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Kummer varieties and their Brauer groups

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Dedicated to Yuri Ivanovich Manin on his 80th birthday,
with admiration and gratitude

Abstract: We study Kummer varieties attached to 2-coverings of abelian varieties of arbitrary dimension. Over a number field we show that the subgroup of odd order elements of the Brauer group does not obstruct the Hasse principle. Sufficient conditions for the triviality of the Brauer group are given, which allow us to give an example of a Kummer K3 surface of geometric Picard rank 17 over the rationals with trivial Brauer group. We establish the non-emptyness of the Brauer–Manin set of everywhere locally soluble Kummer varieties attached to 2-coverings of products of hyperelliptic Jacobians with large Galois action on 2-torsion.

1. Introduction

In [12, 13] Yu.I. Manin introduced what is now called the Brauer–Manin obstruction. To an element of the Brauer–Grothendieck group of a variety X over a number field k he attached a global reciprocity condition on the adelic points of X which is satisfied when an adelic point comes from a k-point. In this paper we study the Brauer–Manin obstruction on Kummer varieties, which are higher-dimensional generalisations of classical Kummer K3 surfaces.

Over complex numbers, Kummer varieties in dimension greater than 2 were introduced in 1890 by W. Wirtinger [31]. Their topological and geometric properties were studied by A. Andreotti, E. Spanier and K. Ueno, see [27], [28], [29].

Over non-closed fields, Kummer varieties come not only from the quotients of abelian varieties by the antipodal involution, but also from the quotients of certain torsors. More precisely, let A be an abelian variety of dimension $g \geq 2$ over a field k of characteristic not equal to 2. Let Y be a k-torsor for A whose class in $\mathrm{H}^1(k, A)$ has order at most 2. Classically such

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torsors are referred to as 2-*coverings* of A. Kummer varieties considered in this paper are minimal desingularisations of the quotient Y/ι by the involution $\iota : Y \to Y$ induced by the antipodal involution $[-1] : A \to A$. In the case $g = 2$ we obtain Kummer surfaces, a particular kind of K3 surface. Due to their intimate relation to abelian varieties, Kummer surfaces are a popular testing ground for conjectures on the geometry and arithmetic of K3 surfaces. Rational points and Brauer groups of Kummer surfaces were studied in [23, 25, 8, 6, 2, 30].

Rational points on Kummer varieties of higher dimension feature in the work of D. Holmes and R. Pannekoek [7]. Their result concerns an abelian variety A over a number field k: if the set of k-points of the Kummer variety X attached to A^n is dense in the Brauer–Manin set of X, then there is a quadratic twist of A over k of rank at least n. More recently, a Hasse principle for Kummer varieties, that are sufficiently general in an appropriate arithmetic sense, was established conditionally on the finiteness of relevant Shafarevich–Tate groups by Y. Harpaz and one of the present authors [6]. Somewhat surprisingly, the Brauer group does not show up in that statement.

Our aim in this paper is twofold. In Section 2 we establish geometric properties of Kummer varieties analogous to similar properties of Kummer surfaces. We show, among other things, that the geometric Picard group $\mathrm{Pic}(\overline{X})$ is a finitely generated free abelian group (Corollary 2.4). In the characteristic zero case we describe a natural isomorphism of Galois modules between the geometric Brauer group of a Kummer variety and the geometric Brauer group of the corresponding abelian variety (Proposition 2.7). From our previous result [24] we then deduce the finiteness of the quotient of $\mathrm{Br}(X)$ by $\mathrm{Br}_0(X) = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(X)]$ when k is finitely generated over $\mathbb{Q}$; see Corollary 2.8. Note, however, that the canonical class of a Kummer variety of dimension $g \geq 3$ is represented by an effective divisor (Proposition 2.6), thus higher-dimensional Kummer varieties are not Calabi–Yau. Yonatan Harpaz asked if this could be relevant for the tension which exists, in the light of the result of Holmes and Pannekoek, between the heuristics for the ranks of elliptic curves over $\mathbb{Q}$ [18] and the conjecture that $\mathbb{Q}$-points of K3 surfaces are dense in the Brauer–Manin set [22, p. 77], [24, p. 484].

The main goal of this paper is to study the Brauer group and the Brauer–Manin obstruction on Kummer varieties. We prove the following general result.

Theorem 3.3 *Let A be an abelian variety of dimension > 1 over a number field k. Let X be the Kummer variety attached to a 2-covering of A such that $X(\mathbb{A}_k) \neq \emptyset$. Then $X(\mathbb{A}_k)^{\mathrm{Br}(X)_{\mathrm{odd}}} \neq \emptyset$, where $\mathrm{Br}(X)_{\mathrm{odd}} \subset \mathrm{Br}(X)$ is the subgroup of elements of odd order.*

In Theorem 4.3 we give sufficient conditions on an abelian variety A which guarantee that the 2-torsion subgroup of $\mathrm{Br}(X)$ is contained in the algebraic Brauer group $\mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})]$ and, moreover, $\mathrm{Br}_1(X) = \mathrm{Br}_0(X)$. The conditions of Theorem 4.3 are satisfied for the Kummer variety X attached to a 2-covering of the Jacobian of the hyperelliptic curve $y^2 = f(x)$, where $f(x) \in k[x]$ is a separable polynomial of odd degree $d \geq 5$ whose Galois group is the symmetric or alternating group on d letters. See Theorem 5.1, where we also treat products of Jacobians assuming that the splitting fields of the corresponding polynomials are linearly disjoint over k. This implies the following

Corollary 5.2 *Let k be a number field. Let A be the product of Jacobians of elliptic or hyperelliptic curves $y^2 = f_i(x)$, where $f_i(x) \in k[x]$ is a separable polynomial of odd degree $m_i \geq 3$ whose Galois group is the symmetric group on d letters. Assume that $\dim(A) > 1$ and the splitting fields of the $f_i(x)$ are linearly disjoint over k. If the Kummer variety X attached to a 2-covering of A is everywhere locally soluble, then $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.*

This explains the absence of the Brauer group from the statements of the Hasse principle for K3 surfaces in Theorems A and B of [6].

As a by-product of our calculations, we use a result of L. Dieulefait [3] to construct a Kummer K3 surface over $\mathbb{Q}$ of geometric Picard rank 17 with trivial Brauer group; see the examples at the end of the paper. Previously known K3 surfaces with this property have geometric Picard rank 18, 19 and 20, see [25, 9, 8].

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2. Kummer varieties and Kummer lattices

Let k be a field of characteristic different from 2 with an algebraic closure $\bar{k}$ and the Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k) := \mathrm{Aut}(\bar{k}/k)$. For a variety X over k we write $\overline{X} = X \times_k \bar{k}$. Let A be an abelian variety over k of dimension $g \geq 2$. We write A^t for the dual abelian variety of A.

Let T be a k-torsor for the group k-scheme $A[2]$. We define the attached 2-covering of A as the quotient $Y = (A \times_k T)/A[2]$ by the diagonal action of $A[2]$. The first projection defines a morphism $f : Y \to A$ which is a torsor for $A[2]$ such that $T = f^{-1}(0)$. The natural action of A on Y makes Y a k-torsor for A. In particular, there is an isomorphism of varieties $\overline{Y} \cong \overline{A}$. Alternatively, Y is the twisted form of A defined by a 1-cocycle with coefficients in $A[2]$ representing the class of T in $\mathrm{H}^1(k, A[2])$, where $A[2]$ acts on A by translations.

We have an exact sequence of Γ-modules

$$0 \longrightarrow A^t(\bar{k}) \longrightarrow \mathrm{Pic}(\overline{Y}) \longrightarrow \mathrm{NS}\,(\overline{Y}) \longrightarrow 0. \tag{1}$$

The abelian groups $\mathrm{NS}\,(\overline{Y})$ and $\mathrm{NS}\,(\overline{A})$ are isomorphic. In fact, $\mathrm{NS}\,(\overline{Y})$ and $\mathrm{NS}\,(\overline{A})$ are also isomorphic as Γ-modules because translations by the elements of $A(\bar{k})$ act trivially on $\mathrm{NS}\,(\overline{A})$, see [16].

The antipodal involution $\iota_A = [-1] : A \to A$ induces an involution $\iota_Y : Y \to Y$. It acts on $\mathrm{Pic}^0(\overline{Y}) = A^t(\bar{k})$ as $[-1]$, which implies that

$$\mathrm{H}^0(\langle \iota_Y \rangle, A^t(\bar{k})) = A^t[2], \quad \mathrm{H}^1(\langle \iota_Y \rangle, A^t(\bar{k})) = 0,$$

where we used the divisibility of $A^t(\bar{k})$. Taking the invariants of the action of ι_Y on the terms of (1) we obtain an exact sequence of Γ-modules

$$0 \longrightarrow A^t[2] \longrightarrow \mathrm{Pic}(\overline{Y})^{\iota_Y} \longrightarrow \mathrm{NS}\,(\overline{Y}) \longrightarrow 0. \tag{2}$$

Let $\sigma : Y' \to Y$ be the blowing-up of the 2^{2g}-point closed subscheme $T \subset Y$. The involution $\iota_Y : Y \to Y$ preserves T and so gives rise to an involution $\iota_{Y'} : Y' \to Y'$.

Definition 2.1. *The* **Kummer variety** *attached to a 2-covering Y of an abelian vareity A is the quotient $X = Y'/\iota_{Y'}$.*

By definition $\dim(X) = g \geq 2$. The fixed point set of $\iota_{Y'}$ is the exceptional divisor $E = \sigma^{-1}(T)$, which is a smooth divisor in Y'. A standard local calculation shows that X is smooth. Thus the natural surjective morphism $\pi : Y' \to X$ is a double covering whose branch locus is E. The divisor $\overline{E} = \sigma^{-1}(\overline{T})$ is the disjoint union of 2^{2g} copies of $\mathbb{P}^{g-1}_{\bar{k}}$. Let $D = \pi(E) \subset X$.

Let us now pause and describe some known facts about Kummer varieties over $k = \mathbb{C}$. Spanier showed that these varieties are simply connected [27, Thm. 1]. He also showed that their integral cohomology groups are torsion-free, and computed the Betti numbers [27, Thm. 2]:

$$b_0 = b_{2g} = 1, \qquad b_{2i+1} = 0, \qquad b_{2i} = \binom{2g}{2i} + 2^{2g}, \text{ where } 0 < i < n.$$

The canonical class of X was calculated by K. Ueno:

$$K_X = \frac{g-2}{2}[D],$$

see [29, Lemma 16.11.1] or Proposition 2.6 below. Since $K_X \geq 0$, a theorem of K. Ueno [28, Prop. 3], [29, Thm. 16.2] says that the Kodaira dimension of X is 0.

We now return to the assumption that k is an arbitrary field of characteristic different from 2.

Lemma 2.2. *The subgroup of* $\mathrm{Pic}(\overline{X})$ *generated by the classes of the irreducible components of* $\overline{D}$ *is a free abelian group of rank* 2^{2g} *whose generators canonically correspond to the* $\bar{k}$*-points of* T.

Proof. Let E_i, for $i = 1, \ldots, 2^{2g}$, be the irreducible components of $\overline{E} \subset \overline{Y}'$. Choose a line $L_i \cong \mathbb{P}^1_{\bar{k}}$ in each E_i. We define $D_i = \pi(E_i) \subset \overline{X}$, where $i = 1, \ldots, 2^{2g}$. The restriction of π to E_i is an isomorphism $E_i \to D_i$.

For $i \neq j$ we have $D_i \cap D_j = \emptyset$, hence $([D_i].[\pi(L_j)])_{\overline{X}} = 0$. The normal bundle N to $E_i \cong \mathbb{P}^{g-1}_{\bar{k}}$ in $\overline{Y}'$ is $\mathcal{O}(-1)$. By the standard formula [4, Prop. 2.6 (c)] for each $i = 1, \ldots, 2^{2g}$ we have

$$([E_i].[L_i])_{\overline{Y}'} = (c_1(N).[L_i])_{E_i} = (\mathcal{O}(-1).[L_i])_{\mathbb{P}^{g-1}_{\bar{k}}} = -1.$$

Since $\pi^*[D_i] = 2[E_i]$, by the projection formula we have

$$([D_i].[\pi(L_i)])_{\overline{X}} = (\pi^*[D_i].[L_i])_{\overline{Y}'} = -2.$$

Thus no non-trivial linear combination of the classes $[D_i]$ is zero in $\mathrm{Pic}(\overline{X})$. ☐

We write $\mathbb{Z}[T] \subset \mathrm{Pic}(\overline{X})$ for the subgroup described in Lemma 2.2. For $x \in T(\bar{k})$ we denote the corresponding generator of $\mathbb{Z}[T]$ by e_x. Define Π as the saturation of $\mathbb{Z}[T]$ in $\mathrm{Pic}(\overline{X})$:

$$\Pi = \{x \in \mathrm{Pic}(\overline{X}) \,|\, nx \in \mathbb{Z}[T] \text{ for some non-zero } n \in \mathbb{Z}\}.$$

For $g = 2$ Nikulin proved in [17, §1] that Π is a lattice in $\mathbb{Q}[T] = \mathbb{Z}[T] \otimes \mathbb{Q}$ generated by $\mathbb{Z}[T]$ and the vectors $\frac{1}{2}\sum_{x\in H} e_x$, where H is a subset of $T(\bar{k}) \simeq A[2](\bar{k})$ given by $L(x) = c$ for some $L \in \mathrm{Hom}(A[2], \mathbb{F}_2)$ and $c \in \mathbb{F}_2$. (This set of generators does not depend on the choice of an isomorphism $T(\bar{k}) \simeq A[2](\bar{k})$ of $\bar{k}$-torsors for $A[2]$.) We generalise this result to $g \geq 2$. In doing so we show that $\mathrm{Pic}(\overline{X})$ is torsion-free for any $g \geq 2$, see Proposition 2.3 below. In particular, Π is also torsion-free, so Π can be called the *Kummer lattice*.

