geq k$ for each i and each j, and also $|S_t^r| = k+1$ and $|S_\ell^c| = k+1$ for some t and some ℓ. Now since $S - \{(t,\ell)\}$ is a kTDS of $K_{k+1} \Box K_{k+1}$, which contradicts the minimality of S, we obtain $\Gamma_{\times k,t}(K_{k+1} \Box K_{k+1}) = k(k+1)$. See Figure 2 for an example. $\square$

In 1963, more formally in 1968, Vizing [25] made an elegant conjecture that has subsequently become one the most famous open problems in domination theory.

Conjecture 1 (Vizing's Conjecture)**.** *For any graphs G and H,*

$$\gamma(G) \cdot \gamma(H) \leq \gamma(G \Box H).$$

Over more than fifty years (see [1] and references therein), Vizing's Conjecture has been shown to hold for certain restricted classes of graphs, and furthermore, upper and lower bounds on the inequality have gradually tightened. Additionally, research has explored inequalities (including Vizing-like inequalities) for different forms of domination [12]. A significant breakthrough occurred in 2000, when Clark and Suen [7] proved that

$$\gamma(G) \cdot \gamma(H) \leq 2\gamma(G\Box H)$$

which led to the discovery of a Vizing-like inequality for total domination [15, 16], i.e.,

$$\gamma_t(G) \cdot \gamma_t(H) \leq 2\gamma_t(G\Box H), \tag{1}$$

as well as for paired [4, 5, 17], and fractional domination [8], and the $\{k\}$-domination function (integer domination) [3, 6, 18], and total $\{k\}$-domination function [18]. In 1996, Nowakowski and Rall in [23] made the following Vizing-like conjecture for the upper domination of Cartesian products of graphs.

Conjecture 2 (Nowakowski-Rall's Conjecture)**.** *For any graphs G and H,*

$$\Gamma(G) \cdot \Gamma(H) \leq \Gamma(G \Box H).$$

A beautiful proof of the Nowakowski-Rall's Conjecture was found by Brešar [2]. Also Paul Dorbec et al. in [10] proved that for any graphs G and H with no isolated vertices,

$$\Gamma_t(G) \cdot \Gamma_t(H) \leq 2\Gamma_t(G \Box H), \tag{2}$$

We guess (2) can be extended as follows:

Conjecture 3. (**Vizing-like conjecture for upper k-tuple total domination**)
For any integer $k \geq 2$ and any graphs G and H with minimum degrees at least k,

$$\Gamma_{\times k,t}(G) \cdot \Gamma_{\times k,t}(H) \leq \frac{k+1}{k} \cdot \Gamma_{\times k,t}(G \Box H).$$

Let $G_1, G_2, \cdots, G_n$ and $H_1, H_2, \cdots, H_m$ be respectively the all connected components of two graphs G and H which have minimum degrees at least $k \geq 2$. Then $G\Box H$ is a disconnected graph with the connected components $G_i\Box H_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. By the truth of Conjecture 3 for connected graphs, since

$$\begin{array}{rcl}\Gamma_{\times k,t}(G\Box H) & = & \sum_{1\leq i\leq n}\sum_{1\leq j\leq m}\Gamma_{\times k,t}(G_i\Box H_j)\\ & \geq & \sum_{1\leq i\leq n}\sum_{1\leq j\leq m}\frac{k}{k+1}\cdot\Gamma_{\times k,t}(G_i)\cdot\Gamma_{\times k,t}(H_j)\\ & = & \frac{k}{k+1}\cdot(\sum_{1\leq i\leq n}\Gamma_{\times k,t}(G_i))\cdot(\sum_{1\leq j\leq m}\Gamma_{\times k,t}(H_j))\\ & = & \frac{k}{k+1}\cdot\Gamma_{\times k,t}(G))\cdot\Gamma_{\times k,t}(H)\end{array}$$

we may conclude that Conjecture 3 is true for disconnected graphs. Proposition 6 shows the bound in Conjecture 3, if true, is best possible. Theorem 6, which is obtained by Theorem 5, shows that Conjecture 3 is true for a family of graphs.

