

Explicit Gromov-Hausdorff compactifications of moduli spaces of Kähler-Einstein Fano manifolds

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Dedicated to Sir Simon Donaldson on his 60th birthday

Abstract: We exhibit the first non-trivial concrete examples of Gromov-Hausdorff compactifications of moduli spaces of Kähler-Einstein Fano manifolds in all complex dimensions bigger than two (Fano K-moduli spaces). We also discuss potential applications to explicit study of moduli spaces of K-stable Fano manifolds with large anti-canonical volume. Our arguments are based on recent progress about the geometry of metric tangent cones and on related ideas about the algebro-geometric study of singularities of K-stable Fano varieties.

1. Introduction

The understanding of *moduli spaces of Einstein metrics* on smooth manifolds, together with the process of formation of singularities, is a difficult task still far from being fully achieved. However, in certain cases the picture become clearer. Under non-collapsing assumptions, Cheeger-Colding theory of limit spaces [9], which generalizes results of Anderson [2], Tian [64] and others previously obtained in real four dimension, provides important insights on the limit singularities. Moreover, when the Einstein metrics are also Kähler, one can use information from complex geometry to study such moduli spaces.

Moduli spaces of biholomorphic isometry classes of complex n -dimensional Kähler-Einstein (KE) manifolds with *positive* scalar curvature, which we denote by \mathcal{E}^+M , are always pre-compact in the Gromov-Hausdorff (GH) topology. By Cheeger-Colding theory, degenerate limits are non-collapsed spaces, singular in Hausdorff codimension at least four. Moreover, such limits are naturally *projective KE Fano varieties*, that is, normal algebraic varieties with positive anticanonical divisor carrying singular KE metrics where the metric and algebraic singularities match precisely [21]. Thus we can define a fairly abstract *GH compactification* $\overline{\mathcal{E}^+M}^{GH}$ parameterizing certain smoothable Fano

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varieties. More precisely, thanks to the equivalence between the existence of KE metrics and the algebro-geometric notion of K-stability [13], which also holds in the singular smoothable setting [62], one can define by forgetting the metric structure a natural “Hitchin-Kobayashi map”

$$\phi : \overline{\mathcal{E}^+M}^{GH} \longrightarrow \overline{\mathcal{KM}},$$

where the set $\overline{\mathcal{KM}}$ denotes the isomorphism classes of (\mathbb{Q} -Gorenstein smoothable) K-polystable Fano varieties. It can be shown that $\overline{\mathcal{KM}}$ admits a natural structure of complex analytic space for which the above map ϕ become a homeomorphism [43, 53, 54], and [61] for a survey.

These abstract *Fano K-moduli spaces* $\overline{\mathcal{KM}}$ generalize analogous compactifications of moduli of varieties with positive canonical bundle (KSBA compactifications [38]), and thus they are important spaces to study also for purely algebro-geometric reasons.

However, very few *explicit examples* of such compact Fano KE/K-moduli spaces have been known: the only non-trivial ones are the complex two dimensional orbifold compactifications pioneered in [46] and completed in [54]. In this article, we begin an investigation of some higher dimensional cases. We are mainly interested in two situations. One is the hypersurfaces/complete intersections cases: here, on one hand, there are known results on classical GIT constructions of moduli compactifications and, on the other hand, we often know that the KE moduli is non-empty. The other situation is the three dimensional case, in which there is a complete classification of *smooth* Fanos by works of Iskovskikh, Mori and Mukai (see the reference book [36]).

More precisely, first natural examples to consider in higher dimensions are given by *del Pezzo manifolds*, i.e., n -dimensional Fano manifolds such that $K_X^{-1} = L^{n-1}$ with L an ample line bundle, in other words, with *Fano index* $n-1$. Smooth del Pezzos and, as it will be important later on, mildly singular ones, are classified by T. Fujita [29]. When the del Pezzo degree $d := c_1^n(L)$ is less than or equal to four there are non-trivial moduli deformations. For $d = 4$ all such smooth del Pezzo manifolds are realized as smooth complete intersection of two quadrics $X = Q_1 \cap Q_2$ in $\mathbb{C}\mathbb{P}^{n+2}$, giving an n -dimensional generalization of degree four Del Pezzo surfaces studied in [46, 54]. Such Fanos form, up to biholomorphism, a family of deformations which we will denote as $M_{dP_4^n}$ and they *all* admit KE metrics by [4]. An algebro-geometric compactification of $M_{dP_4^n}$ is given by a standard GIT quotient $\overline{M}_{dP_4^n}^{GIT}$ naturally associated to their defining embedding [5] (compare Section 4). Its boundary GIT polystable points are well-understood.

We are now ready to state our first main result.

Theorem 1.1. *For any dimension $n \geq 2$, the GH compactification $\overline{\mathcal{E}^+ M_{dP_4^n}^{GH}}$ of the moduli space of KE del Pezzo manifolds of degree four is naturally identified with the classical GIT quotient $\overline{M}_{dP_4^n}^{GIT}$. Hence, $\overline{M}_{dP_4^n}^{GIT} \cong \overline{KM}_{dP_4^n}$.*

We will discuss some properties of these compact moduli spaces and of the varieties they parameterize in Section 4. In particular, this result implies an explicit classification of which possibly singular complete intersections of two quadrics in $\mathbb{C}P^{n+2}$ admit KE metrics (see Theorem 4.9), thus extending and reproving the results for the smooth case in [4]. Moreover, thanks to the matching between algebraic and metric singularities, it gives a concrete description of all singularities in the GH limits (see Corollary 4.10). However, the precise asymptotic metric behaviors near the singular loci remain unsettled, but conjecturally clear.

Similarly to [54], the main idea behind the proof of Theorem 1.1 is based on finding a good preliminary *a-priori* understanding of GH limits via the study of their singularities, and then use some classification results for Fano varieties combined with stability comparison arguments to show that a GH limit of KE intersections of quadrics must be a standard GIT-polystable intersection too. Thus the main content of this paper is to find the right constraints on the singularities (in general not of quotient type) of the metric limits. To obtain this, we make use of the recent algebro-geometric understanding of metric tangent cones via the two steps construction [22]

$$\begin{aligned} \text{sing. germ } (Z, p) &\rightsquigarrow \text{weighted algebraic tangent cone } W \\ &\rightsquigarrow \text{metric tangent cone } C(Y), \end{aligned}$$

combined with the improved volume bound for K-stable Fano varieties recently obtained by Liu [44] (after works of Berman [6], K. Fujita [25] and Li [40]) in term of normalized volume of valuations.

Another motivation for the writing to this article is related to finding *explicit examples* of Kähler-Einstein Fano manifolds. By the result in [13] this amounts to verify the K-stability for a given Fano manifold. There have been important recent progresses towards this, see for example [28, 26, 27, 55] as well as older results based on α -invariant/log canonical threshold and multiplier ideal sheaves techniques, for example [59, 64, 66, 50, 19, 12, 14], or with large symmetries, e.g., [68, 18, 35]. However, in general, a systematic understanding is not yet accomplished.

The study in this paper leads to a different strategy via moduli spaces, which originates from [54] and, when it works, it produces not only one example of K-stable Fano manifold, but a *whole explicit family of them*. This

strategy seems particularly appealing for Fano 3-folds, thanks to the above mentioned complete classification of smooth ones [36]. In essence the basic strategy consists in first showing that a given Fano in a family is KE and then run a “moduli continuity method” to show that the set of KE Fano varieties is both open (via small deformations argument) and closed (via a-priori bound on GH limits and stability comparison) in a given explicit GIT quotient or, more generally, glueings of GIT quotients.

We make some first steps along this direction, mostly under the natural hypothesis of an explicit *gap conjecture* on the volume density of Ricci-flat Kähler cones, which is the ratio between the volume of a ball of radius one centered at the vertex and the corresponding ball in the Euclidean space of the same dimension. We believe that such conjecture holds, especially for small values of n . Notice that, for our interest, in this paper we always assume the dimension to be $n \geq 3$.

Conjecture 1.2 (ODP volume gap conjecture in dimension k). *Let $C(Y)$ be a Ricci-flat Kähler cone in dimension k with an isolated singularity at the vertex. Then the volume density of $C(Y)$ is at most the volume of the k -dimensional Stenzel’s cone, which is the ordinary double point (ODP) $\sum_{i=1}^{k+1} x_i^2 = 0$ in \mathbb{C}^{k+1} endowed with the explicit Ricci-flat Kähler cone metric.*

The standard Bishop-Gromov theorem implies that the volume density is at most 1, and it is also well-known that there is a definite but abstract gap from 1 (compare Lemma 5.1). One can also reinterpret the statement as that the link of the ODP singularity has the maximal volume of among all *smooth* Sasaki-Einstein manifolds in dimension $2k - 1$ that are not isometric to the sphere. We will say more about this in Section 5 and also formulate a corresponding algebro-geometric statement.

Theorem 1.3. *Assuming the Conjecture 1.2 is true for all $k \leq n$, then it holds:*

1. *For smooth KE Fano n -folds with anti-canonical volume $c_1^n(-K_X) > \left(\frac{n^2-1}{n}\right)^n$, the GH moduli closure must contain only Fano varieties with at worst Gorenstein canonical singularities.*
2. *The GIT moduli of cubic n -folds coincides with the KE/ K -moduli. In particular, all smooth cubics admit KE metrics.*

We remark that the above results will be direct consequences of a more general statement independent of the conjecture (compare Theorem 5.2).

In dimension three there is a partial extension of the classification to certain Gorenstein Fano 3-folds by Mukai [49]. It seems possible that one can use

similar results in combination with our discussion to understand further examples of Fano K-moduli spaces. Most examples of Fano 3-folds are within the above volume bound, e.g., the moduli space of Iskovskikh-Mukai-Umemura Fano 3-folds. It is an interesting open problem to understand *on their whole moduli* which of them admit KE metrics. As we will discuss in Section 5, the moduli techniques explained in this article suggest a possible strategy for this and other similar situations.

Postscript Remarks: After the first version of this paper was posted on the ArXiv, two related preprints appeared. In [45] the authors showed that for $n = 3$, the GAP Conjecture 5.5 (the algebro geometric version of the above Conjecture 1.2) holds true using classifications results of three dimensional singularities, and, using the moduli continuity method as in [54], they similarly argue the K-polystability of GIT-polystable cubic threefolds. In [42], it is proved that the equality in Proposition 2.2 holds in general; as a consequence, even Conjecture 1.2 (hence Theorem 1.3) holds for $n = 3$.

2. Metric tangent cones, volume densities and valuations

In this section we review and collect some previously known results. Let Z be a GH limit of an n -dimensional KE Fano manifolds (X_i, ω_i) with fixed volume $V = c_1^n(-K_{X_i})$. By general theory [21], we know that Z is a naturally smoothable \mathbb{Q} -Fano variety, that is, a normal variety whose canonical bundle is \mathbb{Q} -Cartier and with at worst Kawamata log terminal (klt) singularities. Moreover, it carries a weak KE metric ω_∞ in the sense of pluri-potential theory [23] with $Vol(Z) = c_1^n(-K_Z) = V$.