Write $Y_0 = Y \setminus T$ and $X_0 = \pi(\sigma^{-1}(Y_0))$. Then Y_0 is the complement to a finite set in a smooth, proper and geometrically integral variety of dimension at least 2, so we have

$$\bar{k}[Y_0] = \bar{k}, \quad \mathrm{Pic}(\overline{Y}_0) = \mathrm{Pic}(\overline{Y}), \quad \mathrm{Br}(\overline{Y}_0) = \mathrm{Br}(\overline{Y}), \tag{3}$$

where the last property follows from [5, Cor. 6.2, p. 136].

The involution ι_Y acts on Y_0 without fixed points, hence $\pi : Y_0 \to X_0 = Y_0/\iota_Y$ is a torsor for $\mathbb{Z}/2$. There is a Hochschild–Serre spectral sequence [14, Thm. III.2.20]

$$\mathrm{H}^p(\mathbb{Z}/2, \mathrm{H}^q_{\text{ét}}(\overline{Y}_0, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\overline{X}_0, \mathbb{G}_m). \tag{4}$$

Using (3) we deduce an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pic}(\overline{X}_0) \xrightarrow{\sigma_*\pi^*} \mathrm{Pic}(\overline{Y})^{\iota_Y} \longrightarrow 0, \tag{5}$$

where the last zero is due to the fact that $\mathrm{H}^2(\mathbb{Z}/2, \bar{k}^*) = \bar{k}^*/\bar{k}^{*2} = 0$ as $\mathrm{char}(k) \neq 2$. Using the fact that $\mathrm{NS}\,(\overline{Y}) \cong \mathrm{NS}\,(\overline{A})$ is torsion-free, we deduce from (5) and (2) a commutative diagram of Γ-modules with exact rows and columns

(6)
$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathrm{NS}\,(\overline{Y}) \arrow[u] \arrow[r, equal] & \mathrm{NS}\,(\overline{Y}) \arrow[u] & \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathrm{Pic}(\overline{X}_0) \arrow[r] \arrow[u] & \mathrm{Pic}(\overline{Y})^{\iota_Y} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[u, equal] & \mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}} \arrow[r] \arrow[u] & A^t[2] \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since X is smooth, the natural restriction map $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{X}_0)$ is surjective; thus $\mathrm{Pic}(\overline{X}_0) = \mathrm{Pic}(\overline{X})/\mathbb{Z}[T]$. This implies $\mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}} = \Pi/\mathbb{Z}[T]$, so we obtain a commutative diagram of Γ-modules with exact rows and columns

(7)

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathrm{NS}\,(\overline{Y}) \arrow[u] \arrow[r, equal] & \mathrm{NS}\,(\overline{Y}) \arrow[u] & \\
0 \arrow[r] & \mathbb{Z}[T] \arrow[r] & \mathrm{Pic}(\overline{X}) \arrow[u] \arrow[r] & \mathrm{Pic}(\overline{X}_0) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}[T] \arrow[u, equal] \arrow[r] & \Pi \arrow[u] \arrow[r] & \mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}} \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

For future reference we write the middle column of (7) as an exact sequence of Γ-modules

$$0 \longrightarrow \Pi \longrightarrow \mathrm{Pic}(\overline{X}) \xrightarrow{\sigma_*\pi^*} \mathrm{NS}\,(\overline{Y}) \longrightarrow 0. \tag{8}$$

Proposition 2.3. *Let X be a Kummer variety over a field of characteristic different from* 2. *Then the abelian group* $\mathrm{Pic}(\overline{X})$ *is torsion-free. There is an isomorphism of abelian groups* $\mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}} \cong A^t[2] \oplus \mathbb{Z}/2$.

Proof. The statements concern varieties over $\bar{k}$, so we can assume that X is attached to the trivial 2-covering $Y = A$. The translations by points of order 2 commute with the antipodal involution $[-1] : A \to A$. This implies that the finite commutative group k-scheme $\mathcal{G} = A[2] \times_k \mathbb{Z}/2$ acts on A so that the elements of $A[2]$ act as translations and the generator of $\mathbb{Z}/2$ acts as $[-1]$. It is easy to see that $\mathcal{G}$ acts freely on $A_1 = A \setminus A[4]$ with quotient $A_1/\mathcal{G} = X_0$. Hence the quotient morphism $f : A_1 \to X_0$ is a torsor for $\mathcal{G}$. Since $g \geq 2$, we have $\bar{k}[A_1] = \bar{k}$ and $\mathrm{Pic}(\overline{A}_1) = \mathrm{Pic}(\overline{A})$. The Cartier dual $\widehat{\mathcal{G}}$ is isomorphic to $A^t[2] \times \mathbb{Z}/2$, so the exact sequence [21, (2.5), p. 17] gives an injective map $A^t[2] \oplus \mathbb{Z}/2 \hookrightarrow \mathrm{Pic}(\overline{X}_0)$. The bottom exact sequence of (6) shows that the cardinality of $A^t[2] \oplus \mathbb{Z}/2$ equals the cardinality of $\mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}}$, so we obtain an isomorphism of abelian groups $A^t[2] \oplus \mathbb{Z}/2 \xrightarrow{\sim} \mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}}$.

Since $\mathbb{Z}[T]$ is torsion-free, the natural map $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{X}_0)$ induces an injective map of torsion subgroups. In particular, a non-zero torsion element of $\mathrm{Pic}(\overline{X})$ is annihilated by 2 and corresponds to a connected unramified double covering of $\overline{X}$. A double covering of $\overline{X}$ is uniquely determined by its restriction to $\overline{X}_0$. Therefore, it is enough to show that any connected unramified double covering of $\overline{X}_0$ is a restriction of a *ramified* double covering of $\overline{X}$. By the previous paragraph any such covering of $\overline{X}_0$ is of the form $A_1/\mathcal{H}$, where $\mathcal{H}\subset\mathcal{G}$ is a subgroup of index 2.

If $\mathcal{H}=A[2]$, then $\overline{A}_1/A[2]=\overline{A}\setminus A[2]=\overline{A}_0$. Write $\sigma: A'\to A$ for the blowing-up of $A[2]$ in A. Then the unramified double covering $\overline{A}_0\to\overline{X}_0$ extends to the double covering $\overline{A}'\to\overline{X}$ ramified exactly in the exceptional locus $\sigma^{-1}(A[2])$.

If $\mathcal{H}\neq A[2]$, then there is a non-zero $\phi\in\mathrm{Hom}(A[2],\mathbb{Z}/2)=A^t[2]$ such that $\mathcal{H}$ is the kernel of the homomorphism $A[2]\oplus\mathbb{Z}/2\to\mathbb{Z}/2$ given by $(x,y)\mapsto\phi(x)$ or by $(x,y)\mapsto\phi(x)+y$. Define $A_\phi=\overline{A}/\mathrm{Ker}(\phi)$. Choose $a\in A[2](\bar{k})$ such that $\phi(a)\neq 0$. Then $\overline{A}_1/\mathcal{H}$ is the quotient of A_ϕ with $A_\phi[2]$ and $[2]^{-1}(\phi(a))$ removed, by the involution $x\mapsto -x$ in the first case and $x\mapsto\phi(a)-x$ in the second case. It follows that the unramified double covering $\overline{A}_1/\mathcal{H}\to\overline{X}_0$ is the restriction of the double covering of $\overline{X}$ ramified in $\sigma^{-1}(A[2]\setminus\mathrm{Ker}(\phi))$ in the first case and in $\sigma^{-1}(\mathrm{Ker}(\phi))$ in the second case. □

Corollary 2.4. *Any Kummer variety X of dimension $g\geq 2$ over a field k of characteristic not equal to 2 satisfies the following properties:*

(i) $\mathrm{Pic}^0(\overline{X})=0$;

(ii) $\mathrm{Pic}(\overline{X})=\mathrm{NS}\,(\overline{X})$ *is torsion-free of rank* $2^{2g}+\mathrm{rk}(\mathrm{NS}\,(\overline{A}))$;

(iii) $\mathrm{H}^1_{\text{ét}}(\overline{X},\mathbb{Z}_\ell)=0$ *for any prime* $\ell\neq\mathrm{char}(k)$;

(iv) $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell)$ *is torsion-free for any prime* $\ell\neq\mathrm{char}(k)$.

Proof. Since $\mathrm{Pic}(\overline{X})$ is torsion-free, we immediately obtain (i) and $\mathrm{Pic}(\overline{X})=\mathrm{NS}\,(\overline{X})$. From diagram (7) we see that the rank of this group is $2^{2g}+\mathrm{rk}(\mathrm{NS}\,(\overline{Y}))=2^{2g}+\mathrm{rk}(\mathrm{NS}\,(\overline{A}))$. The Kummer sequence gives well-known isomorphisms

$$\mathrm{H}^1_{\text{ét}}(\overline{X},\mu_{\ell^n})=\mathrm{Pic}(\overline{X})[\ell^n]=0,\quad n\geq 1,$$

which imply (iii), by passing to the limit in n. The Kummer sequence also implies the well-known fact that the torsion subgroup of $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$ coincides with the torsion subgroup of $\mathrm{NS}\,(\overline{X})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$. This gives (iv). □

Corollary 2.5. *The Galois cohomology group* $\mathrm{H}^1(k,\mathrm{Pic}(\overline{X}))$ *is finite. The kernel of the natural map* $\mathrm{H}^1(k,\mathrm{Pic}(\overline{X}))\to\mathrm{H}^1(k,\mathrm{NS}\,(\overline{Y}))$ *is annihilated by* 2. *If the order of the finite group* $\mathrm{H}^1(k,\mathrm{NS}\,(\overline{A}))$ *is a power of* 2, *in particular, if* $\mathrm{NS}\,(\overline{A})$ *is a trivial* Γ*-module, then every element of odd order in* $\mathrm{Br}_1(X)$ *is contained in* $\mathrm{Br}_0(X)$.

Proof. The finiteness of $\mathrm{H}^1(k,\mathrm{Pic}(\overline{X}))$ follows from the first statement of Proposition 2.3.

The second statement of Proposition 2.3 implies that $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}})$ is annihilated by 2. Since $\mathbb{Z}[T]$ is a permutation Γ-module we have $\mathrm{H}^1(k, \mathbb{Z}[T]) = 0$. By diagram (7) this implies that $\mathrm{H}^1(k, \Pi)$ is a subgroup of $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}})$ and so is also annihilated by 2. This proves the second statement.

Recall that $\mathrm{NS}\,(\overline{Y})$ and $\mathrm{NS}\,(\overline{A})$ are isomorphic as Γ-modules, so $\mathrm{H}^1(k, \mathrm{NS}\,(\overline{Y})) = \mathrm{H}^1(k, \mathrm{NS}\,(\overline{A}))$. When the order of this group is a power of 2, the order of $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$ is also a power of 2. The last statement is now immediate from the well known inclusion of the quotient $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ into $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$. $\Box$

We define $\Pi_1 \subset \Pi$ as the kernel of the composed surjective map

$$\mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{Pic}(\overline{X}_0) \longrightarrow \mathrm{Pic}(\overline{Y})^{\iota_Y}.$$

Then $\mathbb{Z}[T]$ is a subgroup of Π_1 of index 2. It is easy to see that Π_1 is generated by $\mathbb{Z}[T]$ and $\frac{1}{2}\sum_{x \in T(\bar{k})} e_x$. We thus have a canonical filtration

$$\mathbb{Z}[T] \subset \Pi_1 \subset \Pi \subset \mathrm{Pic}(\overline{X})$$

with successive factors $\mathbb{Z}/2$, $A^t[2]$, $\mathrm{NS}\,(\overline{Y}) = \mathrm{NS}\,(\overline{A})$. This filtration is respected by the action of $A[2]$ on $\overline{Y}$ and $\overline{X}$, as well as by the action of the Galois group Γ.

We summarise our discussion in the form of the following commutative diagram with exact rows and columns, where all arrows are group homomorphisms which respect the actions of Γ and $A[2]$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Pi_1 \arrow[d] \arrow[r, phantom, "="] & \Pi_1 \arrow[d] & & \\
0 \arrow[r] & \Pi \arrow[r] \arrow[d] & \mathrm{Pic}(\overline{X}) \arrow[r, "\sigma_*\pi^*"] \arrow[d] & \mathrm{NS}\,(\overline{Y}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A^t[2] \arrow[r] \arrow[d] & \mathrm{Pic}(\overline{Y})^{\iota_Y} \arrow[r] \arrow[d] & \mathrm{NS}\,(\overline{Y}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{9}$$

Remark 1 It is clear that $\mathbb{Z}[T]$ is a permutation Γ-module. Now consider the particular case when T is a trivial torsor, i.e. $T \cong A[2]$ as k-torsors. The action of Γ on the set $A[2]$ fixes 0. It follows that not just $\mathbb{Z}[A[2]]$ but also Π_1 is a permutation Γ-module. Indeed, Π_1 has a Γ-stable $\mathbb{Z}$-basis consisting of

e_x for $x \in A[2] \setminus \{0\}$ and $\frac{1}{2}\sum_{x\in A[2]} e_x$. Note, however, that this basis is not $A[2]$-stable.

