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Remarks on Donaldson's symplectic submanifolds

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In [Do1], S. Donaldson proved the following:

Theorem 0 (Donaldson)**.** *Let V be a closed manifold and ω a symplectic form on V with integral periods. Then, for every sufficiently large positive integer k, there exists a symplectic submanifold W of codimension 2 in (V, ω) whose homology class is Poincaré dual to $k[\omega]$ and whose inclusion into V is an $(n-1)$-connected map, where $n := \frac{1}{2} \dim_{\mathbb{R}} V$.*

We recall that a continuous map $Y \to X$ between topological spaces is *m-connected* if it induces a bijection $\pi_j Y \to \pi_j X$ for $0 \leq j \leq m-1$, and a surjection $\pi_m Y \to \pi_m X$ (see [FR], for instance).

This result highlights analogies between symplectic geometry and Kähler geometry which were quite unexpected at the time, and actually the ideas and the methods introduced by Donaldson in [Do1, Do2] provide a new insight into both fields. When V is a complex projective manifold and ω a Kähler form with integral periods, the above theorem is a classical result that follows from the works of Bertini, Kodaira and Lefschetz. In this case, $W \subset V$ is a complex hypersurface obtained as a transversal hyperplane section $V \cap H$ of V, where V is holomorphically embedded into a projective space $\mathbb{CP}^m$ and $H \subset \mathbb{CP}^m$ is a hyperplane. As a consequence, $V - W \subset \mathbb{CP}^m - H \simeq \mathbb{C}^m$ is a smooth affine variety and, in particular, a Stein manifold of finite type. Moreover, $\omega{\restriction}_{V-W} = d\,d^{\mathbb{C}}\phi$ for some exhausting function $\phi \colon V - W \to \mathbb{R}$ having no critical points near W. Explicitly, $\phi := -\frac{1}{2k\pi} \log|s|$ where s is the restriction to $V \subset \mathbb{CP}^m$ of the complex linear function (a holomorphic section of $\mathcal{O}(1)$) defining H. (Note that the operator $d^{\mathbb{C}}$ here is given by $d^{\mathbb{C}}\phi(v) := -d\phi(iv)$ for any tangent vector v.)

Our main purpose in this paper is to show that any closed integral symplectic manifold has a very similar structure:

Theorem 1 (Stein Complements)**.** *Let V be a closed manifold and ω a symplectic form on V with integral periods. Then, for every sufficiently large positive integer k, there exist:*

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- *a symplectic submanifold W of codimension 2 in (V, ω) whose homology class is Poincaré dual to $k[\omega]$, and*
- *a complex structure J on $V - W$ such that $\omega\restriction_{V-W} = d\,d^J\phi$ for some exhausting function $\phi\colon V - W \to \mathbb{R}$ having no critical points near W; in particular, $(V - W, J)$ is a Stein manifold of finite type.*

Of course, the difference with the Kähler case is that, in general, the complex structure J (which depends on k) does not extend over the submanifold W. To make the above statement less mysterious, we need to recall a few pieces of terminology.

A *Liouville domain* is a domain[1] F endowed with a *Liouville form*, namely, a 1-form λ with the following properties:

- $d\lambda$ is a symplectic form on F, and
- λ induces a contact form on $K := \partial W$ orienting K as the boundary of $(F, d\lambda)$; equivalently, the *Liouville vector field* $\underset{\to}{\lambda}$ given by $\underset{\to}{\lambda} \lrcorner\, d\lambda = \lambda$ points transversely outwards along K.

A Liouville domain (F, λ) is a *Weinstein domain* if the Liouville field $\underset{\to}{\lambda}$ is gradientlike for some Morse function $\phi\colon F \to \mathbb{R}$, meaning that

$$\underset{\to}{\lambda} \cdot \phi \geq c\,|\underset{\to}{\lambda}|^2,$$

where the norm is computed with respect to any auxiliary metric and c is a positive number depending on that metric. (Obviously, the function ϕ can be further adjusted to be constant on ∂F.)

Not every Liouville domain is a Weinstein domain. In fact, no restriction is known for the topology of a Liouville domain while the topology of a Weinstein domain is strongly constrained. More explicitly, the topology of a Liouville domain (F, λ) is largely concentrated in its skeleton (also called core, or spine), namely the union $\mathrm{Sk}(F, \lambda)$ of all the orbits of $\underset{\to}{\lambda}$ which do not exit through ∂F. Indeed, the whole domain retracts into an arbitrary small neighborhood of $\mathrm{Sk}(F, \lambda)$. Due to the dilation properties of $\underset{\to}{\lambda}$ (its flow expands λ exponentially), the closed subset $\mathrm{Sk}(F, \lambda) \subset F$ has measure zero (for the volume form $(d\lambda)^n$, where $n := \frac{1}{2}\dim F$), but for instance there are Liouville domains (F, λ) for which $\mathrm{Sk}(F, \lambda)$ is a stratified subset of codimension 1 [Mc, Ge, MNW]. In contrast, if (F, λ) is a Weinstein domain, $\mathrm{Sk}(F, \lambda)$ consists of the stable submanifolds of the critical points of the Lyapunov function ϕ. Then the same dilation properties as above force these submanifolds to be isotropic for $d\lambda$, and so the critical indices of ϕ cannot exceed n. In

[1] In this text, the word *domain* means "compact manifold with boundary."

particular, the inclusion $\partial F \to F$ is an $(n-1)$-connected map. Actually, the main examples of Weinstein domains are Stein domains, *i.e.*, sublevel sets of exhausting $\mathbb{C}$-convex[2] functions, and the work of Cieliebak-Eliashberg [CE] shows that Weinstein and Stein domains are essentially the same objects. As for the relationships between Weinstein and Liouville domains, they remain quite mysterious.