Let p be a singular point in the GH limit, recalling that the set of singular points of the underlying algebraic \mathbb{Q} -Fano variety coincides with the set where the limit metric is not smooth. A *metric tangent cone* $C(Y)$ at p is by definition a pointed GH limit centered at p of $(Z, a_i \omega_\infty, p)$ for some increasing sequence of positive numbers $a_i \rightarrow +\infty$. A priori $C(Y)$ may depend on the chosen scaling sequence, but in this case it has been proved in [22] that the tangent cone is actually unique, and it can be realized by some affine complex variety admitting a singular CY cone metric, on the smooth part written as $g_{C(Y)} = dr^2 + r^2 g_Y$, with Y its $2n-1$ real dimensional link, carrying a (possibly singular) Sasaki-Einstein metric. Similarly one can talk about *iterated tangent cones*, that is, metric tangent cones at singular points of the tangent cones, and further iterations. The theory described below applies to these as well.

The cone $C(Y)$ can be obtained from the analytic germ $(Z, p) \subset Z$ near p via the following two steps procedure, which we now recall.

Let $r_{KE}(x) = d_{KE}(x, p)$ be the distance function from p with respect to the limit KE metric. Then one can define the order of vanishing with respect to the KE metric of a germ of holomorphic function $f \in \mathcal{O}_p$ to be equal to

$$\nu_{KE}(f) = \limsup_{r \rightarrow 0^+} \frac{\log \left(\sup_{r_{KE} \leq r} |f| \right)}{\log(r)}.$$

It is proved in [22], that the above is actually a limit, and $\nu_{KE}(f)$ takes values in a discrete set $S(C(Y)) \subseteq \mathbb{R}_{\geq 0}$ called *holomorphic spectrum* of the tangent cone $C(Y)$. Ordering the elements of $S(C(Y)) \subseteq \mathbb{R}_{\geq 0}$, one has a filtration of ideals $\mathcal{O}_p = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \dots$ and an associated ring $R_p = \bigoplus_k \mathcal{I}_k / \mathcal{I}_{k+1}$. Then $W := \text{Spec}(R_p)$ is a well defined normal affine complex variety. Moreover, the analytic germ (Z, p) admits a flat degeneration to W , which further admits an equivariant degeneration to $C(Y)$. Furthermore, it is conjectured in [22] that both W and $C(Y)$ should be uniquely characterized in terms of the algebraic geometry of the germ at p .

Going back to more Riemannian consideration, thanks to Bishop-Gromov monotonicity, which also holds for GH limits and their (iterated) tangent cones by Colding’s volume convergence theorem [15], we can define the *volume density* at any point $p \in Z$:

$$\Theta(Z, p) = \lim_{r \rightarrow 0^+} \frac{\text{Vol}(B_p^{KE}(r))}{\omega_{2n} r^{2n}} \in (0, 1],$$

where ω_{2n} denotes the volume of the unit ball in the flat \mathbb{C}^n . Here are some well-known properties:

- $\Theta(Z, p) = \Theta(C(Y)) = \frac{\text{Vol}(Y)}{\text{Vol}(S^{2n-1}(1))}$, where Y is the link of the (unique) metric tangent Calabi-Yau cone $C(Y)$ at p .
- $\Theta(Z, p) = 1$ if and only if $p \in Z$ is smooth.
- $\Theta(Z, p)$ is a lower semi-continuous function of p . The same is true for densities on the (iterated) metric tangent cones.

Next we need to introduce further algebro-geometric inputs (see [40] for details). Given a *valuation* ν centered at $p \in Z$, one can consider its *normalized volume*

$$\widehat{\text{vol}}(\nu) = A^n(\nu) \lim_{k \rightarrow +\infty} \frac{n!}{k^n} \dim \mathcal{O}_p / \mathcal{J}_k,$$

where the ideal $\mathcal{J}_k = \{f \in \mathcal{O}_p \mid \nu(f) \geq k\}$, and $A(\nu)$ the log-discrepancy of ν .

Generalizing K. Fujita’s bound of the volume of smooth KE manifolds [25] (algebro-geometrically proved via the notion of Ding semistability) to

the singular setting, Liu has recently shown the following optimal volume bound in term of the normalized volume of valuations.

Theorem 2.1 (Liu’s volume estimate, [44]). *Let X be a K -semistable \mathbb{Q} -Fano variety. Then*

$$c_1^n(-K_X) \leq \left(1 + \frac{1}{n}\right)^n \widehat{vol}(\nu),$$

for any valuation ν centered at $p \in X$.

In particular, one can consider in the above inequality the valuation induced by the order of vanishing ν_{KE} , which appeared in the above recalled definition of weighted tangent cone W , since a GH limit Z is indeed K -polystable by Berman’s result [6]. It is conjectured in [40], and partially verified [41], that such valuation minimizes the normalized volume.

A crucial observation for us is that the volume of such KE valuation ν_{KE} is related with the volume density, as shown in [33], Appendix C.

Proposition 2.2. [33] *Let p a point in KE \mathbb{Q} -Fano variety Z . Then the following inequality holds:*

$$n^n \Theta(Z, p) \geq \widehat{vol}(\nu_{KE}).$$

We remark that the above inequality is actually an equality if the metric tangent cone $C(Y)$ at p is quasi-regular, and it is conjectured in [33] that it is always an equality even if the metric tangent cone is irregular.

Thus the above two results combine to provide an estimate of the total volume of a n -dimensional KE \mathbb{Q} -Fano variety Z in terms of the volume of $\mathbb{C}\mathbb{P}^n$, with the Fubini-Study metric with the same Einstein constant:

$$(2.1) \quad \Theta(Z, p) \geq \frac{Vol(Z)}{Vol(\mathbb{C}\mathbb{P}^n)} = \frac{c_1^n(-K_Z)}{c_1^n(-K_{\mathbb{P}^n})},$$

for any $p \in Z$.

By semi-continuity of densities, the same inequality also holds with the left hand side equal to the volume densities $\Theta(C(Y'))$ for any iterated tangent cone $C(Y')$ at $q \in C(Y)$, and further iteration. Strict inequality between densities holds if the iterated tangent cone is not isometric to the original tangent cone itself by the rigidity of Bishop-Gromov, see Lemma 4.5.

Finally, we remark that such estimate should be compared to the analogous one we could obtain using differential geometric argument based on Bishop-Gromov theorem. However, such estimate (based on comparison with the $2n$ -dimensional round sphere) is weaker than the above one obtained via algebro-geometric techniques. For example, one can check that in our

three dimensional case, a differential geometric estimate would be given by $c_1^3(-K_Z) \leq 100 \Theta(Z, p)$, but clearly $c_1^3(-K_{\mathbb{C}P^3}) = 64 < 100$. Having this improved bound is essential for our next investigation.

3. Local Cartier index bound and volume density

Let V and r be positive integers, and denote by $\mathcal{K}_{n,V,r}$ be the set of all n -dimensional Kähler-Einstein Fano manifolds (X, ω) with $c_1^n(-K_X) = V$ and $K_X^{-1} = L^r$ for an ample line bundle L . Let Z be a GH limit of elements in $\mathcal{K}_{n,V,r}$.

Since we have smooth convergence on the smooth locus Z^{sm} of Z , there is a limit Hermitian holomorphic line bundle L_Z on $Z^{sm} \subset Z$ with $L_Z^r = K_Z^{-1}$. Moreover, by [21], L_Z extends as a \mathbb{Q} -line bundle on the whole Z . Namely, there is an integer \tilde{r} such that $L_Z^{\tilde{r}}$ extends as a holomorphic line bundle to Z . In other words, $L_Z^{\tilde{r}}$ is Cartier on Z . At each singular point p of Z , we define the local *Cartier index* $ind(p, L_Z)$ to be the smallest integer \tilde{r} such that $L_Z^{\tilde{r}}$ is Cartier near p , and let $ind(p) = ind(p, L_Z) / gcd(r, ind(p, L_Z))$ be the local *Gorenstein index* of p , so that $K_Z^{-ind(p)}$ is Cartier near p .

More generally, if we choose a sequence of points $p_i \in X_i$ that converge naturally to p under the GH convergence, and a sequence of positive integers a_i , we can discuss *pointed rescaled GH limits* of $(X_i, a_i \omega_i, p_i)$. It is shown in [22], Theorem 1.4 that such a rescaled limit (Z', p') if non-compact (i.e., when $a_i \rightarrow \infty$), is an affine algebraic variety with log terminal singularities which is endowed with a weak Ricci-flat Kähler metric. On Z' , there is also a limit Hermitian \mathbb{Q} -line bundle $L_{Z'}$ which is a genuine line bundle on the smooth locus, and one can similarly define the local Cartier index and the Gorenstein index at a singular point of Z' . Notice that if $a_i \rightarrow \infty$, then the limit Chern connection on $L_{Z'}$ over the smooth locus of Z' is flat (since the curvature of the connection is fixed but the Kähler metric is rescaled by a_i). In general the flat connection may not be trivial since we do not know a-priori its holonomy, which depends on the local topology around the singular locus. This is a key technical issue that has to be overcome in the proof of the main result in [21], and the idea there was to raise the power of the line bundle to make the holonomy sufficiently small, which suffices for [21]. For our main purpose here however, we need to get *explicit* estimates of the power needed.

Let (Z', p) be a pointed rescaled GH limit, which may be Z itself (i.e., when a_i is bounded). One can take a metric tangent cone $C_p := C(Y)$ at p . Notice that, by a diagonal argument, C_p itself is also a pointed rescaled limit. Let O_p be the vertex of C_p , then we have:

Lemma 3.1. $ind(p, L_{Z'}) = ind(O_p, L_{C_p})$.

Proof. We first prove that $ind(p, L_{Z'}) \geq ind(O_p, L_{C_p})$. Let $r_1 = ind(p, L_{Z'})$, then there is a local non-vanishing holomorphic section s of $L_{Z'}^{r_1}$ in a neighborhood of p . Using the local gradient estimate of holomorphic sections as in [22] we know $|\nabla s|$ is uniformly bounded in a neighborhood of p . As we rescale the metric, we see the derivative of s tends to zero, so that s gives rise to a parallel section of $L_{C_p}^{r_1}$ on the smooth part of C_p . In particular $ind(O_p, L_{C_p}) \leq r_1$.

