Pure and Applied Mathematics Quarterly
Volume 13, Number 2, 291–335, 2017

Self-dual Grassmannian, Wronski map, and representations of $\mathfrak{gl}_N$, $\mathfrak{sp}_{2r}$, $\mathfrak{so}_{2r+1}$

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Dedicated to Yuri Ivanovich Manin on the occasion of his 80th birthday

Abstract: We define a $\mathfrak{gl}_N$-stratification of the Grassmannian of N planes $\mathrm{Gr}(N,d)$. The $\mathfrak{gl}_N$-stratification consists of strata $\Omega_{\boldsymbol{\Lambda}}$ labeled by unordered sets $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ of nonzero partitions with at most N parts, satisfying a condition depending on d, and such that $(\otimes_{i=1}^{n} V_{\lambda^{(i)}})^{\mathfrak{sl}_N} \neq 0$. Here $V_{\lambda^{(i)}}$ is the irreducible $\mathfrak{gl}_N$-module with highest weight $\lambda^{(i)}$. We show that the closure of a stratum $\Omega_{\boldsymbol{\Lambda}}$ is the union of the strata $\Omega_{\boldsymbol{\Xi}}$, $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(m)})$, such that there is a partition $\{I_1, \dots, I_m\}$ of $\{1, 2, \dots, n\}$ with $\mathrm{Hom}_{\mathfrak{gl}_N}(V_{\xi^{(i)}}, \otimes_{j \in I_i} V_{\lambda^{(j)}}) \neq 0$ for $i = 1, \dots, m$. The $\mathfrak{gl}_N$-stratification of the Grassmannian agrees with the Wronski map.

We introduce and study the new object: the self-dual Grassmannian $\mathrm{sGr}(N,d) \subset \mathrm{Gr}(N,d)$. Our main result is a similar $\mathfrak{g}_N$-stratification of the self-dual Grassmannian governed by representation theory of the Lie algebra $\mathfrak{g}_{2r+1} := \mathfrak{sp}_{2r}$ if $N = 2r+1$ and of the Lie algebra $\mathfrak{g}_{2r} := \mathfrak{so}_{2r+1}$ if $N = 2r$.

1. Introduction

The Grassmannian $\mathrm{Gr}(N,d)$ of N-dimensional subspaces of the complex d-dimensional vector space has the standard stratification by Schubert cells Ω_λ labeled by partitions $\lambda = (d - N \geqslant \lambda_1 \geqslant \dots \geqslant \lambda_N \geqslant 0)$. A Schubert cycle is the closure of a cell Ω_λ. It is well known that the Schubert cycle $\overline{\Omega}_\lambda$ is the union of the cells Ω_ξ such that the Young diagram of λ is inscribed into the Young diagram of ξ. This stratification depends on a choice of a full flag in the d-dimensional space.

In this paper we introduce a new stratification of $\mathrm{Gr}(N,d)$ governed by representation theory of $\mathfrak{gl}_N$ and called the $\mathfrak{gl}_N$-*stratification*, see Theorem 3.5. The $\mathfrak{gl}_N$-strata $\Omega_{\boldsymbol{\Lambda}}$ are labeled by unordered sets $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ of

Received 11 May 2017.

nonzero partitions $\lambda^{(i)} = (d - N \geqslant \lambda_1^{(i)} \geqslant \ldots \geqslant \lambda_N^{(i)} \geqslant 0)$ such that

$$(1.1) \qquad (\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{sl}_N} \neq 0, \qquad \sum_{i=1}^n \sum_{j=1}^N \lambda_j^{(i)} = N(d - N),$$

where $V_{\lambda^{(i)}}$ is the irreducible $\mathfrak{gl}_N$-module with highest weight $\lambda^{(i)}$. We have $\dim \Omega_{\boldsymbol{\Lambda}} = n$. We call the closure of a stratum $\Omega_{\boldsymbol{\Lambda}}$ in $\mathrm{Gr}(N, d)$ a $\mathfrak{gl}_N$-*cycle*. The $\mathfrak{gl}_N$-cycle $\overline{\Omega}_{\boldsymbol{\Lambda}}$ is an algebraic set in $\mathrm{Gr}(N, d)$. We show that $\overline{\Omega}_{\boldsymbol{\Lambda}}$ is the union of the strata $\Omega_{\boldsymbol{\Xi}}$, $\boldsymbol{\Xi} = (\xi^{(1)}, \ldots, \xi^{(m)})$, such that there is a partition $\{I_1, \ldots, I_m\}$ of $\{1, 2, \ldots, n\}$ with $\mathrm{Hom}_{\mathfrak{gl}_N}(V_{\xi^{(i)}}, \otimes_{j \in I_i} V_{\lambda^{(j)}}) \neq 0$ for $i = 1, \ldots, m$, see Theorem 3.8.

Thus we have a partial order on the set of sequences of partitions satisfying (1.1). Namely $\boldsymbol{\Lambda} \geqslant \boldsymbol{\Xi}$ if there is a partition $\{I_1, \ldots, I_m\}$ of $\{1, 2, \ldots, n\}$ with $\mathrm{Hom}_{\mathfrak{gl}_N}(V_{\xi^{(i)}}, \otimes_{j \in I_i} V_{\lambda^{(j)}}) \neq 0$ for $i = 1, \ldots, m$. An example of the corresponding graph is given in Example 3.9. The $\mathfrak{gl}_N$-stratification can be viewed as the geometrization of this partial order.

Let us describe the construction of the strata in more detail. We identify the Grassmannian $\mathrm{Gr}(N, d)$ with the Grassmannian of N-dimensional subspaces of the d-dimensional space $\mathbb{C}_d[x]$ of polynomials in x of degree less than d. In other words, we always assume that for $X \in \mathrm{Gr}(N, d)$, we have $X \subset \mathbb{C}_d[x]$. Set $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Then, for any $z \in \mathbb{P}^1$, we have the osculating flag $\mathcal{F}(z)$, see (3.3), (3.4). Denote the Schubert cells corresponding to $\mathcal{F}(z)$ by $\Omega_\lambda(\mathcal{F}(z))$. Then the stratum $\Omega_{\boldsymbol{\Lambda}}$ consists of spaces $X \in \mathrm{Gr}(N, d)$ such that X belongs to the intersection of Schubert cells $\Omega_{\lambda^{(i)}}(\mathcal{F}(z_i))$ for some choice of distinct $z_i \in \mathbb{P}^1$:

$$\Omega_{\boldsymbol{\Lambda}} = \bigcup_{\substack{z_1, \ldots, z_n \\ z_i \neq z_j}} \Big(\bigcap_{i=1}^n \Omega_{\lambda^{(i)}}(\mathcal{F}(z_i)) \Big) \subset \mathrm{Gr}(N, d).$$

A stratum $\Omega_{\boldsymbol{\Lambda}}$ is a ramified covering over $(\mathbb{P}^1)^n$ without diagonals quotient by the free action of an appropriate symmetric group, see Proposition 3.4. The degree of the covering is $\dim(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{sl}_N}$.

For example, if $N = 1$, then $\mathrm{Gr}(1, d)$ is the $(d-1)$-dimensional projective space of the vector space $\mathbb{C}_d[x]$. The strata $\Omega_{\boldsymbol{m}}$ are labeled by unordered sets $\boldsymbol{m} = (m_1, \ldots, m_n)$ of positive integers such that $m_1 + \cdots + m_n = d - 1$. A stratum $\Omega_{\boldsymbol{m}}$ consists of all polynomials $f(x)$ which have n distinct zeros of multiplicities $m_1, \ldots, m_n$. In this stratum we also include the polynomials of degree $d - 1 - m_i$ with $n - 1$ distinct roots of multiplicities $m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n$. We interpret these polynomials as having a zero of multiplicity m_i at infinity.

The stratum $\Omega_{(1,\dots,1)}$ is open in $\mathrm{Gr}(1,d)$. The union of other strata is classically called the *swallowtail* and the $\mathfrak{gl}_1$-stratification is the standard stratification of the swallowtail; see, for example, Section 2.5 of Part 1 of [AGV].

The $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$ agrees with the Wronski map

$$\mathrm{Wr}:\mathrm{Gr}(N,d)\to\mathrm{Gr}(1,N(d-N)+1)$$

which sends an N-dimensional subspace of polynomials to its Wronskian $\det(d^{i-1}f_j/dx^{i-1})_{i,j=1}^N$, where $f_1(x),\dots,f_N(x)$ is a basis of the subspace. For any $\mathfrak{gl}_1$-stratum $\Omega_{\boldsymbol{m}}$ of $\mathrm{Gr}(1,N(d-N)+1)$, the preimage of $\Omega_{\boldsymbol{m}}$ under the Wronski map is the union of $\mathfrak{gl}_N$-strata of $\mathrm{Gr}(N,d)$ and the restriction of the Wronski map to each of those strata $\Omega_{\boldsymbol{\Lambda}}$ is a ramified covering over $\Omega_{\boldsymbol{m}}$ of degree $b(\boldsymbol{\Lambda})\dim(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{sl}_N}$, where $b(\boldsymbol{\Lambda})$ is some combinatorial symmetry coefficient of $\boldsymbol{\Lambda}$, see (3.9).

The main goal of this paper is to develop a similar picture for the new object $\mathrm{sGr}(N,d)\subset\mathrm{Gr}(N,d)$, called *self-dual Grassmannian*. Let $X\in\mathrm{Gr}(N,d)$ be an N-dimensional subspace of polynomials in x. Let $X^\vee$ be the N-dimensional space of polynomials which are Wronski determinants of $N-1$ elements of X:

$$X^\vee=\{\det\left(d^{i-1}f_j/dx^{i-1}\right)_{i,j=1}^{N-1},\ f_j(x)\in X\}.$$

The space X is called *self-dual* if $X^\vee=g\cdot X$ for some polynomial $g(x)$, see [MV1]. We define $\mathrm{sGr}(N,d)$ as the subset of $\mathrm{Gr}(N,d)$ of all self-dual spaces. It is an algebraic set.

The main result of this paper is the stratification of $\mathrm{sGr}(N,d)$ governed by representation theory of the Lie algebras $\mathfrak{g}_{2r+1}:=\mathfrak{sp}_{2r}$ if $N=2r+1$ and $\mathfrak{g}_{2r}:=\mathfrak{so}_{2r+1}$ if $N=2r$. This stratification of $\mathrm{sGr}(N,d)$ is called the $\mathfrak{g}_N$*-stratification*, see Theorem 4.11.

The $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$ consists of $\mathfrak{g}_N$-strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ labeled by unordered sets of dominant integral $\mathfrak{g}_N$-weights $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$, equipped with nonnegative integer labels $\boldsymbol{k}=(k_1,\dots,k_n)$, such that $(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{g}_N}\neq 0$ and satisfying a condition similar to the second equation in (1.1); see Section 4.3. Here $V_{\lambda^{(i)}}$ is the irreducible $\mathfrak{g}_N$-module with highest weight $\lambda^{(i)}$. Different liftings of an $\mathfrak{sl}_N$-weight to a $\mathfrak{gl}_N$-weight differ by a vector $(k,\dots,k)$ with integer k. Our label k_i is an analog of this parameter in the case of $\mathfrak{g}_N$.