The following proposition shows that the canonical class of a Kummer variety of dimension $g \geq 3$ is represented by an effective divisor, so such varieties are not Calabi–Yau. In the case $\mathrm{char}(k) = 0$, this was proved in [29, Lemma 16.11.1].

Proposition 2.6. *We have* $K_{\overline{X}} = \frac{g-2}{2}[\overline{D}] = \frac{g-2}{2}\sum_{x\in T(\bar{k})} e_x$.

Proof. The natural map $\pi^* : \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{Y}')$ is injective. Indeed, its kernel is contained in Π_1, by the exactness of the middle column of (9). In the notation of the proof of Lemma 2.2 we have $\pi^*[D_i] = 2[E_i]$, hence π^* is injective on Π_1. Since $K_{\overline{Y}} = 0$, the standard formulae give $K_{\overline{Y}'} = (g-1)\sum[E_i]$ and $K_{\overline{Y}'} = \pi^*K_{\overline{X}} + \sum[E_i]$. Now our statement follows from the injectivity of $\pi^* : \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{Y}')$. □

Proposition 2.7. *Assume that the characteristic of k is zero. Then the morphisms $\pi : Y' \to X$ and $\sigma : Y' \to Y$ induce isomorphisms of Γ-modules*

$$\mathrm{Br}(\overline{X})\xrightarrow{\sim}\mathrm{Br}(\overline{Y}')\xleftarrow{\sim}\mathrm{Br}(\overline{Y}) \cong \mathrm{Br}(\overline{A}).$$

Proof The last isomorphism is due to the fact that Y is the twist of A by a 1-cocycle with coefficients in $A[2]$, but the induced action of $A[2]$ on $\mathrm{Br}(\overline{A})$ is trivial. In fact, the whole group $A(\bar{k})$ acts trivially on the finite group $\mathrm{Br}(\overline{A})[n]$ for every integer n, because any homomorphism from the divisible group $A(\bar{k})$ to the finite group $\mathrm{Aut}(\mathrm{Br}(\overline{A})[n])$ is trivial.

The middle isomorphism is a consequence of the birational invariance of the Brauer group of a smooth and projective variety over a field of characteristic zero.

The natural map $\pi^* : \mathrm{Br}(\overline{X}) \to \mathrm{Br}(\overline{Y}')$ is a map of Γ-modules. To prove that it is an isomorphism we can work over an algebraically closed field of characteristic zero and so assume that $Y = A$. We remark that Grothendieck's exact sequence [5, Cor. 6.2, p. 137] gives an exact sequence

$$0 \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \mathrm{Br}(\overline{X}_0) \longrightarrow \bigoplus \mathrm{H}^1(\mathbb{P}^{g-1}_{\bar{k}}, \mathbb{Q}/\mathbb{Z}),$$

where the terms in the direct sum are numbered by the 2^{2g} points of $A[2](\bar{k})$. We have $\mathrm{H}^1(\mathbb{P}^{g-1}_{\bar{k}}, \mathbb{Z}/n) = 0$ for any positive integer n, so the natural map $\mathrm{Br}(\overline{X})\xrightarrow{\sim}\mathrm{Br}(\overline{X}_0)$ is an isomorphism. By (3) there is a natural isomorphism $\mathrm{Br}(\overline{A})\xrightarrow{\sim}\mathrm{Br}(\overline{A}_0)$.

We analyse the map $\pi^* : \mathrm{Br}(\overline{X}_0) \to \mathrm{Br}(\overline{A}_0)$ using the spectral sequence (4). We have already seen that $\mathrm{H}^2(\mathbb{Z}/2, \bar{k}^*) = 0$. We have a natural isomorphism $\mathrm{Pic}(\overline{A}_0) = \mathrm{Pic}(\overline{A})$ and we claim that $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{Pic}(\overline{A})) = 0$. In view of the exact sequence (1) it is enough to prove that $\mathrm{H}^1(\mathbb{Z}/2, A^t) = \mathrm{H}^1(\mathbb{Z}/2, \mathrm{NS}\,(\overline{A})) = 0$. The torsion-free group $\mathrm{NS}\,(\overline{A})$ is contained in the group $\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}_\ell(1))$ for any prime ℓ. The involution $[-1]$ acts trivially on $\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}_\ell(1))$ and hence on $\mathrm{NS}\,(\overline{A})$. It follows that $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{NS}\,(\overline{A})) = 0$. On the other hand, $[-1]$ acts on $\mathrm{Pic}^0(\overline{A}) \cong A^t(\bar{k})$ as $[-1]$, hence $\mathrm{H}^1(\mathbb{Z}/2, A^t) = 0$. The spectral sequence (4) now gives an injective map $\mathrm{Br}(\overline{X}) \hookrightarrow \mathrm{Br}(\overline{A})$.

By the well known Grothendieck's computation [5, §8] we have $\mathrm{Br}(\overline{A}) \cong (\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$, where $b_2 = g(2g-1)$ is the dimension of $\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Q}_\ell(1))$ and $\rho = \mathrm{rk}(\mathrm{NS}\,(\overline{A}))$. To complete the proof it is enough to show that the corank of the divisible part of $\mathrm{Br}(\overline{X})$ is $g(2g-1)-\rho$. (Indeed, any injective homomorphism $(\mathbb{Q}/\mathbb{Z})^r \to (\mathbb{Q}/\mathbb{Z})^r$ is an isomorphism.) By Corollary 2.4 (ii) $\mathrm{Pic}(\overline{X})$ is torsion-free of rank $\rho + 2^{2g}$. Thus it remains to show that the dimension of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ is $g(2g-1) + 2^{2g}$ for any prime ℓ. The Gysin sequence gives an exact sequence

$$0 \longrightarrow (\mathbb{Q}_\ell)^{2^{2g}} \longrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}_0, \mathbb{Q}_\ell(1)) \longrightarrow 0.$$

The spectral sequence $\mathrm{H}^p(\mathbb{Z}/2, \mathrm{H}^q_{\text{ét}}(\overline{A}_0, \mathbb{Q}_\ell)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\overline{X}_0, \mathbb{Q}_\ell)$ degenerates because each $\mathrm{H}^q_{\text{ét}}(\overline{A}_0, \mathbb{Q}_\ell)$ is a vector space over a field of characteristic 0. We obtain

$$\mathrm{H}^n_{\text{ét}}(\overline{X}_0, \mathbb{Q}_\ell) = \mathrm{H}^n_{\text{ét}}(\overline{A}_0, \mathbb{Q}_\ell)^{[-1]^*}$$

for all $n \geq 0$. In particular, the dimension of $\mathrm{H}^2_{\text{ét}}(\overline{X}_0, \mathbb{Q}_\ell(1))$ is $g(2g-1)$, as required. □

Corollary 2.8. *Let k be a field finitely generated over $\mathbb{Q}$. Let X be the Kummer variety attached to a 2-covering of an abelian variety. Then the groups* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *and* $\mathrm{Br}(\overline{X})^\Gamma$ *are finite.*

Proof. By the spectral sequence $\mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$ and Corollary 2.5 the finiteness of $\mathrm{Br}(\overline{X})^\Gamma$ implies the finiteness of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$. By Proposition 2.7 this follows from the finiteness of $\mathrm{Br}(\overline{A})^\Gamma$ which is established in [24]. □

Remark 2 Assume that $\mathrm{char}(k) = 0$. The commutative diagram

$$\begin{tikzcd}
\mathrm{Br}(\overline{X}) \arrow[r, "\sim"] & \mathrm{Br}(\overline{Y}) \\
\mathrm{Br}(X) \arrow[u] \arrow[r, "\sigma_*\pi^*"] & \mathrm{Br}(Y) \arrow[u]
\end{tikzcd}$$

identifies $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ with a subgroup of $\mathrm{Br}(Y)/\mathrm{Br}_1(Y)$.

3. When the Hasse principle is unobstructed

Let n be an *odd* integer and let k be a field of characteristic coprime to $2n$. Let Λ be a Γ-module such that $n\Lambda = 0$.

If A be an abelian variety over k, then $[-1]$ acts on $\mathrm{H}^q_{\text{ét}}(\overline{A},\Lambda)$ by $(-1)^q$, where $q \geq 0$. Hence for a 2-covering Y of A the involution ι_Y acts on $\mathrm{H}^q_{\text{ét}}(\overline{Y},\Lambda)$ by $(-1)^q$.

For $m \geq 0$ let $\mathrm{H}^m_{\text{ét}}(Y,\Lambda)^+$ be the ι_Y-invariant subgroup of $\mathrm{H}^m_{\text{ét}}(Y,\Lambda)$. Let $\mathrm{H}^m_{\text{ét}}(Y,\Lambda)^-$ be the ι_Y-anti-invariant subgroup, i.e. the group of elements on which ι_Y acts by -1. Since n is odd, we can write

$$\mathrm{H}^m_{\text{ét}}(Y,\Lambda) = \mathrm{H}^m_{\text{ét}}(Y,\Lambda)^+ \oplus \mathrm{H}^m_{\text{ét}}(Y,\Lambda)^-. \tag{10}$$

Proposition 3.1. *Let Y be a 2-covering of an abelian variety A. Then we have a canonical decomposition of abelian groups*

$$\mathrm{H}^2_{\text{ét}}(Y,\Lambda) = \mathrm{H}^2(k,\Lambda) \oplus \mathrm{H}^1(k,\mathrm{H}^1_{\text{ét}}(\overline{Y},\Lambda)) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y},\Lambda)^\Gamma$$

compatible with the natural action of the involution ι_Y on $\mathrm{H}^2_{\text{ét}}(Y,\Lambda)$, so that

$$\mathrm{H}^2_{\text{ét}}(Y,\Lambda)^+ = \mathrm{H}^2(k,\Lambda) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y},\Lambda)^\Gamma \quad \textit{and} \quad \mathrm{H}^2_{\text{ét}}(Y,\Lambda)^- = \mathrm{H}^1(k,\mathrm{H}^1(\overline{Y},\Lambda)).$$

Proof. Let $m \geq 1$. The morphisms $Y \to \mathrm{Spec}(k)$ and $\overline{Y} \to Y$ induce the ι_Y-equivariant maps

$$\alpha_m : \mathrm{H}^m(k,\Lambda) \longrightarrow \mathrm{H}^m_{\text{ét}}(Y,\Lambda), \quad \beta_m : \mathrm{H}^m_{\text{ét}}(Y,\Lambda) \longrightarrow \mathrm{H}^m_{\text{ét}}(\overline{Y},\Lambda)^\Gamma, \quad \beta_m\alpha_m = 0.$$

Since ι_Y acts trivially on $\mathrm{H}^m(k,\Lambda)$ and on $\mathrm{H}^2_{\text{ét}}(\overline{Y},\Lambda)$, we have

$$\mathrm{Im}(\alpha_m) \subset \mathrm{H}^m_{\text{ét}}(Y,\Lambda)^+, \qquad \mathrm{H}^2_{\text{ét}}(Y,\Lambda)^- \subset \mathrm{Ker}(\beta_2).$$

We claim that it is enough to show that

$\alpha_m : \mathrm{H}^m(k,\Lambda) \to \mathrm{H}^m_{\text{ét}}(Y,\Lambda)^+$ has a retraction, for every $m \geq 0$;

$\beta_2 : \mathrm{H}^2_{\text{ét}}(Y,\Lambda)^+ \to \mathrm{H}^2_{\text{ét}}(\overline{Y},\Lambda)^\Gamma$ has a section.

Indeed, if this is true, then $\mathrm{H}^m(k,\Lambda)$ is a direct summand of $\mathrm{H}^m_{\text{ét}}(Y,\Lambda)^+$ for $m \geq 0$. Moreover, $\mathrm{H}^2_{\text{ét}}(\overline{Y},\Lambda)^\Gamma$ is a direct summand of $\mathrm{H}^2_{\text{ét}}(Y,\Lambda)^+$, so that

$$\mathrm{H}^2_{\text{ét}}(Y,\Lambda) = \mathrm{H}^2(k,\Lambda) \oplus \mathrm{Ker}(\beta_2)/\mathrm{Im}(\alpha_2) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y},\Lambda)^\Gamma.$$

Note that the maps α_m and β_m are the canonical maps in the spectral sequence

$$\mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(\overline{Y}, \Lambda)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(Y, \Lambda). \tag{11}$$

From (11) we obtain the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\beta_2)/\mathrm{Im}(\alpha_2) \longrightarrow \mathrm{H}^1(k, \mathrm{H}^1(\overline{Y}, \Lambda)) \longrightarrow \mathrm{Ker}(\alpha_3) = 0.$$

Since ι_Y acts on $\mathrm{H}^1(\overline{Y}, \Lambda)$ by -1 and the Galois group Γ commutes with ι_Y, we get

$$\mathrm{Ker}(\beta_2)/\mathrm{Im}(\alpha_2) = \mathrm{H}^1(k, \mathrm{H}^1(\overline{Y}, \Lambda)) \subset \mathrm{H}^2_{\text{ét}}(Y, \Lambda)^-.$$

Now the claim follows from (10).