Theorem 5. *For any two $\Gamma_{\times k,t}$-external graphs G and H with minimum degree at least $k\geq 2$,*

$$\Gamma_{\times k,t}(G\,\Box\,H)\geq\max\{\Gamma_{\times k,t}(G)\cdot|V(H)|,\Gamma_{\times k,t}(H)\cdot|V(G)|\}.$$

Proof. Let G and H be two $\Gamma_{\times k,t}$-external graphs with minimum degrees at least $k\geq 2$, and let $\Gamma_{\times k,t}(G)\cdot|V(H)|=\max\{\Gamma_{\times k,t}(G)\cdot|V(H)|,\Gamma_{\times k,t}(H)\cdot|V(G)|\}$. Assume D_G is a $\Gamma_{\times k,t}(G)$-set in which every vertex of it has an external (D_G,k)-opn. Obviously, $D=D_G\times V(H)$ is a kTDS of $G\,\Box\,H$. To prove that D is minimal, let $v'\in opn_k(v;D_G)\cap(V(G)-D_G)$. Then $N_G(v')\cap D_G=\{v,v_1,v_2,\cdots,v_{k-1}\}$ for some vertices $v_1,v_2,\cdots,v_{k-1}\in D_G$, and so

$$\begin{array}{rcl}N_{G\,\Box\,H}((v',w))\cap D & = & ((N_G(v')\times\{w\})\cup(\{v'\}\times N_H(w))\cap D\\ & = & (N_G(v')\cap D_G)\times\{w\}\cup(\emptyset\times N_H(w)\\ & = & \{(v,w),(v_1,w),(v_2,w),\cdots,(v_{k-1},w)\},\end{array}$$

which implies $opn_k((v,w);D)\neq\emptyset$ for every vertex $(v,w)\in D$, that is, D is minimal. Hence

$$\begin{array}{rcl}\Gamma_{\times k,t}(G\,\Box\,H) & \geq & |D|\\ & \geq & \Gamma_{\times k,t}(G)\cdot|V(H)|\\ & = & \max\{\Gamma_{\times k,t}(G)\cdot|V(H)|,\Gamma_{\times k,t}(H)\cdot|V(G)|\}.\end{array}$$

□

Theorem 6. *Let G be a $\Gamma_{\times k,t}$-external graph with $\delta(G)\geq k\geq 2$. Then for any graph H with $\delta(H)\geq k$,*

$$\Gamma_{\times k,t}(G\,\Box\,H)\geq\Gamma_{\times k,t}(G)\cdot\Gamma_{\times k,t}(H).$$

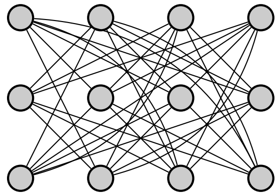

Figure 3: The $K_3 \times K_4$.

The proof of Theorem 5 with Proposition 1 and Theorem 3 imply next theorem.

Theorem 7. *Let G be a $\Gamma_{\times k,t}$-external graph, and let H be an arbitrary graph. Then the following statements hold.*

i. *If $\delta(H) \geq k+1$, then $\Gamma_{\times k,t}(G \,\Box\, H) \geq \Gamma_{\times k,t}(G)(\Gamma_{\times k,t}(H) + \delta(H) - k)$.*

ii. *If H is k-regular, then $\Gamma_{\times k,t}(G \,\Box\, H) \geq \Gamma_{\times k,t}(G) \cdot \Gamma_{\times k,t}(H)$.*

iii. *If H is not k-regular and $\delta(H) = k$, then $\Gamma_{\times k,t}(G \,\Box\, H) \geq \Gamma_{\times k,t}(G) \times (\Gamma_{\times k,t}(H) + 1)$.*

5. The cross product of graphs

In this section, we study the upper k-tuple total domination number of the cross product of two graphs. First we recall that the *cross product* (also known as the *direct product*, *tensor product*, *categorical product*, and *conjunction* in the literature) $G \times H$ has $V(G) \times V(H)$ as vertex set and two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if $(g_1, g_2) \in E(G)$ and $(h_1, h_2) \in E(H)$. For example see Figure 3.