A $\mathfrak{g}_N$-stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is a ramified covering over $(\mathbb{P}^1)^n$ without diagonals quotient by the free action of an appropriate symmetric group. The degree of the covering is $\dim(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{g}_N}$ and, in particular, $\dim\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}=n$; see

Proposition 4.9. We call the closure of a stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ in $\mathrm{sGr}(N,d)$ a $\mathfrak{g}_N$-*cycle*. The $\mathfrak{g}_N$-cycle $\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is an algebraic set. We show that $\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is the union of the strata $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$, $\boldsymbol{\Xi}=(\xi^{(1)},\dots,\xi^{(m)})$, such that there is a partition $\{I_1,\dots,I_m\}$ of $\{1,2,\dots,n\}$ satisfying $\mathrm{Hom}_{\mathfrak{g}_N}(V_{\xi^{(i)}},\otimes_{j\in I_i}V_{\lambda^{(j)}})\neq 0$ for $i=1,\dots,m$, and the appropriate matching of labels; see Theorem 4.13.

If $N=2r$, there is exactly one stratum of top dimension $2(d-N)=\dim \mathrm{sGr}(N,d)$. For example, the $\mathfrak{so}_5$-stratification of $\mathrm{sGr}(4,6)$ consists of 9 strata of dimensions 4, 3, 3, 3, 2, 2, 2, 2, 1, see the graph of adjacencies in Example 4.14. If $N=2r+1$, there are many strata of top dimension $d-N$ (except in the trivial cases of $d=2r+1$ and $d=2r+2$). For example, the $\mathfrak{sp}_4$-stratification of $\mathrm{sGr}(5,8)$ has four strata of dimension 3; see Section 4.7. In all cases we have exactly one one-dimensional stratum corresponding to $n=1$, $\boldsymbol{\Lambda}=(0)$, and $\boldsymbol{k}=(d-N)$.

Essentially, we obtain the $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$ by restricting the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$ to $\mathrm{sGr}(N,d)$.

For $X\in \mathrm{sGr}(N,d)$, the multiplicity of every zero of the Wronskian of X is divisible by r if $N=2r$ and by N if $N=2r+1$. We define the reduced Wronski map $\overline{\mathrm{Wr}}:\mathrm{sGr}(N,d)\to \mathrm{Gr}(1,2(d-N)+1)$ if $N=2r$ and $\overline{\mathrm{Wr}}:\mathrm{sGr}(N,d)\to \mathrm{Gr}(1,d-N+1)$ if $N=2r+1$ by sending X to the r-th root of its Wronskian if $N=2r$ and to the N-th root if $N=2r+1$. The $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$ agrees with the reduced Wronski map and swallowtail $\mathfrak{gl}_1$-stratification of $\mathrm{Gr}(1,2(d-N)+1)$ or $\mathrm{Gr}(1,d-N+1)$. For any $\mathfrak{gl}_1$-stratum $\Omega_{\boldsymbol{m}}$ the preimage of $\Omega_{\boldsymbol{m}}$ under $\overline{\mathrm{Wr}}$ is the union of $\mathfrak{g}_N$-strata (see Proposition 4.17) and the restriction of the reduced Wronski map to each of those strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is a ramified covering over $\Omega_{\boldsymbol{m}}$; see Proposition 4.18.

Our definition of the $\mathfrak{gl}_N$-stratification is motivated by the connection to the Gaudin model of type A; see Theorem 3.2. Similarly, our definition of the self-dual Grassmannian and of the $\mathfrak{g}_N$-stratification is motivated by the connection to the Gaudin models of types B and C; see Theorem 4.5.

It is interesting to study the geometry and topology of strata, cycles, and of self-dual Grassmannian; see Section 4.7.

The exposition of the material is as follows. In Section 2 we introduce the $\mathfrak{gl}_N$ Bethe algebra. In Section 3 we describe the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$. In Section 4 we define the $\mathfrak{g}_N$-stratification of the self-dual Grassmannian $\mathrm{sGr}(N,d)$. In Section 5 we recall the interrelations of the Lie algebras $\mathfrak{sl}_N$, $\mathfrak{so}_{2r+1}$, $\mathfrak{sp}_{2r}$. In Section 6 we discuss $\mathfrak{g}$-opers and their relations to self-dual spaces. Section 7 contains proofs of theorems formulated in Sections 3 and 4. In Appendix A we describe the bijection between the self-dual spaces and the set of $\mathfrak{gl}_N$ Bethe vectors fixed by the Dynkin diagram automorphism of $\mathfrak{gl}_N$.

Acknowledgments

The authors thank V. Chari, A. Gabrielov, and L. Rybnikov for useful discussions. A.V. was supported in part by NSF grants DMS-1362924, DMS-1665239, and Simons Foundation grant #336826. E.M. was supported in part by Simons Foundation grant #353831.

2. Lie algebras

2.1. Lie algebra $\mathfrak{gl}_N$

Let e_{ij}, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$, satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js}e_{ik} - \delta_{ik}e_{sj}$. We identify the Lie algebra $\mathfrak{sl}_N$ with the subalgebra of $\mathfrak{gl}_N$ generated by the elements $e_{ii} - e_{jj}$ and e_{ij} for $i \neq j$, $i, j = 1, \dots, N$.

Let M be a $\mathfrak{gl}_N$-module. A vector $v \in M$ has weight $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{ii}v = \lambda_i v$ for $i = 1, \dots, N$. A vector v is called *singular* if $e_{ij}v = 0$ for $1 \leqslant i < j \leqslant N$.

We denote by $(M)_\lambda$ the subspace of M of weight λ, by $(M)^{\mathrm{sing}}$ the subspace of M of all singular vectors and by $(M)^{\mathrm{sing}}_\lambda$ the subspace of M of all singular vectors of weight λ.

Denote by V_λ the irreducible $\mathfrak{gl}_N$-module with highest weight λ.

The $\mathfrak{gl}_N$-module $V_{(1,0,\dots,0)}$ is the standard N-dimensional vector representation of $\mathfrak{gl}_N$, which we denote by L.

A sequence of integers $\lambda = (\lambda_1, \dots, \lambda_N)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N \geqslant 0$ is called *a partition with at most N parts*. Set $|\lambda| = \sum_{i=1}^N \lambda_i$. Then it is said that λ is a partition of $|\lambda|$. The $\mathfrak{gl}_N$-module $L^{\otimes n}$ contains the module V_λ if and only if λ is a partition of n with at most N parts.

Let λ, μ be partitions with at most N parts. We write $\lambda \subseteq \mu$ if and only if $\lambda_i \leqslant \mu_i$ for $i = 1, \dots, N$.

2.2. Simple Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{i,j})_{i,j=1}^r$. Let $D = \mathrm{diag}\{d_1, \dots, d_r\}$ be the diagonal matrix with positive relatively prime integers d_i such that DA is symmetric.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the Cartan decomposition. Fix simple roots $\alpha_1, \dots, \alpha_r$ in $\mathfrak{h}^*$. Let $\check{\alpha}_1, \dots, \check{\alpha}_r \in \mathfrak{h}$ be the corresponding coroots. Fix a nondegenerate invariant bilinear form $(,)$ in $\mathfrak{g}$ such that $(\check{\alpha}_i, \check{\alpha}_j) = a_{i,j}/d_j$. The corresponding invariant bilinear form in $\mathfrak{h}^*$ is given by $(\alpha_i, \alpha_j) = d_i a_{i,j}$. We have $\langle \lambda, \check{\alpha}_i \rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ for $\lambda \in \mathfrak{h}^*$. In particular, $\langle \alpha_j, \check{\alpha}_i \rangle = a_{i,j}$. Let $\omega_1, \dots, \omega_r \in \mathfrak{h}^*$ be the fundamental weights, $\langle \omega_j, \check{\alpha}_i \rangle = \delta_{i,j}$.

Let $\mathcal{P} = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \check{\alpha}_i \rangle \in \mathbb{Z},\ i = 1, \dots, r\}$ and $\mathcal{P}^+ = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \check{\alpha}_i \rangle \in \mathbb{Z}_{\geqslant 0},\ i = 1, \dots, r\}$ be the weight lattice and the cone of dominant integral weights.

For $\lambda \in \mathfrak{h}^*$, let V_λ be the irreducible $\mathfrak{g}$-module with highest weight λ. We denote $\langle \lambda, \check{\alpha}_i \rangle$ by λ_i and sometimes write $(\lambda_1, \lambda_2, \dots, \lambda_r)$ for λ.

Let M be a $\mathfrak{g}$-module. Let $(M)^{\mathrm{sing}} = \{v \in M \mid \mathfrak{n}_+ v = 0\}$ be the subspace of singular vectors in M. For $\mu \in \mathfrak{h}^*$ let $(M)_\mu = \{v \in M \mid hv = \mu(h)v,\ \text{for all } h \in \mathfrak{h}\}$ be the subspace of M of vectors of weight μ. Let $(M)_\mu^{\mathrm{sing}} = (M)^{\mathrm{sing}} \cap (M)_\mu$ be the subspace of singular vectors in M of weight μ.

Given a $\mathfrak{g}$-module M, denote by $(M)^{\mathfrak{g}}$ the subspace of $\mathfrak{g}$-invariants in M. The subspace $(M)^{\mathfrak{g}}$ is the multiplicity space of the trivial $\mathfrak{g}$-module in M. The following facts are well known. Let λ, μ be partitions with at most N parts, $\dim(V_\lambda \otimes V_\mu)^{\mathfrak{sl}_N} = 1$ if $\lambda_i = k - \mu_{N+1-i}$, $i = 1, \dots, N$, for some integer $k \geqslant \mu_1$ and 0 otherwise. Let λ, μ be $\mathfrak{g}$-weights, $\dim(V_\lambda \otimes V_\mu)^{\mathfrak{g}} = \delta_{\lambda,\mu}$ for $\mathfrak{g} = \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}$.

For any Lie algebra $\mathfrak{g}$, denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

2.3. Current algebra $\mathfrak{g}[t]$

Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{g}$-valued polynomials with the pointwise commutator. We call it the *current algebra* of $\mathfrak{g}$. We identify the Lie algebra $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes 1$ of constant polynomials in $\mathfrak{g}[t]$. Hence, any $\mathfrak{g}[t]$-module has the canonical structure of a $\mathfrak{g}$-module. The standard generators of $\mathfrak{gl}_N[t]$ are $e_{ij} \otimes t^p$, $i, j = 1, \dots, N$, $p \in \mathbb{Z}_{\geqslant 0}$. They satisfy the relations $[e_{ij} \otimes t^p, e_{sk} \otimes t^q] = \delta_{js} e_{ik} \otimes t^{p+q} - \delta_{ik} e_{sj} \otimes t^{p+q}$. It is convenient to collect elements of $\mathfrak{g}[t]$ in generating series of a formal variable x. For $g \in \mathfrak{g}$, set

$$g(x) = \sum_{s=0}^{\infty} (g \otimes t^s) x^{-s-1}. \tag{2.1}$$

For $\mathfrak{gl}_N[t]$ we have $(x_2 - x_1)[e_{ij}(x_1), e_{sk}(x_2)] = \delta_{js}(e_{ik}(x_1) - e_{ik}(x_2)) - \delta_{ik}(e_{sj}(x_1) - e_{sj}(x_2))$. For each $a \in \mathbb{C}$, there exists an automorphism τ_a of

$\mathfrak{g}[t]$, $\tau_a : g(x) \to g(x-a)$. Given a $\mathfrak{g}[t]$-module M, we denote by $M(a)$ the pull-back of M through the automorphism τ_a. As $\mathfrak{g}$-modules, M and $M(a)$ are isomorphic by the identity map.