For the other direction it seems likely to follow from general algebro-geometric facts, but here we point out that it also follows directly from the results of [21, 22]. Let $r_0 = ind(O_p, L_{C_p})$. Given a local non-vanishing holomorphic section s of $L_{C_p}^{r_0}$ in a neighborhood of O_p , one can apply the Hörmander construction to obtain a local non-vanishing holomorphic section of $(L_{Z'})^{r_0}$ around p . This can be proved in exactly the same way as in Lemma 2.5 and Proposition 2.9 of [22]. One can simply replace the holomorphic function there by the holomorphic section s . The point is that locally one can realize (Z', p) as GH limits of smooth Kähler-Einstein manifolds, and one can use s to construct local holomorphic sections with uniform lower bound on these smooth manifolds by L^2 -estimate on Stein manifolds. Then by applying a diagonal argument we obtain the required section of $L_{Z'}^{r_0}$ on a neighborhood of p in Z' . \square

Remark 3.2. *We have indeed proved more: on the tangent cone C_p , the flat connection on $L_{C_p}^{r_1}$ is trivial and in particular $L_{C_p}^{r_1}$ is holomorphically trivial. Here $r_1 = ind(p, L_{Z'})$.*

Let \mathcal{I}_p be the set of all iterated metric tangent cones at p , i.e., we include C_p , the set of all the tangent cones of all points in C_p , all the further tangent cones, and so on. As we explained in Section 2, we have that $\Theta(C) \geq V/(n+1)^n$ for all C in \mathcal{I}_p . Recall from general convergence theory of Riemannian manifolds that each element in \mathcal{I}_p is a geodesic length space, meaning that given any two points there is a *length minimizing geodesic* whose length agrees with the distance between the two points. By Cheeger-Colding theory, any element $C(Y)$ in \mathcal{I}_p admits a regular-singular decomposition which is invariant under radial dilation. By the result of Colding-Naber [17] the regular locus of $C(Y)$ is *geodesically convex*, namely, any two regular points can be connected by a length minimizing geodesic that consists only of regular points. Indeed, a stronger statement is proved in [17]: given one regular point and another point, then one can find a length minimizing geodesic connecting them, all points on which are regular except possibly one end point. For convenience we call this property *strong geodesic convexity*.

Lemma 3.3. *For any cone $C \in \mathcal{I}_p$ which is isometrically of the form $\mathbb{C}^k \times C(Y)$ where $k \geq 0$, we have $\pi_1(Y)$ is finite and its order is bounded by $\Theta(C)^{-1}$.*

Proof. It is elementary to see that given any two point q_1, q_2 in Y with distance D smaller than π , a length minimizing geodesic connecting $(0, q_1)$ and $(0, q_2)$ in $\mathbb{C}^k \times C(Y)$ is of the form $(0, c(t)\gamma(t))$ for a length minimizing geodesic $\gamma(t)$ in Y connecting q_1 and q_2 and a function $c(t)$ that depends only on D (which can be determined by considering the model case $Y = S^1$). This implies that the regular set of Y is strongly geodesically convex.

Let $\pi : \tilde{Y} \rightarrow Y$ be the universal cover. Then \tilde{Y} is also naturally a geodesic length space, and a length minimizing geodesic on \tilde{Y} locally maps to length minimizing geodesic in Y . We claim that the regular part of \tilde{Y} is also strongly geodesically convex. To see this, take a regular point q_1 and an arbitrary point q_2 in \tilde{Y} , and let $\tilde{\gamma}(t) (t \in [0, 1])$ be a length minimizing geodesic connecting them. If $\tilde{\gamma}(t) (t \in [0, 1])$ does not contain entirely of regular points, then since the regular set is open, we can find the first time $T \in (0, 1)$ such that $\tilde{\gamma}(T)$ becomes singular. Let $\delta > 0$ be such that any ball of radius 2δ in \tilde{Y} is mapped isometrically to a ball in Y . Then we consider $\tilde{\gamma}(t)$ for $t \in [T - \delta, T + \delta] \cap [0, 1]$. Denote by $\tilde{T} = \min(T + \delta, 1)$. We know $\pi(\tilde{\gamma}(T - \delta))$ is regular, so by the strong geodesic convexity of Y it follows that we can find a possibly different length minimizing geodesic $\gamma'(t)$ connecting $\pi(\tilde{\gamma}(T - \delta))$ and $\pi(\tilde{\gamma}(\tilde{T}))$ so that γ' is regular in the interior. Now by our choice of δ we can lift γ' to a length minimizing geodesic $\tilde{\gamma}'$ in \tilde{Y} connecting $\tilde{\gamma}(T - \delta)$ and $\tilde{\gamma}(\tilde{T})$. So we can replace the portion of $\tilde{\gamma}(t)$ over $[T - \delta, \tilde{T}]$ by $\tilde{\gamma}'$, and still obtain a length minimizing geodesic but on which the first singular time is at least \tilde{T} . Now we can continue this process and since $\delta > 0$ is a fixed number we can in the end reach a length minimizing geodesic on which there is no singular point in the interior. This proves the claim.

By Cheeger-Colding-Tian [10] it follows that $k \leq n - 2$, and the regular locus of \tilde{Y} is Einstein with Ricci curvature $2(n - k)$. Now since the singular set of \tilde{Y} has zero volume (since it has Hausdorff codimension at least 2), we can adapt the usual proof of Bishop-Gromov volume comparison theorem to \tilde{Y} , and conclude that the volume of a ball centered at a regular point $q \in \tilde{Y}$ is at most the volume of the corresponding ball in $S^{2n-2k-1}$. Therefore the volume of \tilde{Y} is at most the volume of $S^{2n-2k-1}$. On the other hand, it is easy to see by definition that the volume of Y is equal to $\Theta(C)Vol(S^{2n-2k-1})$, so we conclude that the degree of the cover is finite, and is bounded above by $\Theta(C)^{-1}$. □

Remark 3.4. *Indeed the above arguments prove more. Take a singular point \tilde{q} of \tilde{Y} , with image $q \in Y \subset C(Y)$, then it holds that*

$$(3.1) \quad Vol(\tilde{Y}) \leq \Theta(C(Y), q)Vol(S^{2n-2k-1}),$$

where $\Theta(C(Y), q)$ is the density of the tangent cone of $C(Y)$ at q . This follows from the fact that for fixed $r < \delta$, $\text{Vol}(B(q, r))$ is a continuous function of q in \tilde{Y} , and applying Bishop-Gromov volume comparison on balls centered at a regular point of \tilde{Y} .

Remark 3.5. We mention that for $n = 3$ it is proved in [21] that the link of tangent cones are five dimensional Sasaki-Einstein orbifolds, in which case the above arguments can be made more direct since the universal cover \tilde{Y} is also a Sasaki-Einstein orbifold.

We can now use the above proposition to study the Cartier index on iterated tangent cones via an induction argument based on the stratification of the singularities.

Lemma 3.6. *If $\Theta(Z, p) > 1/2$, then the \mathbb{Q} -line bundle L_C is Cartier for all $C \in \mathcal{I}_p$.*

Proof. We prove this by induction on k where $n - k$ is the maximum number of flat factor \mathbb{C} that a tangent cone can split off holomorphically isometrically. When $k \leq 1$ by [10] the tangent cone is \mathbb{C}^n and there is nothing to prove. Suppose the conclusion holds for all $k \leq l$. Now suppose one cone C is of the form $\mathbb{C}^{n-l-1} \times C'$. For each non-vertex point q of C' , the metric tangent cone C_q would split off at least a factor \mathbb{C}^{n-l} , by the splitting theorem [10]. So by induction assumption and Lemma 3.1 we know $L_{C'}$ is Cartier near q , hence L_C is Cartier on the complement of $\mathbb{C}^{n-l-1} \times \{O_{C'}\}$. It then suffices to prove that L_C is holomorphically trivial. By Remark 3.2 we know $L_C^{r_1}$ is holomorphically trivial with $r_1 = \text{ind}(p, L_{Z'})$. Let m be the smallest integer so that L_C^m is holomorphically trivial, and let s be a trivializing section. Then we can define a degree m covering of $C \setminus \mathbb{C}^{n-l-1} \times \{O_{C'}\}$ as the subset of (l, x) where $l \in (L_C)_x$ with $l^m = s_x$. This is clearly well-defined on the regular locus and it also extends naturally over the singular locus since L_C is Cartier. The covering is connected by our choice of m . By the volume bound we know $\Theta(C) \geq \Theta(Z, p) > 1/2$, so by Lemma 3.3 the set $C \setminus \mathbb{C}^{n-l-1} \times \{O_{C'}\}$ is simply connected hence m must be equal to 1. Hence we have finished the induction proof. □

Proposition 3.7. *If $\Theta(Z, p) > 1/2$, then L_Z is a genuine line bundle near p , and p is a Gorenstein canonical singularity.*

Proof. By the density estimate, we know $\Theta(C) > 1/2$ for all possibly iterated tangent cones. Applying Lemmas 3.1 and 3.6 we see that $\text{ind}(p, L_Z) = 1$. Hence L_Z is Cartier. Then so is $-K_X$, hence the Gorenstein index of Z is one. Since Z has log terminal singularities, the discrepancies of any log

resolution, which must be integers in our case, are bigger than -1 , so they are non-negative. This shows Z has canonical singularities. It is well-known that canonical singularities are Cohen-Macaulay so must be Gorenstein. \square

Using (2.1) we immediately obtain the following result on GH limits of KE Fanos with large volumes.

Corollary 3.8. *Suppose $V > \frac{1}{2}(n + 1)^n$, then a Gromov-Hausdorff limit Z of KE Fanos in $\mathcal{K}_{n,V,r}$ has Gorenstein canonical singularities and $K_Z = L_Z^{-r}$ for a genuine line bundle L_Z .*

As we will see below, such condition is not empty since del Pezzo manifold of degree four are within that bound. However, it is not known to us if there exist other Fano n -folds ($n \geq 4$) which have *non-trivial K -moduli* and which satisfy the above volume inequality. Thus, we ask the following question:

Question 3.9. *Beside the degree four del Pezzo case, are there any other examples of not rigid K -stable Fano manifolds whose volume is bigger than half the volume of the projective space?*

In any case, one can use exactly the same argument to obtain the following more general Cartier index bound.

Proposition 3.10. *The local Cartier index satisfies $\text{ind}(p, L_Z) \leq \Lambda^{n-1}$, where $\Lambda = \lfloor 1/\Theta_p \rfloor$.*

Proof. By general results from Cheeger-Colding-Tian theory we know that if a metric cone in \mathcal{I}_p splits off isometrically a factor \mathbb{C}^{n-1} then it must be isometric to \mathbb{C}^n . Using this, we can easily follow the proof of Lemma 3.6. \square

Remark 3.11. *Notice this bound is certainly not optimal. One expects the exponent $n - 1$ should not appear. We need it here due to purely technical reason since we used the induction argument. In general at a singular point p , similar to the proof of Lemma 3.6, one can always take the covering of the regular part of C_p with degree equal to $\text{ind}(O_p, L_{C_p})$, and then this yields a bound of the Cartier index if one understands the metric completion of the covering well-enough to show the Bishop-Gromov inequality holds. This is a non-trivial question and we leave it for future study.*

4. KE moduli of degree four del Pezzos

Let X be a smooth n -dimensional del Pezzo manifolds of del Pezzo degree $d = 4$. Then, by classification, X is realized as a smooth complete intersection of two quadrics $X = Q_1 \cap Q_2$ in $\mathbb{C}\mathbb{P}^{n+2}$.

Now we recall that, more generally, a *Del Pezzo variety* is a \mathbb{Q} -Fano variety with Gorenstein canonical singularities such that $K_X^{-1} = H^{n-1}$ for some ample line bundle H . Then clearly $c_1^n(-K_X) = d(n-1)^n$. The classification theory of T. Fujita extends also to this singular case [29]. In particular when $d = 4$ we still get intersection of two quadrics in $\mathbb{C}\mathbb{P}^{n+2}$.