Returning to our closed integral symplectic manifold (V, ω), we will call *hyperplane section of degree k in* (V, ω) any submanifold W of codimension 2 in V whose homology class is Poincaré dual to $k[\omega]$. A preliminary remark is that the complement of a symplectic hyperplane section W of arbitrary degree in (V, ω) is isomorphic to the interior of a Liouville domain (cf. Proposition 5). There is no general evidence that the Liouville domains obtained in this way have peculiar topological properties, but this may happen under additional assumptions on (V, ω). Revisiting a construction due to Auroux [Au1], we will illustrate this by discussing the case of symplectic hyperplane sections in tori (see Propositions 9 and 10). As for the symplectic hyperplane sections provided by Donaldson's construction, we have (see [Gi, Proposition 11]):

Theorem 2 (Weinstein Complements)**.** *Let V be a closed manifold and ω a symplectic form on V with integral periods. Then, for every sufficiently large positive integer k, there exist a Weinstein domain (F, λ) and a map $q\colon F \to V$ with the following properties:*

- *$q(\partial F)$ is a symplectic hyperplane section W of degree k in (V, ω) and ∂F is the normal circle bundle of W projecting to W by q;*
- *$q\restriction_{F-\partial F}\colon F - \partial F \to V - W$ is a diffeomorphism, with $q^*\omega = d\lambda$.*

Theorem 1 is then a corollary of Theorem 2 and the results of [CE].

Remark 3 (About Headings of Theorems). The proofs of Theorems 1 and 2 are variants of Donaldson's proof of Theorem 0. In particular, with the terminology used by Auroux in [Au2], the symplectic hyperplane sections they produce are the zero sets of "asymptotically holomorphic and uniformly transverse sections" of certain prequantization line bundles. It then follows from Auroux's uniqueness theorem [Au2, Theorem 2] that, for every sufficiently large integer k, these various symplectic hyperplane sections lie in the same Hamiltonian isotopy class. Thus, Theorems 1 and 2 can essentially be rephrased by saying that the symplectic hyperplane sections given by Donaldson's construction have Stein, resp. Weinstein, complements.

[2]We use the term $\mathbb{C}$-convex — or J-convex, if we want to refer to a specific complex structure J — to mean "strictly plurisubharmonic."

In [Bi], Biran adopted a very fruitful new viewpoint on the decomposition of a complex projective manifold V described at the beginning of this paper. Instead of regarding V as decomposed into a complex hyperplane section W and the affine variety $V - W$, he considered V as consisting of the skeleton of $V - W$ (this Stein manifold can be compactified to a Weinstein domain) and its complement. His key observation is that the latter is a simple symplectic object that he calls a "standard symplectic disk bundle" over W (see the discussion preceding Corollary 8 for a precise definition). As a byproduct of Theorem 2, we can extend Theorem 1.A of [Bi] as follows:

Corollary 4 (Generalization of Biran's Decomposition)**.** *Let V be a closed manifold and ω a symplectic form on V with integral periods. Then, for every sufficiently large positive integer k, there exists an isotropic skeleton $\Delta \subset V$ whose complement $V - \Delta$ has the structure of a standard symplectic disk bundle of area $1/k$ over a symplectic manifold W.*

Actually, one can take for Δ the skeleton of any Weinstein domain as in Theorem 2. We refer the reader to [Bi] for applications of Corollary 4 to intersection problems.

A. Symplectic hyperplane sections and Liouville domains

We begin with a simple observation:

Proposition 5 (Liouville Complements)**.** *Let V be a closed manifold, ω a symplectic form on V with integral periods and $W \subset V$ a symplectic hyperplane section of degree k. Then there exists a Liouville domain (F, λ) and a map $q \colon F \to V$ with the following properties:*

- *$q(\partial F) = W$ is the symplectic hyperplane section, ∂F is the normal circle bundle of W projecting to W by q, and $-2k\pi i\lambda$ defines a unitary connection on ∂F with curvature form $-2k\pi i\omega\restriction_W$;*
- *$q\restriction_{F-\partial F} \colon F - \partial F \to V - W$ is a diffeomorphism, and $q^*\omega = d\lambda$.*

A Liouville domain as above will be called a *Liouville compactification* of $V - W$.

Remark 6 (Liouville Domains and Symplectic Hyperplane Sections). Conversely, take a Liouville domain (F, λ) whose boundary ∂F has the structure of a principal circle bundle over a manifold W, and assume that $-2k\pi i\lambda$, for some positive integer k, induces a (unitary) connection form on ∂F. Then the quotient V of F by the equivalence relation which collapses every fiber of $\partial F \to W$ to a point is an integral symplectic manifold in which W sits as a symplectic hyperplane section of degree k (see the definition of a "standard symplectic disk bundle" below).

Proof. Let $L \to V$ be a Hermitian line bundle whose Chern class is a lift of $k[\omega]$, and denote by $P \subset L$ the unit circle bundle with projection $p\colon P \to V$. By standard obstruction theory, L has a section s whose zero set equals W and is cut out transversely. Then $u = s/|s|$ is a section of P over $V - W$, and the set

$$F = u(V - W) \cup p^{-1}(W) = \mathrm{Clos}\big(u(V - W)\big) \subset P$$

is a smooth compact submanifold of P with boundary $K := p^{-1}(W)$, which can be viewed as the result of a "real oriented blowup" of V along W.

Fix a unitary connection ∇ on L with curvature form $-2k\pi i\omega$. On the principal U_1-bundle P, the connection ∇ is given by a 1-form $-2k\pi i\alpha$ where α is a real contact form such that $d\alpha = p^*\omega$. Thus, the 1-form λ induced by α on F restricts to a contact form on K, and satisfies

$$u^*d\lambda = u^*d\alpha = u^*(p^*\omega) = (p \circ u)^*\omega = \omega.$$

Therefore, (F, λ) is essentially the required Liouville domain, except that $d\lambda$ degenerates along $K = \partial F$ (the kernel of $d\alpha$ is spanned by the vector field generating the U_1-action, and hence is tangent to K). Lemma 7 below explains how to solve this problem by attaching the boundary differently. □

Now recall that the *symplectization* of a contact manifold (K, ξ) is the symplectic submanifold SK of T^*K consisting of the non-zero covectors $\beta_x \in T^*_xK$, $x \in K$, whose cooriented kernel is ξ_x (all contact structures are cooriented in this paper). This is an $\mathbb{R}_{>0}$-principal bundle over K whose sections are the global Pfaff equations of ξ. Thus, any such contact form α determines a splitting

$$SK = \big\{s\alpha_x \in T^*K : (s, x) \in \mathbb{R}_{>0} \times K\big\} \simeq \mathbb{R}_{>0} \times K.$$

We denote by $K_\alpha \subset SK$ the graph of α, and by $SK_{<\alpha}$ (resp. $SK_{\leq\alpha}$) the subset of SK given by the condition $s < 1$ (resp. $s \leq 1$).