We have the evaluation homomorphism, $\mathrm{ev} : \mathfrak{g}[t] \to \mathfrak{g}$, $\mathrm{ev} : g(x) \to gx^{-1}$. Its restriction to the subalgebra $\mathfrak{g} \subset \mathfrak{g}[t]$ is the identity map. For any $\mathfrak{g}$-module M, we denote by the same letter the $\mathfrak{g}[t]$-module, obtained by pulling M back through the evaluation homomorphism. For each $a \in \mathbb{C}$, the $\mathfrak{g}[t]$-module $M(a)$ is called an *evaluation module*.

For $\mathfrak{g} = \mathfrak{sl}_N$, $\mathfrak{sp}_{2r}$, $\mathfrak{so}_{2r+1}$, it is well known that finite-dimensional irreducible $\mathfrak{g}[t]$-modules are tensor products of evaluation modules $V_{\lambda^{(1)}}(z_1) \otimes \dots \otimes V_{\lambda^{(n)}}(z_n)$ with dominant integral $\mathfrak{g}$-weights $\lambda^{(1)}, \dots, \lambda^{(n)}$ and distinct evaluation parameters $z_1, \dots, z_n$.

2.4. Bethe algebra

Let S_l be the permutation group of the set $\{1, \dots, l\}$. Given an $N \times N$ matrix B with possibly noncommuting entries b_{ij}, we define its *row determinant* to be

$$\mathrm{rdet}\, B = \sum_{\sigma \in S_N} (-1)^\sigma b_{1\sigma(1)} b_{2\sigma(2)} \dots b_{N\sigma(N)}.$$

Define *the universal differential operator* $\mathcal{D}^{\mathcal{B}}$ by

$$\mathcal{D}^{\mathcal{B}} = \mathrm{rdet}(\delta_{ij}\partial_x - e_{ji}(x))_{i,j=1}^N. \tag{2.2}$$

It is a differential operator in variable x, whose coefficients are formal power series in x^{-1} with coefficients in $\mathcal{U}(\mathfrak{gl}_N[t])$,

$$\mathcal{D}^{\mathcal{B}} = \partial_x^N + \sum_{i=1}^N B_i(x)\partial_x^{N-i}, \tag{2.3}$$

where

$$B_i(x) = \sum_{j=i}^\infty B_{ij} x^{-j}$$

and $B_{ij} \in \mathcal{U}(\mathfrak{gl}_N[t])$, $i = 1, \dots, N$, $j \in \mathbb{Z}_{\geqslant i}$. We call the unital subalgebra of $\mathcal{U}(\mathfrak{gl}_N[t])$ generated by $B_{ij} \in \mathcal{U}(\mathfrak{gl}_N[t])$, $i = 1, \dots, N$, $j \in \mathbb{Z}_{\geqslant i}$, the *Bethe algebra* of $\mathfrak{gl}_N$ and denote it by $\mathcal{B}$.

The Bethe algebra $\mathcal{B}$ is commutative and commutes with the subalgebra $\mathcal{U}(\mathfrak{gl}_N) \subset \mathcal{U}(\mathfrak{gl}_N[t])$, see [T]. As a subalgebra of $\mathcal{U}(\mathfrak{gl}_N[t])$, the algebra $\mathcal{B}$ acts on any $\mathfrak{gl}_N[t]$-module M. Since $\mathcal{B}$ commutes with $\mathcal{U}(\mathfrak{gl}_N)$, it preserves the subspace of singular vectors $(M)^{\mathrm{sing}}$ as well as weight subspaces of M. Therefore, the subspace $(M)^{\mathrm{sing}}_{\lambda}$ is $\mathcal{B}$-invariant for any weight λ.

We denote $M(\infty)$ the $\mathfrak{gl}_N$-module M with the trivial action of the Bethe algebra $\mathcal{B}$. More generally, for a $\mathfrak{gl}_N[t]$-module M', we denote by $M' \otimes M(\infty)$ the $\mathfrak{gl}_N$-module where we define the action of $\mathcal{B}$ so that it acts trivially on $M(\infty)$. Namely, the element $b \in \mathcal{B}$ acts on $M' \otimes M(\infty)$ by $b \otimes 1$.

Note that for $a \in \mathbb{C}$ and $\mathfrak{gl}_N$-module M, the action of $e_{ij}(x)$ on $M(a)$ is given by $e_{ij}/(x-a)$ on M. Therefore, the action of series $B_i(x)$ on the module $M' \otimes M(\infty)$ is the limit of the action of the series $B_i(x)$ on the module $M' \otimes M(z)$ as $z \to \infty$ in the sense of rational functions of x. However, such a limit of the action of coefficients B_{ij} on the module $M' \otimes M(z)$ as $z \to \infty$ does not exist.

Let $M = V_\lambda$ be an irreducible $\mathfrak{gl}_N$-module and let M' be an irreducible finite-dimensional $\mathfrak{gl}_N[t]$-module. Let c be the value of the $\sum_{i=1}^N e_{ii}$ action on M'.

Lemma 2.1. *We have an isomorphism of vector spaces:*

$$\pi: \ (M' \otimes V_\lambda)^{\mathfrak{sl}_N} \to (M')^{\mathrm{sing}}_{\bar{\lambda}}, \text{ where } \bar{\lambda}_i = \frac{c + |\lambda|}{N} - \lambda_{N+1-i},$$

given by the projection to a lowest weight vector in V_λ. The map π is an isomorphism of $\mathcal{B}$-modules $(M' \otimes V_\lambda(\infty))^{\mathfrak{sl}_N} \to (M')^{\mathrm{sing}}_{\bar{\lambda}}$. □

Consider $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$. Set

$$\mathring{\mathbb{P}}_n := \{\boldsymbol{z} = (z_1, \dots, z_n) \in (\mathbb{P}^1)^n \mid z_i \neq z_j \ \text{ for } \ 1 \leqslant i < j \leqslant n\},$$

$$\mathbb{R}\mathring{\mathbb{P}}_n := \{\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n \mid z_i \in \mathbb{R} \text{ or } z_i = \infty, \quad \text{for} \quad 1 \leqslant i \leqslant n\}.$$

We are interested in the action of the Bethe algebra $\mathcal{B}$ on the tensor product $\bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s)$, where $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is a sequence of partitions with at most N parts and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$. By Lemma 2.1, it is sufficient to consider spaces of invariants $(\bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s))^{\mathfrak{sl}_N}$. For brevity, we write $V_{\boldsymbol{\Lambda},\boldsymbol{z}}$ for the $\mathcal{B}$-module $\bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s)$ and $V_{\boldsymbol{\Lambda}}$ for the $\mathfrak{gl}_N$-module $\bigotimes_{s=1}^n V_{\lambda^{(s)}}$.

Let $v \in V_{\boldsymbol{\Lambda},\boldsymbol{z}}$ be a common eigenvector of the Bethe algebra $\mathcal{B}$, $B_i(x)v = h_i(x)v$, $i = 1, \dots, N$. Then we call the scalar differential operator

$$\mathcal{D}_v = \partial_x^N + \sum_{i=1}^N h_i(x)\partial_x^{N-i}$$

the *differential operator associated with the eigenvector* v.

3. The $\mathfrak{gl}_N$-stratification of Grassmannian

Let N, $d \in \mathbb{Z}_{>0}$ such that $N \leqslant d$.

3.1. Schubert cells

Let $\mathbb{C}_d[x]$ be the space of polynomials in x with complex coefficients of degree less than d. We have $\dim \mathbb{C}_d[x] = d$. Let $\mathrm{Gr}(N, d)$ be the Grassmannian of all N-dimensional subspaces in $\mathbb{C}_d[x]$. The Grassmannian $\mathrm{Gr}(N, d)$ is a smooth projective complex variety of dimension $N(d - N)$.

Let $\mathbb{R}_d[x] \subset \mathbb{C}_d[x]$ be the space of polynomials in x with real coefficients of degree less than d. Let $\mathrm{Gr}^{\mathbb{R}}(N, d) \subset \mathrm{Gr}(N, d)$ be the set of subspaces which have a basis consisting of polynomials with real coefficients. For $X \in \mathrm{Gr}(N, d)$ we have $X \in \mathrm{Gr}^{\mathbb{R}}(N, d)$ if and only if $\dim_{\mathbb{R}}(X \cap \mathbb{R}_d[x]) = N$. We call such points X *real*.

For a full flag $\mathcal{F} = \{0 \subset F_1 \subset F_2 \subset \cdots \subset F_d = \mathbb{C}_d[x]\}$ and a partition $\lambda = (\lambda_1, \dots, \lambda_N)$ such that $\lambda_1 \leqslant d - N$, the Schubert cell $\Omega_\lambda(\mathcal{F}) \subset \mathrm{Gr}(N, d)$ is given by

$$\begin{aligned}\Omega_\lambda(\mathcal{F}) = \{X \in \mathrm{Gr}(N, d) \mid \ & \dim(X \cap F_{d-j-\lambda_{N-j}}) = N - j,\\ & \dim(X \cap F_{d-j-\lambda_{N-j}-1}) = N - j - 1\}.\end{aligned}$$

We have $\operatorname{codim} \Omega_\lambda(\mathcal{F}) = |\lambda|$.

The Schubert cell decomposition associated to a full flag $\mathcal{F}$, see for example [GH], is given by

$$\mathrm{Gr}(N, d) = \bigsqcup_{\lambda,\ \lambda_1 \leqslant d-N} \Omega_\lambda(\mathcal{F}). \tag{3.1}$$

The Schubert cycle $\overline{\Omega}_\lambda(\mathcal{F})$ is the closure of a Schubert cell $\Omega_\lambda(\mathcal{F})$ in the Grassmannian $\mathrm{Gr}(N, d)$. Schubert cycles are algebraic sets with very rich geometry and topology. It is well known that Schubert cycle $\overline{\Omega}_\lambda(\mathcal{F})$ is described by the formula

$$\overline{\Omega}_\lambda(\mathcal{F}) = \bigsqcup_{\substack{\lambda \subseteq \mu,\\ \mu_1 \leqslant d-N}} \Omega_\mu(\mathcal{F}). \tag{3.2}$$

Given a partition $\lambda = (\lambda_1, \dots, \lambda_N)$ such that $\lambda_1 \leqslant d - N$, introduce a new partition

$$\bar{\lambda} = (d - N - \lambda_N, d - N - \lambda_{N-1}, \dots, d - N - \lambda_1).$$

We have $|\lambda| + |\bar{\lambda}| = N(d-N)$.

Let $\mathcal{F}(\infty)$ be the full flag given by

$$\mathcal{F}(\infty) = \{0 \subset \mathbb{C}_1[x] \subset \mathbb{C}_2[x] \subset \cdots \subset \mathbb{C}_d[x]\}. \tag{3.3}$$

The subspace X is a point of $\Omega_\lambda(\mathcal{F}(\infty))$ if and only if for every $i = 1, \dots, N$, it contains a polynomial of degree $\bar{\lambda}_i + N - i$.

For $z \in \mathbb{C}$, consider the full flag

$$\mathcal{F}(z) = \{0 \subset (x-z)^{d-1}\mathbb{C}_1[x] \subset (x-z)^{d-2}\mathbb{C}_2[x] \subset \cdots \subset \mathbb{C}_d[x]\}. \tag{3.4}$$

The subspace X is a point of $\Omega_\lambda(\mathcal{F}(z))$ if and only if for every $i = 1, \dots, N$, it contains a polynomial with a root at z of order $\lambda_i + N - i$.