By (2.1), we see that if $n > 3$, $d = 4$, then $d(n-1)^n > \frac{1}{2}(n+1)^n$. So by Corollary 3.8 we know any GH limit of Kähler-Einstein del Pezzo manifolds of degree 4 in dimension $n > 3$ must be again a del Pezzo variety of degree 4, in the above sense. In combination with T. Fujita’s classification [29] of Gorenstein canonical Fano variety with large Fano index, we immediately obtain:

Corollary 4.1. *For any dimension $n > 3$, a GH limit of KE del Pezzo manifolds of degree 4 is always an intersection of two quadrics in $\mathbb{C}\mathbb{P}^{n+2}$.*

The case $n = 3$ does not follow immediately, since we hit the equality case in the volume estimate 2.1. To deal with it, some ad hoc consideration, based on rigidity results, is required.

4.1. GH limits in the three dimensional case

Our next goal is to show that Corollary 4.1 also holds for $n = 3$. Namely:

Proposition 4.2. *Let Z be a GH limit of three dimensional smooth Kähler-Einstein del Pezzo manifolds of degree 4, then Z is a del Pezzo 3-fold of degree 4.*

Again by Fujita’s classification it suffices to show that on any GH limit Z , the corresponding limit bundle L_Z is a genuine line bundle on the whole Z .

If $p \in Z$ is a singular point in the GH limit with density $\Theta(Z, p) > 1/2$, then Proposition 3.7 still apply, giving that L_Z is a line bundle near p and that the singularity is Gorenstein canonical.

We remark that in this three dimensional case there is another argument. Note that in such case the link of the tangent cone Y (clearly necessarily smooth, by semicontinuity of densities) is simply connected (again by volume comparison). Moreover by Cheeger-Tian [11] and Colding-Minicozzi [16]) we have $Y \times \mathbb{R}$ is diffeomorphic a punctured neighborhood of the singularity, so to $M \times \mathbb{R}$, where M is to the classical Milnor’s link obtained by intersecting the germ with a small Euclidean sphere in an embedding of the singularity. Hence M is also simply connected, and this is also enough to have our local index bound at p .

Remark 4.3. *It may be interesting to note that, even if a-priori it is not clear that the link of the cone Y and the Milnor’s link M at the singularity are homeomorphic, the fact that they are actually diffeomorphic follows easily (see uniqueness statement in the main Theorem of [8]) by using the h -cobordism Theorem. We thank Marcel Bökstedt for pointing such reference to us.*

We now assume that $\Theta(Z, p) = 1/2$. Here there are two possible sub-cases:

Case 1: $\Theta(Z, p) = \frac{1}{2}$ and $C(Y)$ has an isolated singularity; The same argument as before works to imply that the singularity is Gorenstein canonical and L is a line bundle near p , due to the following lemma.

Lemma 4.4. *In this case the link Y is simply connected.*

Proof. Applying the Bishop-Gromov volume comparison to the universal cover of Y we obtain that either Y is simply connected, or $\pi_1(Y) = \mathbb{Z}_2$. In the second case the universal cover of Y is isometric to the round sphere S^5 with radius one. So the tangent cone $C(Y)$ is the orbifold $\mathbb{C}^3/\mathbb{Z}_2$ where the action is free on S^5 . But this is impossible by Schlessinger’s rigidity of quotient singularities [58], since the cone is a degeneration of W , which is itself a degeneration of the smoothable local germ Z at p . □

Case 2: $\Theta(Z, p) = \frac{1}{2}$ and $C(Y)$ has non-isolated singularity; The first observation is based on the following general rigidity splitting Lemma for CY tangent cones we mention in Section 2.

Lemma 4.5. *Let $q(\neq 0) \in C(Y)$ be a point where $\Theta(C(Y), q) = \Theta(C(Y), 0)$. Then $C(Y)$ splits isometrically as $C(Y') \times \mathbb{C}$.*

Proof. Consider the volume ratio $f(r) = \frac{Vol_q^{CY}(B(r))}{\omega_{2n} r^{2n}}$ centered at q . Then f is non-increasing and $\Theta(C(Y), q) = \lim_{r \rightarrow 0^+} f(r)$. On the other end, the volume density at infinity is always equal to $\lim_{r \rightarrow +\infty} f(r) = \frac{Vol(Y)}{Vol(S^{2n-1}(1))}$. Thus $f(r)$ is identically constant, i.e., there is a *volume cone* centered at q . But, by Cheeger-Colding theory, any volume cone is actually a metric cone. Thus there is a *line* going through q and hence, by the almost splitting Theorem, we obtain the result. □

Applying the Lemma to our case, we have that $C(Y)$ splits as $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_2)$, i.e., the cone is locally analytically given by the equation $x^2 + y^2 + z^2 = 0$ in \mathbb{C}^4 .

We now prove the following more general proposition, which will be useful also in the discussion of Section 5.

Proposition 4.6. *Let Z' be a pointed rescaled Gromov-Hausdorff limit of a sequence of elements in $\mathcal{K}_{n,V,r}$ (as in the beginning of Section 3), suppose a point $p \in Z'$ has metric tangent cone $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^{n-2}$, then the limit $L_{Z'}$ is Cartier near p and hence p is a Gorenstein canonical singularity of Z' .*

We begin with the following easy Lemma, which is a direct consequence of the arguments in [22]. We think such statement should be well-known (for an algebro-geometric version see for example Exercise II.5.8 and Proposition II.8.7 in [31]), but we gave an argument for readers' convenience.

Recall the embedding dimension (denoted by $Embdim(Z, p)$) of a germ of a singularity (Z, p) is the minimal integer $N > 0$ such that there is a complex-analytic embedding $F : (Z, p) \rightarrow (\mathbb{C}^N, 0)$. By general theory (see Grauert-Remmert [30]), suppose (Z, p) is embedded in some \mathbb{C}^M with $M > Embdim(Z, p)$, then there is a holomorphic function f in a neighborhood of the origin in \mathbb{C}^M that vanishes on Z and has non-vanishing derivative, i.e., $f \in \mathcal{I}_Z \setminus \mathfrak{m}_0^2$.

Lemma 4.7.

$$Embdim(Z, p) \leq Embdim(W, 0) \leq Embdim(C(Y), 0)$$

Proof. We first prove the the first inequality. Notice W is given by the weighted tangent of (Z, p) under some analytic embedding $F : (Z, p) \rightarrow \mathbb{C}^k$, with respect to some weight $w \in (\mathbb{R}^+)^k$. If $k = Embdim(W, 0)$ then we are done. Otherwise, by the above discussion, we find a function $f \in \mathcal{I}_W \setminus \mathfrak{m}_0^2$. Using the grading on W we may further assume that f is homogenous with respect to the weight w , and f has non-vanishing linear term. From the definition of weighted tangent cone, we may find a holomorphic function $g \in \mathcal{I}_Z$ such that f is the initial term of g with respect to w . Clearly g also has non-vanishing linear term. Say this is z_k . Then we may project (Z, p) and W to the z_1, \dots, z_{k-1} plane, and these projections are indeed holomorphic equivalence onto their images. Hence we have realized W as weighted tangent cone of (Z, p) in a smaller \mathbb{C}^{k-1} . We can repeat this until we reach the embedding dimension of W , and we are done.

For the second inequality, we notice that there is an equivariant test configuration degenerating W to $C(Y)$. One can work purely in a polynomial level, and essential argue as in the above: if there is a function vanishing on $C(Y)$ with non-vanishing linear term, then there must be another function vanishing on W also with non-vanishing linear term. \square

A corollary of Lemma 4.7 is that if $C(Y)$ is a hypersurface, then Z is also a hypersurface singularity. Note that it is in general *not* the case that the

GH limit Z is also locally isomorphic to $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ near p : as suggested by [22] and [32], a Kähler-Einstein metric on the three dimensional (isolated) A_k singularity $x^2 + y^2 + z^2 + w^{k+1} = 0$ in \mathbb{C}^4 should have metric tangent cone $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ when $k \geq 3$.

By [22] and the above lemma we can realize $C(Y)$ as the hypersurface $x^2 + y^2 + z^2 = 0$ in \mathbb{C}^4 and embed a neighborhood of p in Z analytically into \mathbb{C}^4 (with p mapped to 0), such that W is given by the weighted tangent cone of Z at p with respect to the weight $(2, 2, 2, 1)$ (which corresponds to the Reeb vector field of $C(Y)$), and $C(Y)$ is a further equivariant degeneration limit of W . Now, by taking a generic hyperplane H in \mathbb{C}^4 passing 0, we obtain a deformation of $S_\infty = H \cap C(Y)$, which is locally isomorphic to the 2 dimensional ordinary double point $x^2 + y^2 + z^2 = 0$ in \mathbb{C}^3 , by the corresponding slices in Z . Since the versal deformation of 2d ordinary double point is given by $x^2 + y^2 + z^2 = \epsilon$ which is one dimensional, it follows that a generic hyperplane slice S_Z at p in Z must also have an ordinary double point singularity at p , see for example [37].

Now we consider the convergence of X_i to Z inside a projective space [21]. We can fit X_i in a flat family (say, over a certain Hilbert scheme) \mathcal{X} with central fiber Z . Similar argument as in the proof of the lemma shows that by restricting to the germ around p , we may obtain a corresponding family of germs of hypersurfaces in \mathbb{C}^4 , which we still denote by \mathcal{X} , that gives rise to a deformation of the germ Z at p . By taking generic hyperplane sections again we may assume that the corresponding slices in \mathcal{X} gives a deformation of S_Z . Again by versality of deformations we conclude that for i large by shrinking the size, the slice S_i in X_i must be isomorphic to a neighborhood in the smoothing $x^2 + y^2 + z^2 = 1$, given by intersecting with a Euclidean ball in \mathbb{C}^3 .

The important fact here is that such a neighborhood is topologically isomorphic to a neighborhood of the zero section in the cotangent bundle of an S^2 (the vanishing sphere), hence it must be simply connected.

Now we can prove that L_Z must be a genuine line bundle. As in Lemma 3.1 it suffices to show that $L_{C(Y)}$ is the trivial line bundle. To see this we first recall by the result of [21] that there is a fixed integer $N > 0$ such that L_Z^N is a line bundle and it is naturally the limit of L_i^N over X_i under the GH convergence. We can take holomorphic sections s_i of L_i^N in a neighborhood in X_i , that converges to a non-vanishing section s of L_Z^N . As in the proof of Lemma 3.1 this further gives rise to a parallel section $s_{C(Y)}$ of $L_{C(Y)}^N$. Since S_i is simply connected, and we can take a N -th root of s_i , to obtain a holomorphic section σ_i of L_i over S_i . Under the natural convergence of S_i to the slice S_∞ (first take limit to S_Z , then dilate, and we can take a diagonal

sequence), we know σ_i converges naturally a holomorphic section σ_∞ of $L_{C(Y)}$ restricted to the smooth locus of S_∞ (which is homotopic to the link S^3/\mathbb{Z}_2), with $\sigma_\infty^{\otimes N} = s_\infty$. Indeed, because of the scaling procedure the section σ_∞ (also s_∞) is parallel section of the flat bundle $L_{C(Y)}$. This implies the flat connection on $L_{C(Y)}$ restricted to one link S^3/\mathbb{Z}_2 has trivial holonomy, hence it is the trivial flat connection outside the singular locus of $C(Y)$. So the corresponding holomorphic line bundle $L_{C(Y)}$ is also trivial. This concludes the proof of Proposition 4.6.