Theorem 8. *If G and H are graphs satisfying $\delta(G) \geq k \geq 1$ and $\delta(H) \geq \ell \geq 1$, then*

$$\Gamma_{\times k\ell,t}(G \times H) \geq \Gamma_{\times k,t}(G) \cdot \Gamma_{\times \ell,t}(H).$$

Proof. Let D_G and D_H be two $\Gamma_{\times k,t}$-sets of G and H, respectively. For a vertex $(u, v) \in V(G \times H)$, let $D_{G,u} = D_G \cap N_G(u)$ and $D_{H,v} = D_H \cap N_H(v)$. Since D_G is a kTDS of G and D_H is a ℓTDS of H, we have $|D_{G,u}| \geq k$ and $|D_{H,v}| \geq \ell$, and so $|D_{G,u} \times D_{H,v}| \geq k\ell$. Now by knowing

$$\begin{aligned} D_{G,u} \times D_{H,v} &\subseteq N_G(u) \times N_H(v) \\ &= N_{G \times H}((u, v)), \end{aligned}$$

we conclude the Cartesian product $D_G \times D_H$ of D_G and D_H is a $k\ell$TDS of $G \times H$. To prove the minimality of $D_G \times D_H$ let $(a, b) \in D_G \times D_H$. Then $a \in D_G$

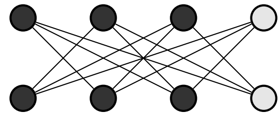

Figure 4: The dark vertices highlight a minimal 2TDS of $K_4 \times K_2$ with maximum cardinality.

and $b \in D_H$ and the minimality of D_G and D_H imply $N_G(a') \cap D_G = S_a$ and $N_H(b') \cap D_H = S_b$ for some vertices $a', b' \in V(G)$ and some k-subset $S_a \subseteq D_G$ and some ℓ-subset $S_b \subseteq D_H$. Hence

$$N_{G\times H}((a', b')) \cap (D_G \times D_H) = S_a \times S_b$$

for the vertex $(a', b') \in V(G \times H)$ and the $k\ell$-subset $S_a \times S_b$. Hence $D_G \times D_H$ is a minimal $k\ell$TDS of $G \times H$, and so

$$\begin{aligned}\Gamma_{\times k\ell,t}(G \times H) &\geq |D_G \times D_H|\\ &= |D_G| \cdot |D_H|\\ &= \Gamma_{\times k,t}(G) \cdot \Gamma_{\times \ell,t}(H).\end{aligned}$$

□

Corollary 2. *If G and H are graphs satisfying $\delta(G) \geq \delta(H) \geq k \geq 1$, then*

$$\Gamma_{\times k,t}(G \times H) \geq \max\{\Gamma_{\times k,t}(G) \cdot \Gamma_t(H), \Gamma_{\times k,t}(H) \cdot \Gamma_t(G)\}.$$

Next proposition shows that the bound given in Theorem 8 is tight.

Proposition 7. *For any integers $1 \leq k \leq n-1$, $\Gamma_{\times k,t}(K_n \times K_2) = 2k + 2$.*