Let us construct a retraction of α_m. Recall that T is a 0-dimensional subscheme of Y, and so we have a restriction map $\mathrm{H}^m_{\text{ét}}(Y, \Lambda) \to \mathrm{H}^m_{\text{ét}}(T, \Lambda)$. Write T as a disjoint union of closed points

$$T = \bigsqcup_{i=1}^{r} \mathrm{Spec}(k_i),$$

where each k_i is a finite field extension of k. Then $\mathrm{H}^m_{\text{ét}}(T, \Lambda)$ is the direct sum of the Galois cohomology groups $\mathrm{H}^m(k_i, \Lambda)$ for $i = 1, \ldots, r$. The composition of the restriction map $\mathrm{H}^m(k, \Lambda) \to \mathrm{H}^m(k_i, \Lambda)$ and the corestriction map $\mathrm{H}^m(k_i, \Lambda) \to \mathrm{H}^m(k, \Lambda)$ is the multiplication by $[k_i : k]$. The direct sum of these corestriction maps is a map $\mathrm{H}^m_{\text{ét}}(T, \Lambda) \to \mathrm{H}^m(k, \Lambda)$ whose composition with the natural restriction map $\mathrm{H}^m(k, \Lambda) \to \mathrm{H}^m_{\text{ét}}(T, \Lambda)$ is the multiplication by $|T(\bar{k})| = 2^{2g}$. Since n is odd, there is an integer r such that $2^{2g} r \equiv 1 \bmod n$. Thus the composition

$$\mathrm{H}^m_{\text{ét}}(Y, \Lambda) \longrightarrow \mathrm{H}^m_{\text{ét}}(T, \Lambda) \longrightarrow \mathrm{H}^m(k, \Lambda) \xrightarrow{[r]} \mathrm{H}^m(k, \Lambda) \tag{12}$$

is a retraction of α_m. Since $T \subset Y$ is the fixed point set of ι_Y, this retraction is ι_Y-equivariant.

Let us construct a section of β_2. The translations by the points of $A(\bar{k})$ act trivially on $\mathrm{H}^q_{\text{ét}}(\overline{A}, \Lambda)$ for any $q \geq 0$, so we have canonical isomorphisms of Γ-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{Y}, \Lambda) = \mathrm{H}^2_{\text{ét}}(\overline{A}, \Lambda) = \mathrm{Hom}(\wedge^2 A[n], \Lambda). \tag{13}$$

Since $[Y] \in \mathrm{H}^1(k, A)[2]$ and n is odd, the multiplication by n on A defines a morphism $[n] : Y \to Y$ which is a torsor with structure group $A[n]$. We denote this torsor by $\mathcal{T}_n$. The class of this torsor is an element $[\mathcal{T}_n] \in \mathrm{H}^1_{\text{ét}}(Y, A[n])$. Using cup-product we obtain a class

$$\wedge^2[\mathcal{T}_n] \in \mathrm{H}^2_{\text{ét}}(Y, \wedge^2 A[n]).$$

The isomorphisms (13) give rise to a pairing

$$\mathrm{H}^2_{\text{ét}}(Y, \wedge^2 A[n]) \times \mathrm{H}^2_{\text{ét}}(\overline{Y}, \Lambda)^\Gamma \longrightarrow \mathrm{H}^2_{\text{ét}}(Y, \Lambda).$$

Let $s : \mathrm{H}^2_{\text{ét}}(\overline{Y}, \Lambda)^\Gamma \to \mathrm{H}^2_{\text{ét}}(Y, \Lambda)$ be the map defined by pairing with the class $\wedge^2[\mathcal{T}_n]$. As n is an odd integer, the same proof as in [25, Prop. 2.2] (where we treated the case $Y = A$) shows that s is a section of the natural map $\mathrm{H}^2_{\text{ét}}(Y, \Lambda) \to \mathrm{H}^2_{\text{ét}}(\overline{Y}, \Lambda)^\Gamma$.

Since $[-1] \cdot [n] = [-n] = [n] \cdot [-1]$ we see that $\iota_Y^*[\mathcal{T}_n] = [\mathcal{T}_{-n}]$, and the torsor $\mathcal{T}_{-n}$ is obtained from $\mathcal{T}_n$ by applying the automorphism $[-1]$ to the structure group $A[n]$. Hence $\iota_Y^*[\mathcal{T}_n] = -[\mathcal{T}_n]$. It follows that $\iota_Y^*(\wedge^2[\mathcal{T}_n]) = \wedge^2[\mathcal{T}_n]$. We conclude that s is ι_Y-equivariant, and so is a section of the map $\beta_2 : \mathrm{H}^2_{\text{ét}}(Y, \Lambda)^+ \to \mathrm{H}^2_{\text{ét}}(\overline{Y}, \Lambda)^\Gamma$. □

Remark 3 The restriction of the morphism $[n] : Y \to Y$ to T has a section given by the identity map $T \tilde{\longrightarrow} T$. Thus the restriction of $\mathcal{T}_n$ to T is trivial. It follows that $s(\mathrm{H}^2_{\text{ét}}(\overline{Y}, \Lambda)^\Gamma)$ is contained in the kernel of the restriction map $\mathrm{H}^2_{\text{ét}}(Y, \Lambda) \to \mathrm{H}^2_{\text{ét}}(T, \Lambda)$.

Recall that the Kummer variety X attached to Y is defined as follows. Let $\sigma : Y' \to Y$ be the blowing-up of T in Y. Then $\pi : Y' \to X$ is the double cover which is the quotient by a natural involution on Y' compatible with ι_Y. We note that the same variety X can also be obtained from any quadratic twist of Y. More precisely, let F be an étale k-algebra of dimension 2, i.e. $k \oplus k$ or a quadratic extension of k. We denote by A_F the quadratic twist of the abelian variety A by F, defined as the quotient of $A \times_k \mathrm{Spec}(F)$ by the simultaneous action of $\mathbb{Z}/2$ such that the generator of $\mathbb{Z}/2$ acts on A as $[-1]$ and on $\mathrm{Spec}(F)$ as $c \in \mathrm{Gal}(F/k)$, $c \neq 0$. (In the case of $F = k \oplus k$ the action of c permutes the factors of $\mathrm{Spec}(F)$, so that $A_F = A$ in this case.) We define Y_F similarly, replacing $[-1]$ with ι_Y. Since $[-1]$ commutes with translations by the elements of $A[2]$, we have a morphism $Y_F \to Y_F/A[2] = A_F$, which is a 2-covering of A_F defined by the same k-torsor T for $A[2] = A_F[2]$. The blowing-up $\sigma_F : Y'_F \to Y_F$ of the closed subscheme $T \subset Y_F$ has an involution compatible with ι_{Y_F}. It gives rise to the double covering $\pi_F : Y'_F \to X$.

Let $Y_{F0} = Y_F \setminus T$. We have a commutative diagram

(14)
$$\begin{tikzcd}
\mathrm{H}^2_{\text{ét}}(X,\Lambda) \arrow[r, "\pi_F^*"] \arrow[d] & \mathrm{H}^2_{\text{ét}}(Y'_F,\Lambda) \arrow[r, "\sigma_{F*}"] \arrow[d] & \mathrm{H}^2_{\text{ét}}(Y_F,\Lambda) \arrow[d, equal] \\
\mathrm{H}^2_{\text{ét}}(X_0,\Lambda) \arrow[r, "\pi_F^*"] & \mathrm{H}^2_{\text{ét}}(Y_{F0},\Lambda) \arrow[r, phantom, "="] & \mathrm{H}^2_{\text{ét}}(Y_{F0},\Lambda)
\end{tikzcd}$$

The restriction map $\mathrm{H}^2_{\text{ét}}(Y_F,\Lambda)\xrightarrow{\sim}\mathrm{H}^2_{\text{ét}}(Y_{F0},\Lambda)$ is an isomorphism by the purity of étale cohomology [14, Remark VI.5.4 (b)] as $\mathrm{codim}_Y(T) \geq 2$. The map σ_{F*} is the composition of the restriction to the open set $Y_{F0} \subset Y'_F$ and the inverse of $\mathrm{H}^2_{\text{ét}}(Y_F,\Lambda)\xrightarrow{\sim}\mathrm{H}^2_{\text{ét}}(Y_{F0},\Lambda)$. In particular, the composition

$$\mathrm{H}^2_{\text{ét}}(Y_F,\Lambda) \xrightarrow{\sigma_F^*} \mathrm{H}^2_{\text{ét}}(Y'_F,\Lambda) \xrightarrow{\sigma_{F*}} \mathrm{H}^2_{\text{ét}}(Y_F,\Lambda)$$

is the identity map.

For the sake of completeness we note that the Hochschild–Serre spectral sequence [14, Thm. III.2.20]

$$\mathrm{H}^p(\mathbb{Z}/2, \mathrm{H}^q_{\text{ét}}(Y_{F0},\Lambda)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X_0,\Lambda)$$

gives canonical isomorphisms

$$\mathrm{H}^p_{\text{ét}}(X_0,\Lambda)\xrightarrow{\sim}\mathrm{H}^p_{\text{ét}}(Y_{F0},\Lambda)^+.$$

Indeed, $\mathrm{H}^p(\mathbb{Z}/2, \mathrm{H}^q_{\text{ét}}(Y_{F0},\Lambda)) = 0$ for $p \geq 1$, since 2 and n are coprime.

Proposition 3.2. *Let X be the Kummer variety attached to a 2-covering Y of an abelian variety of dimension at least 2. Let $n \geq 1$ be an odd integer. For any $x \in \mathrm{H}^2_{\text{ét}}(X,\mu_n)$ there exists an $a_0 \in \mathrm{H}^2(k,\mu_n)$ such that for any étale k-algebra F of dimension 2 we have*

$$\sigma_{F*}\pi_F^*(x) - a_0 \in s(\mathrm{H}^2_{\text{ét}}(\overline{Y}_F,\mu_n)^\Gamma),$$

where s is the section of the natural map $\mathrm{H}^2_{\text{ét}}(Y_F,\mu_n) \to \mathrm{H}^2_{\text{ét}}(\overline{Y}_F,\mu_n)^\Gamma$ constructed in the proof of Proposition 3.1.

Proof. Since $\sigma_{F*}\pi_F^*(x)$ is ι_{Y_F}-invariant, by Proposition 3.1 we have $\sigma_{F*}\pi_F^*(x) = a_0 + s(a)$ for some $a_0 \in \mathrm{H}^2(k,\mu_n)$ and $a \in \mathrm{H}^2_{\text{ét}}(\overline{Y}_F,\mu_n)^\Gamma$. We need to show that a_0 does not depend on F. Recall that

$$X \setminus X_0 \cong Y'_F \setminus Y_{F0} = \mathbb{P}^{g-1}_T = \mathbb{P}^{g-1}_k \times_k T,$$

and the natural morphism $Y'_F\backslash Y_{F0}\to Y_F\backslash Y_{F0}=T$ is the structure morphism $\mathbb{P}^{g-1}_T\to T$. We have a commutative diagram, where the vertical arrows are the natural restriction maps

$$\begin{tikzcd}
\mathrm{H}^2_{\text{ét}}(X,\mu_n) \arrow[r, "\pi_F^*"] \arrow[d, "\tau_1"] & \mathrm{H}^2_{\text{ét}}(Y'_F,\mu_n) \arrow[d, "\tau_2"] & \mathrm{H}^2_{\text{ét}}(Y_F,\mu_n) \arrow[l, "\sigma_F^*"'] \arrow[d, "\tau_3"] \\
\mathrm{H}^2_{\text{ét}}(\mathbb{P}^{g-1}_T,\mu_n) \arrow[r, "\mathrm{id}"] & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^{g-1}_T,\mu_n) \arrow[r, shift right, "\rho^*"'] & \mathrm{H}^2_{\text{ét}}(T,\mu_n) \arrow[l, shift right]
\end{tikzcd} \tag{15}$$

A choice of a k-point in $\mathbb{P}^{g-1}_k$ defines a section ρ of the structure morphism $\mathbb{P}^{g-1}_T\to T$, and we denote by ρ^* the induced map $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^{g-1}_T,\mu_n)\to\mathrm{H}^2_{\text{ét}}(T,\mu_n)$.

Recall from (12) that a_0 is obtained by applying to $\tau_3\sigma_{F*}\pi_F^*(x)$ the corestriction map $\mathrm{H}^2_{\text{ét}}(T,\mu_n)\to\mathrm{H}^2(k,\mu_n)$ followed by the multiplication by r.