Lemma 7 (Boundary Degenerations of Liouville Domains)**.** *Let F be a domain and λ a 1-form on F which is a positive contact form on $K := \partial F$ and whose differential $d\lambda$ is a symplectic form on $F - K$ but may degenerate along K. Then the singular foliation spanned by $\underrightarrow{\lambda}$ in $F - K$ extends to a foliation of F transverse to K and, denoting by U the open collar consisting of all orbits which exit through K, there exists a unique smooth homeomorphism*

$$h = h_\lambda\colon U \to SK_{\leq\alpha}$$

such that:

- *h is the identity on $K \cong K_\alpha$ and induces a diffeomorphism between $U - K$ and $SK_{<\alpha}$;*
- *$\lambda\restriction_U = h^*\lambda^\xi$ where λ^ξ is the canonical 1-form on SK.*

Furthermore, the singularities of h are exactly the points of K where $d\lambda$ degenerates and, in particular, the points where the $2n$-form $(d\lambda)^n$ vanishes transversely (with $n := \frac{1}{2}\dim F$) correspond to folds.

As a result, one can change (F, λ) to a genuine Liouville domain just by gluing $F - K$ with $SK_{\le\alpha}$ along $U - K \cong SK_{<\alpha}$.

Proof. Let μ be an arbitrary positive volume form on F and consider the function $v := (d\lambda)^n/\mu$. We shall show that the vector field ν given by $\nu \lrcorner \mu = n\lambda \wedge (d\lambda)^{n-1}$ has the following properties:

- ν is non-singular along K and points transversely outwards;
- $\nu = v\underset{\rightarrow}{\lambda}$ at every point where $d\lambda$ is non-degenerate;
- the flow f_t of ν is defined for all $t \le 0$ and the diffeomorphism
$$f\colon \mathbb{R}_- \times K \to U, \quad (t, x) \mapsto f_t(x),$$
satisfies $f^*\lambda = e^w\alpha$ where $w(t, x) = \int_0^t v(f_s(x))\, ds$.

The first two properties show that ν generates a foliation transverse to K which extends the foliation spanned by $\underset{\rightarrow}{\lambda}$. The third property implies that the map $h\colon U \to SK_{\le\alpha}$ defined by
$$h \circ f(t, x) = e^{w(t,x)}\alpha_x$$
is a smooth homeomorphism with the desired behavior. Moreover, h is unique since the identity is the only homeomorphism of $SK_{\le\alpha}$ which fixes K_α pointwise and induces a diffeomorphism of $SK_{<\alpha}$ preserving λ^ξ.

The contact property of λ means that $\lambda \wedge (d\lambda)^{n-1}$ induces a positive volume form on K, so ν is non-singular along K and points transversely outwards. Next, at any point where $d\lambda$ is symplectic,
$$\underset{\rightarrow}{\lambda} \lrcorner\, d\lambda^n = v\underset{\rightarrow}{\lambda} \lrcorner\, \mu = n\lambda \wedge (d\lambda)^{n-1} = \nu \lrcorner\, \mu,$$
so $\nu = v\underset{\rightarrow}{\lambda}$. In particular, $\nu \lrcorner\, d\lambda = v\lambda$ and this equality holds everywhere on F by continuity.

To compute the form $f^*\lambda$, note that it vanishes on ∂_t, $t \in \mathbb{R}_-$, because $Df(\partial_t) = \nu$ and $\nu \lrcorner\, \lambda = 0$. Thus $f^*\lambda$ at a point (t, x) is just (the pullback of) $f_t^*\lambda$ at point x. Furthermore, $f_t^*\lambda$ satisfies the linear differential equation
$$\tfrac{d}{dt} f_t^*\lambda = f_t^*(\nu \cdot \lambda) = f_t^*(\nu \lrcorner\, d\lambda) = f_t^*(v\lambda) = (v \circ f_t)\, f_t^*\lambda.$$

Since $f_0^*\lambda = \alpha$, we obtain

$$f_t^*\lambda = \exp\left(\int_0^t (v \circ f_s)\, ds\right) \alpha,$$

as claimed. □

We now briefly describe the notion of standard symplectic disk bundle, referring to [Bi, Subsection 2.1] for a more detailed discussion. The most relevant approach here is as follows. Consider a closed integral symplectic manifold (W, ω_W) and denote by $p\colon K \to W$ a principal U_1-bundle whose Chern/Euler class is an integral lift of $[\omega_W]$. Fix any connection 1-form $-2\pi i\alpha$ on K such that $d\alpha = p^*\omega_W$. Then α is a contact form on K and the quotient of the manifold $SK_{\leq\alpha}$ that we obtain by collapsing each circle fiber in $K = K_\alpha$ to a point has the structure of an open disk bundle U over W: the smooth structure of the disk fibers is defined by polar coordinates (r, θ), where $d\theta$ is the form induced by α and $1-r$ is the symplectization coordinate. Moreover, the manifold U inherits a symplectic form ω_U from SK whose restriction to the zero section W is ω_W, and each fiber of the map $U \to W$ is a symplectic disk of area 1 (by Stokes' theorem). The symplectic manifold (U, ω_U) is what Biran calls a *standard symplectic disk bundle of area* 1 over W (see [Bi, Remarks 2.1]). For convenience, when the periods of ω_W are integer multiples of some integer $k \geq 1$, we also call *standard symplectic disk bundle of area* $1/k$ the integral symplectic manifold $(U, \frac{1}{k}\omega_U)$. Thus, the area of a standard symplectic disk bundle is, by definition, of the form $1/k$, and it determines the Chern class of the disk bundle.

Given a Liouville domain (F, λ) with boundary $K := \partial F$, the manifold $F - \mathrm{Sk}(F, \lambda)$, equipped with the 1-form λ, is isomorphic to $SK_{\leq\lambda\restriction_K}$ with its canonical 1-form. Thus, as a consequence of Proposition 5, we have:

Corollary 8 (Standard Disk Bundles in Symplectic Manifolds)**.** *Let V be a closed manifold, ω a symplectic form on V with integral periods, W a symplectic hyperplane section of degree k and (F, λ) a Liouville compactification of $V - W$. Then the complement of $Sk(F, \lambda)$ in (V, ω) has full measure and is a standard symplectic disk bundle of area $1/k$.*

Corollary 4 follows readily from Theorem 2 and Corollary 8.