Finally, we remark that there is another argument to conclude such 3-dimensional case. If p is an isolated singularity of Z , then by Milnor’s Theorem on the homotopy groups of hypersurfaces singularities [48] Theorem 5.2 we know the topological link of p has trivial fundamental group. Then one can argue similarly as we did several times before to see that L must be holomorphically trivial on the link, hence must be a line bundle near p . If p is an non-isolated singularity of Z , then again by the above slice argument we know a generic nearby singularity must locally be given by $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. Now we can apply the rigidity case of the Liu’s estimate in [44] Theorem 3, to conclude that $Z = \mathbb{C}\mathbb{P}^3/\mathbb{Z}_2$, with $[x : y : z : t] \simeq [x : y : -z : -t]$. Moreover $\mathbb{C}\mathbb{P}^3/\mathbb{Z}_2$ embeds in $\mathbb{C}\mathbb{P}^5$ via the map $[x : y : z : t] \mapsto [x^2 : y^2 : xy : z^2 : t^2 : zt]$ as the unique, up to linear transformation, GIT-polystable intersection of two quadrics with non-isolated singularities.

In conclusion, by combining the results of this section, we have proved Proposition 4.2.

4.2. Moduli identification

In the previous sections we have shown that all GH limits Z of smooth KE intersection of quadrics are themselves embedded in $\mathbb{C}\mathbb{P}^{n+2}$ as intersection of two quadrics (and the limit root L of the anticanonical bundle is very ample).

By considering the pencil spanned by the two quadrics (i.e., taking the second Hilbert point) one can consider the following GIT picture for the natural Plücker linearization:

$$SL(n + 3, \mathbb{C}) \curvearrowright Gr(2, Sym^2(\mathbb{C}^{n+3})) \hookrightarrow \mathbb{P}(\wedge^2 Sym^2(\mathbb{C}^{n+3})).$$

Let $\overline{M}_{dP_4^n}^{GIT}$ such GIT quotient. It is well-known [5], that strictly GIT stable points correspond precisely to smooth quadric intersections, thus forming a moduli space $M_{dP_4^n}$, itself of dimension n , while more general polystable

boundary points corresponds to certain simultaneously diagonalizable intersections of two quadrics. The study of intersection of quadrics, and questions of simultaneous diagonalization (see for example Reid thesis [57]), is an old subject. Here we mention that the study of the above GIT quotient is in fact related to the study of binary forms, which can be naturally associated to a pencil by taking its discriminant (see [46] and [5]).

Our next goal is to show that all GH limits of smooth KE intersections of quadrics correspond to GIT polystable points. Using K-polystability of the limit Z , which holds by [6], we can say more, via a stability comparison argument based on the CM line bundle [56]. Since the proof is the same as Theorems 3.4 and 4.2 in [54], we just sketch the main points.

Moreover, note that we know $\mathcal{E}^+M_{dP_4^n}$ is not empty [50].

Lemma 4.8. *Z is GIT-polystable.*

Proof. Being the Picard group of the Grassmanian parameter space isomorphic to \mathbb{Z} and the action an $SL(n + 3, \mathbb{C})$ action, the CM line bundle (whose weight is the Donaldson-Futaki invariant) on such Grassmanian is equivariantly equivalent up to positive scaling to the Plücker linearization, since we know that there is at least one K-polystable point. Hence K-polystability implies GIT-polystability¹. □

For readers' convenience, let us sketch the moduli continuity type argument used to prove our main theorem. By the above results we can now define a natural map

$$\phi_{dP_4^n} : \overline{\mathcal{E}^+M_{dP_4^n}^{GH}} \rightarrow \overline{M_{dP_4^n}^{GIT}}.$$

Such map is continuous with respect to the GH, and analytic topology of the target. This follows by Luna's slice Theorem (or [43, 53]): if (X_i, ω_i) GH converge to X_∞ , then $[X_i]$ are represented by GIT polystable points near $[X_\infty]$, since the varieties X_i converge to X_∞ . Hence $[X_i] \rightarrow [X_\infty]$. The continuity of $\phi_{dP_4^n}$ extends to the boundary. Now we claim that $\phi_{dP_4^n}$ is actually a homeomorphism. Injectivity follows by Bando-Mabuchi uniqueness and its generalization to the singular setting [7]. Since $Aut(X)$ is finite for a smooth X , by Implicit Function Theorem $\phi_{dP_4^n}(\mathcal{E}^+M_{dP_4^n})$ is open in $M_{dP_4^n}$. Moreover $\phi_{dP_4^n}(\mathcal{E}^+M_{dP_4^n})$ is also closed: let $[X_i] \rightarrow [X_\infty]$ in $M_{dP_4^n}$, with X_i KE. Then, by continuity of the map, $\phi_{32}([X_{i_j}, \omega_{i_j}]) \rightarrow \phi_{dP_4^n}([Y_\infty, \omega_\infty]) = [Y_\infty] \in \overline{M_{dP_4^n}^{GIT}}$, with Y_∞ a (a-priori possibly singular) KE GH limit. Since $\overline{M_{dP_4^n}^{GIT}}$ is Hausdorff

¹The CM line bundle is not defined on all the Grassmanian due to the presence of points representing non-equidimensional intersection of quadrics. The proof requires some further checking (see the Appendix).

and by definition $\phi_{dP_4^n}([X_{i_j}, \omega_{i_j}]) = [X_{i_j}]$, the limit is unique, hence $X_\infty \cong Y_\infty$ admits a KE metric. $M_{dP_4^n}$ is also connected, hence $\phi_{dP_4^n}(\mathcal{E}^+M_{dP_4^n}) = M_{dP_4^n}$, recovering [4]. Finally $\phi_{dP_4^n}(\overline{\mathcal{E}^+M_{dP_4^n}}^{GH})$ is a compact set containing the dense subset $M_{dP_4^n}$. Hence $\phi_{dP_4^n}$ has to be onto. Being $\phi_{dP_4^n}$ a continuous map between a compact and an Hausdorff space, $\phi_{dP_4^n}$ is also open. This concludes the proof of our main Theorem 1.1.

We now describe some properties of such KE moduli spaces and of their boundary points. By the classification of GIT polystable intersections of two quadrics [5] (Theorem 4.2 and its proof), we get the following result on the explicit existence of (weak) KE metrics on intersections of two quadrics in $\mathbb{C}P^{n+2}$.

Theorem 4.9. *A possibly singular complete intersection of two quadrics in $\mathbb{C}P^{n+2}$ admits a KE metric if and only if the two quadrics can be simultaneously diagonalized, the discriminant of their pencil admits no roots of multiplicity $> \frac{n+3}{2}$ and if there is a root of multiplicity exactly equal to $\frac{n+3}{2}$ then $Q_1 \cap Q_2$ is actually isomorphic to $\{\sum_{i=0}^{\frac{n+1}{2}} x_i^2 = \sum_{i=\frac{n+3}{2}}^{n+2} x_i^2 = 0\}$.*

The above just says that, by changing basis, KE intersections are cut by $Q_1 = Id$ and Q_2 diagonal with eigenvalues whose multiplicities are no greater than $\frac{n+3}{2}$. Note that the equality $\frac{n+3}{2}$ case for the multiplicity can occur for odd dimensional intersections of two quadrics only.

In dimension three, in particular, we can say that GH limits have at most isolated singularities or, if not, they coincide with the orbifold $\mathbb{C}P^3/\mathbb{Z}_2$, which is singular along two disjoint smooth rational curves (this a rigidity case for Liu’s volume estimate). It is also easy to see that the number of ODP singularities is always even, their maximum number is six, and this is achieved by the unique toric KE intersection $xy = zt = uv$. Moreover, thanks to the relation with the well-understood invariants of binary sextics, we have $\overline{\mathcal{E}^+M_{dP_4^3}}^{GH} \cong \overline{\mathcal{KM}}_{dP_4^3} \cong \mathbb{C}P(1, 2, 3, 5)$ as topological spaces, with boundary divisor given by $t = 0$, where t is the weight 5 coordinate in the weighted projective space.

More generally, from Theorem 4.9, we can get the following understanding of the singular set:

Corollary 4.10. *The singular set of GH limits of KE intersections of two quadrics in $\mathbb{C}P^{n+2}$ is at most of complex dimension $\lfloor \frac{n-1}{2} \rfloor$. Moreover, the algebraic stratification of the singular set consists of disjoint smooth strata which are bundles of ODP singularities, i.e., locally analytically of type $\mathbb{C}^k \times \mathbb{A}_1^{n-k}$ with $k \in \{0, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. In particular, there are no quotient singularities as soon as $n \geq 4$.*

Clearly the description is so explicit that one can even compute the number of connected components of the singular set, as well as describe their topology.

Finally a couple of remarks are needed.

Remark 4.11. *It may be interesting to observe that the above KE spaces have isolated ODP singularities which degenerate to non-isolated singularities (here still of type $\mathbb{C}^k \times A_1^{n-k}$) in their GH limits: locally analytically this corresponds to deformations of singularities like:*

$$V_t : x_0^2 + x_1^2 + x_2^2 + tx_3^2 = 0.$$

Such behavior shares some similarities with the jumping of metric tangent cones described in Section 2.

The above phenomenon already happens for certain deformations of the three dimensional KE orbifold $\mathbb{CP}^3/\mathbb{Z}_2$. Note also that, in such case, stability excludes the possibility of obtaining a KE smoothing of this KE orbifold with only one of the two rational curves of singularities smoothed out. Clearly, similar behaviors occur in higher dimensions too.

Remark 4.12. *It is natural to conjecture that the above singular KE metrics admit asymptotic expansions in suitable local holomorphic gauge to products of flat times Stenzel's cone metrics (i.e., they are conically singular metrics). We should remark that by [33] similar results hold for smoothable Calabi-Yau varieties with isolated ODP singularities. It is natural to expect that similar arguments may work also in the KE Fano case, and we leave this for future work.*

4.3. Relations with moduli of hyperelliptic curves

Finally, we briefly emphasize that smooth intersections of two quadrics are also deeply related to moduli spaces of hyperelliptic curves: to any such *smooth* Fano n -fold one can naturally associate a genus two curve by taking the ramified $2 : 1$ cover branching over the zeros of the discriminant of the pencil.

When $n = 3$, one can further recover the original Fano as a moduli of stable rank 2 vector bundles of odd degree on the curve ([52, 51]). This gives a well-know instance of a “Fourier-Mukai type duality” between moduli space of genus two curves and moduli space of our Fano manifolds.

However (see [5]), this 1-1 relation breaks at the boundaries of their two canonical compactifications: the Deligne-Mumford compactification \overline{M}_2^{DM}

(which is the exactly equal to the K-compactification in the case of curves) is a (smooth) blow-up of our K-compactification $\overline{\mathcal{KM}}_{dP^3_4}$. In particular, $\overline{\mathcal{KM}}_{dP^3_4}$ is more related to a different algebro-geometric compactification of genus two curves, namely to the so-called $\overline{M}_2[A_2]$ compactification (e.g., the survey [24]). Such compactification naturally appears in the Hassett-Keel birational study of the moduli space of curves.