A point $z \in \mathbb{C}$ is called a *base point* for a subspace $X \subset \mathbb{C}_d[x]$ if $g(z) = 0$ for every $g \in X$.

3.2. Intersection of Schubert cells

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of partitions with at most N parts and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$. Set $|\boldsymbol{\Lambda}| = \sum_{s=1}^n |\lambda^{(s)}|$.

The following lemma is elementary.

Lemma 3.1. *If* $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} > 0$*, then* $|\boldsymbol{\Lambda}|$ *is divisible by* N*. Suppose further* $|\boldsymbol{\Lambda}| = N(d-N)$*, then* $\lambda_1^{(s)} \leqslant d - N$ *for* $s = 1, \dots, n$*.* □

Assuming $|\boldsymbol{\Lambda}| = N(d-N)$, denote by $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ the intersection of the Schubert cells:

$$\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}} = \bigcap_{s=1}^{n} \Omega_{\lambda^{(s)}}(\mathcal{F}(z_s)). \tag{3.5}$$

Note that due to our assumption, $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is a finite subset of $\mathrm{Gr}(N,d)$. Note also that $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is non-empty if and only if $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} > 0$.

Theorem 3.2. *Suppose* $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} > 0$*. Let* $v \in (V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ *be an eigenvector of the Bethe algebra* $\mathcal{B}$*. Then* $\operatorname{Ker}\mathcal{D}_v \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$*. Moreover, the assignment* $\kappa : v \mapsto \operatorname{Ker}\mathcal{D}_v$ *is a bijective correspondence between the set of eigenvectors of the Bethe algebra in* $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ *(considered up to multiplication by nonzero scalars) and the set* $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$*.*

Proof. The first statement is Theorem 4.1 in [MTV3] and the second statement is Theorem 6.1 in [MTV4]. □

We also have the following lemma, see for example [MTV1].

Lemma 3.3. *Let $\boldsymbol{z}$ be a generic point in $\mathring{\mathbb{P}}_n$. Then the action of the Bethe algebra $\mathcal{B}$ on $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ is diagonalizable. In particular, this statement holds for any sequence $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$.* □

3.3. The $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N, d)$

The following definition plays an important role in what follows.

Define a partial order $\geqslant$ on the set of sequences of partitions with at most N parts as follows. Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$, $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(m)})$ be two sequences of partitions with at most N parts. We say that $\boldsymbol{\Lambda} \geqslant \boldsymbol{\Xi}$ if and only if there exists a partition $\{I_1, \dots, I_m\}$ of the set $\{1, 2, \dots, n\}$ such that

$$\mathrm{Hom}_{\mathfrak{gl}_N}(V_{\xi^{(i)}}, \bigotimes_{j\in I_i} V_{\lambda^{(j)}}) \neq 0, \quad i = 1, \dots, m.$$

Note that $\boldsymbol{\Lambda}$ and $\boldsymbol{\Xi}$ are comparable only if $|\boldsymbol{\Lambda}| = |\boldsymbol{\Xi}|$.

We say that $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is *nontrivial* if and only if $(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} \neq 0$ and $|\lambda^{(s)}| > 0$, $s = 1, \dots, n$. The sequence $\boldsymbol{\Lambda}$ will be called *d-nontrivial* if $\boldsymbol{\Lambda}$ is nontrivial and $|\boldsymbol{\Lambda}| = N(d - N)$.

Suppose $\boldsymbol{\Xi}$ is d-nontrivial. If $\boldsymbol{\Lambda} \geqslant \boldsymbol{\Xi}$ and $|\lambda^{(s)}| > 0$ for all $s = 1, \dots, n$, then $\boldsymbol{\Lambda}$ is also d-nontrivial.

Recall that $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is the intersection of Schubert cells for each given $\boldsymbol{z}$, see (3.5), define $\Omega_{\boldsymbol{\Lambda}}$ by the formula

$$\Omega_{\boldsymbol{\Lambda}} := \bigcup_{\boldsymbol{z}\in\mathring{\mathbb{P}}_n} \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}} \subset \mathrm{Gr}(N, d). \tag{3.6}$$

By definition, $\Omega_{\boldsymbol{\Lambda}}$ does not depend on the order of $\lambda^{(s)}$ in the sequence $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$. Note that $\Omega_{\boldsymbol{\Lambda}}$ is a constructible subset of the Grassmannian $\mathrm{Gr}(N, d)$ in Zariski topology. We call $\Omega_{\boldsymbol{\Lambda}}$ with a d-nontrivial $\boldsymbol{\Lambda}$ a *$\mathfrak{gl}_N$-stratum* of $\mathrm{Gr}(N, d)$.

Let $\mu^{(1)}, \dots, \mu^{(a)}$ be the list of all distinct partitions in $\boldsymbol{\Lambda}$. Let n_i be the number of occurrences of $\mu^{(i)}$ in $\boldsymbol{\Lambda}$, $i = 1, \dots, a$, then $\sum_{i=1}^{a} n_i = n$. Denote $\boldsymbol{n} = (n_1, \dots, n_a)$. We shall write $\boldsymbol{\Lambda}$ in the following order: $\lambda^{(i)} = \mu^{(j)}$ for $\sum_{s=1}^{j-1} n_s + 1 \leqslant i \leqslant \sum_{s=1}^{j} n_s$, $j = 1, \dots, a$.

Let $S_{\boldsymbol{n};n_i}$ be the subgroup of the symmetric group S_n permuting $\{n_1 + \cdots + n_{i-1} + 1, \dots, n_1 + \cdots + n_i\}$, $i = 1, \dots, a$. Then the group $S_{\boldsymbol{n}} = S_{\boldsymbol{n};n_1} \times S_{\boldsymbol{n};n_2} \times \cdots \times S_{\boldsymbol{n};n_a}$ acts freely on $\mathring{\mathbb{P}}_n$ and on $\mathbb{R}\mathring{\mathbb{P}}_n$. Denote by $\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$ and $\mathbb{R}\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$ the sets of orbits.

Proposition 3.4. *Suppose* $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ *is d-nontrivial. The stratum* $\Omega_{\boldsymbol{\Lambda}}$ *is a ramified covering of* $\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$*. Moreover, the degree of the covering is equal to* $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N}$*. In particular,* $\dim \Omega_{\boldsymbol{\Lambda}} = n$*. Over* $\mathbb{R}\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$*, this covering is unramified of the same degree, moreover all points in fibers are real.*

Proof. The statement follows from Theorem 3.2, Lemma 3.3, and Theorem 1.1 of [MTV3]. □

Clearly, we have the following theorem.

Theorem 3.5. *We have*

$$\mathrm{Gr}(N,d) = \bigsqcup_{d\text{-nontrivial } \boldsymbol{\Lambda}} \Omega_{\boldsymbol{\Lambda}}. \tag{3.7}$$

□

Next, for a d-nontrivial $\boldsymbol{\Lambda}$, we call the closure of $\Omega_{\boldsymbol{\Lambda}}$ inside $\mathrm{Gr}(N,d)$, a $\mathfrak{gl}_N$*-cycle*. The $\mathfrak{gl}_N$-cycle $\overline{\Omega}_{\boldsymbol{\Lambda}}$ is an algebraic set. We describe the $\mathfrak{gl}_N$-cycles as unions of $\mathfrak{gl}_N$-strata.

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ and $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(n-1)})$ be such that $\boldsymbol{\Xi} \leqslant \boldsymbol{\Lambda}$. We call $\Omega_{\boldsymbol{\Xi}}$ a *simple degeneration* of $\Omega_{\boldsymbol{\Lambda}}$ if and only if both $\boldsymbol{\Lambda}$ and $\boldsymbol{\Xi}$ are d-nontrivial. In view of Theorem 3.2, taking a simple degeneration is equivalent to making two coordinates of $\boldsymbol{z}$ collide.

Theorem 3.6. *If* $\Omega_{\boldsymbol{\Xi}}$ *is a simple degeneration of* $\Omega_{\boldsymbol{\Lambda}}$*, then* $\Omega_{\boldsymbol{\Xi}}$ *is contained in the* $\mathfrak{gl}_N$*-cycle* $\overline{\Omega}_{\boldsymbol{\Lambda}}$*.*

Theorem 3.6 is proved in Section 7.1.

Suppose $\boldsymbol{\Theta} = (\theta^{(1)}, \dots, \theta^{(l)})$ is d-nontrivial and $\boldsymbol{\Lambda} \geqslant \boldsymbol{\Theta}$. Then, it is clear that $\Omega_{\boldsymbol{\Theta}}$ is obtained from $\Omega_{\boldsymbol{\Lambda}}$ by a sequence of simple degenerations. We call $\Omega_{\boldsymbol{\Theta}}$ a *degeneration* of $\Omega_{\boldsymbol{\Lambda}}$.

Corollary 3.7. *If* $\Omega_{\boldsymbol{\Theta}}$ *is a degeneration of* $\Omega_{\boldsymbol{\Lambda}}$*, then* $\Omega_{\boldsymbol{\Theta}}$ *is contained in the* $\mathfrak{gl}_N$*-cycle* $\overline{\Omega}_{\boldsymbol{\Lambda}}$*.* □

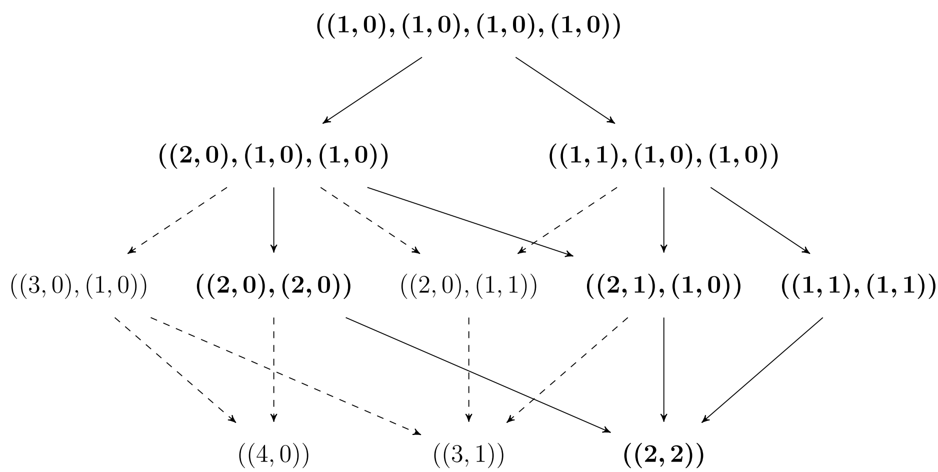
Theorem 3.8. *For d-nontrivial* $\boldsymbol{\Lambda}$*, we have*

$$\overline{\Omega}_{\boldsymbol{\Lambda}} = \bigsqcup_{\substack{\boldsymbol{\Xi} \leqslant \boldsymbol{\Lambda}, \\ d\text{-nontrivial } \boldsymbol{\Xi}}} \Omega_{\boldsymbol{\Xi}}. \tag{3.8}$$

Theorem 3.8 is proved in Section 7.1.

Theorems 3.5 and 3.8 imply that the subsets $\Omega_{\boldsymbol{\Lambda}}$ with d-nontrivial $\boldsymbol{\Lambda}$ give a stratification of $\mathrm{Gr}(N,d)$. We call it the $\mathfrak{gl}_N$*-stratification of* $\mathrm{Gr}(N,d)$.