In the remainder of this section, we make a couple of remarks on the topology of symplectic hyperplane sections in tori. We begin with an observation of Auroux [Au1, Au4] which shows that the Liouville domains given by Proposition 5 need not be Weinstein domains:

Proposition 9 (Auroux)**.** *In the standard symplectic torus of dimension* 4*, there exist disconnected symplectic hyperplane sections of arbitrarily large even degrees.*

In particular, the complements of these symplectic hyperplane sections have Liouville compactifications which are not Weinstein domains.

Interestingly enough, Auroux's argument can be "reversed" in higher dimensions to prove the following:

Proposition 10 (Connectedness in Higher Dimensional Tori)**.** *In the standard symplectic torus of dimension* $2n \geq 6$*, every symplectic hyperplane section is connected.*

Proofs of Propositions 10 and 9. The main underlying remark is that, if a closed integral symplectic manifold (V, ω) of dimension $2n$ contains a disconnected symplectic hyperplane section $W = W_1 \sqcup W_2$, then the cohomology class w Poincaré dual to $[W] = [W_1] + [W_2]$ splits as the sum $w_1 + w_2$ of two non-zero integral classes which satisfy $w_1 \smile w_2 = 0$ and $w_i \smile w^{n-1} > 0$, $i \in \{1, 2\}$. It follows that $w^n = w_1^n + w_2^n$, so at least one of the summands w_1^n, w_2^n is non-zero (and positive). We assume below that $w_1^n > 0$.

If $V = \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, its cohomology algebra can be identified with the exterior algebra of $\mathbb{R}^{2n}$. In this identification, w_1 and w_2 become exterior 2-forms ω_1 and ω_2, and the hypothesis that $w_1^n > 0$ means that ω_1 is a linear symplectic form. But then, by a classical result of Lefschetz, multiplication by ω_1 defines a map $\bigwedge^2 \mathbb{R}^{2n} \to \bigwedge^4 \mathbb{R}^{2n}$ which is injective for $n \geq 3$. Since $\omega_1 \wedge \omega_2 = 0$, we get to the conclusion that $\omega_2 = 0$, which contradicts our assumption that w_1 and w_2 are non-zero. This proves Proposition 10.

To prove Proposition 9 (following Auroux [Au1, Au4]), we first notice that the symplectic form $\omega := dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ on $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ can be written as $\omega = \frac{1}{2}(\omega_1 + \omega_2)$ where ω_1, ω_2 are positive linear symplectic forms with integral periods whose product $\omega_1 \wedge \omega_2$ is zero. For instance, one can take

$$\begin{aligned} \omega_1 &:= dx_1 \wedge (dx_2 - dx_3) + (dx_3 + dx_2) \wedge dx_4, \\ \omega_2 &:= dx_1 \wedge (dx_2 + dx_3) + (dx_3 - dx_2) \wedge dx_4. \end{aligned}$$

Next, we observe that the homology classes Poincaré dual to $[\omega_1]$ and $[\omega_2]$ are represented by the following immersed oriented submanifolds $\widehat{W}_1(a)$ and $\widehat{W}_2(b)$, respectively, for any $a, b \in \mathbb{T}^4$:

$$\begin{aligned} \widehat{W}_1(a) := \{x \in \mathbb{T}^4 : x_1 - a_1 = x_2 - x_3 - a_2 = 0\} \\ \cup \{x \in \mathbb{T}^4 : x_3 + x_2 - a_3 = x_4 - a_4 = 0\}, \end{aligned}$$

$$\begin{aligned}\widehat{W}_2(b) := \{x \in \mathbb{T}^4 : x_1 - b_1 = x_2 + x_3 - b_3 = 0\} \\ \cup \{x \in \mathbb{T}^4 : x_3 - x_2 + b_2 = x_4 - b_4 = 0\}.\end{aligned}$$

(The ordered set of equations given for each piece determines the orientation.) Each cycle $\widehat{W}_1(a)$ consists of two linear tori which are both symplectic for ω_1 and Lagrangian for ω_2, and which intersect positively (in exactly two points). Thus, $\widehat{W}_1(a)$ is an immersed symplectic submanifold in $(\mathbb{T}^4, \omega)$ with positive transverse double points. By an embedded surgery localized near each double point (see below), $\widehat{W}_1(a)$ can be desingularized to an embedded and homologous symplectic surface $W_1(a)$ in $(\mathbb{T}^4, \omega)$. Similarly, $\widehat{W}_2(b)$ can be desingularized to an embedded symplectic surface $W_2(b)$ in $(\mathbb{T}^4, \omega)$. Moreover, since $\widehat{W}_1(a)$ and $\widehat{W}_2(b)$ are disjoint for $a \neq b$, so are $W_1(a)$ and $W_2(b)$. Therefore, if $a \neq b$, the union $W := W_1(a) \cup W_2(b)$ is a disconnected symplectic submanifold of $(\mathbb{T}^4, \omega)$ whose homology class is Poincaré dual to $2[\omega]$; in other words, W is a symplectic hyperplane section of degree 2.

To obtain a symplectic hyperplane section of degree $2k$, we just replace each linear torus involved in the definition of $\widehat{W}_1(a)$ and $\widehat{W}_2(b)$ by k parallel copies.

To complete the argument, we explain how to desingularize the transverse double points in the construction. To keep formulas simple, we reset our notations. We consider in $\mathbb{C}^2$, with coordinates (z_1, z_2) where $z_j = x_j + iy_j$, the complex curve $\widehat{W} := \{z_1 z_2 = 0\}$ (namely, the two coordinate axes, which intersect transversely at 0) and the two positive symplectic forms

$$\omega_1 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \quad \text{and} \quad \omega_2 := dx_1 \wedge dx_2 - dy_1 \wedge dy_2.$$

The immersed surface $\widehat{W}$ is symplectic for ω_1 but Lagrangian for ω_2. The desingularization trick consists in replacing $\widehat{W}$ by an embedded surface of the form

$$W := \Big\{ z_1 z_2 = \varepsilon^2 \, \chi\big(|z_1/\varepsilon|^2\big) \, \chi\big(|z_2/\varepsilon|^2\big) \Big\},$$

where ε is a positive number and $\chi \colon \mathbb{R} \to [0,1]$ a smooth function with compact support equal to 1 over $[0,1]$. A direct calculation shows that, if ε is sufficiently small, the smooth surface W is still symplectic for ω_1 and Lagrangian for ω_2. Hence, it is symplectic with respect to $\omega_1 + \omega_2$. □