More precisely, the coarse varieties $\overline{\mathcal{KM}}_{dP^3_4}$ and $\overline{M}_2[A_2]$ are *isomorphic* to each other and are both categorical quotients of some good moduli spaces in Alper’s sense: of the *KE-K moduli stack* of K-semistable degree four del Pezzo threefolds (see [43, 53, 54] for details on the definitions), and of the stack of pseudostable curves (nodal curves where elliptic tails are replaced by cusps singularities) for $\overline{M}_2[A_2]$. In particular, there is a set theoretical 1-1 map between closed points in the stacks (the relation become more complicated on the full semistable strata). For example, the orbifold $\mathbb{CP}^3/\mathbb{Z}_2$ corresponds to the *bicuspidal rational curve*, i.e., the rational curve with only two cuspidal singularities of type $x^2 = y^3$. It is not clear how the vector bundle relation extends to the boundary. We think that such correspondences deserve further investigation.

For higher dimensions, here we just recall that, by the well-known theorem of Desale and Ramanan [20] there is still an identification between moduli spaces of stable rank two vector bundles with fixed determinant on smooth hyperelliptic curves of genus g and the variety of $g - 2$ dimensional linear subspaces of the corresponding simultaneously diagonalized smooth intersection of two quadrics.

5. Further KE moduli examples: conjectures

In this last section we discuss further examples of KE moduli under the hypothesis of a natural gap conjecture about volume of CY cones/Sasaki-Einstein manifolds. Moreover, we formulate an analogous conjecture on the algebro-geometric counterpart.

5.1. ODP volume gap conjecture

As it is apparent from the discussion in previous sections, the volume of the link of tangent cones plays an important role. In dimension two, since the singularities occurring in GH limits are of orbifold type, it is obvious that such volume densities are always of type $1/|G_p|$ where $|G_p|$ is the order of the orbifold group at p . In particular, one sees immediately that there is a *gap* between smooth or singular points where the density is always $\Theta(Z, p) \leq \frac{1}{2}$

and the value $\frac{1}{2}$ precisely occurs for the A_1 singularity, i.e., the ordinary double point singularity, which can be considered in some sense as “the simplest singularity” in dimension two.

In higher dimensions non-quotient singularities may appear in GH limits, and the metric tangent cone may not be isomorphic to the singularity itself. Nonetheless one can still consider the volume density at a singular point p , and we still have the following well-known abstract gap result:

Lemma 5.1. *There is a constant $\delta < 1$ depending only on n , such that for any GH limit Z of KE Fano manifolds in dimension n , we have $\Theta(Z, p) \leq \delta$ for all singular points $p \in Z$.*

Proof. If the tangent cone $C(Y)$ has smooth cross section Y , then Y is Sasaki-Einstein and this follows from Anderson’s rigidity result [3]. In general one needs to use in addition Colding’s theorem on volume convergence under GH limits [15]. Indeed, if p has volume density sufficiently close to 1, then we can find a ball centered at p with radius r sufficiently small such that $\text{Vol}(B(p, r))/\omega_{2n}r^{2n}$ is also close to 1. By volume convergence, we can find corresponding balls $B(p_i, r)$ on the smooth manifolds before we take limits, with $\text{Vol}(B(p_i, r))/\omega_{2n}r^{2n}$ close to 1, hence by Anderson’s gap result one obtains that $B(p_i, r)$ converges smoothly to $B(p, r)$ (see Theorem 0.8 in [15]). \square

Now for $n \geq 2$ we define $\mathcal{A}(n)$ to be the supremum of $\Theta(Z, p)$ for all GH limits Z of KE Fano manifolds in dimension n , and for all singular points $p \in Z$. By the above lemma $\mathcal{A}(n) < 1$, and $1 - \mathcal{A}(n)$ can be viewed as the optimal volume gap in dimension n . Again by Colding’s volume convergence theorem $\mathcal{A}(n)$ can be realized as the density of some pointed Gromov-Hausdorff limit metric cone of a sequence of smooth KE Fano manifolds.

For our purpose here we only need a slightly different notion, without referring to singular GH limits. Namely we define $\mathcal{A}'(n)$ to be the supremum of $\Theta(C(Y))$, where $C(Y)$ is a cone of the form $\mathbb{C}^{n-k} \times C(Y')$ for $C(Y')$ a k dimensional CY cone which is singular precisely at the vertex. Clearly $\mathcal{A}(n) \leq \mathcal{A}'(n) \leq 1$, and $\mathcal{A}'(m) \leq \mathcal{A}'(n)$ for $m \leq n$. Moreover, by [47], $\mathcal{A}'(n)$ is always an algebraic number.

For the simplest possible singularity, namely, the A_1 ODP singularity $\sum_{i=1}^{n+1} x_i^2 = 0$ (which is not of quotient type as soon as $n > 2$), we know it admits a natural Stenzel’s CY cone metric. The corresponding Sasaki-Einstein manifold is a circle bundle over the quadric hypersurface in \mathbb{P}^n equipped with a homogeneous Kähler-Einstein metric. It is easy to calculate that the volume

density is $2(1 - \frac{1}{n})^n$. Hence we have a lower bound

$$(5.1) \quad \mathcal{A}'(n) \geq 2(1 - \frac{1}{n})^n.$$

Theorem 5.2. *Let Z be a GH limit of KE Fano manifolds X_i with Fano index r and*

$$c_1^n(-K_{X_i}) > \frac{1}{2}\mathcal{A}'(n)(n + 1)^n,$$

then Z has Gorenstein canonical singularities and $-K_Z = rL_Z$ for some Cartier divisor L_Z .

Proof. The proof follows similar lines as in Section 3 and 4. Below we only sketch the new technical points.

Lemma 5.3. *For any iterated tangent cone of the form $\mathbb{C}^k \times C(Y)$ with $k \geq 0$, $\pi_1(Y)$ is trivial unless the cone is a quotient of the form \mathbb{C}^n/G for a finite subgroup $G \subset U(n)$.*

Proof. Suppose $\pi_1(Y)$ is not trivial, then arguing as in the proof of Lemma 3.3, we obtain the universal cover \tilde{Y} with $Vol(\tilde{Y}) > \mathcal{A}'(n)Vol(S^{2n-2k-1})$. We claim that \tilde{Y} must be smooth. Otherwise, by Remark 3.4 there is a point $q \in Y \subset C(Y)$ with $\Theta(C(Y), q) > \mathcal{A}'(n)$. By passing to an iterated tangent cone we obtain a cone of the of form $\mathbb{C}^l \times C(Y')$ with Y' smooth but volume density bigger than $\mathcal{A}'(n)$. Contradiction. Now given the claim, by our gap assumption we conclude that the cone over \tilde{Y} must be the flat \mathbb{C}^n , hence $C(Y)$ is a quotient of \mathbb{C}^n . \square

By (5.1), $\mathcal{A}'(n) > 2/3$ for $n \geq 6$ and $\mathcal{A}'(n) > 1/2$ for all n , it follows that the above subgroup G can only be either \mathbb{Z}_2 or \mathbb{Z}_3 . Now one can follow the induction proof of Lemma 3.6. If at p all the iterated tangent cones are not equal to quotient of \mathbb{C}^n , then the proof is exactly the same. If one iterated tangent cone is a quotient \mathbb{C}^n/G , then for the induction argument to go through, we need to follow the proof of Proposition 4.6. Notice such quotient is necessarily of the form $\mathbb{C}^{n-l} \times \mathbb{C}^l/G$, where G acts non-trivially in the sphere $S^{2l-1} \subset \mathbb{C}^l$. We claim $l = 2$. This follows by taking generic sections as in the proof of Proposition 4.6, and using Schlessinger’s rigidity of quotient singularities in dimension greater than two [58]. Now for our induction to work by the proof of Proposition 4.6, it suffices to show that on a smoothing of \mathbb{C}^2/G with $G = \mathbb{Z}_2$ or \mathbb{Z}_3 , there is a Stein neighborhood of the vanishing cycle that has trivial fundamental group. This is clearly the case when $G \subset SU(2)$. So we are left with the case $G = \mathbb{Z}_3$ acts diagonally on \mathbb{C}^2 with

weight (ζ_3, ζ_3) . We claim this case never occurs. Notice it is the non- \mathbb{Q} -Gorenstein smoothable cone over a rational normal curve of degree 3. Its versal deformation space (coinciding with the Artin component [63]) is given by a two dimensional space which parameterizes smoothings admitting Stein neighborhoods homeomorphic to the total space of the $\mathcal{O}(-3)$ bundle over \mathbb{P}^1 , hence again simply connected. So by the proof of Proposition 4.6 it follows that \mathbb{C}^2/G has to be Gorenstein, so we have proved the claim. \square

When we use the rough bound $\mathcal{A}'(n) \leq 1$ the above result reduces to Corollary 3.8. In general for practical applications of Theorem 5.2 it is important to understand $\mathcal{A}'(n)$ better. It is not unreasonable to expect the following:

Conjecture 5.4. *For all n ,*

$$\mathcal{A}'(n) \leq 2\left(1 - \frac{1}{n}\right)^n.$$

One expects the equality only holds for the n dimensional Stenzel cone. Now it is easy to see that Conjecture 1.2 implies Conjecture 5.4, and the first part of Theorem 1.3 follows exactly from Theorem 5.2.

Volume of links of CY cones have been studied by Martelli-Sparks-Yau in the context of AdS-CFT correspondence (see, e.g., [47]), where it is shown that, CY cones are minimizers of the volume functional on the space of Reeb vector fields. From a physics point of view, here we just remark that, in dimension three, such volume densities are supposed to be inverses of quantities related to central charges of certain dual superconformal field theory (*a-maximization*).

As we have recalled in Section 2, thanks to the description of the volume minimization in terms of the language of valuations [40] and their relation to densities (still conjecturally in the irregular case), it is natural to formulate a purely algebro-geometric conjecture analogous to the density gap above described.

Conjecture 5.5 (ODP volume gap conjecture, algebraic version). *Let (V, p) be a non-smooth germ of a n -dimensional klt singularity. Then the infimum of the normalized volume of valuations centered at p satisfies*

$$\inf_{\nu} \widehat{\text{vol}}(\nu) \leq 2(n-1)^n,$$

with equality realized only by the ordinary double point singularity $\sum_{i=1}^{n+1} x_i^2 = 0$.

- Remark 5.6.** 1. *From discussions with Kento Fujita and Yuchen Liu, we are informed that even the weaker inequality $\inf_\nu \widehat{\text{vol}}(\nu) \leq n^n$ is not yet proved in complete generality. Notice from a metric point of view, if (V, p) is a germ of a pointed rescaled Gromov-Hausdorff limit of KE Fano manifolds, then by Proposition 2.2 this weaker inequality holds by Bishop-Gromov volume comparison theorem. On the other hand, Conjecture 5.4 follows from Conjecture 5.5 if the equality case of Proposition 2.2 holds (which is known to be true when the metric tangent cone is quasi-regular).*
2. *When $n = 3$, it is possible that such densities are always $\leq \frac{1}{2}$ except the cases of A_1 singularity, when it is equal to $16/27$ or A_2 singularity, when it is equal to $125/243$ (note that they are equal to $1/2$ for any other A_k singularity for $k \geq 3$, corresponding to the expected jumping picture for the metric tangent cone to the flat cone $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$). In fact, we do not know any other CY cones (even among known quasi-regular or irregular ones) which have higher values of densities.*

5.2. Del Pezzo of degree three (cubic hypersurfaces)

Now we prove part (2) of Theorem 1.3.