Let $y=\sigma_F^*\sigma_{F*}\pi_F^*(x)-\pi_F^*(x)\in\mathrm{H}^2_{\text{ét}}(Y'_F,\mu_n)$. We claim that for any closed point $i:\mathrm{Spec}(K)\hookrightarrow Y'_F$ we have

$$i^*(y)=0\in\mathrm{H}^2(K,\mu_n).$$

Indeed, $\sigma_{F*}\sigma_F^*=\mathrm{id}$ implies that $\sigma_{F*}(y)=0$ and hence y goes to 0 under the restriction map $\mathrm{H}^2_{\text{ét}}(Y'_F,\mu_n)\to\mathrm{H}^2_{\text{ét}}(Y_{F0},\mu_n)$. The natural injective map of étale sheaves $\mu_n\to\mathbb{G}_m$ gives rise to the canonical maps $\mathrm{H}^2_{\text{ét}}(Y'_F,\mu_n)\to\mathrm{Br}(Y'_F)$ and $\mathrm{H}^2_{\text{ét}}(Y_{F0},\mu_n)\to\mathrm{Br}(Y_{F0})$. By Grothendieck's purity theorem for the Brauer group [5, III, §6] the natural restriction map $\mathrm{Br}(Y'_F)\to\mathrm{Br}(Y_{F0})$ is injective. Hence the image of y in $\mathrm{Br}(Y'_F)$ is zero. On the other hand, the map $\mathrm{H}^2(K,\mu_n)\to\mathrm{Br}(K)$ is injective by Hilbert's Theorem 90. This implies $i^*(y)=0$.

In particular, we have $\rho^*\tau_2(y)=0$, hence $\rho^*\tau_2\pi_F^*(x)=\rho^*\tau_2\sigma_F^*\sigma_{F*}\pi_F^*(x)$. The commutativity of the right hand square of (15) and the fact that ρ is a section of the structure morphism $\mathbb{P}^{g-1}_T\to T$ imply that $\rho^*\tau_2\sigma_F^*=\tau_3$. Hence $\rho^*\tau_2\pi_F^*(x)=\tau_3\sigma_{F*}\pi_F^*(x)$. By the commutativity of the left hand square of (15) we have $\tau_2\pi_F^*(x)=\tau_1(x)$. Hence $\rho^*\tau_2\pi_F^*(x)=\rho^*\tau_1(x)$, which does not depend on F. We conclude that $\tau_3\sigma_{F*}\pi_F^*(x)$, and hence also a_0, do not depend on F. □

Now let k be a number field. We write $\mathbb{A}_k$ for the ring of adèles of k. If X is a proper variety over k we have $X(\mathbb{A}_k)=\prod X(k_v)$, where v ranges over all places of k. The Brauer–Manin pairing $X(\mathbb{A}_k)\times\mathrm{Br}(X)\to\mathbb{Q}/\mathbb{Z}$ is given by the sum of local invariants of class field theory, see [21, §5.2]. For a subgroup $B\subset\mathrm{Br}(X)$ we denote by $X(\mathbb{A}_k)^B\subset X(\mathbb{A}_k)$ the orthogonal complement to B under this pairing.

Theorem 3.3. *Let A be an abelian variety of dimension $g \geq 2$ over a number field k. Let X be the Kummer variety attached to a 2-covering of A such that $X(\mathbb{A}_k) \neq \emptyset$. Then $X(\mathbb{A}_k)^{\mathrm{Br}(X)_{\mathrm{odd}}} \neq \emptyset$, where $\mathrm{Br}(X)_{\mathrm{odd}} \subset \mathrm{Br}(X)$ is the subgroup of elements of odd order.*

Proof. By Corollary 2.8 the group $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. It follows that $\mathrm{Br}(X)_{\mathrm{odd}}$ is generated by finitely many elements modulo $\mathrm{Br}(X)_{\mathrm{odd}} \cap \mathrm{Br}_0(X)$. Hence there is an odd integer n such that the images of $\mathrm{Br}(X)_{\mathrm{odd}}$ and $\mathrm{Br}(X)[n]$ in $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ are equal. Since the sum of local invariants of an element of $\mathrm{Br}_0(X)$ is always zero, this implies that $X(\mathbb{A}_k)^{\mathrm{Br}(X)_{\mathrm{odd}}} = X(\mathbb{A}_k)^{\mathrm{Br}(X)[n]}$.

We have the natural maps

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2_{\text{ét}}(Y', \mathbb{G}_m) \xrightarrow{\sigma_*} \mathrm{H}^2_{\text{ét}}(Y, \mathbb{G}_m).$$

Here σ_* is the composition of the restriction to the open set $Y_0 \subset Y'$ and the inverse of the restriction $\mathrm{H}^2_{\text{ét}}(Y, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(Y_0, \mathbb{G}_m)$, which is an isomorphism by Grothendieck's purity theorem for the Brauer group [5, III, Cor. 6.2] as $\mathrm{codim}_Y(T) \geq 2$. These maps are compatible with the similar maps (14) with finite coefficients $\Lambda = \mu_n$. Now the Kummer sequences for X and Y give rise to the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2_{\text{ét}}(Y, \mu_n) \arrow[r] & \mathrm{Br}(Y)[n] \arrow[r] & 0 \\
\mathrm{H}^2_{\text{ét}}(X, \mu_n) \arrow[r] \arrow[u, "\sigma_*\pi^*"] & \mathrm{Br}(X)[n] \arrow[r] \arrow[u, "\sigma_*\pi^*"] & 0
\end{tikzcd}$$

The same considerations apply if we replace Y by any quadratic twist Y_F.

Take any $\mathcal{A} \in \mathrm{Br}(X)[n]$ and lift it to some $x \in \mathrm{H}^2_{\text{ét}}(X, \mu_n)$. By the commutativity of the previous diagram $\sigma_{F*}\pi_F^*(\mathcal{A}) \in \mathrm{Br}(Y_F)[n]$ comes from $\sigma_{F*}\pi_F^*(x) \in \mathrm{H}^2_{\text{ét}}(Y_F, \mu_n)$. By Proposition 3.2 there is $a_0 \in \mathrm{H}^2(k, \mu_n)$ such that $\sigma_{F*}\pi_F^*(x) - a_0 \in s(\mathrm{H}^2_{\text{ét}}(\overline{Y}_F, \mu_n)^\Gamma)$.

Now we can complete the proof of the theorem. Let $(P_v) \in X(\mathbb{A}_k)$. For each v there is a class $\alpha_v \in \mathrm{H}^1(k_v, \mu_2) = k_v^*/k_v^{*2}$ such that P_v lifts to a k_v-point on the quadratic twist $Y'_{k_v(\sqrt{\alpha_v})}$, which is a variety defined over k_v. By weak approximation in k we can assume that α_v comes from $\mathrm{H}^1(k, \mu_2) = k^*/k^{*2}$, and hence assume that $Y'_{k_v(\sqrt{\alpha_v})} \cong Y'_F \times_k k_v$ for some étale k-algebra F of dimension 2.

It follows that $Y'_F(k_v) \neq \emptyset$ and hence $Y_F(k_v) \neq \emptyset$. Since Y_F is smooth, the non-empty set $Y_F(k_v)$ is Zariski dense in Y_F. Thus there is a k_v-point

$R_v \in Y_F(k_v)$ such that the point $M_v = [n]R_v \in Y_F(k_v)$ is contained in the open subset Y_{F0}. The specialisation of $\mathcal{T}_n$ at M_v contains a k_v-point, hence is a trivial torsor. It follows that the specialisation of the class $\wedge^2[\mathcal{T}_n] \in \mathrm{H}^2(Y_F, \wedge^2 A[n])$ at M_v is zero. By the construction of the section s in the proof of Proposition 3.1 we obtain that $s(a) \in \mathrm{H}^2(Y_F, \mu_n)$ evaluated at M_v is zero for any a. Therefore, $\sigma_{F*}\pi_F^*(x)$ evaluated at M_v is the image of a_0 in $\mathrm{H}^2(k_v, \mu_n)$, and hence $\sigma_{F*}\pi_F^*(\mathcal{A})(M_v) \in \mathrm{Br}(k_v)$ comes from a global element $a_0 \in \mathrm{H}^2(k, \mu_n) = \mathrm{Br}(k)[n]$.

Since $M_v \in Y_{F0}(k_v)$ there exists a unique point $M'_v \in Y'_F(k_v)$ such that $\sigma_F(M'_v) = M_v$. Let $Q_v = \pi_F(M'_v) \in X(k_v)$. By the projection formula we have $\mathcal{A}(Q_v) = \sigma_{F*}\pi_F^*(\mathcal{A})(M_v)$. Since this is the image of $a_0 \in \mathrm{Br}(k)$ under the restriction map to $\mathrm{Br}(k_v)$, the sum of local invariants of $\mathcal{A}$ evaluated at the adelic point $(Q_v) \in X(\mathbb{A}_k)$ is zero. Thus $(Q_v) \in X(\mathbb{A}_k)^{\mathrm{Br}(X)[n]} = X(\mathbb{A}_k)^{\mathrm{Br}(X)_{\mathrm{odd}}}$. □

4. Kummer varieties attached to products of abelian varieties

For an abelian group G we denote by $G\{\ell\}$ the ℓ-primary subgroup of G.

Proposition 4.1. *Let k be a field and let ℓ be a prime different from the characteristic of k. Let $A_1, \ldots, A_n$ be principally polarised abelian varieties over k such that the fields $k(A_i[\ell])$ are pairwise linearly disjoint over k, where $i = 1, \ldots, n$. Assume that each Γ-module $A_i[\ell]$ is simple, and, moreover, if* $\dim(A_i) > 1$, *then it is absolutely simple. For any 2-covering Y of $A = \prod_{i=1}^n A_i$ we have* $\mathrm{Br}(Y)\{\ell\} \subset \mathrm{Br}_1(Y)$, *in particular,* $\mathrm{Br}(A)\{\ell\} \subset \mathrm{Br}_1(A)$. *If* $\mathrm{char}(k) = 0$ *and* $\dim(A) \geq 2$, *for the Kummer variety X attached to Y we have* $\mathrm{Br}(X)\{\ell\} \subset \mathrm{Br}_1(X)$.

Proof. Let m be a positive integer. The Kummer sequences for Y and $\overline{Y}$ give a commutative diagram of abelian groups with exact rows

$$\begin{tikzcd} 0 \arrow[r] & (\mathrm{NS}\,(\overline{Y})/\ell^m)^\Gamma \arrow[r] & \mathrm{H}^2(\overline{Y}, \mu_{\ell^m})^\Gamma \arrow[r] & \mathrm{Br}(\overline{Y})[\ell^m]^\Gamma & \\ 0 \arrow[r] & \mathrm{Pic}(Y)/\ell^m \arrow[r] \arrow[u] & \mathrm{H}^2(Y, \mu_{\ell^m}) \arrow[r] \arrow[u] & \mathrm{Br}(Y)[\ell^m] \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{16}$$

If $(\mathrm{NS}\,(\overline{Y})/\ell^m)^\Gamma \to \mathrm{H}^2(\overline{Y}, \mu_{\ell^m})^\Gamma$ is an isomorphism, then $\mathrm{H}^2(\overline{Y}, \mu_{\ell^m})^\Gamma \to \mathrm{Br}(\overline{Y})[\ell^m]^\Gamma$ is the zero map. In this case from the commutativity of the right hand square of (16) and the surjectivity of $\mathrm{H}^2(Y, \mu_{\ell^m}) \to \mathrm{Br}(Y)[\ell^m]$ we see that $\mathrm{Br}(Y)[\ell^m] \to \mathrm{Br}(\overline{Y})[\ell^m]^\Gamma$ is the zero map. This shows that $\mathrm{Br}(Y)[\ell^m]$ is contained in $\mathrm{Br}_1(Y)$ for any m, hence $\mathrm{Br}(Y)\{\ell\} \subset \mathrm{Br}_1(Y)$. In the particular case $Y = A$ we get $\mathrm{Br}(A)\{\ell\} \subset \mathrm{Br}_1(A)$.

The variety Y is obtained by twisting A by a cocycle with coefficients in $A[2]$ acting on A by translations. The argument used in the proof of Proposition 2.7 shows that the divisible group $A(\bar{k})$, which contains $A[2]$, acts trivially on the finite group $\mathrm{H}^2(\overline{A}, \mu_{\ell^m})$. Since $\mathrm{NS}\,(\overline{Y})$ is canonically isomorphic to $\mathrm{NS}\,(\overline{A})$ as a Γ-module, we have an isomorphism of Γ-modules $\mathrm{NS}\,(\overline{Y})/\ell^m \cong \mathrm{NS}\,(\overline{A})/\ell^m$ compatible with the cycle class map to $\mathrm{H}^2(\overline{Y}, \mu_{\ell^m}) \cong \mathrm{H}^2(\overline{A}, \mu_{\ell^m})$. Thus the injective map $(\mathrm{NS}\,(\overline{Y})/\ell^m)^\Gamma \to \mathrm{H}^2(\overline{Y}, \mu_{\ell^m})^\Gamma$ is the same as $(\mathrm{NS}\,(\overline{A})/\ell^m)^\Gamma \to \mathrm{H}^2(\overline{A}, \mu_{\ell^m})^\Gamma$. It remains to show that this last map is an isomorphism.