Proof. For integers $1 \leq k \leq n-1$ let $K_n \times K_2$ be the cross product of K_n and K_2 with $V(K_n \times K_2) = V_1 \cup V_2$ in which $V_i = \{1, 2, \cdots, n\} \times \{i\}$ for $i = 1, 2$. For a minimal kTDS S of $K_n \times K_2$ with maximum cardinality, let $S_i = S \cap V_i$ for $i = 1, 2$, and $|S_1| \geq |S_2|$. Obviously $|S_i| \geq k$ for each i, and the minimality of S implies $|S_2| \leq k + 1$. Furthermore, since S has maximum cardinality, $|S_2| = k + 1$. If $|S_1| > k + 1$, then for any vertex $v \in S_1 - S_2'$ the set $S - \{v\}$ is a kTDS of $K_n \times K_2$ in which $S_2' = \{(a, 1)|(a, 2) \in S_2\}$, a contradiction. Hence $|S_1| = |S_2| = k + 1$, and so $\Gamma_{\times k,t}(K_n \times K_2) \leq 2k + 2$. Now the equality can be obtained by Corollary 2. Figure 4 shows a minimal 2TDS of $K_4 \times K_2$ with maximum cardinality. □

As a natural question we may ask the next question.

Question 1. *For any* $n, m \geq 2$ *such that* $\max\{n, m\} \geq k+1$, *whether* $\Gamma_{\times k,t}(K_n \times K_m) = 2k+2$?

Now we present a lower bound for the upper k-tuple total domination number of the cross product of two complete multipartite graphs.

Proposition 8. *Let* $G \times H$ *be the cross product of two complete multipartite graphs* $G = K_{t_1,t_2,\cdots,t_m}$ *and* $H = K_{s_1,s_2,\cdots,s_n}$ *with* $\delta(G \times H) \geq k$. *If*

$$\sum_{1\leq \ell\leq n} t_i s_\ell \geq \sum_{1\leq \ell\leq n} t_j s_\ell \geq 2k \quad \textit{for some} \quad 1 \leq i \neq j \leq m, \quad \textit{or}$$

$$\sum_{1\leq i\leq m} s_\ell t_i \geq \sum_{1\leq i\leq m} s_r t_i \geq 2k \quad \textit{for some} \quad 1 \leq \ell \neq r \leq m,$$

then $\Gamma_{\times k,t}(G \times H) \geq 4k$.

Proof. Let $G = K_{t_1,t_2,\cdots,t_m}$ be a complete m-partite graphs which has the partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_m$ to the disjoint independent sets X_1, X_2, $\cdots$, X_m in which $|X_i| = t_i$ for each i. Similarly, let $H = K_{s_1,s_2,\cdots,s_n}$ be a complete n-partite graphs which has the partition $V(H) = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ to the disjoint independent sets Y_1, Y_2, $\cdots$, Y_n in which $|Y_i| = s_i$ for each i. Then $V(G \times H) = \bigcup_{1\leq i\leq m,\ 1\leq j\leq n}(X_i \times Y_j)$ is the partition of the vertex set of $G \times H$ to the independent sets $X_i \times Y_j$. Without loss of generality, we may assume $m \geq n \geq 2$ and $\sum_{1\leq \ell\leq n} t_1 s_\ell \geq \sum_{1\leq \ell\leq n} t_2 s_\ell \geq 2k$. For $1 \leq i \leq r$, let $k_i \leq \min\{t_1 s_i, t_2 s_i, t_1 s_{i+r}, t_2 s_{i+r}\}$ be a positive integer such that $k = k_1 + \cdots + k_r$. Now we choose a subset S of $V(G \times H)$ such that $|S \cap (X_1 \times Y_i)| = k_i$ for each i. It can be easily seen that S is a minimal kTDS of $G \times H$, and so $\Gamma_{\times k,t}(G \times H) \geq 4k$. $\square$

We think that finding some complete multipartite graphs G and H with $\Gamma_{\times k,t}(G \times H) = 4k$ is a good problem to work.

6. Upper k-transversal in hypergraphs

In this section, we show that the problem of finding upper k-tuple total dominating sets in graphs can be translated to the problem of finding upper k-transversal in hypergraphs. We recall that H_G denotes the open neighborhood hypergraph of a graph G.

Theorem 9. *If* G *is a graph with* $\delta(G) \geq k \geq 1$, *then* $\Gamma_{\times k,t}(G) = \Upsilon_k(H_G)$.