Let Z be a GH limit of smooth Kähler-Einstein cubic hypersurfaces in dimension $n \geq 3$. Since the volume is given by $3(n-1)^n \geq (1 - \frac{1}{n})^n (n+1)^n$, by Theorem 5.2 we know Z has at worst Gorenstein canonical singularities and $K_Z = L_Z^{n-1}$ for some ample line bundle L_Z . Hence by Fujita's classification [29] Z must be a cubic hypersurface too. Notice also by [65] (or [50] for $n \leq 4$) a Fermat cubic hypersurface $x_0^3 + \dots + x_n^3 = 0$ admits a Kähler-Einstein metric so is K-stable. By the same discussion as in [54] this implies that the CM line bundle over the GIT moduli is a positive multiple of the standard polarization $\mathcal{O}(1)$, hence is ample. By [6] we know Z is K-polystable, so Z is a GIT polystable cubic hypersurface. Now we argue as in [54] and conclude that the GIT compactification agrees with KE compactification. This finishes the proof of Theorem 1.3, (2).

Now we recall that in dimension 3 and 4 the GIT moduli space has been extensively studied. GIT of cubic threefolds is done by Allcock [1]. We recall the result

- Theorem 5.7** ([1]).
- *A cubic threefold is GIT stable if and only if it has only isolated singularities of type A_k , $k \leq 4$.*
 - *A cubic threefold is GIT polystable with non-discrete stabilizer if and only if it is isomorphic to $F_\Delta = x_0x_1x_2 + x_3^3 + x_4^3$ which has exactly*

three D_4 singularities; or it is isomorphic to $F_{A,B} = Ax_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 + Bx_1x_2x_3$ where at least one of A, B is non-zero, which has at worst isolated A_1 and A_5 singularities when $4A \neq B^2$, and which has a transverse A_1 singularity along a rational normal curve when $4A = B^2$ (the chordal cubic).

In particular it is easy to see that a GIT polystable cubic in dimension 3 must have canonical singularities.

GIT of cubic fourfolds has been studied in detail by Laza in [39]. In particular, it is known that a cubic fourfold with ADE singularities is GIT stable, and singularities of a polystable cubic fourfold corresponding to a generic point on the GIT quotient compactification are classified (see the longer list of such possible singularities in Table 3 of [39]).

As far as we are aware, there is no explicit study of higher dimensional GIT quotients of cubics hypersurfaces. However, our previous discussion leads to some purely GIT questions that may be worth studying: as a direct consequence of Theorem 1.3, (2), we notice that if Conjecture 5.4 holds, then the GIT moduli agrees with KE moduli. However, by [21] every GH limit in the KE moduli must have log terminal singularities, and if is again a cubic hypersurface then it must have Gorenstein canonical singularities. So a natural question is:

Question 5.8. *Is it true that a GIT polystable cubic in \mathbb{P}^n has at worst Gorenstein canonical singularities? In particular, does it always have normal singularities?*

This is intimately related to Conjecture 5.4. If the answer to this question is negative, then it would mean Conjecture 5.4 is false.

We observe that even without assuming Conjecture 5.4 we have:

Proposition 5.9. *A GH limit Z of smooth Kähler-Einstein cubic threefolds has only Gorenstein canonical singularities, and $K_Z^{-1} = L_Z^2$ for some \mathbb{Q} -line bundle L_Z .*

This follows from similar arguments as in the proof of Lemma 3.6 and Proposition 4.6, using the fact that $K_{X_i}^{-1} = L_{X_i}^2$. If one can prove L_Z is a genuine line bundle, then it would follow from the classification of Fujita [29] that Z must be a cubic hypersurface in \mathbb{P}^4 and hence the KE moduli agree with GIT moduli. It would also be interesting to see if Fujita's classification can be extended to our setting.

5.3. Other Fano manifolds

We expect that, in order to proceed further, an extension of the classification of mildly singular smoothable Fano 3-folds with small Gorenstein index and with bounds on invariant of the singularities given by differential geometric considerations as above, would be useful for studying both moduli space of Fano 3-folds and the existence of KE metrics on them. We should recall that there is a classification of Gorenstein Fano 3-folds with assumptions on decompositions of the anticanonical linear system by Mukai [49].

Our discussion gives the following picture regarding a-priori bounds on the Gorenstein index of GH limits of KE Fano 3-folds:

Corollary 5.10. *Assuming Conjecture 5.4, GH limits of smooth KE Fano 3-folds whose volume is ≥ 20 are always Gorenstein with canonical singularities. Without assuming Conjecture 5.4, the Gorenstein index of GH limits is at most two as long as the volume is ≥ 22 and in this case the canonical divisor is Gorenstein away from at most finitely many points.*

Among the 105 deformation types of smooth Fano 3-folds, 75 have volume bigger than or equal to 20, and four deformation classes have volume equal to 22.

Here we recall that the volume of a Fano 3-fold is always even, and as long as the volume is bigger than 32 we have mostly rigid examples.

Interesting examples include the case of Iskovskikh-Mukai-Umemura Fano 3-folds of degree 22 and Picard rank one, since arguments along these lines can solve the problem of understanding which smooth ones are KE, even “far away” from the special KE Mukai-Umemura 3-fold. Here it would be interesting to show that all GH limits are still given by intersections of three sections of the tautological vector bundle over the Grassmanian as the smooth ones do. Then (see discussion in [60]), it seems very plausible that one can consider a GIT picture similar to the previous case of degree four del Pezzos to conclude.

Another example (with much smaller anticanonical volume) we like to briefly discuss is the case of quartic hypersurfaces in $\mathbb{C}\mathbb{P}^4$. By very recent work of Fujita [26], it is known that all smooth quartics admit KE metrics. However, by classification of smooth Fano 3-folds, it is known that quartic hypersurfaces do not form a complete family, since they can be deformed to the “hyperelliptic” Fanos given by double covers of a smooth quadric in \mathbb{P}^4 ramified over a smooth divisor of degree eight obtained as intersection with a quartic. In such situation, it is also known that all such Fanos admit KE metrics [4]. Thus the GH moduli compactification is *not* equal to the GIT quotient of quartics, since we need (at least) to blow-up such GIT quotient

at the non-reduced double conic, similarly to the case of degree 2 del Pezzo considered in [54]. We think that a concrete algebro-geometric analysis of such situation, by trying to explicitly construct a compactification made only by \mathbb{Q} -Fano varieties, is indeed very interesting.

Similarly, moduli of del Pezzos of low degree and of other Fano 3-folds could be possibly analyzed by considering GIT gluings coming from certain hypersurfaces in weighted projective spaces. In general, it is clearly very interesting to investigate precisely which Fano 3-folds are known to have a non-empty KE moduli spaces.

Appendix by Yuji Odaka, Cristiano Spotti & Song Sun

Theorem 3.4 of [54] cannot be applied directly to conclude that a K -polystable double complete intersection of quadrics, that is a del Pezzo surface of degree four, is GIT-polystable. This was overlooked in [54, Theorem 4.2] and Lemma 4.8 of this paper at the time of writing. In this appendix we provide a fix to this problem.

Let us explain the issue first. The Grassmannian $S = Gr(2, \text{Sym}^2 \mathbb{C}^{n+1})$ parametrizes the space of all complete intersections of two quadrics in \mathbb{P}^n for $n \geq 4$. To apply Theorem 3.4 in [54] we need to make the CM line bundle well-defined over S which is *not* the case since there is a special locus Σ in S parametrizing subschemes of \mathbb{P}^n which contain some hypersurfaces, hence violating the hypothesis of Theorem 3.4 of [54].

First, the following easy lemma identifies the non-equidimensional locus $\Sigma \subseteq S$ of the family.

Lemma A.1. *A quadric intersection parameterized by S is not of (pure) codimension 2, if and only if it is isomorphic to $\{x_0x_1 = x_0x_2 = 0\}$ or to $\{x_0^2 = x_0x_1 = 0\}$ via a (projective) linear transformation. Here, x_0, \dots, x_n is a homogeneous coordinate system of \mathbb{P}^n .*

Proof. Let F, G form a basis of the 2-plane in $H^0(\mathbb{P}^n, \mathcal{O}(2))$. If $F = 0$ is irreducible then G must be not divided by F and thus $F = G = 0$ is of codimension 2. If $F = 0$ is reducible, it must split into two hyperplanes, say $F = x_0y$. If $F = G = 0$ is still $(n - 1)$ -dimensional, then $G = 0$ must contain one of the two hyperplanes, say $x_0 = 0$. Then the other factor of G is of degree one and $G = x_0w$ after a linear transformation, thus proving the lemma. \square

Thus Σ is the $PGL(n + 1; \mathbb{C})$ orbits of the points in S corresponding to the subschemes $\{x_0x_1 = x_0x_2 = 0\}$ and $\{x_0^2 = x_0x_1 = 0\}$. The CM line bundle λ_{CM} is well-defined over $S' = S \setminus \Sigma$. Since S is smooth and the codimension

of Σ is at least 2, we know λ_{CM} extends to a line bundle on the whole S and $\lambda_{CM} \simeq \mathcal{O}(a)$ for some integer a since $\text{Pic}(S) \simeq \mathbb{Z}$. Being $PGL(n + 1; \mathbb{C})$ connected without non-trivial characters, it follows that this isomorphism can be taken to be equivariant. Moreover, we know $a > 0$ since in the context of [54] and of this paper, S contains at least one K -polystable point. It follows that if we consider a degeneration *within* S' , then the sign of the Donaldson-Futaki invariant agrees with the sign of the GIT weight. In order to fix the issue it is then sufficient to check the following.

Proposition A.2. • *Suppose an element $P \in S'$ is GIT unstable, then there is a one-parameter subgroup $\lambda(t)$ such that the corresponding limit $P_0 = \lim_{t \rightarrow 0} \lambda(t).P$ has negative GIT weight and is contained in S' .*
 • *If an element $P \in S'$ is GIT semistable but not GIT polystable, then there is a one-parameter subgroup $\lambda(t)$ such that the corresponding limit P_0 is GIT polystable hence is contained in S' .*

Corollary A.3. *If the associated quadric intersection to a point $P \in S'$ is K -semistable (resp., K -polystable), then P is GIT semistable (resp., GIT polystable).*

Proof. This follows from the proposition and the comparison of K -stability and GIT-stability using the CM line bundle as in Theorem 3.4 in [54]. □

Proof of proposition A.2. For the second item, we know by standard GIT that for such P , there is a one-parameter subgroup $\lambda(t)$ such that P_0 is GIT polystable. Since the GIT instability of Σ easily follows from its definition by the vanishing of the discriminant $\Delta(P)$ (cf., also [5, Proposition 3.1]), P_0 must be contained in S' . So we just need to prove the first item of Proposition A.2.