We have canonical isomorphisms of Γ-modules

$$\wedge^2_{\mathbb{Z}/\ell^m}(\oplus_{i=1}^n A_i[\ell^m]) = (\oplus_{i=1}^n \wedge^2_{\mathbb{Z}/\ell^m} A_i[\ell^m]) \oplus (\oplus_{i<j}(A_i[\ell^m] \otimes_{\mathbb{Z}/\ell^m} A_j[\ell^m]))$$

and

$$\mathrm{Hom}(A_i[\ell^m] \otimes_{\mathbb{Z}/\ell^m} A_j[\ell^m], \mu_{\ell^m}) = \mathrm{Hom}(A_i[\ell^m], \mathrm{Hom}(A_j[\ell^m], \mu_{\ell^m})).$$

Since each A_i is principally polarised, the Γ-modules $\mathrm{Hom}(A_j[\ell^m], \mu_{\ell^m}) = A_j^t[\ell^m]$ and $A_j[\ell^m]$ are isomorphic. Hence the Γ-module

$$\mathrm{H}^2(\overline{A}, \mu_{\ell^m}) = \wedge^2_{\mathbb{Z}/\ell^m}\mathrm{H}^1(\overline{A}, \mathbb{Z}/\ell^m)(1) = \mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell^m}A[\ell^m], \mu_{\ell^m}) \tag{17}$$

is isomorphic to the Γ-module

$$\bigoplus_{i=1}^n \mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell^m}(A_i[\ell^m]), \mu_{\ell^m}) \oplus \bigoplus_{i<j} \mathrm{Hom}(A_i[\ell^m], A_j[\ell^m]). \tag{18}$$

For $i \neq j$ the Γ-modules $A_i[\ell]$ and $A_j[\ell]$ are simple and non-isomorphic, hence $\mathrm{Hom}_\Gamma(A_i[\ell], A_j[\ell]) = 0$. We claim that $\mathrm{Hom}_\Gamma(A_i[\ell^m], A_j[\ell^m]) = 0$ for any $m \geq 1$ when $i \neq j$. For $m > 1$ the exact sequence of Γ-modules

$$0 \longrightarrow A_i[\ell] \longrightarrow A_i[\ell^m] \xrightarrow{[\ell]} A_i[\ell^{m-1}] \longrightarrow 0,$$

gives rise to an exact sequence of $\mathbb{Z}/\ell^m$-modules

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\Gamma(A_i[\ell^{m-1}], A_j[\ell^m]) &\longrightarrow \mathrm{Hom}_\Gamma(A_i[\ell^m], A_j[\ell^m]) \\ &\longrightarrow \mathrm{Hom}_\Gamma(A_i[\ell], A_j[\ell^m]). \end{aligned}$$

It is clear that

$$\mathrm{Hom}_\Gamma(A_i[\ell^{m-1}], A_j[\ell^m]) = \mathrm{Hom}_\Gamma(A_i[\ell^{m-1}], A_j[\ell^{m-1}]),$$

and

$$\mathrm{Hom}_\Gamma(A_i[\ell], A_j[\ell^m]) = \mathrm{Hom}_\Gamma(A_i[\ell], A_j[\ell]).$$

We obtain an exact sequence

$$\begin{aligned}&0 \to \mathrm{Hom}_\Gamma(A_i[\ell^{m-1}], A_j[\ell^{m-1}]) \to \mathrm{Hom}_\Gamma(A_i[\ell^m], A_j[\ell^m])\\ &\quad\to \mathrm{Hom}_\Gamma(A_i[\ell], A_j[\ell]).\end{aligned} \tag{19}$$

The induction assumption now implies $\mathrm{Hom}_\Gamma(A_i[\ell^m], A_j[\ell^m]) = 0$ when $i \neq j$.

If $\dim(A_i) = 1$, then $\mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell^m}(A_i[\ell^m]), \mu_{\ell^m})$ is the trivial Γ-module $\mathbb{Z}/\ell^m$.

Now assume $\dim(A_i) > 1$. Since the Γ-modules $\mathrm{Hom}(A_i[\ell^m], \mu_{\ell^m})$ and $A_i[\ell^m]$ are isomorphic, the Γ-module $\mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell^m} A_i[\ell^m], \mu_{\ell^m})$ is a submodule of $\mathrm{End}(A_i[\ell^m])$. Since the Γ-module $A_i[\ell]$ is absolutely simple, we have $\mathrm{End}_\Gamma(A_i[\ell]) = \mathbb{F}_\ell \cdot \mathrm{Id}$. We claim that $\mathrm{End}_\Gamma(A_i[\ell^m]) = \mathbb{Z}/\ell^m \cdot \mathrm{Id}$ for any $m \geq 1$. We argue by induction in m and assume that

$$\mathrm{End}_\Gamma(A_i[\ell^{m-1}]) = \mathbb{Z}/\ell^{m-1} \cdot \mathrm{Id}.$$

In particular, the order of $\mathrm{End}_\Gamma(A_i[\ell^{m-1}])$ equals ℓ^{m-1}. The exact sequence (19) in the case $i = j$ implies that the order of $\mathrm{End}_\Gamma(A_i[\ell^m])$ divides $\ell^{m-1} \cdot \ell = \ell^m$. However, $\mathrm{End}_\Gamma(A_i[\ell^m])$ contains the subgroup $\mathbb{Z}/\ell^m \cdot \mathrm{Id}$ of order ℓ^m. This implies that $\mathrm{End}_\Gamma(A_i[\ell^m]) = \mathbb{Z}/\ell^m \cdot \mathrm{Id}$, which proves our claim.

From (17) and (18) we now conclude that $\mathrm{H}^2(\overline{A}, \mu_{\ell^m})^\Gamma \subset (\mathbb{Z}/\ell^m)^n$.

The principal polarisation of each A_i defines a non-zero class in $\mathrm{NS}\,(\overline{A}_i)^\Gamma$. It is well known that the Γ-module $\oplus_{i=1}^n \mathrm{NS}\,(\overline{A}_i)$ is a direct summand of $\mathrm{NS}\,(\overline{A})$; see, e.g. [26, Prop. 1.7]. Hence $\mathrm{NS}\,(\overline{A})$ contains the trivial Γ-module $\mathbb{Z}^n$ as a full sublattice. Thus $(\mathrm{NS}\,(\overline{A})/\ell^m)^\Gamma$ contains a subgroup isomorphic to $(\mathbb{Z}/\ell^m)^n$. It follows that the map $(\mathrm{NS}\,(\overline{A})/\ell^m)^\Gamma \to \mathrm{H}^2(\overline{A}, \mu_{\ell^m})^\Gamma$ is an isomorphism for any $m \geq 1$.

We have proved that $\mathrm{Br}(Y)\{\ell\} \subset \mathrm{Br}_1(Y)$. An equivalent statement is that the natural map $\mathrm{Br}(Y)\{\ell\} \to \mathrm{Br}(\overline{Y})$ is zero. In the characteristic zero case Remark 2 in Section 2 implies that the natural map $\mathrm{Br}(X)\{\ell\} \to \mathrm{Br}(\overline{X})$ is zero. Equivalently, $\mathrm{Br}(X)\{\ell\} \subset \mathrm{Br}_1(X)$. □

Under additional assumptions we can prove a bit more.

Proposition 4.2. *Let k be a field of characteristic 0. Let ℓ be a prime. Let $A_1, \ldots, A_n$ be principally polarised abelian varieties over k satisfying the following conditions.*

(a) *The fields $k(A_i[\ell])$, where $i = 1, \ldots, n$, are linearly disjoint over k.*

(b) *The* Γ*-module* $A_i[\ell]$ *is absolutely simple for each* $i=1,\ldots,n$.

(c) $\mathrm{NS}\,(\overline{A}_i)\cong\mathbb{Z}$ *for each* $i=1,\ldots,n$.

(d) *For each* $i=1,\ldots,n$ *the group* $\mathrm{Gal}(k(A_i[\ell])/k)$ *contains a subgroup* H_i *such that* H_i *has no normal subgroup of index* ℓ*, the* H_i*-module* $A_i[\ell]$ *is simple, and, moreover, the* H_i*-module* $A_i[\ell]$ *is absolutely simple if* $\dim(A_i)>1$.

Let $A=\prod_{i=1}^n A_i$*. Then* $\mathrm{Br}(\overline{A})[\ell]^\Gamma=0$*. When* $\dim(A)\geq 2$*, for the Kummer variety* X *attached to a* 2*-covering of* A *we have* $\mathrm{Br}(\overline{X})[\ell]^\Gamma=0$.

Proof. We claim that $\mathrm{NS}\,(\overline{A})\cong\oplus_{i=1}^n\mathrm{NS}\,(\overline{A}_i)$. It is well known that this is equivalent to the condition $\mathrm{Hom}(\overline{A}_i,\overline{A}_j)=0$ for all $i\neq j$, see, e.g. [26, Prop. 1.7]. In view of (a) and (b) this condition holds by [33, Thm. 2.1]. Now (c) implies that $\mathrm{NS}\,(\overline{A})$ is isomorphic to the trivial Γ-module $\mathbb{Z}^n$.

The properties $\mathrm{Br}(\overline{A})[\ell]^\Gamma=0$ and $\mathrm{Br}(\overline{X})[\ell]^\Gamma=0$ can be proved over any extension k' of k contained in $\bar{k}$. Let k' be the compositum of $k(A_i[\ell])^{H_i}$ for $i=1,\ldots,n$, and let $H=\prod_{i=1}^n H_i$. Then $\mathrm{Gal}(k'(A[\ell])/k')=H$ and the fields $k'(A_i[\ell])$ are linearly disjoint over k'. By assumption (d) each $A_i[\ell]$ is a simple $\mathrm{Gal}(\bar{k}/k')$-module and is absolutely simple whenever $\dim(A_i)>1$. Thus the assumptions of Proposition 4.1 are satisfied for the abelian varieties $A_1,\ldots,A_n$ over k'. In the rest of the proof we write k for k' and Γ for $\mathrm{Gal}(\bar{k}/k')$.

The Kummer sequence gives an exact sequence of Γ-modules

$$0\longrightarrow\mathrm{NS}\,(\overline{A})/\ell\longrightarrow\mathrm{H}^2(\overline{A},\mu_\ell)\longrightarrow\mathrm{Br}(\overline{A})[\ell]\longrightarrow 0. \tag{20}$$

In view of (17), Γ acts on the terms of (20) via its quotient H. In particular, $\mathrm{Br}(\overline{A})[\ell]^\Gamma=\mathrm{Br}(\overline{A})[\ell]^H$. We obtain an exact sequence of cohomology groups of H:

$$0\to(\mathrm{NS}\,(\overline{A})/\ell)^H\to\mathrm{H}^2(\overline{A},\mu_\ell)^H\to\mathrm{Br}(\overline{A})[\ell]^H\to\mathrm{H}^1(H,\mathrm{NS}\,(\overline{A})/\ell). \tag{21}$$

The proof of Proposition 4.1 shows that the second arrow in (21) is an isomorphism. Since $\mathrm{NS}\,(\overline{A})/\ell$ is the trivial H-module $(\mathbb{F}_\ell)^n$, we have

$$\mathrm{H}^1(H,\mathrm{NS}\,(\overline{A})/\ell)=\mathrm{Hom}(H,(\mathbb{F}_\ell)^n)=0,$$

because by assumption H has no normal subgroup of index ℓ. We conclude that $\mathrm{Br}(\overline{A})[\ell]^\Gamma=\mathrm{Br}(\overline{A})[\ell]^H=0$. The second claim follows from Proposition 2.7. □

Note that the condition that H has no normal subgroup of index ℓ cannot be removed. See the remark on [25, p. 20] for an example of an abelian surface

A for which a Galois-invariant element in $\mathrm{Br}(\overline{A})[2]$ does not come from a Galois-invariant element of $\mathrm{H}^2(\overline{A},\mu_2)$. (In this example $H=\mathrm{GL}(2,\mathbb{F}_2)=\mathbf{S}_3$, the symmetric group on three letters.)

Here is one of the main results of this paper.

Theorem 4.3. *Let k be a field of characteristic zero. Let $A_1,\ldots,A_n$ be principally polarised abelian varieties over k such that for $i=1,\ldots,n$ we have $\mathrm{NS}\,(\overline{A}_i)\cong\mathbb{Z}$, the Γ-modules $A_i[2]$ are absolutely simple, the fields $k(A_i[2])$ are linearly disjoint over k, and $\mathrm{H}^1(G_i,A_i[2])=0$, where $G_i=\mathrm{Gal}(k(A_i[2])/k)$. Let $A=\prod_{i=1}^n A_i$. If $g=\dim(A)\geq 2$, then for the Kummer variety X attached to any 2-covering of A we have the following isomorphisms of abelian groups:*

(i) $\mathrm{Pic}(\overline{X})\cong\mathbb{Z}^{2^{2g}+n}$;

(ii) $\mathrm{Br}(X)\{2\}=\mathrm{Br}_1(X)\{2\}$;

(iii) $\mathrm{Br}_1(X)=\mathrm{Br}_0(X)$.