Example 3.9. We give an example of the $\mathfrak{gl}_2$-stratification for $\mathrm{Gr}(2,4)$ in the following picture. In the picture, we simply write $\mathbf{\Lambda}$ for $\Omega_{\mathbf{\Lambda}}$. We also write tuples of numbers with bold font for 4-nontrivial tuples of partitions, solid arrows for simple degenerations between 4-nontrivial tuples of partitions. The dashed arrows go between comparable sequences where the set $\Omega_{\mathbf{\Xi}}$ corresponding to the smaller sequence is empty.



In particular, $\Omega_{((1,0),(1,0),(1,0),(1,0))}$ is dense in $\mathrm{Gr}(2,4)$.

Remark 3.10. In general, for $\mathrm{Gr}(N,d)$, let $\epsilon_1 = (1,0,\dots,0)$ and let

$$\mathbf{\Lambda} = (\underbrace{\epsilon_1, \epsilon_1, \dots, \epsilon_1}_{N(d-N)}).$$

Then $\mathbf{\Lambda}$ is d-nontrivial, and $\Omega_{\mathbf{\Lambda}}$ is dense in $\mathrm{Gr}(N,d)$. Clearly, $\Omega_{\mathbf{\Lambda}}$ consists of spaces of polynomials whose Wronskian (see Section 3.4) has only simple roots.

Remark 3.11. The group of affine translations acts on $\mathbb{C}_d[x]$ by changes of variable. Namely, for $a \in \mathbb{C}^*, b \in \mathbb{C}$, we have a map sending $f(x) \mapsto f(ax+b)$ for all $f(x) \in \mathbb{C}_d[x]$. This group action preserves the Grassmannian $\mathrm{Gr}(N,d)$ and the strata $\Omega_{\mathbf{\Lambda}}$.

3.4. The case of $N = 1$ and the Wronski map

We show that the decomposition in Theorems 3.5 and 3.8 respects the Wronski map.

From now on, we use the convention that $x - z_s$ is considered as the constant function 1 if $z_s = \infty$.

Consider the Grassmannian of lines $\mathrm{Gr}(1, \tilde{d})$. By Theorem 3.5, the decomposition of $\mathrm{Gr}(1, \tilde{d})$ is parameterized by unordered sequences of positive integers $\boldsymbol{m} = (m_1, \dots, m_n)$ such that $|\boldsymbol{m}| = \tilde{d} - 1$.

Let $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$. We have $\mathbb{C}f \in \Omega_{\boldsymbol{m},\boldsymbol{z}}$ if and only if

$$f(x) = a \prod_{s=1}^{n} (x - z_s)^{m_s}, \quad a \neq 0.$$

In other words, the stratum $\Omega_{\boldsymbol{m}}$ of the $\mathfrak{gl}_1$-stratification (3.7) of $\mathrm{Gr}(1, \tilde{d})$ consists of all points in $\mathrm{Gr}(1, \tilde{d})$ whose representative polynomials have n distinct roots (one of them can be ∞) of multiplicities $m_1, \dots, m_n$.

Therefore the $\mathfrak{gl}_1$-stratification is exactly the celebrated swallowtail stratification.

For $g_1(x), \dots, g_l(x) \in \mathbb{C}[x]$, denote by $\mathrm{Wr}(g_1(x), \dots, g_l(x))$ the *Wronskian*,

$$\mathrm{Wr}(g_1(x), \dots, g_l(x)) = \det(d^{i-1} g_j / dx^{i-1})_{i,j=1}^{l}.$$

Let $X \in \mathrm{Gr}(N, d)$. The Wronskians of two bases of X differ by a multiplication by a nonzero number. We call the monic polynomial representing the Wronskian the *Wronskian* of X and denote it by $\mathrm{Wr}(X)$. It is clear that $\deg_x \mathrm{Wr}(X) \leqslant N(d - N)$.

The *Wronski* map

$$\mathrm{Wr} : \mathrm{Gr}(N, d) \to \mathrm{Gr}(1, N(d - N) + 1)$$

is sending $X \in \mathrm{Gr}(N, d)$ to $\mathbb{C}\mathrm{Wr}(X)$.

The Wronski map is a finite algebraic map; see, for example, Propositions 3.1 and 4.2 in [MTV5], of degree $\dim(L^{\otimes N(d-N)})^{\mathfrak{sl}_n}$, which is explicitly given by

$$(N(d - N))! \; \frac{0!\ 1!\ 2! \dots (d - N - 1)!}{N!\ (N + 1)!\ (N + 2)! \dots (d - 1)!},$$

see [S].

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be d-nontrivial and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$. If $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$, then one has

$$\mathrm{Wr}(X) = \prod_{s=1}^{n} (x - z_s)^{|\lambda^{(s)}|}.$$

Set $\tilde{d} = N(d - N) + 1$. Therefore, we have the following proposition.

Proposition 3.12. *The preimage of the stratum $\Omega_{\boldsymbol{m}}$ of* $\mathrm{Gr}(1, N(d-N)+1)$ *under the Wronski map is a union of all d-nontrivial strata $\Omega_{\boldsymbol{\Lambda}}$ of* $\mathrm{Gr}(N, d)$ *such that $|\lambda^{(s)}| = m_s$, $s = 1, \dots, n$.* □

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be an unordered sequence of partitions with at most N parts. Let a be the number of distinct partitions in $\boldsymbol{\Lambda}$. We can assume that $\lambda^{(1)}, \dots, \lambda^{(a)}$ are all distinct and let $n_1, \dots, n_a$ be their multiplicities in $\boldsymbol{\Lambda}$, $n_1 + \dots + n_a = n$. Define the *symmetry coefficient* of $\boldsymbol{\Lambda}$ as the product of multinomial coefficients:

$$b(\boldsymbol{\Lambda}) = \prod_i \frac{\left(\sum_{s=1,\dots,a,\ |\lambda^{(s)}|=i} n_s\right)!}{\prod_{s=1,\dots,a,\ |\lambda^{(s)}|=i} (n_s)!}. \tag{3.9}$$

Proposition 3.13. *Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be d-nontrivial. Then the Wronski map $\mathrm{Wr}|_{\Omega_{\boldsymbol{\Lambda}}} : \Omega_{\boldsymbol{\Lambda}} \to \Omega_{\boldsymbol{m}}$ is a ramified covering of degree $b(\boldsymbol{\Lambda}) \dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N}$.*

Proof. The statement follows from Theorem 3.2, Lemma 3.3, and Proposition 3.12. □

In other words, the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N, d)$ given by Theorems 3.5 and 3.8, is adjacent to the swallowtail $\mathfrak{gl}_1$-stratification of $\mathrm{Gr}(1, N(d-N)+1)$ and the Wronski map.

4. The $\mathfrak{g}_N$-stratification of self-dual Grassmannian

It is convenient to use the notation: $\mathfrak{g}_{2r+1} = \mathfrak{sp}_{2r}$, and $\mathfrak{g}_{2r} = \mathfrak{so}_{2r+1}$, $r \geqslant 2$. We also set $\mathfrak{g}_3 = \mathfrak{sl}_2$. The case of $\mathfrak{g}_3 = \mathfrak{sl}_2$ is discussed in detail in Section 4.6.

4.1. Self-dual spaces

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a tuple of partitions with at most N parts such that $|\boldsymbol{\Lambda}| = N(d-N)$ and let $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$.

Define a tuple of polynomials $\boldsymbol{T} = (T_1, \dots, T_N)$ by

$$T_i(x) = \prod_{s=1}^{n} (x - z_s)^{\lambda_i^{(s)} - \lambda_{i+1}^{(s)}}, \quad i = 1, \dots, N, \tag{4.1}$$

where $\lambda_{N+1}^{(s)} = 0$. We say that $\boldsymbol{T}$ is *associated with* $\boldsymbol{\Lambda}, \boldsymbol{z}$.

Let $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and $g_1, \dots, g_i \in X$. Define the *divided Wronskian* $\mathrm{Wr}^\dagger$ *with respect to* $\boldsymbol{\Lambda}, \boldsymbol{z}$ by

$$\mathrm{Wr}^\dagger(g_1, \dots, g_i) = \mathrm{Wr}(g_1, \dots, g_i) \prod_{j=1}^{i} T_{N+1-j}^{j-i-1}, \quad i = 1, \dots, N.$$

Note that $\mathrm{Wr}^\dagger(g_1,\dots,g_i)$ is a polynomial in x.

Given $X \in \mathrm{Gr}(N,d)$, define the *dual space* $X^\dagger$ of X by

$$X^\dagger = \{\mathrm{Wr}^\dagger(g_1,\dots,g_{N-1}) \mid g_i \in X,\ i=1,\dots,N-1\}.$$

Lemma 4.1. *If $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$, then $X^\dagger \in \Omega_{\tilde{\boldsymbol{\Lambda}},\boldsymbol{z}} \subset \mathrm{Gr}(N,\tilde d)$, where*

$$\tilde d = \sum_{s=1}^{n} \lambda_1^{(s)} - d + 2N,$$

and $\tilde{\boldsymbol{\Lambda}} = (\tilde\lambda^{(1)},\dots,\tilde\lambda^{(n)})$ is a sequence of partitions with at most N parts such that

$$\tilde\lambda_i^{(s)} = \lambda_1^{(s)} - \lambda_{N+1-i}^{(s)}, \quad i=1,\dots,N, \quad s=1,\dots,n. \qquad \square$$

Note that we always have $\tilde\lambda_N^{(s)} = 0$ for every $s=1,\dots,n$, hence $X^\dagger$ has no base points.

Given a space of polynomials X and a rational function g in x, denote by $g\cdot X$ the space of rational functions of the form $g\cdot f$ with $f \in X$.

A self-dual space is called a *pure self-dual space* if $X = X^\dagger$. A space of polynomials X is called *self-dual* if $X = g\cdot X^\dagger$ for some polynomial $g \in \mathbb{C}[x]$. In particular, if $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is self-dual, then $X = T_N \cdot X^\dagger$, where T_N is defined in (4.1). Note also, that if X is self-dual then $g\cdot X$ is also self-dual.

It is obvious that every point in $\mathrm{Gr}(2,d)$ is a self-dual space.

Let $\mathrm{sGr}(N,d)$ be the set of all self-dual spaces in $\mathrm{Gr}(N,d)$. We call $\mathrm{sGr}(N,d)$ the *self-dual Grassmannian*. The self-dual Grassmannian $\mathrm{sGr}(N,d)$ is an algebraic subset of $\mathrm{Gr}(N,d)$.

Let $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ be the finite set defined in (3.5) and $\Omega_{\boldsymbol{\Lambda}}$ the set defined in (3.6). Denote by $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ the set of all self-dual spaces in $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and by $\mathrm{s}\Omega_{\boldsymbol{\Lambda}}$ the set of all self-dual spaces in $\Omega_{\boldsymbol{\Lambda}}$:

$$\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}} = \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}} \bigcap \mathrm{sGr}(N,d) \quad \text{and} \quad \mathrm{s}\Omega_{\boldsymbol{\Lambda}} = \Omega_{\boldsymbol{\Lambda}} \bigcap \mathrm{sGr}(N,d).$$

We call the sets $\mathrm{s}\Omega_{\boldsymbol{\Lambda}}$ $\mathfrak{g}_N$*-strata* of the self-dual Grassmannian. A stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda}}$ does not depend on the order of the set of partitions $\boldsymbol{\Lambda}$. Note that each $\mathrm{s}\Omega_{\boldsymbol{\Lambda}}$ is a constructible subset of the Grassmannian $\mathrm{Gr}(N,d)$ in Zariski topology.