B. Symplectic hyperplane sections and Weinstein domains

This section is devoted to the proof of Theorem 2, and we will assume that the reader is familiar with the techniques introduced by Donaldson in [Do1, Do2] and further developed by Auroux, notably in [Au2, Au3]. Actually, the proof of Theorem 2 is a variation on Donaldson's proof of Theorem 0 and we will only explain the extra arguments we need (a sketch of proof can already be found in [Gi]). We recall the setting:

- V is a closed manifold, ω a symplectic form on V with integral periods, J an ω-compatible almost complex structure and g the metric given by $g(.,.) := \omega(., J.)$;
- $L \to V$ is a Hermitian line bundle whose Chern class is a lift of $[\omega]$ and ∇ is a unitary connection on L with curvature form $-2\pi i\omega$;
- ∇', ∇'' are the J-linear and J-antilinear components of ∇, respectively;
- L^k, for any integer k, is the k-th tensor power of L endowed with the connection induced by ∇, which we still write $\nabla = \nabla' + \nabla''$ and whose curvature form is $-2k\pi i\omega$;
- g_k, for $k \geq 1$, is the rescaled metric $g_k := kg$.

In [Do1], each symplectic hyperplane section of Theorem 0 is obtained as the zero set $W := \{s_k = 0\}$ of a section $s_k \colon V \to L^k$, where the sections s_k, $k \gg 0$, satisfy the following properties (that we formulate using Auroux's terminology [Au2]):

- The sections $s_k \colon V \to L^k$ are *asymptotically holomorphic*. This means that there is a positive constant R such that, for every k, at every point of V and for $0 \leq j \leq 2$,[3]

  $$|s_k| \leq R, \quad |\nabla^{j+1} s_k|_{g_k} \leq R \quad \text{and} \quad |\nabla^j \nabla'' s_k|_{g_k} \leq R k^{-1/2}.$$

  Note that the derivatives $\nabla^{j+1} s_k$ and $\nabla^j \nabla'' s_k$ with $j > 0$ involve both the connection ∇ on L^k and the Levi-Civita connection of the metric g_k (or g).
- The sections $s_k \colon V \to L^k$ are *uniformly transverse* (to 0). This means that there is a positive constant η such that, for every sufficiently large integer k,

  $$|\nabla s_k(x)|_{g_k} \geq \eta \quad \text{at every point } x \text{ where } |s_k(x)| \leq \eta.$$

A key point here is that any section $s_k \colon V \to L^k$ satisfying the above estimates with $k > 4R^2/\eta^2$ automatically also satisfies $|\nabla'' s_k| < |\nabla' s_k|$ at every point of $W = \{s_k = 0\}$, and this inequality guarantees that W is a symplectic submanifold. To prove Theorem 2, we will need a similar inequality all over V:

[3] Given any positive integer r, we can actually impose similar bounds for $0 \leq j \leq r$; we can even take $r = \infty$ provided R is allowed to depend on j. The value $r = 2$ is the minimum we need to prove Donaldson's theorem and our results.

Definition 11 (Quasiholomorphic Sections)**.** Let $\kappa \in [0,1)$. We will say that a section $s_k \colon V \to L^k$ is *κ-quasiholomorphic* if $|\nabla'' s_k| \leq \kappa\, |\nabla' s_k|$ at every point of V.

The geometric significance of this notion is the following:

Lemma 12 (Quasiholomorphic Sections and Symplectic Convexity)**.** *Let W be the zero set of a κ-quasiholomorphic section $s \colon V \to L^k$, $\kappa \in [0,1)$. Then the function*

$$\phi := -\frac{1}{2\pi} \log|s| \colon V - W \to \mathbb{R}$$

admits a Liouville pseudogradient, namely the vector field $\underrightarrow{\lambda}$ where $-2k\pi i\lambda$ is the 1-form defining ∇ in the trivialization $s/|s|$ on $V - W$.

As a consequence, if s vanishes transversely and if $\phi := -\log|s|$ is a Morse function, then W is a symplectic hyperplane section and the Liouville compactification of $V - W$ (see Proposition 5) is a Weinstein domain.

Proof. Setting $r := |s|$ and using the definition of λ, we can write (twice) the partial covariant derivatives of s over $V - W$ in the form

$$2\nabla' s = dr - 2\pi J^*\lambda r - i\, J^*(dr - 2\pi J^*\lambda r),$$
$$2\nabla'' s = dr + 2\pi J^*\lambda r + i\, J^*(dr + 2\pi J^*\lambda r),$$

so

$$|\nabla' s| = \tfrac{1}{2}\, |dr - 2\pi J^*\lambda r|,$$
$$|\nabla'' s| = \tfrac{1}{2}\, |dr + 2\pi J^*\lambda r|.$$

Since s is κ-quasiholomorphic, we have $|\nabla'' s| \leq \kappa\, |\nabla' s|$ and we obtain (after dividing by $2\pi r$):

$$|\lambda + J^* d\phi| \leq \kappa\, |\lambda - J^* d\phi|.$$

Now the derivative of ϕ along the Liouville field $\underrightarrow{\lambda}$ is equal to the inner product $g_k(\lambda, d^J\phi)$. Thus, for $\kappa \in [0,1)$, the above inequality implies that

$$\underrightarrow{\lambda} \cdot \phi \geq \tfrac{1}{2} \frac{1-\kappa^2}{1+\kappa^2} \left(|\lambda|_{g_k}^2 + |d\phi|_{g_k}^2 \right).$$

This shows that $\underrightarrow{\lambda}$ is a pseudogradient of ϕ. ☐

With this lemma in mind, it suffices to show:

Proposition 13 (Construction of Quasiholomorphic Sections)**.** *Let s_k^0 be asymptotically holomorphic and uniformly transverse sections $V \to L^k$, and* *let κ be any number in $(0,1)$. Then there exist κ-quasiholomorphic sections $s_k\colon V \to L^k$ such that, for every sufficiently large integer k:*

- *the section s_k vanishes transversely and the symplectic hyperplane section $W := \{s_k = 0\}$ is Hamiltonian isotopic to $W^0 := \{s_k^0 = 0\}$;*
- *the function $-\log|s_k|\colon V - W \to \mathbb{R}$ is a Morse function.*

The quasiholomorphic sections s_k will not be asymptotically holomorphic anymore (see below). The main step in the proof is the next Lemma which provides asymptotically holomorphic sections of L^k satisfying an additional uniform transversality condition.