Proof. Let Y be the 2-covering of A to which X is attached. It is clear that $Y=\prod_{i=1}^n Y_i$, where Y_i is a 2-covering of A_i for $i=1,\ldots,n$. Using the principal polarisation of A_i we identify A_i with its dual abelian variety A_i^t. By [33, Thm. 2.1] we have $\mathrm{Hom}(\overline{A}_i,\overline{A}_j)=0$ for any $i\neq j$. A well known consequence of this (see e.g., [26, Prop. 1.7]) gives canonical isomorphisms of Γ-modules

$$\mathrm{Pic}(\overline{Y})\cong\bigoplus_{i=1}^n\mathrm{Pic}(\overline{Y}_i),\qquad \mathrm{NS}\,(\overline{Y})\cong\bigoplus_{i=1}^n\mathrm{NS}\,(\overline{Y}_i)\cong\mathbb{Z}^n, \tag{22}$$

where Γ acts trivially on $\mathbb{Z}^n$. Now (i) follows from the exact sequence (8).

Part (ii) follows from Proposition 4.1, so it remains to establish part (iii).

For each $i=1,\ldots,n$ we have $Y_i=(A_i\times_k T_i)/A_i[2]$, where T_i is a torsor for $A_i[2]$, so that $Y=(A\times_k T)/A[2]$ with $T=\prod_{i=1}^n T_i$. Write $K_i=k(A_i[2])$ for the field of definition of the 2-torsion subgroup of A_i so that $G_i=\mathrm{Gal}(K_i/k)$. Let $k(T_i)$ be the smallest subfield of $\bar{k}$ over which all $\bar{k}$-points of T_i are defined. Then Γ acts on $T_i(\bar{k})\cong A_i[2]$ through $\mathrm{Gal}(k(T_i)/k)$.

If T_i is a trivial torsor, then $T_i\cong A_i[2]$ and $\mathrm{Gal}(k(T_i)/k)=G_i$. If T_i is a non-trivial torsor, in our assumptions [6, Prop. 3.6] gives us that $\mathrm{Gal}(k(T_i)/k)=A_i[2]\rtimes G_i$, where $A_i[2]$ acts on itself by translations and G_i acts on $A_i[2]$ by linear transformations.

The following lemma is a version of [6, Prop. 3.12].

Lemma 4.4. *In the assumptions of Theorem 4.3 the Galois extensions $k(T_1),\ldots,k(T_n)$ of k are linearly disjoint over k.*

Proof. Let $m \leq n$ be the cardinality of $I \subseteq \{1, \ldots, n\}$ such that T_i is a non-trivial torsor if and only if $i \in I$. We proceed by double induction in $n \geq 1$ and $m \geq 0$. The statement holds when $n = 1$ (trivially) or when $m = 0$ (by assumption). Suppose that $n \geq 2$ and $m \geq 1$ and the statement is proved for $(n, m-1)$ and for $(n-1, m-1)$. Without loss of generality we can assume that T_n is non-trivial. By inductive assumption for $(n-1, m-1)$ the fields $k(T_i)$, $i = 1, \ldots, n-1$, are linearly disjoint over k. Let L be the compositum of these fields, and let $E = L \cap k(T_n)$. Each field $k(T_i)$ is Galois over k. To check our statement it is enough to show that $E = k$. By [6, Cor. 3.9] the fact that T_n is non-trivial implies that $E \subset K_n$ or $K_n \subset E$. By inductive assumption for $(n, m-1)$ we have $L \cap K_n = k$. Thus $E \subset K_n$ implies $E = k$. On the other hand, $K_n \subset E$ implies $K_n = k$, which is incompatible with our assumption that $A_n[2]$ is a simple Γ-module. $\square$

Without loss of generality we can assume that T_i is non-trivial for $i = 1, \ldots, m$ and T_i is trivial for $i = m+1, \ldots, n$. Lemma 4.4 implies that the image of the action of Γ on $T(\bar{k}) = \prod_{i=1}^{n} T_i(\bar{k})$ is the direct product

$$P = \prod_{i=1}^{m} (A_i[2] \rtimes G_i) \times \prod_{i=m+1}^{n} G_i.$$

Write $G = \prod_{i=1}^{n} G_i$. If we define $B = \prod_{i=1}^{m} A_i$, then $P = B[2] \rtimes G$.

To prove the desired property $\mathrm{Br}_1(X) = \mathrm{Br}_0(X)$ it is enough to prove that $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) = 0$. The abelian groups in the exact sequence (8) are torsion-free, hence

$$\mathrm{Pic}(\overline{X}) \subset \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q} \cong \mathbb{Q}[T] \oplus (\mathrm{NS}\,(\overline{Y}) \otimes \mathbb{Q}) \cong \mathbb{Q}[T] \oplus \mathbb{Q}^n,$$

where $\mathbb{Q}[T]$ is the vector space with basis $T(\bar{k})$ and a natural action of Γ. It follows that the image of the action of Γ on $\mathrm{Pic}(\overline{X})$ is P. Thus it is enough to prove that

$$\mathrm{H}^1(P, \mathrm{Pic}(\overline{X})) = 0. \tag{23}$$

As an abelian group, Π_1 is generated by $\mathbb{Z}[T]$ and one half of the sum of the canonical generators of $\mathbb{Z}[T]$. This gives an exact sequence of $A[2] \rtimes G$-modules

$$0 \longrightarrow \mathbb{Z}[T] \longrightarrow \Pi_1 \longrightarrow \mathbb{Z}/2 \longrightarrow 0. \tag{24}$$

By Shapiro's lemma $\mathrm{H}^1(P, \mathbb{Z}[T]) = 0$, because $\mathbb{Z}[T]$ is a permutation P-module. The cohomology exact sequences of (24) considered with respect to the action of P and G give rise to the following commutative diagram with exact upper row, where the vertical arrows are given by restriction to the subgroup $G \subset P$:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(P, \Pi_1) \arrow[r] \arrow[d] & \mathrm{Hom}(P, \mathbb{Z}/2) \arrow[d] \\ & 0 = \mathrm{H}^1(G, \Pi_1) \arrow[r] & \mathrm{Hom}(G, \mathbb{Z}/2) \end{tikzcd} \tag{25}$$

Here $\mathrm{H}^1(G, \Pi_1) = 0$, as Π_1 is a permutation G-module, see Remark 1 in Section 2. For $i = 1, \ldots, m$ we see from [6, Lemma 3.2 (ii)] that any subgroup of $A_i[2] \rtimes G_i$ of index 2 has the form $A_i[2] \rtimes H$ for a subgroup $H \subset G_i$ of index 2. Hence the right vertical arrow in (25) is an isomorphism. The commutativity of (25) now implies that $\mathrm{H}^1(P, \Pi_1) = 0$. The exact sequence of the cohomology groups of P defined by the middle column of (9) shows that to prove (23) it is enough to prove that $\mathrm{H}^1(P, \mathrm{Pic}(\overline{Y})^{\iota_Y}) = 0$.

In view of the decomposition (22), we must prove that for each $i = 1, \ldots, n$ we have $\mathrm{H}^1(P, \mathrm{Pic}(\overline{Y}_i)^{\iota_Y}) = 0$. For Y_i the exact sequence (2) takes the form

$$0 \longrightarrow A_i^t[2] \longrightarrow \mathrm{Pic}(\overline{Y}_i)^{\iota_Y} \longrightarrow \mathrm{NS}\,(\overline{A}_i) \longrightarrow 0. \tag{26}$$

Since $\mathrm{NS}\,(\overline{A}_i) \cong \mathbb{Z}$ we have $\mathrm{H}^1(P, \mathrm{NS}\,(\overline{A}_i)) = 0$.

We first consider the case when T_i is a trivial torsor. By assumption $\mathrm{H}^1(G_i, A_i[2]) = 0$. We have $A_i[2]^{G_i} = 0$, because $A_i[2]$ is a simple G_i-module with a non-trivial action of G_i. The restriction-inflation sequence for the normal subgroup $G_i \subset P$ acting on $A_i[2]$ shows that $\mathrm{H}^1(P, A_i[2]) = 0$, hence $\mathrm{H}^1(P, \mathrm{Pic}(\overline{Y}_i)^{\iota_Y}) = 0$.

Now suppose that the torsor T_i is non-trivial. The Galois group Γ acts on $A_i[2]$ via its image G_i, hence so does $\mathrm{Gal}(k(T_i)/k) = A_i[2] \rtimes G_i$. We have $\mathrm{H}^1(A_i[2] \rtimes G_i, A_i[2]) = \mathbb{F}_2$, see [6, Prop. 3.6]. This group is naturally a subgroup of $\mathrm{H}^1(k, A_i[2])$ and contains the class $[T_i]$, because this class goes to zero under the restriction map $\mathrm{H}^1(k, A_i[2]) \to \mathrm{H}^1(k(T_i), A_i[2])$. Thus $[T_i]$ is the unique non-zero element of $\mathrm{H}^1(A_i[2] \rtimes G_i, A_i[2])$.

Using the fact that $A_i[2]^{G_i} = 0$, the Hochschild–Serre spectral sequence for the normal subgroup $A_i[2] \rtimes G_i \subset P$ gives $\mathrm{H}^1(P, A_i[2]) = \mathbb{F}_2$. The same argument as above shows that $[T_i]$ is the unique non-zero element of this group.

The principal polarisation $\lambda \in \mathrm{NS}\,(\overline{A}_i)^\Gamma = \mathrm{NS}\,(\overline{A}_i)$ gives rise to an isomorphism $\varphi_\lambda : A_i \tilde{\longrightarrow} A_i^t$ which induces an isomorphism of Γ-modules

$\varphi_{\lambda *} : A_i[2] \xrightarrow{\sim} A_i^t[2]$. Since the Γ-modules $\mathrm{NS}\,(\overline{Y}_i)$ and $\mathrm{NS}\,(\overline{A}_i)$ are canonically isomorphic, we can think of λ as a generator of the trivial Γ-module $\mathrm{NS}\,(\overline{Y}_i) \cong \mathbb{Z}$.

Consider the exact sequence (26) as a sequence of P-modules. We claim that the differential $\mathrm{NS}\,(\overline{Y}_i) \to \mathrm{H}^1(P, A_i^t[2])$ sends the principal polarisation λ to $\varphi_{\lambda *}[T_i]$, so this differential is surjective. This implies that the first map in the exact sequence

$$\mathrm{H}^1(P, A_i^t[2]) \longrightarrow \mathrm{H}^1(P, \mathrm{Pic}(\overline{Y}_i)^{\iota}) \longrightarrow \mathrm{H}^1(P, \mathbb{Z}) = 0$$

is zero, hence $\mathrm{H}^1(P, \mathrm{Pic}(\overline{Y}_i)^{\iota}) = 0$.

To finish the proof of the theorem it remains to justify our claim. In the particular case of a trivial 2-covering the exact sequence of Γ-modules (2) takes the form

$$0 \longrightarrow A_i^t[2] \longrightarrow \mathrm{Pic}(\overline{A}_i)^{[-1]^*} \longrightarrow \mathrm{NS}\,(\overline{A}_i) \longrightarrow 0. \tag{27}$$

Following [19] we shall write c_λ for the image of λ under the differential $\mathrm{NS}\,(\overline{A}_i)^{\Gamma} \to \mathrm{H}^1(k, A_i^t[2])$ attached to (27). By [19, Lemma 3.6 (a)] we know that c_λ lies in the kernel of the restriction map $\mathrm{H}^1(k, A_i^t[2]) \to \mathrm{H}^1(K_i, A^t[2])$. We have $G_i = \mathrm{Gal}(K_i/k)$, so the restriction-inflation sequence shows that c_λ belongs to the subgroup $\mathrm{H}^1(G_i, A_i^t[2]) \subset \mathrm{H}^1(k, A_i^t[2])$. However, our assumptions imply that this group is zero, hence $c_\lambda = 0$. Thus λ is the image of some Γ-invariant $\mathcal{L} \in \mathrm{Pic}(\overline{A}_i)^{[-1]^*}$, hence (27) is a *split* exact sequence of Γ-modules.

The exact sequence (26) is obtained by twisting the exact sequence (27) by a 1-cocycle $\tau : \Gamma \to A_i[2]$ representing $[T_i] \in \mathrm{H}^1(k, A_i[2])$. By the definition of φ_λ the translation by $x \in A_i(\bar{k})$ acts on $\mathrm{Pic}(\overline{A}_i)$ by sending $y \in \mathrm{Pic}(\overline{A}_i)$ to $y + \varphi_\lambda(x)$. Since Y_i is the twist of A_i by τ with respect to the action of $A_i[2]$ by translations, we see that $g \in \Gamma$ acts on $\mathcal{L}$, understood as an element of $\mathrm{Pic}(\overline{Y}_i)$, by sending it to $\mathcal{L} + \varphi_{\lambda *}(\tau(g))$. By a standard explicit description of the differential $\mathrm{NS}\,(\overline{Y}_i) \to \mathrm{H}^1(k, A_i^t[2])$ we see that λ goes to $\varphi_{\lambda *}[T_i]$, as claimed. □

Remark 4 If k is finitely generated over $\mathbb{Q}$, then $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite by Corollary 2.8. The 2-primary subgroup of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is the image of the 2-primary subgroup of $\mathrm{Br}(X)$, and hence Theorem 4.3 implies that the order of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is odd.