A partition λ with at most N parts is called N*-symmetric* if $\lambda_i - \lambda_{i+1} = \lambda_{N-i} - \lambda_{N-i+1}$, $i=1,\dots,N-1$. If the stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda}}$ is nonempty, then all partitions $\lambda^{(s)}$ are N-symmetric; see also Lemma 4.4 below.

The self-dual Grassmannian is related to the Gaudin model in types B and C, see [MV1] and Theorem 4.5 below. We show that $\mathrm{sGr}(N,d)$ also has a remarkable stratification structure similar to the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$, governed by representation theory of $\mathfrak{g}_N$; see Theorems 4.11 and 4.13.

Remark 4.2. The self-dual Grassmannian also has a stratification induced from the usual Schubert cell decomposition (3.1), (3.2). For $z \in \mathbb{P}^1$, and an N-symmetric partition λ with $\lambda_1 \leqslant d-N$, set $\mathrm{s}\Omega_\lambda(\mathcal{F}(z)) = \Omega_\lambda(\mathcal{F}(z)) \cap \mathrm{sGr}(N,d)$. Then it is easy to see that

$$\mathrm{sGr}(N,d) = \bigsqcup_{\substack{N-\text{symmetric } \mu,\\ \mu_1 \leqslant d-N}} \mathrm{s}\Omega_\mu(\mathcal{F}(z)) \quad \text{and}$$

$$\overline{\mathrm{s}\Omega}_\lambda(\mathcal{F}(z)) = \bigsqcup_{\substack{N-\text{symmetric } \mu,\\ \mu_1 \leqslant d-N,\ \lambda \subseteq \mu}} \mathrm{s}\Omega_\mu(\mathcal{F}(z)).$$

4.2. Bethe algebras of types B and C and self-dual Grassmannian

The Bethe algebra $\mathcal{B}$ (the algebra of higher Gaudin Hamiltonians) for a simple Lie algebras $\mathfrak{g}$ were described in [FFR]. The Bethe algebra $\mathcal{B}$ is a commutative subalgebra of $\mathcal{U}(\mathfrak{g}[t])$ which commutes with the subalgebra $\mathcal{U}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g}[t])$. An explicit set of generators of the Bethe algebra in Lie algebras of types B, C, and D was given in [M]. Such a description in the case of $\mathfrak{gl}_N$ is given above in Section 2.4. For the case of $\mathfrak{g}_N$ we only need the following fact.

Recall our notation $g(x)$ for the current of $g \in \mathfrak{g}$, see (2.1).

Proposition 4.3 ([FFR, M]). *Let $N > 3$. There exist elements $F_{ij} \in \mathfrak{g}_N$, $i,j = 1,\ldots,N$, and polynomials $G_s(x)$ in $d^kF_{ij}(x)/dx^k$, $s = 1,\ldots,N$, $k = 0,\ldots,N$, such that the Bethe algebra of $\mathfrak{g}_N$ is generated by coefficients of $G_s(x)$ considered as formal power series in x^{-1}.* □

Similar to the $\mathfrak{gl}_N$ case, for a collection of dominant integral $\mathfrak{g}_N$-weights $\mathbf{\Lambda} = (\lambda^{(1)},\ldots,\lambda^{(n)})$ and $\boldsymbol{z} = (z_1,\ldots,z_n) \in \mathring{\mathbb{P}}_n$, we set $V_{\mathbf{\Lambda},\boldsymbol{z}} = \bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s)$, considered as a $\mathcal{B}$-module. Namely, if $\boldsymbol{z} \in \mathbb{C}^n$, then $V_{\mathbf{\Lambda},\boldsymbol{z}}$ is a tensor product of evaluation $\mathfrak{g}_N[t]$-modules and therefore a $\mathcal{B}$-module. If, say, $z_n = \infty$, then $\mathcal{B}$ acts trivially on $V_{\lambda^{(n)}}(\infty)$. More precisely, in this case, $b \in \mathcal{B}$ acts by $b \otimes 1$ where the first factor acts on $\bigotimes_{s=1}^{n-1} V_{\lambda^{(s)}}(z_s)$ and 1 acts on $V_{\lambda^{(n)}}(\infty)$.

We also denote $V_{\mathbf{\Lambda}}$ the module $V_{\mathbf{\Lambda},\boldsymbol{z}}$ considered as a $\mathfrak{g}_N$-module.

Let μ be a dominant integral $\mathfrak{g}_N$-weight and $k \in \mathbb{Z}_{\geqslant 0}$. Define an N-symmetric partition $\mu_{A,k}$ with at most N parts by the rule: $(\mu_{A,k})_N = k$

and

$$(4.2)\qquad (\mu_{A,k})_i-(\mu_{A,k})_{i+1}=\begin{cases}\langle\mu,\check\alpha_i\rangle, & \text{if } 1\leqslant i\leqslant [\frac{N}{2}],\\ \langle\mu,\check\alpha_{N-i}\rangle, & \text{if } [\frac{N}{2}]<i\leqslant N-1.\end{cases}$$

We call $\mu_{A,k}$ the partition *associated with weight* μ *and integer* k.

Let $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$ be a sequence of dominant integral $\mathfrak{g}_N$-weights and let $\boldsymbol{k}=(k_1,\dots,k_n)$ be an n-tuple of nonnegative integers. Then denote $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}=(\lambda^{(1)}_{A,k_1},\dots,\lambda^{(n)}_{A,k_n})$ the sequence of partitions associated with $\lambda^{(s)}$ and k_s, $s=1,\dots,n$.

We use notation $\mu_A=\mu_{A,0}$ and $\boldsymbol{\Lambda}_A=\boldsymbol{\Lambda}_{A,(0,\dots,0)}$.

Lemma 4.4. *If* Ξ *is a d-nontrivial sequence of partitions with at most* N *parts and* $\mathrm{s}\Omega_\Xi$ *is nonempty, then* Ξ *has the form* $\Xi=\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ *for a sequence of dominant integral* $\mathfrak{g}_N$*-weights* $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$ *and an* n*-tuple* $\boldsymbol{k}$ *of nonnegative integers. The pair* $(\boldsymbol{\Lambda},\boldsymbol{k})$ *is uniquely determined by* Ξ*. Moreover, if* $N=2r$*, then* $\sum_{s=1}^n\langle\lambda^{(s)},\check\alpha_r\rangle$ *is even.*

Proof. The first statement follows from Lemma 4.1. If $N=2r$ is even, the second statement follows from the equality

$$N(d-N)=|\Xi|=\sum_{s=1}^n r\Big(2\sum_{i=1}^{r-1}\langle\lambda^{(s)},\check\alpha_i\rangle+\langle\lambda^{(s)},\check\alpha_r\rangle\Big)+N\sum_{s=1}^n k_s. \qquad\square$$

Therefore the strata are effectively parameterized by sequences of dominant integral $\mathfrak{g}_N$-weights and tuples of nonnegative integers. In what follows we write $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ for $\mathrm{s}\Omega_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}}}$ and $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ for $\mathrm{s}\Omega_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$.

Define a formal differential operator

$$\mathcal{D}^{\mathcal{B}}=\partial_x^N+\sum_{i=1}^N G_i(x)\partial_x^{N-i}.$$

For a $\mathcal{B}$-eigenvector $v\in V_{\boldsymbol{\Lambda},\boldsymbol{z}}$, $G_i(x)v=h_i(x)v$, we denote $\mathcal{D}_v=\partial_x^N+\sum_{i=1}^N h_i(x)\partial_x^{N-i}$ the corresponding scalar differential operator.

Theorem 4.5. *Let* $N>3$.

There exists a choice of generators $G_i(x)$ *of the* $\mathfrak{g}_N$ *Bethe algebra* $\mathcal{B}$ *(see Proposition 4.3), such that for any sequence of dominant integral* $\mathfrak{g}_N$*-weights* $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$*, any* $\boldsymbol{z}\in\mathring{\mathbb{P}}_n$*, and any* $\mathcal{B}$*-eigenvector* $v\in(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$*, we have* Ker $((T_1\dots T_N)^{1/2}\cdot\mathcal{D}_v\cdot(T_1\dots T_N)^{-1/2})\in\mathrm{s}\Omega_{\boldsymbol{\Lambda}_A,\boldsymbol{z}}$*, where* $\boldsymbol{T}=(T_1,\dots,T_N)$ *is associated with* $\boldsymbol{\Lambda}_A,\boldsymbol{z}$*.*

Moreover, if $|\mathbf{\Lambda}_A| = N(d-N)$*, then this defines a bijection between the joint eigenvalues of* $\mathcal{B}$ *on* $(V_{\mathbf{\Lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$ *and* $\mathrm{s}\Omega_{\mathbf{\Lambda}_A,\boldsymbol{z}} \subset \mathrm{Gr}(N,d)$*.*

Proof. Theorem 4.5 is deduced from [R] in Section 7.2. □

The second part of the theorem also holds for $N=3$; see Section 4.6.

Remark 4.6. In particular, Theorem 4.5 implies that if $\dim(V_{\mathbf{\Lambda}})^{\mathfrak{g}_N} > 0$, then $\dim(V_{\mathbf{\Lambda}_{A,\boldsymbol{k}}})^{\mathfrak{sl}_N} > 0$. This statement also follows from Lemma A.2 given in the Appendix.

We also have the following lemma from [R].

Lemma 4.7. *Let* $\boldsymbol{z}$ *be a generic point in* $\mathring{\mathbb{P}}_n$*. Then the action of the* $\mathfrak{g}_N$ *Bethe algebra on* $(V_{\mathbf{\Lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$ *is diagonalizable and has simple spectrum. In particular, this statement holds for any sequence* $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$*.* □

4.3. Properties of the strata

We describe simple properties of the strata $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}}$.

Given $\mathbf{\Lambda},\boldsymbol{k},\boldsymbol{z}$, define $\tilde{\mathbf{\Lambda}},\tilde{\boldsymbol{k}},\tilde{\boldsymbol{z}}$ by removing all zero components, that is the ones with both $\lambda^{(s)}=0$ and $k_s=0$. Then $\mathrm{s}\Omega_{\tilde{\mathbf{\Lambda}},\tilde{\boldsymbol{k}},\tilde{\boldsymbol{z}}} = \mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ and $\mathrm{s}\Omega_{\tilde{\mathbf{\Lambda}},\tilde{\boldsymbol{k}}} = \mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}}$. Also, by Remark 4.6, if $(V_{\mathbf{\Lambda}})^{\mathfrak{g}_N} \neq 0$, then $\dim(V_{\mathbf{\Lambda}_{A,\boldsymbol{k}}})^{\mathfrak{sl}_N} > 0$, thus $|\mathbf{\Lambda}_{A,\boldsymbol{k}}|$ is divisible by N.

We say that $(\mathbf{\Lambda},\boldsymbol{k})$ is *d-nontrivial* if and only if $(V_{\mathbf{\Lambda}})^{\mathfrak{g}_N} \neq 0$, $|\lambda^{(s)}_{A,k_s}| > 0$, $s=1,\dots,n$, and $|\mathbf{\Lambda}_{A,\boldsymbol{k}}| = N(d-N)$.

If $(\mathbf{\Lambda},\boldsymbol{k})$ is d-nontrivial then the corresponding stratum $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ is nonempty, see Proposition 4.9 below.