An element in S is the same as a pencil of quadrics, and we can define its discriminant $\Delta(P)$, which is a binary form of degree $n + 1$. For terminology and background see [5] and [34]. We divide the proof into a few cases:

Case $r_0 \neq 0$ in the canonical form of [34, pag. 289]. This is an easier case, since P has non-finite isotropy group and one can explicitly write down a destabilizing one parameter subgroup that fixes P with negative GIT weight (since we have the freedom to choose weight for the last r_0 variables).

Non degenerate case. First suppose $\Delta(P)$ is not identically zero, i.e. P is non-degenerate.

Then due to a classical result of Weierstrass and C. Segre, one can put P into a canonical form (c.f. [5]), so that $P = \lambda A + \mu B$, where A and B are block diagonal, each block being of the form $A = \text{diag}(P_i), B = \text{diag}(Q_i)$ as in [5]. We recall the following (for a self-contained proof see the end of this note).

Proposition A.4 (cf., [67, §4(a)], [46, Theorem 6.1], [5, Theorem 3.1(a)]). *If P is non-degenerate and unstable, then $\Delta(P)$ has a root of multiplicity*

$$(A.2) \quad d > \frac{n+1}{2}.$$

Thus, changing (λ, μ) by an element in $SL(2; \mathbb{C})$, we may assume this multiple root is $(1 : 0)$. Denote by $\mu^{d_1}, \dots, \mu^{d_r}$ ($d_1 \geq d_2 \geq \dots \geq d_r$) the elementary divisors of the pencil at this root. Then we have the first r blocks of the form

$$P_{1j} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$Q_{1j} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

where for each $j \in \{1, \dots, r\}$, Q_{1j} and P_{1j} are both $d_j \times d_j$ square matrices, and $d = \sum_j d_j$. Let all the other elementary divisors of the pencil be of the form $(\lambda - a_\alpha \mu)^{e_\alpha}$, $\alpha = 1, \dots, k$. Write $e = \sum_\alpha e_\alpha$, then we have $d > e$ and $d + e = n + 1$. The rest of blocks are of the form:

$$P_\alpha = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_\alpha \\ 0 & \cdots & 0 & a_\alpha & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_\alpha & 1 & \cdots & 0 \\ a_\alpha & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$Q_\alpha = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where for $\alpha \in \{1, \dots, k\}$, P_α and Q_α are both $e_\alpha \times e_\alpha$ square matrices, and a_α non necessarily distinct.

Now we choose a diagonal one parameter subgroup $\lambda(t) \subset SL(n + 1; \mathbb{C})$ such that the weights on the blocks corresponding to P_{1j} and Q_{1j} ($j = 1, \dots, r$) are given by

$$w_{j,s} = \eta_j + 2\left(s - \frac{1 + d_j}{2}\right), \quad s = 1, \dots, d_j$$

and the weights on the blocks corresponding to P_α and Q_α ($\alpha = 1, \dots, k$) are given by

$$w_{\alpha,s} = \xi_\alpha - 2\left(s - \frac{1 + e_\alpha}{2}\right), \quad s = 1, \dots, e_\alpha$$

We choose η_j and e_α to satisfy the following constraints

$$(A.3) \quad \eta_j = c \quad \forall j,$$

$$(A.4) \quad \xi_\alpha = c + 1 \quad \forall \alpha,$$

$$(A.5) \quad \left(\sum_{1 \leq j \leq k} d_j\right)c + \left(\sum_{\alpha} e_\alpha\right)(c + 1) = 0,$$

for a uniform constant c , which does not depend on j nor α . Equation (A.5) makes sure that $\lambda(t) \subset SL(n + 1; \mathbb{C})$, while Equation (A.3), (A.4) imply that the limit as $t \rightarrow 0$ of the pencil $\lambda(t).P$ is still non-degenerate and is given by similar form to the above, except with $a_\alpha = 0$ for all $\alpha = 1, \dots, k$. In particular, the limit $\lim_{t \rightarrow 0} \lambda(t).P$ is not in Σ . These constraints uniquely determine c hence all the weights. Then the corresponding GIT weight is $-(4\xi_\alpha - 2) = -(4c + 2)$, which is negative because of (A.3), (A.4), (A.5), and (A.2).

Purely degenerate case $r_0 = 0$ and $A = B = 0$. In this case, we use the canonical form in [34, pag. 289]. We also have k blocks of quadratic forms of the form

$$P = 2(x_0x_1 + \dots + x_{2r_a-2}x_{2r_a-1})$$

$$Q = 2(x_1x_2 + \dots + x_{2r_a-1}x_{2r_a}).$$

Since we assume the pencil is not in Σ , $r_a \neq 1$. First we treat the case $k = 1$. To destabilize we choose weights $w_i, i = 0, \dots, 2r_1$ to satisfy

$$w_{2i} = \zeta + i, i = 0, \dots, r_1 - 1$$

$$w_{2i+1} = \xi - i, i = 0, \dots, r_1 - 1$$

$$w_{2r_1} = -r_1(\zeta + \xi)$$

Then P is homogeneous under the one parameter subgroup with weight $\zeta + \xi$, and Q is homogeneous with weight $\zeta + \xi + 1$ except the last term has weight $\xi - r_1 + 1 + w_{2r_1}$. If we further choose ζ, ξ, w_{2r_1} so that

1. $2(\zeta + \xi) + 1 > 0$,
2. $w_{2r} = -r_1(\zeta + \xi) > \zeta + r_1$,

then we get the limit of Q is by at worst making the last term of it disappear. On the other hand, the GIT weight is $-(2(\zeta + \xi) + 1) < 0$. So we get a good de-stabilizer. We finally confirm that the above conditions (1), (2) are satisfiable, e.g., set $\xi := -\zeta + \text{const}$ with $\zeta \ll 0$.

For general k case we just make sure that $\zeta_j + \xi_j$ do not depend on j , and satisfy the above constraints (1), (2). Then we will get the same conclusion.

Mixed general case $r_0 = 0$, but $A, B \neq 0$. Following [34, pag. 289], we have

- the purely degenerate blocks of size $2r_a + 1$ for $i = a, \dots, k$,
- non-degenerate blocks (corresponding to elementary divisors) $P_{1,1}, \dots, P_{1,j}, \dots$ at root $\infty = [1 : 0]$ of size $d_j (j = 1, \dots)$ and
- non-degenerate blocks (corresponding to elementary divisors) Q_α with roots at finite numbers (i.e., other than ∞) of size $e_\alpha (\alpha = 1, \dots)$.

We put weights rational w_i on non-degenerate variables x_i for $i \geq \sum_{a=1}^k (2r_a + 1)$ following the previous construction (A.3), (A.4), with some rational number constant c to be fixed later. Note that unlike in the non-degenerate situation, unstability does not necessarily implies $\sum d_j > \sum e_\alpha$. Denote the total sum of the temporary weights $\sum_{i=\sum_{a=1}^k (2r_a+1)+1}^{n+1} w_i$ as kW for some constant W .

Imitating Case 5.3, but with doubled weights, we also put weights $w_i \in \mathbb{Q}$ on the variables $x_i (i = 0, 1, \dots, 2r_1)$ on the first degenerate block as

$$(A.6) \quad w_{2i} = 2(\zeta_1 + i) (i = 0, \dots, r_1 - 1)$$

$$(A.7) \quad w_{2i+1} = 2(\xi_1 - i) (i = 0, \dots, r_1 - 1),$$

$$(A.8) \quad w_{2r_1} = -2r_1c - W,$$

with $\xi_1 = -\zeta_1 + c, \zeta_1 \ll 0$ as before, and do the same for other degenerate blocks of each size r_a for $a = 2, \dots, k$. (A.8) is designed to make the whole one parameter subgroup satisfy SL condition. Since the weights w 's are *rational* numbers in general, we normalize them by multiplying to the exponent

a sufficiently divisible positive integer constant m , so that they all become integers. As before, the weights are chosen so that the limit $\lim_{t \rightarrow 0} \lambda(t).P$ is not in Σ , and the GIT weight is equal $-m(2c + (2c + 2)) = -2m(2c + 1) < 0$ for a suitable choice of c . \square

Remark A.5. • *For del Pezzo surfaces in dimension two note that Bishop-Gromov estimate already implies that KE orbifolds must have at worst A_1 -singularities. Further simplifications can also be given in higher dimensions by showing first, using L^2 -embeddings similarly to [46], that GH limits must be simultaneously diagonalized. However, we decided to present above a more elementary proof that does not make use of further information about GH limits.*

- *Issues similar to the one considered in such note appear also for more general complete intersections, which would then need a more carefully check of GIT unstability to apply the comparison of K-stability and GIT-stability using the CM line bundle. Along the lines of conjectures in this paper, it would be interesting to know for a specific GIT moduli problem, whether one can always find an equidimensional destabilizer for an equidimensional unstable point, and more generally what is the best geometric property satisfied by the destabilizers of an unstable point.*

Proof. of proposition A.4. As in [5] we set

$$V := H^0(\mathbb{P}^n, \mathcal{O}(2)) = \text{Sym}^2(\mathbb{C}^{n+1}),$$

$$S_{n+1} := \{(A, B) \in V \oplus V \mid \text{rank}(\langle A, B \rangle) \subset V = 2\}.$$

S_{n+1} is a principle $GL(2)$ -bundle over the Grassmanian S . Let us define \tilde{S} as the geometric $SL(2)$ -quotient of S_{n+1} , so that $\tilde{S} \rightarrow S$ is a principle $(\mathbb{C}^*/\pm 1)$ -bundle. Since $(\mathbb{C}^*/\pm 1) \simeq \mathbb{C}^*$, it corresponds to a line bundle over S (note also that $\text{Pic}(S) \simeq \mathbb{Z}$.)

We consider the discriminant map $d: V \oplus V \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(n+1))$, which is clearly $SL(n+1) \times GL(2)$ -equivariant. (The $SL(n+1)$ -action on the target $H^0(\mathbb{P}^1, \mathcal{O}(n+1))$ is trivial.) As in [46, Proposition 6.2], take a $SL(2)$ -invariant regular function f on $H^0(\mathbb{P}^1, \mathcal{O}(n+1))$. Then

$$\tilde{f} := f \circ d: V \oplus V \rightarrow \mathbb{C}$$

is also $SL(n+1) \times SL(2)$ -invariant. Hence, by the universality of geometric quotient it descends to a regular function \bar{f} on \tilde{S} .

Now suppose $(A, B) \in V \oplus V$ satisfies the conditions that $d(A, B) \neq 0$ and that $[d(A, B)] \in \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(n+1)))$ is GIT-semistable. Then there is

a $SL(2)$ -invariant function f on $H^0(\mathbb{P}^1, \mathcal{O}(n+1))$ which does not vanish at $[d(A, B)]$. Then, obviously $\bar{f}(A, B) \neq 0$. By decomposing \bar{f} into the finite homogeneous eigenvectors with respect to the given $(\mathbb{C}^*/\pm 1)$ -action, we get a regular function g which is a non-vanishing eigenvector with respect to the $(\mathbb{C}^*/\pm 1)$ -action, i.e., corresponding to the element of $H^0(S, L^{\otimes l})$ where L is the ample generator of $\text{Pic}(S)$ and l some positive integer. Thus $\langle A, B \rangle$ is semistable by the definition of semistability. \square

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