Corollary 4.5. *Let k be a number field and let X be a Kummer variety satisfying the assumptions of Theorem 4.3. If $X(\mathbb{A}_k) \neq \emptyset$, then $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.*

Proof. In view of Remark 4 this is a formal consequence of Theorems 3.3 and 4.3. $\square$

This corollary explains the absence of the Brauer–Manin obstruction from the statement of the (conditional) Hasse principle for Kummer varieties recently established in [6, Thm. 2.2], once we impose the additional condition $\mathrm{NS}\,(\overline{A}_i)\cong\mathbb{Z}$ for $i=1,\ldots,n$. In the next section we give examples related to hyperelliptic curves where this condition holds.

5. Kummer varieties attached to products of Jacobians of hyperelliptic curves

In this section we consider the case when each factor of $A=\prod_{i=1}^n A_i$ is the Jacobian of a hyperelliptic curve given by a polynomial of odd degree ≥ 3 with a large Galois group. It will be convenient to include elliptic curves as a particular case of hyperelliptic curves, so we shall adopt this terminology here without further mention.

We write $\mathbf{S}_n$ for the symmetric group on n letters, and $\mathbf{A}_n\subset\mathbf{S}_n$ for the alternating group on n letters.

Theorem 5.1. *Let k be a field of characteristic zero. Let A be the product of Jacobians of the hyperelliptic curves $y^2=f_i(x)$, where $f_i(x)\in k[x]$ is a separable polynomial of odd degree $d_i\geq 5$ with Galois group $\mathbf{S}_{d_i}$ or $\mathbf{A}_{d_i}$, or a separable polynomial of degree 3 with Galois group $\mathbf{S}_3$, for $i=1,\ldots,n$. Assume that $g=\sum_{i=1}^n(d_i-1)/2\geq 2$ and the splitting fields of the polynomials $f_i(x)$, $i=1,\ldots,n$, are linearly disjoint over k. Then the conclusions of Theorem 4.3 hold for the Kummer variety X attached to any 2-covering of the abelian variety A. Moreover, $\mathrm{Br}(\overline{X})[2]^\Gamma=0$.*

Proof. For $i=1,\ldots,n$ let C_i be the smooth and projective curve given by the equation $y^2=f_i(x)$ and let A_i be the Jacobian of C_i. Let $A=\prod_{i=1}^n A_i$, and let Y be a 2-covering of A such that X is the Kummer variety attached to Y.

Since A_i is canonically principally polarised, we have an isomorphism $A_i\xrightarrow{\sim}A_i^t$. It is well known that $\mathrm{NS}\,(\overline{A}_i)$ is isomorphic to the subgroup of self-dual endomorphisms in $\mathrm{End}(\overline{A}_i)=\mathrm{Hom}(\overline{A}_i,\overline{A}_i^t)$. If $\deg(d_i)\geq 5$, by [32, Thm. 2.1] we have $\mathrm{End}(\overline{A}_i)\cong\mathbb{Z}$, hence $\mathrm{NS}\,(\overline{A}_i)\cong\mathbb{Z}$. If $\deg(d_i)=3$, then we obviously have $\mathrm{NS}\,(\overline{A}_i)\cong\mathbb{Z}$.

Let $W_i\subset C_i$ be the subscheme given by $f_i(x)=0$. The double covering $C_i\to\mathbb{P}^1_k$ is ramified precisely at the $\bar{k}$-points of $W_i\cup\{\infty\}$. It is well known that the Γ-module $A_i[2]$ is isomorphic to the zero sum subspace of the $\mathbb{F}_2$-vector space with basis $W_i(\bar{k})$, with the action of Γ defined by the natural action of Γ on $W(\bar{k})$. In particular, the splitting field of $f_i(x)$ is $k(A_i[2])$, and the Galois group of $f_i(x)$ is $G_i = \mathrm{Gal}(k(A_i[2])/k)$. This implies that the fields $k(A_i[2])$ are linearly disjoint over k. Since d_i is odd, the permutation G_i-module $\mathbb{F}_2^{d_i}$ whose canonical generators are given by the $\bar{k}$-points of W, is the direct sum $\mathbb{F}_2 \oplus A_i[2]$. We note that for $d_i \geq 5$ the standard representation of $\mathbf{A}_{d_i} \subset \mathrm{Sp}(d_i - 1, \mathbb{F}_2)$ in $\mathbb{F}_2^{d_i-1}$ is absolutely irreducible [15] (see also [32, Lemma 5.2]). The same is true if we replace $\mathbf{A}_{d_i}$ by $\mathbf{S}_{d_i}$. If $d_i = 3$, then the standard 2-dimensional representation of $\mathbf{S}_3$ is absolutely irreducible [15]. This implies that in all our cases $\mathrm{End}_{G_i}(A_i[2]) = \mathbb{F}_2$, cf. [32, Thm. 5.3]. We conclude that the Γ-module $A_i[2]$ is absolutely simple, for $i = 1, \ldots, n$.

By Shapiro's lemma we have $\mathrm{H}^1(\mathbf{A}_{d_i}, \mathbb{F}_2[\mathbf{A}_{d_i}/\mathbf{A}_{d_i-1}]) = \mathrm{H}^1(\mathbf{A}_{d_i-1}, \mathbb{F}_2) = 0$ for $d_i \geq 5$, because $\mathbf{A}_{d_i-1}$ is generated by the elements of order 3, and so contains no subgroup of index 2. Hence $\mathrm{H}^1(\mathbf{A}_{d_i}, A_i[2]) = 0$. Similarly, we have $\mathrm{H}^1(\mathbf{S}_{d_i}, \mathbb{F}_2[\mathbf{S}_{d_i}/\mathbf{S}_{d_i-1}]) = \mathrm{H}^1(\mathbf{S}_{d_i-1}, \mathbb{F}_2) = \mathbb{F}_2$ for $d_i \geq 3$, because $\mathbf{A}_{d_i-1}$ is the unique subgroup of $\mathbf{S}_{d_i-1}$ of index 2. This implies $\mathrm{H}^1(\mathbf{S}_{d_i}, A_i[2]) = 0$ (cf. [6, Lemma 2.1]). Thus $\mathrm{H}^1(G_i, A_i[2]) = 0$ for all $i = 1, \ldots, n$.

We have checked that all the assumptions of Theorem 4.3 are satisfied. In particular, conditions (a), (b), (c) of Proposition 4.2 are satisfied. Condition (d) is also satisfied if we take $H_i = \mathbf{A}_{d_i}$ for $i = 1, \ldots, n$. Indeed, each $A_i[2]$ is a simple $\mathbf{A}_{d_i}$-module for all odd $d_i \geq 3$ and is absolutely simple if $d_i \geq 5$. Finally, $\mathbf{A}_{d_i}$ has no subgroup of index 2 as it is generated by the elements of order 3. An application of Proposition 4.2 gives that $\mathrm{Br}(\overline{X})[2]^\Gamma = 0$. □

Corollary 5.2. *Let k be a number field. Let A be the product of Jacobians of the hyperelliptic curves $y^2 = f_i(x)$, where $f_i(x) \in k[x]$ is a separable polynomial of odd degree $d_i \geq 5$ with Galois group $\mathbf{S}_{d_i}$ or $\mathbf{A}_{d_i}$, or of degree 3 with Galois group $\mathbf{S}_3$, for $i = 1, \ldots, n$. Assume that $g = \sum_{i=1}^n (d_i - 1)/2 \geq 2$ and the splitting fields of the polynomials $f_i(x)$, $i = 1, \ldots, n$, are linearly disjoint over k. If the Kummer variety X attached to a 2-covering of A is everywhere locally soluble, then $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.*

Proof. In view of Remark 4 at the end of Section 4 this is a formal consequence of Theorems 3.3 and 5.1. □

Example 1 L. Dieulefait shows in [3, Thm. 5.8] that for $k = \mathbb{Q}$ and $f(x) = x^5 - x + 1$ the image of the Galois group $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Aut}(A[\ell])$, where A is the Jacobian of the hyperelliptic curve $y^2 = f(x)$, is $\mathrm{GSp}(4, \mathbb{F}_\ell)$ for each prime $\ell \geq 3$. (In [3] this result was conditional on the Serre conjectures [20], which have been later proved by C. Khare and J.-P. Wintenberger [11].) A verification with `magma` gives that the Galois group of x^5-x+1 is $\mathbf{S}_5$, so Theorem 5.1 can be applied. Thus for the Kummer surface X attached to a 2-covering of A we have $\mathrm{Br}(\overline{X})[2]^\Gamma=0$ and $\mathrm{Br}_1(X)=\mathrm{Br}_0(X)$. On the other hand, Proposition 4.2 can be applied for each prime $\ell\geq 3$ with $H=\mathrm{Sp}(4,\mathbb{F}_\ell)$. Indeed, for $\ell\geq 3$ the group $\mathrm{PSp}(4,\mathbb{F}_\ell)$ is simple non-abelian [1, Thm. 5.2, p. 177] of order $\ell^4(\ell^4-1)(\ell^2-1)/2>\ell$, so H contains no normal subgroups of index ℓ. The tautological representation of $\mathrm{Sp}(4,\mathbb{F}_\ell)$ is well known to be absolutely irreducible. We obtain that the Kummer surface X attached to a 2-covering of A is a K3 surface of geometric Picard rank 17 such that $\mathrm{Br}(\overline{X})^\Gamma=0$. Hence $\mathrm{Br}(X)=\mathrm{Br}_0(X)$.

Example 2 R. Jones and J. Rouse consider the Jacobian A of the curve of genus 2 given by $y^2=f(x)$, where $f(x)=4x^6-8x^5+4x^4+4x^2-8x+5$ is a polynomial over $\mathbb{Q}$ with Galois group $\mathbf{S}_6$ and discriminant quadratic extension $\mathbb{Q}(\sqrt{-3\cdot 13\cdot 31})$, see [10, Example 6.4, pp. 787–788]. They show that the image of $\Gamma=\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Aut}(A[\ell])$ is $\mathrm{GSp}(4,\mathbb{F}_\ell)$ for all primes ℓ. For odd ℓ the only non-trivial isomorphic quotients of $\mathrm{GSp}(4,\mathbb{F}_2)\cong\mathbf{S}_6$ and $\mathrm{GSp}(4,\mathbb{F}_\ell)$ are cyclic groups of order 2, namely $\mathbf{S}_6/\mathbf{A}_6$ and $\mathrm{GSp}(4,\mathbb{F}_\ell)/(\mathbb{F}_\ell^{*2}\cdot\mathrm{Sp}(4,\mathbb{F}_\ell))$, respectively. By Goursat's lemma a subgroup of $\mathbf{S}_6\times\mathrm{GSp}(4,\mathbb{F}_\ell)$ that maps surjectively onto each factor is either the whole product or the inverse image of the graph of the unique isomorphism

$$\mathbf{S}_6/\mathbf{A}_6 \xrightarrow{\sim} \mathrm{GSp}(4,\mathbb{F}_\ell)/(\mathbb{F}_\ell^{*2}\cdot\mathrm{Sp}(4,\mathbb{F}_\ell)).$$

Hence such a subgroup contains $\mathbf{A}_6\times\mathrm{Sp}(4,\mathbb{F}_\ell)$. Let α be a root of $f(x)$, and let $k=\mathbb{Q}(\alpha)$ or $k=\mathbb{Q}(\alpha,\sqrt{-3\cdot 13\cdot 31})$. Then the Galois group of $f(x)$ over k is $\mathbf{S}_5$ or $\mathbf{A}_5$, respectively, whereas $\mathrm{Gal}(k(A[\ell])/k)$ contains $\mathrm{Sp}(4,\mathbb{F}_\ell)$ for all $\ell\geq 3$. Now the same arguments as in Example 1 show that for the Kummer surface X over k attached to a 2-covering of A we have $\mathrm{Br}(\overline{X})^\Gamma=0$ and $\mathrm{Br}_1(X)=\mathrm{Br}_0(X)$, hence $\mathrm{Br}(X)=\mathrm{Br}_0(X)$.

Example 3 D. Zywina [35, Thm. 1.1] gives an example of a smooth plane quartic curve over $\mathbb{Q}$ such that the image of $\Gamma=\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the torsion points of its Jacobian A is the full group $\mathrm{GSp}(6,\hat{\mathbb{Z}})$. We have $\mathrm{End}(\overline{A})\cong\mathbb{Z}$, as follows from [34, Thm. 3, p. 577], where one takes $X=A$, $\tilde{G}_2=\mathrm{GSp}(6,\mathbb{F}_2)$ and $G=\mathrm{Sp}(6,\mathbb{F}_2)$. This implies $\mathrm{NS}\,(\overline{A})\cong\mathbb{Z}$. Let $k\subset\mathbb{Q}(A[2])$ be such that $\mathrm{Gal}(\mathbb{Q}(A[2])/k)$ is $\mathbf{S}_7$ or $\mathbf{A}_7$ embedded into $\mathrm{Sp}(6,\mathbb{F}_2)$ in the usual way. We can adapt the proof of Theorem 5.1 to this case and use Proposition 4.2 in the same way as in Example 1. This shows that the Kummer threefold X over k attached to a 2-covering of A has $\mathrm{Br}(\overline{X})^\Gamma=0$ and $\mathrm{Br}_1(X)=\mathrm{Br}_0(X)$, hence $\mathrm{Br}(X)=\mathrm{Br}_0(X)$.

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