We recall that, given a positive number η, a Riemannian manifold M and a Hermitian vector bundle $E \to M$ endowed with a unitary connection ∇, a section $\sigma\colon M \to E$ is *η-transverse* (to 0) if, at every point $x \in M$ with $|\sigma(x)| \le \eta$, the linear map $\nabla\sigma(x)\colon T_xM \to E_x$ is surjective and has a right inverse whose operator norm does not exceed $1/\eta$. If the real rank of E equals the dimension of M, it is equivalent to require that $|\nabla\sigma(x)\cdot v| \ge \eta\,|v|$ for all vectors $v \in T_xM$.

In what follows, we consider sections $\sigma_k\colon V \to E \otimes L^k$, where $E \to V$ is a fixed Hermitian bundle (the bundle of J-linear complex-valued 1-forms on V) and k runs over all sufficiently large integers, and we say that such sections are *uniformly transverse* if they are η-transverse for some positive η independent of k, where the amount of transversality is measured with the metric g_k.

Lemma 14 (Extra Uniform Transversality Condition)**.** *Let s_k^0 be asymptotically holomorphic and uniformly transverse sections $V \to L^k$. For large integers k, the sections s_k^0 are homotopic, through asymptotically holomorphic and uniformly transverse sections, to sections $s_k^1\colon V \to L^k$ whose partial covariant derivatives $\nabla' s_k$ are uniformly transverse.*

Proof of Lemma 14. The proof follows step by step the path opened by Donaldson in [Do1]. We just explain here how to obtain uniform local transversality for sections of the form $\nabla' s_k$. The globalization process elaborated by Donaldson in [Do1] then applies readily to provide the desired sections s_k^1. The sections s_k^1 will be asymptotically holomorphic by construction. Moreover, given any $\delta > 0$, we can arrange that all the differences $s_k^1 - s_k^0$ are bounded by δ in $\mathcal{C}^1$-norm. For δ smaller than the uniform transversality modulus of the sections s_k^0, it follows that, for every $t \in [0,1]$, the sections $(1-t)s_k^0 + ts_k^1$ are still asymptotically holomorphic and uniformly transverse.

To achieve uniform local transversality, we essentially need to show that the derivatives $\nabla' s_k^0$ are represented (in Darboux coordinates independent of k and in balls of fixed g_k-radius) by maps which (on smaller balls) are approximated within ε in $\mathcal{C}^1$-norm by polynomial maps of degree bounded by $C\log(1/\varepsilon)$, where C is a positive constant (independent of k).

We work in complex Darboux coordinates $(z_1,\dots,z_n)$ centered on a point a (meaning that $\omega = \frac{i}{2}\sum_1^n dz_j \wedge d\overline{z}_j$), and we use the trivialization of L^k given by parallel translation along rays. We denote by J_0 the standard complex structure in the coordinates and by ∇'_0, ∇''_0 (resp. d', d'') the J_0-linear and J_0-antilinear components of ∇ (resp. of the usual differential d). Thus we have

$$\nabla' s_k^0 - \nabla'_0 s_k^0 = -\tfrac{i}{2}\nabla s_k^0 \circ (J - J_0)$$

where the right-hand side, measured with the metric g_k on a ball of fixed radius, is bounded by $O(k^{-1/2})$ in $\mathcal{C}^1$-norm. Hence it suffices to make the partial covariant derivatives $\nabla'_0 s_k$ uniformly transverse to 0, and for this we can use the connection of the flat metric rather than that of g_k. Note that there is a little subtlety here: we want $\nabla' s_k$ to be transverse to 0 as a section of $T'V \otimes L^k$ ($T'V$ denoting the bundle of J-linear forms in $T^*V \otimes \mathbb{C}$), but $\nabla' s_k$ and $\nabla'_0 s_k$ are not sections of the same bundle. To derive the transversality of $\nabla' s_k$ from that of $\nabla'_0 s_k$, we observe that transversality between spaces of equal dimensions is a dilation property for all non-zero vectors (under the differential) and this property is stable under $\mathcal{C}^1$-small perturbations.

Let $s_{a,k}$ be the Gaussian section of L^k centered on a and cut off at g_k-distance $O(k^{1/6})$ of a. Since we work in a ball of given radius in the metric g_k, for k sufficiently large,

$$s_{a,k}(z) = \exp(-\pi|z|^2/2).$$

There are two obvious bases in the space of J_0-linear forms, one consisting of the forms $dz_j s_{a,k}$ and one consisting of the forms $\nabla'_0(z_j s_{a,k})$. They are related by

$$\begin{aligned}\nabla'_0(z_i s_{a,k}) &= dz_i s_{a,k} + z_i \nabla'_0 s_{a,k}\\ &= \left(dz_i - \pi z_i \sum_j \overline{z}_j dz_j\right) s_{a,k}\\ &= \sum_j \Phi_{ij}(z)\, dz_j s_{a,k}\end{aligned}$$

where the entries of the matrix

$$\Phi(z) = \big(\Phi_{ij}(z)\big) = \big(\delta_{ij} - \pi z_i \overline{z}_j\big)$$

are (real) polynomials independent of k.

We now represent $\nabla'_0 s^0_k$ by the map $h = (h_1, \dots, h_n)$ (with values in $\mathbb{C}^n$) defined by

$$\nabla'_0 s^0_k = \sum_j h_j \nabla'_0 (z_j s_{a,k}).$$

If $w = (w_1, \dots, w_n)$ is a δ-transverse value of h (meaning that $h - w$ is η-transverse to 0) then the section

$$\nabla'_0 \left(s^0_k - \sum_j w_j z_j s_{a,k} \right) = \sum_j (h_j - w_j) \nabla'_0 (z_j s_{a,k})$$

is η'-transverse to 0 for some η' which is a definite fraction of η. On the other hand, considering the function $f = s^0_k / s_{a,k}$, we have

$$\begin{aligned} \nabla'_0 s^0_k &= d' f \, s_{a,k} + f \, \nabla'_0 s_{a,k} \\ &= \left(d' f - \pi f \sum_i \overline{z}_i dz_i \right) s_{a,k} \\ &= \sum_i (\partial_{z_i} f - \pi f \overline{z}_i) dz_i s_{a,k}. \end{aligned}$$

In other words, if we denote by $u = (u_1, \dots, u_n)$ the map given by

$$u_i := \partial_{z_i} f - \pi \overline{z}_i f, \quad 1 \le i \le n,$$

we get

$$h(z) = \Phi(z)^{-1} u(z).$$

Since the function f is approximately holomorphic and the entries of the matrix Φ^{-1} are analytic functions independent of k, the map h admits the required polynomial approximations (see [Do1] for more details). □

Remark (Cheaper Approach). The above argument appeals (implicitly) to the quantitative version of Sard's theorem given in [Do2, Section 5] or, more accurately, to its real version proved in [Mo1, Section 6]. This is a great result but its proof is difficult and quite technical. One could modify our argument to appeal, instead, to the trick proposed by Auroux in [Au3]. This would definitely make the complete proof of Theorem 2 technically much simpler, but it would make our exposition here more intricate.