Proof. Since every kTDS of G contains at least k vertices from the open neighborhood of each vertex in G, we conclude every kTDS of G is a k-transversal in H_G. On the other hand, we know that every k-transversal in H_G contains at least k vertices from the open neighborhood of each vertex of G, and so it is a kTDS of G. Therefore we have proved that a vertex subset S is a kTDS of G if and only if it is a k-transversal in H_G, and so $\Gamma_{\times k,t}(G) = \Upsilon_k(H_G)$. □

The authors in [13] proved the problem of finding k-tuple total dominating sets in graphs can be translated to the problem of finding k-transversal in hypergraphs, that is, for every integer $k \geq 1$ and every graph G with minimum degree k, $\gamma_{\times k,t}(G) = \tau_k(H_G)$. This fact and Theorem 9 imply the next theorem.

Theorem 10. *For any graph G with $\delta(G) \geq k \geq 1$,*

$$\Gamma_{\times k,t}(G) = \gamma_{\times k,t}(G) \text{ if and only if } \Upsilon_k(H_G) = \tau_k(H_G).$$

In Proposition 3, we have characterized graphs G satisfying $\gamma_{\times k,t}(G) = k+1$. The next theorem characterizes graphs G satisfying $\gamma_{\times k,t}(G) = m$ for each $m \geq k+2 \geq 3$. We note that in the next three theorems, K'_m denotes a simple graph of order m which has minimum degree at least k.

Theorem 11. *Let G be a graph with $\delta(G) \geq k \geq 1$, and let $m \geq k+2$ be an integer. Then $\gamma_{\times k,t}(G) = m$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$ in which m is minimum in*

$$T = \{t \mid G = F' \circ_k K'_t \text{ for some graphs } F' \text{ and } K'_t\},$$

and $F = G - K'_m$.

Proof. Let G be a graph with $\delta(G) \geq k \geq 1$, and let S be a min-kTDS of $G = (V, E)$ with cardinality $m \geq k+2$. Then $G[S] = K'_m$ for some graph K'_m (because every vertex has at least k neighbors in S). If $|V| = m$, then $G = K'_m$. Otherwise, let $F = G[V - S]$. Since every vertex in $V - S$ has at least k neighbors in S, we conclude $G = F \circ_k K'_m$, and by the definition of the k-tuple total domination number, m is minimum in T.

Conversely, let $G = K'_m$ or $G = F \circ_k K'_m$, in which m is minimum in T, and let $F = G - K'_m$. Then $\gamma_{\times k,t}(G) \leq m$ because $V(K'_m)$ is a kTDS with cardinality m. Now if $\gamma_{\times k,t}(G) = m'$ for some $m' < m$, then, by the previous discussion, $G = F' \circ_k K'_{m'}$ for some graph F' and some graph $K'_{m'}$, which contradicts the minimality of m. This implies $\gamma_{\times k,t}(G) = m$. □

By Proposition 3 and Theorem 11, we obtain the next theorem.

Theorem 12. *For any graph G with $\delta(G) \geq k \geq 1$, $\Gamma_{\times k,t}(G) = \gamma_{\times k,t}(G)$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$ in which m is minimum in*

$$T = \{t \mid t \geq m+1,\ G = F' \circ_k K'_t \text{ for some graphs } F' \text{ and } K'_t\},$$

and $F = G - K'_m$.

Now by Theorems 10 and 12, we conclude:

Theorem 13. *For any integer $k \geq 1$ and any hypergraph H, $\Upsilon_k(H) = \tau_k(H)$ if and only if $H = H_G$, in which G is K'_m or $F \circ_k K'_m$ for some graph K'_m and m is minimum in*

$$T = \{t \mid t \geq m+1,\ G = F' \circ_k K'_t \text{ for some graphs } F' \text{ and } K'_t\},$$

and $F = G - K'_m$.

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