Note that $|\mathbf{\Lambda}_{A,\boldsymbol{k}}| = |\mathbf{\Lambda}_A| + N|\boldsymbol{k}|$, where $|\boldsymbol{k}| = k_1+\dots+k_n$. In particular, if $(\mathbf{\Lambda},\mathbf{0})$ is d-nontrivial then $(\mathbf{\Lambda},\boldsymbol{k})$ is $(d+|\boldsymbol{k}|)$-nontrivial. Further, there exists a bijection between $\Omega_{\mathbf{\Lambda}_A,\boldsymbol{z}}$ in $\mathrm{Gr}(N,d)$ and $\Omega_{\mathbf{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$ in $\mathrm{Gr}(N,d+|\boldsymbol{k}|)$ given by

$$\Omega_{\mathbf{\Lambda}_A,\boldsymbol{z}} \to \Omega_{\mathbf{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}, \qquad X \mapsto \prod_{s=1}^{n}(x-z_s)^{k_s}\cdot X. \tag{4.3}$$

Moreover, (4.3) restricts to a bijection of $\mathrm{s}\Omega_{\mathbf{\Lambda}_A,\boldsymbol{z}}$ in $\mathrm{sGr}(N,d)$ and $\mathrm{s}\Omega_{\mathbf{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$ in $\mathrm{sGr}(N,d+|\boldsymbol{k}|)$.

If $(\mathbf{\Lambda},\boldsymbol{k})$ is d-nontrivial then $\mathbf{\Lambda}_{A,\boldsymbol{k}}$ is d-nontrivial. The converse is not true.

Example 4.8. For this example we write the highest weights in terms of fundamental weights, e.g. $(1,0,0,1) = \omega_1 + \omega_4$. We also use $\mathfrak{sl}_N$-modules instead of $\mathfrak{gl}_N$-modules, since the spaces of invariants are the same.

For $N = 4$ and $\mathfrak{g}_4 = \mathfrak{so}_5$ of type B_2, we have

$$\dim(V_{(2,0)} \otimes V_{(1,0)} \otimes V_{(2,0)})^{\mathfrak{g}_4} = 0 \ \text{ and } \ \dim(V_{(2,0,2)} \otimes V_{(1,0,1)} \otimes V_{(2,0,2)})^{\mathfrak{sl}_4} = 2.$$

Let $\boldsymbol{\Lambda} = ((2,0),(1,0),(2,0))$. Then $\boldsymbol{\Lambda}_A$ is 9-nontrivial, but $(\boldsymbol{\Lambda},(0,0,0))$ is not.

Similarly, for $N = 5$ and $\mathfrak{g}_5 = \mathfrak{sp}_4$ of type C_2, we have

$$\dim(V_{(1,0)} \otimes V_{(0,1)} \otimes V_{(0,1)})^{\mathfrak{g}_5} = 0 \ \text{ and } \ \dim(V_{(1,0,0,1)} \otimes V_{(0,1,1,0)} \otimes V_{(0,1,1,0)})^{\mathfrak{sl}_5} = 2.$$

Let $\boldsymbol{\Lambda} = ((1,0),(0,1),(1,0))$. Then $\boldsymbol{\Lambda}_A$ is 8-nontrivial, but $(\boldsymbol{\Lambda},(0,0,0))$ is not.

Let $\mu^{(1)},\dots,\mu^{(a)}$ be all distinct partitions in $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$. Let n_i be the number of occurrences of $\mu^{(i)}$ in $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$, then $\sum_{i=1}^a n_i = n$. Denote $\boldsymbol{n} = (n_1,\dots,n_a)$, we shall write $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ in the following order: $\lambda^{(i)}_{A,k_i} = \mu^{(j)}$ for $\sum_{s=1}^{j-1} n_s + 1 \leqslant i \leqslant \sum_{s=1}^{j} n_s$, $j = 1,\dots,a$.

Proposition 4.9. *Suppose $(\boldsymbol{\Lambda},\boldsymbol{k})$ is d-nontrivial. The set $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is a ramified covering of $\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$. Moreover, the degree of the covering is equal to $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{g}_N}$. In particular, $\dim \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} = n$. Over $\mathbb{R}\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$, this covering is unramified of the same degree, moreover all points in fibers are real.*

Proof. The proposition follows from Theorem 4.5, Lemma 4.7, and Theorem 1.1 of [MTV3]. □

We find strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ of the largest dimension.

Lemma 4.10. *If $N = 2r$, then the d-nontrivial stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ with the largest dimension has $(\lambda^{(s)},k_s) = (\omega_r,0)$, $s = 1,\dots,2(d-N)$. In particular, the dimension of this stratum is $2(d-N)$.*

If $N = 2r+1$, the d-nontrivial strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ with the largest dimension have $(\lambda^{(s)},k_s)$ equal to either $(\omega_{j_s},0)$ with some $j_s \in \{1,\dots,r\}$, or to $(0,1)$, for $s = 1,\dots,d-N$. Each such stratum is either empty or has dimension $d-N$. There is at least one nonempty stratum of this dimension, and if $d > N+1$ then more than one.

Proof. By Proposition 4.9, we are going to find the maximal n such that $(\boldsymbol{\Lambda},\boldsymbol{k})$ is d-nontrivial, where $\boldsymbol{\Lambda} = (\lambda^{(1)},\dots,\lambda^{(n)})$ is a sequence of dominant integral $\mathfrak{g}_N$-weights and $\boldsymbol{k} = (k_1,\dots,k_n)$ is an n-tuple of nonnegative integers. Since $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ is d-nontrivial, it follows that $\lambda^{(s)} \neq 0$ or $\lambda^{(s)} = 0$ and $k_s > 0$, for all $s = 1,\dots,n$.

Suppose $N = 2r$. If $\lambda^{(s)} \neq 0$, we have

$$|\lambda^{(s)}_{A,k_s}| \geqslant |\lambda^{(s)}_{A,0}| = r\Big(2\sum_{i=1}^{r-1}\langle\lambda^{(s)},\check\alpha_i\rangle + \langle\lambda^{(s)},\check\alpha_r\rangle\Big) \geqslant r.$$

If $k_s > 0$, then $|\lambda^{(s)}_{A,k_s}| \geqslant 2rk_s \geqslant 2r$. Therefore, it follows that

$$rn \leqslant \sum_{s=1}^{n} |\lambda^{(s)}_{A,k_s}| = |\boldsymbol{\Lambda}_{A,\boldsymbol{k}}| = (d-N)N.$$

Hence $n \leqslant 2(d-N)$.

If we set $\lambda^{(s)} = w_r$ and $k_s = 0$ for all $s = 1, \dots, 2(d-N)$. Then $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial since

$$\dim(V_{\omega_r} \otimes V_{\omega_r})^{\mathfrak{so}_{2r+1}} = 1.$$

Now let us consider $N = 2r+1$, $r \geqslant 1$. Similarly, if $\lambda^{(s)} \neq 0$, we have

$$|\lambda^{(s)}_{A,k_s}| \geqslant |\lambda^{(s)}_{A,0}| = (2r+1)\sum_{i=1}^{r}\langle\lambda^{(s)},\check\alpha_i\rangle \geqslant 2r+1.$$

If $k_s > 0$, then $|\lambda^{(s)}_{A,k_s}| \geqslant (2r+1)k_s \geqslant 2r+1$. It follows that

$$(2r+1)n \leqslant \sum_{s=1}^{n} |\lambda^{(s)}_{A,k_s}| = |\boldsymbol{\Lambda}_{A,\boldsymbol{k}}| = (d-N)N.$$

Hence $n \leqslant d-N$. Clearly, the equality is achieved only for the $(\boldsymbol{\Lambda}, \boldsymbol{k})$ described in the statement of the lemma. Note that if $(\lambda^{(s)}, k_s) = (0,1)$ for all $s = 1, \dots, d-N$, then $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial and therefore nonempty. If $d > N+1$ we also have d-nontrivial tuples parameterized by $i = 1, \dots, r$, such that $(\lambda^{(s)}, k_s) = (0,1)$, $s = 3, \dots, d-N$, and $(\lambda^{(s)}, k_s) = (\omega_i, 0)$, $s = 1, 2$. □

4.4. The $\mathfrak{g}_N$-stratification of self-dual Grassmannian

The following theorem follows directly from Theorems 3.5 and 4.5.

Theorem 4.11. *We have*

$$\mathrm{sGr}(N,d) = \bigsqcup_{d\text{-nontrivial } (\boldsymbol{\Lambda},\boldsymbol{k})} \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}. \tag{4.4}$$

□

Next, for a d-nontrivial $(\boldsymbol{\Lambda}, \boldsymbol{k})$, we call the closure of $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ inside $\mathrm{sGr}(N, d)$, a $\mathfrak{g}_N$-*cycle*. The $\mathfrak{g}_N$-cycles $\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}}$ are algebraic sets in $\mathrm{sGr}(N, d)$ and therefore in $\mathrm{Gr}(N, d)$. We describe $\mathfrak{g}_N$-cycles as unions of $\mathfrak{g}_N$-strata similar to (3.8).

Define a partial order $\geqslant$ on the set of pairs $\{(\boldsymbol{\Lambda}, \boldsymbol{k})\}$ as follows. Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$, $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(m)})$ be two sequences of dominant integral $\mathfrak{g}_N$-weights. Let $\boldsymbol{k} = (k_1, \dots, k_n)$, $\boldsymbol{l} = (l_1, \dots, l_m)$ be two tuples of nonnegative integers. We say that $(\boldsymbol{\Lambda}, \boldsymbol{k}) \geqslant (\boldsymbol{\Xi}, \boldsymbol{l})$ if and only if there exists a partition $\{I_1, \dots, I_m\}$ of $\{1, 2, \dots, n\}$ such that

$$\mathrm{Hom}_{\mathfrak{g}_N}\Big(V_{\xi^{(i)}}, \bigotimes_{j \in I_i} V_{\lambda^{(j)}}\Big) \neq 0, \qquad |\xi^{(i)}_{A,l_i}| = \sum_{j \in I_i} |\lambda^{(j)}_{A,k_j}|,$$

for $i = 1, \dots, m$.

If $(\boldsymbol{\Lambda}, \boldsymbol{k}) \geqslant (\boldsymbol{\Xi}, \boldsymbol{l})$ are d-nontrivial, we call $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$ a *degeneration* of $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$. If we suppose further that $m = n - 1$, we call $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$ a *simple degeneration* of $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$.

Theorem 4.12. *If* $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$ *is a degeneration of* $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$*, then* $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$ *is contained in the* $\mathfrak{g}_N$*-cycle* $\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}}$.

Theorem 4.12 is proved in Section 7.2.

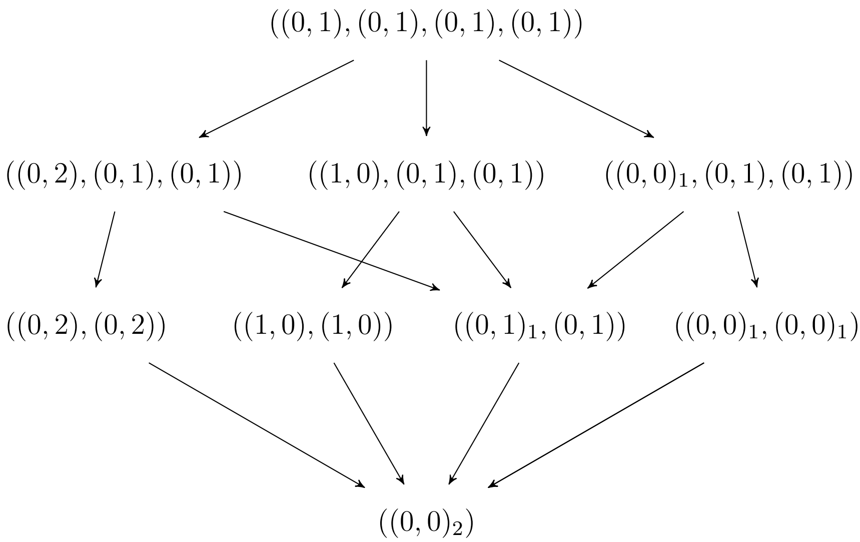
Theorem 4.13. *For d-nontrivial* $(\boldsymbol{\Lambda}, \boldsymbol{k})$*, we have*

$$\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}} = \bigsqcup_{\substack{(\boldsymbol{\Xi},\boldsymbol{l}) \leqslant (\boldsymbol{\Lambda},\boldsymbol{k}), \\ d\text{-nontrivial } (\boldsymbol{\Xi},\boldsymbol{l})}} \mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}. \tag{4.5}$$

Theorem 4.13 is proved in Section 7.2.