Proof of Proposition 13. First observe that, since the sections s_k^0 and s_k^1 are homotopic through asymptotically holomorphic and uniformly transverse sections, their zero sets $W^0 := \{s_k^0 = 0\}$ and $W^1 := \{s_k^1 = 0\}$ are Hamiltonian isotopic. We will now construct κ-quasiholomorphic sections s_k by modifying the sections s_k^1 away from their zero sets. Hence, though asymptotic holomorphicity will not be preserved in this process, the symplectic hyperplane sections $W := \{s_k = 0\} = W^1$ and W^0 will remain Hamiltonian isotopic for every large integer k.

Consider the sets $\Gamma_k \supset \Delta_k$ defined by

$$\Gamma_k = \{x \in V : |\nabla'' s_k^1(x)| \geq \kappa\, |\nabla' s_k^1(x)|\},$$
$$\Delta_k = \{x \in V : \nabla' s_k^1(x) = 0\},$$

where the sections $s_k^1 \colon V \to L^k$ are those given by Lemma 14.

Since s_k^1 vanishes η-transversely, Γ_k avoids a tube of fixed g_k-radius (independent of k) about $W^1 := \{s_k^1 = 0\}$. Moreover, since $\nabla' s_k^1$ vanishes δ-transversely, Δ_k is a discrete (hence finite) set and:

Lemma 15 (Location of Bad Points)**.** *For every sufficiently small positive number ρ and every sufficiently large integer $k \geq k(\rho)$, the balls $B_k(a, \rho)$, $a \in \Delta_k$, are disjoint and cover Γ_k.*

To see this, recall that the sections s_k^1 are asymptotically holomorphic, and so

$$|\nabla' s_k^1(x)|_{g_k} \leq \kappa\, |\nabla'' s_k^1(x)|_{g_k} \leq \kappa R k^{-1/2}$$

at every point $x \in \Gamma_k$. Since $\nabla' s_k^1$ is δ-transverse to 0, the above estimate implies that

$$|\nabla\nabla' s_k^1(x) \cdot v|_{g_k} \geq \delta\, |v|_{g_k} \quad \text{provided } \kappa R k^{-1/2} \leq \delta.$$

Then, as in [Do2, Lemma 8 and Proposition 9], Lemma 15 is a consequence of the following simple fact:

Lemma 16 (Inverse Function Theorem)**.** *Let $\phi \colon \mathbb{D}^n \to \mathbb{R}^n$ be a map $\mathcal{C}^2$-bounded by c and such that*

$$|d\phi(0) \cdot v| \geq \delta\, |v| \quad \textit{for all vectors } v.$$

If $|\phi(0)| \leq \delta\rho/2$ for some $\rho \leq \delta/c$, the equation $\phi(x) = 0$ has a unique solution x in the ball of radius ρ about 0.

To prove Lemma 15, we apply Lemma 16 to the map representing $\nabla' s_k^1$ in the complex Darboux coordinates centered on a point a of Γ_k. At this point,

$$|\nabla' s_k^1(a)| \le \kappa^{-1} |\nabla'' s_k^1(a)| \le R\kappa^{-1} k^{-1/2}$$

so the hypotheses of Lemma 16 are fulfilled once k is sufficiently large.

To complete the proof of the proposition, we will modify s_k^1 near each point $a \in \Delta_k$ (see [Do2, Lemma 10 and the subsequent discussion]). Again, we work in the complex Darboux coordinates centered on a. For any $\rho > 0$, fix a cutoff function $\beta = \beta_\rho$ such that $\beta(z) = 1$ for $|z| \le \rho/2$, $\beta(z) = 0$ for $|z| \ge \rho$, and $|d\beta(z)| \le 3/\rho$ for all z. Write $s_k^1 = f s_{a,k}$ and denote by f_0 the complex polynomial of degree 2 given by

$$f_0(z) = f(0) + \tfrac{1}{2} \sum_{ij} \partial^2_{z_i z_j} f(0) z_i z_j.$$

We then consider the sections s_k defined in the coordinates $(z_1, \dots, z_n)$ by

$$s_k := (\beta f_0 + (1-\beta) f) s_{a,k}.$$

Before comparing the derivatives $\nabla' s_k$ and $\nabla'' s_k$, let us compare the derivatives $\nabla'_0 s_k$ and $\nabla''_0 s_k$. As we already noticed, the closeness of $\nabla' s_k^1$ and $\nabla'_0 s_k^1$ guarantees that the latter derivative is $\eta/2$-transverse to 0 on the ball of radius ρ for k sufficiently large. On the other hand, the identities

$$\begin{aligned} d'' f(0) &= \nabla''_0 s_k^1(0), \\ dd'' f(0) &= \nabla_0 \nabla''_0 s_k^1(0) \end{aligned}$$

(where ∇_0 denotes the connection associated to the flat metric) show that $|d'' f(0)|$ and $|dd'' f(0)|$ are bounded by $Ck^{-1/2}$. Therefore, if k is sufficiently large, the partial derivative $\nabla'_0 (f_0 s_{a,k})$ is so close to $\nabla'_0 s_k^1$ that it is $\eta/4$-transverse to 0. Furthermore, $f_0 s_{a,k}$ is a holomorphic section. Thus, on the ball of radius $\rho/2$ (where $\beta = 1$), we have

$$\nabla''_0 s_k(z) = 0 \quad \text{and} \quad |\nabla'_0 s_k(z)| \ge \frac{\eta}{4} |z|.$$

Hence, on that same ball,

$$|\nabla'' s_k(z)| \le Ck^{-1/2} |z| \quad \text{and} \quad |\nabla'_0 s_k(z)| \ge \Big(\frac{\eta}{4} - Ck^{-1/2} \Big) |z|.$$

In the annular region $\rho/2 \leq |z| \leq \rho$, the calculations above imply that

$$|f(z) - f_0(z)| \leq C(\rho^3 + \rho k^{-1/2})$$

and, since the gradient of β is bounded by $3/\rho$, the same arguments as in [Do2] give the desired inequalities when ρ is sufficiently small.

It remains to show that the function $\phi := -\log|s_k|\colon V - W \to \mathbb{R}$ (where $W := \{s_k = 0\}$) is a Morse function. Since s_k is κ-quasiholomorphic with $\kappa < 1$, the critical points of ϕ are the zeros of $\nabla' s_k$, namely the points of Δ_k. It then follows from the properties of s_k in $B_k(a, \rho/2)$, $a \in \Delta_k$, that ∇s_k vanishes transversely at a, so the critical points of ϕ are non-degenerate. ☐

C. Symplectic hyperplane sections and Stein domains

Here we derive Theorem 1 from Theorem 2. The main ingredient we will use is a special case (a domain is a cobordism with empty bottom boundary) of [CE, Theorem 13.5]:

Theorem 17 (Cieliebak-Eliashberg)**.** *Let (F, λ) be a Weinstein domain and ϕ_0 a function on F with pseudogradient $\underrightarrow{\lambda}$ and regular level set $\partial F = \{\phi_0 = 0\}$. Then there exist a complex structure J and a path of 1-forms λ_t on F $(t \in [0,1])$ with the following properties:*

- *all forms $d\lambda_t$ are symplectic on F, and $\lambda_0 = \lambda$;*
- *all Liouville vector fields $\underrightarrow{\lambda_t}$ are pseudogradients of ϕ_0;*
- *$\lambda_1 = d^J(u \circ \phi_0)$ for some convex increasing function $u\colon \mathbb{R}_{\leq 0} \to \mathbb{R}_{\leq 0}$ with $u(0) = 0$.*

In particular, (F, J) is a Stein domain and $u \circ \phi$ is a J-convex function.

To complete the proof of Theorem 1, we actually need a variant of the above result, namely:

Corollary 18 (Weinstein and Stein Domains)**.** *Let (F, λ) be a Weinstein domain. Then there exist a complex structure J on F and a J-convex Morse function $\phi\colon F \to \mathbb{R}_{\leq 0}$, with regular level set $\partial F = \{\phi = 0\}$, such that $d\lambda = d\,d^J\phi$.*

Proof. Pick an arbitrary function ϕ_0 on F with pseudogradient $\underrightarrow{\lambda}$ and regular level set $\partial F = \{\phi_0 = 0\}$. Consider the complex structure J and the path of 1-forms λ_t (along with the function u) given by Theorem 17. Since the Liouville vector fields $\underrightarrow{\lambda_t}$ are all pseudogradients of ϕ_0, each form λ_t induces a contact form α_t on ∂F. Using Gray's stability theorem and a suitable isotopy extension, we can arrange that the forms λ_t have the same kernel along ∂F, *i.e.* $\lambda_t = v_t \lambda_0$ on ∂F for some function $v_t \colon \partial F \to \mathbb{R}_{>0}$.

Assume temporarily that $v_t = 1$ for all t. Then Moser's argument provides an isotopy h_t of F relative to ∂F such that $h_0 = \mathrm{id}$ and $h_t^* d\lambda_t = d\lambda$. Then the complex structure $h_1^* J$ and the function $h_1^*(u \circ \phi_0)$ have the desired properties.

Therefore it suffices to modify the forms λ_t so that they coincide on (or along) ∂F and still satisfy the conditions of Theorem 17. It is easy to find positive functions w_t on F such that $w_t = 1/v_t$ on ∂F and $\underrightarrow{\lambda_t} \cdot \log w_t > -1$. Then the forms $\widetilde{\lambda}_t := w_t \lambda_t$ agree along ∂F and satisfy the first two conditions of Theorem 17, but $\widetilde{\lambda}_1$ and $d^J(u \circ \phi_0)$ are not equal. Set $\phi_1 = u \circ \phi_0$ and note that

$$\widetilde{\lambda}_1 = w_1 \lambda_1 = w_1 d^J \phi_1.$$

Lemma 19 below provides a function ϕ such that $\widetilde{\lambda}_1 = d^J \phi$, which completes the proof of the corollary. □

Lemma 19 (Rescaling of J-Convex Functions)**.** *Let F be a domain, J a complex structure on F and $\phi_1 \colon F \to \mathbb{R}$ a J-convex Morse function on F with regular level set $\partial F = \{\phi_1 = 0\}$. For every positive function w on ∂F, there exists a J-convex Morse function $\phi \colon F \to \mathbb{R}$ equivalent to ϕ_1 such that $d^J \phi = w\, d^J \phi_1$ along ∂F.*

By "equivalent", we mean that $\phi = u \circ \phi_1 \circ h$, where $u \colon \mathbb{R} \to \mathbb{R}$ is an increasing function while h is a diffeomorphism of F.

Proof. First extend w to a positive function on F and define $\phi_2 := (w + c\phi_1)\phi_1$, where c is a positive constant. Then ∂F is a regular component of the zero-level set of ϕ_2, and $d^J \phi_2 = w\, d^J \phi_1$ at every point of ∂F. Moreover,

$$\begin{aligned} d\, d^J \phi_2 = (w + c\phi_1)\, d\, d^J \phi_1 + \phi_1\, d\, d^J (w + c\phi_1) \\ + dw \wedge d^J \phi_1 + d\phi_1 \wedge d^J w + 2c\, d\phi_1 \wedge d^J \phi_1, \end{aligned}$$

so ϕ_2 is J-convex near ∂F for any sufficiently large constant c. We henceforth fix such a c. Then there exists a number $\delta > 0$ such that $d\phi_2$ is positive on the Liouville field $\underrightarrow{d^J \phi_1}$ in the collar $\{-\delta \le \phi_1 \le 0\}$ (indeed, $d\phi_2 = w\, d\phi_1$ at every point of ∂F). Now set

$$\phi_3 = a\phi_1 + b \quad \text{with} \quad b := \tfrac{1}{2} \sup\{\phi_2(x) : \phi(x) = -\delta\}, \quad a < \frac{b}{\delta}.$$

Clearly, ϕ_3 is J-convex and we obtain the desired function ϕ by smoothing the function $\max(\phi_1, \phi_2)$ (see [CE, Chapter 2] for details on the relevant smoothing technique). □

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