Theorems 4.11 and 4.13 imply that the subsets $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ with d-nontrivial $(\boldsymbol{\Lambda}, \boldsymbol{k})$ give a stratification of $\mathrm{sGr}(N, d)$, similar to the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N, d)$; see (3.7) and (3.8). We call it the $\mathfrak{g}_N$*-stratification of* $\mathrm{sGr}(N, d)$.

Example 4.14. The following picture gives an example for $\mathfrak{so}_5$-stratification of $\mathrm{sGr}(4, 6)$. In the following picture, we write $((\lambda^{(1)})_{k_1}, \dots, (\lambda^{(n)})_{k_n})$ for $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$. We also simply write $\lambda^{(s)}$ for $(\lambda^{(s)})_0$. For instance, $((0, 1)_1, (0, 1))$ represents $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ where $\boldsymbol{\Lambda} = ((0, 1), (0, 1))$ and $\boldsymbol{k} = (1, 0)$. The solid arrows represent simple degenerations. Unlike the picture in Example 3.9 we do not include here the pairs of sequences which are not 6-nontrivial, as there are too many of them.



In particular, the stratum $\mathrm{s}\Omega_{((0,1),(0,1),(0,1),(0,1))}$ is dense in $\mathrm{sGr}(4, 6)$.

Proposition 4.15. *If $N = 2r$ is even, then the stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ with $(\lambda^{(s)}, k_s) = (\omega_r, 0)$, where $s = 1, \dots, 2(d - N)$, is dense in $\mathrm{sGr}(N, d)$.*

Proof. For $N = 2r$, one has the $\mathfrak{g}_N$-module decomposition

$$V_{\omega_r} \otimes V_{\omega_r} = V_{2\omega_r} \oplus V_{\omega_1} \oplus \cdots \oplus V_{\omega_{r-1}} \oplus V_{(0,\dots,0)}. \tag{4.6}$$

It is clear that $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial. It also follows from (4.6) that if $(\boldsymbol{\Xi}, \boldsymbol{l})$ is d-nontrivial then $(\boldsymbol{\Lambda}, \boldsymbol{k}) \geqslant (\boldsymbol{\Xi}, \boldsymbol{l})$. The proposition follows from Theorems 4.11 and 4.13. □

Remark 4.16. The group of affine translations, see Remark 3.11, preserves the self-dual Grassmannian $\mathrm{sGr}(N, d)$ and the strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$.

4.5. The $\mathfrak{g}_N$-stratification of sGr(N, d) and the Wronski map

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of dominant integral $\mathfrak{g}_N$-weights and let $\boldsymbol{k} = (k_1, \dots, k_n)$ be an n-tuple of nonnegative integers. Let $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$.

Recall that $\lambda_i^{(s)} = \langle \lambda^{(s)}, \check{\alpha}_i \rangle$. If $X \in \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$, one has

$$\mathrm{Wr}(X) = \begin{cases} \Big(\prod_{s=1}^{n} (x - z_s)^{\lambda_1^{(s)} + \dots + \lambda_r^{(s)} + k_s} \Big)^N, & \text{if } N = 2r + 1; \\ \Big(\prod_{s=1}^{n} (x - z_s)^{2\lambda_1^{(s)} + \dots + 2\lambda_{r-1}^{(s)} + \lambda_r^{(s)} + 2k_s} \Big)^r, & \text{if } N = 2r. \end{cases}$$

We define the *reduced Wronski map* $\overline{\mathrm{Wr}}$ as follows.

If $N = 2r + 1$, the reduced Wronski map

$$\overline{\mathrm{Wr}} : \mathrm{sGr}(N, d) \to \mathrm{Gr}(1, d - N + 1)$$

is sending $X \in \mathrm{sGr}(N, d)$ to $\mathbb{C}(\mathrm{Wr}(X))^{1/N}$.

If $N = 2r$, the reduced Wronski map

$$\overline{\mathrm{Wr}} : \mathrm{sGr}(N, d) \to \mathrm{Gr}(1, 2(d - N) + 1)$$

is sending $X \in \mathrm{sGr}(N, d)$ to $\mathbb{C}(\mathrm{Wr}(X))^{1/r}$.

The reduced Wronski map is also a finite map.

For $N = 2r$, the degree of the reduced Wronski map is given by $\dim(V_{\omega_r}^{\otimes 2(d-N)})^{\mathfrak{g}_N}$. This dimension is given by, see [KLP],

$$(N-1)!! \prod_{1\leqslant i<j\leqslant r} \big((j-i)(N-i-j+1)\big) \prod_{k=0}^{r-1} \frac{(2(d-N+k))!}{(d-k-1)!(d-N+k)!}. \tag{4.7}$$

Let $\tilde{d} = d - N + 1$ if $N = 2r + 1$ and $\tilde{d} = 2(d - N) + 1$ if $N = 2r$. Let $\boldsymbol{m} = (m_1, \ldots, m_n)$ be an unordered sequence of positive integers such that $|\boldsymbol{m}| = \tilde{d} - 1$.

Similar to Section 3.4, we have the following proposition.

Proposition 4.17. *The preimage of the stratum $\Omega_{\boldsymbol{m}}$ of $\mathrm{Gr}(1, \tilde{d})$ under the reduced Wronski map is a union of all strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ of $\mathrm{sGr}(N, d)$ such that $|\lambda^{(s)}_{A,k_s}| = Nm_s$, $s = 1, \ldots, n$, if N is odd and such that $|\lambda^{(s)}_{A,k_s}| = rm_s$, $s = 1, \ldots, n$, if $N = 2r$ is even.* □

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ be an unordered sequence of dominant integral $\mathfrak{g}_N$-weights and $\boldsymbol{k} = (k_1, \ldots, k_n)$ a sequence of nonnegative integers. Let a be the number of distinct pairs in the set $\{(\lambda^{(s)}, k_s),\ s = 1, \ldots, n\}$. We can assume that $(\lambda^{(1)}, k_1), \ldots, (\lambda^{(a)}, k_a)$ are all distinct, and let $n_1, \ldots, n_a$ be their multiplicities, $n_1 + \cdots + n_a = n$.

Consider the unordered set of integers $\boldsymbol{m} = (m_1, \ldots, m_n)$, where $Nm_s = |\lambda^{(s)}_{A,k_s}|$ if N is odd or $rm_s = |\lambda^{(s)}_{A,k_s}|$ if $N = 2r$ is even. Consider the stratum $\Omega_{\boldsymbol{m}}$ in $\mathrm{Gr}(1, \tilde{d})$, corresponding to polynomials with n distinct roots of multiplicities $m_1, \ldots, m_n$.

Proposition 4.18. *Let $(\boldsymbol{\Lambda}, \boldsymbol{k})$ be d-nontrivial. Then the reduced Wronski map $\overline{\mathrm{Wr}}|_{\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}} : \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \to \Omega_{\boldsymbol{m}}$ is a ramified covering of degree $b(\boldsymbol{\Lambda}_{A,\boldsymbol{k}}) \dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{g}_N}$, where $b(\boldsymbol{\Lambda}_{A,\boldsymbol{k}})$ is given by (3.9).*

Proof. The statement follows from Theorem 4.5, Lemma 4.7, and Proposition 4.17. ☐

In other words, the $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$ given by Theorems 4.11 and 4.13, is adjacent to the swallowtail $\mathfrak{gl}_1$-stratification of $\mathrm{Gr}(1,\tilde{d})$ and the reduced Wronski map.

4.6. Self-dual Grassmannian for $N = 3$

Let $N = 3$ and $\mathfrak{g}_3 = \mathfrak{sl}_2$. We identify the dominant integral $\mathfrak{sl}_2$-weights with nonnegative integers. Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)}, \lambda)$ be a sequence of nonnegative integers and $\boldsymbol{z} = (z_1, \dots, z_n, \infty) \in \mathring{\mathbb{P}}_{n+1}$.

Choose d large enough so that $k := d - 3 - \sum_{s=1}^{n} \lambda^{(s)} - \lambda \geqslant 0$. Let $\boldsymbol{k} = (0, \dots, 0, k)$. Then $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ has coordinates

$$\lambda_A^{(s)} = (2\lambda^{(s)}, \lambda^{(s)}, 0), \quad s = 1, \dots, n,$$

$$\lambda_{A,k} = \Big(d - 3 - \sum_{s=1}^{n} \lambda^{(s)} + \lambda, d - 3 - \sum_{s=1}^{n} \lambda^{(s)}, d - 3 - \sum_{s=1}^{n} \lambda^{(s)} - \lambda\Big).$$

Note that we always have $|\boldsymbol{\Lambda}_{A,\boldsymbol{k}}| = 3(d-3)$ and spaces of polynomials in $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ are pure self-dual spaces.

Theorem 4.19. *There exists a bijection between the common eigenvectors in $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_2}$ of the $\mathfrak{gl}_2$ Bethe algebra $\mathcal{B}$ and $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$.*

Proof. Let $X \in \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$, and let $\boldsymbol{T} = (T_1(x), T_2(x), T_3(x))$ be associated with $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}, \boldsymbol{z}$, then

$$T_1(x) = T_2(x) = \prod_{s=1}^{n} (x - z_s)^{\lambda^{(s)}}.$$

Following Section 6 of [MV1], let $\boldsymbol{u} = (u_1, u_2, u_3)$ be a Witt basis of X, one has

$$\mathrm{Wr}(u_1, u_2) = T_1 u_1, \quad \mathrm{Wr}(u_1, u_3) = T_1 u_2, \quad \mathrm{Wr}(u_2, u_3) = T_1 u_3.$$

Let $y(x,c) = u_1 + c u_2 + \frac{c^2}{2} u_3$, it follows from Lemma 6.15 of [MV1] that

$$\mathrm{Wr}\Big(y(x,c), \frac{\partial y}{\partial c}(x,c)\Big) = T_1 y(x,c).$$

Since X has no base points, there must exist $c' \in \mathbb{C}$ such that $y(x,c')$ and $T_1(x)$ do not have common roots. It follows from Lemma 6.16 of [MV1] that

$y(x,c') = p^2$ and $y(x,c) = (p + (c - c')q)^2$ for suitable polynomials $p(x), q(x)$ satisfying $\mathrm{Wr}(p,q) = 2T_1$. In particular, $\{p^2, pq, q^2\}$ is a basis of X. Without loss of generality, we can assume that $\deg p < \deg q$. Then

$$\deg p = \frac{1}{2}\Big(\sum_{s=1}^{n} \lambda^{(s)} - \lambda\Big), \quad \deg q = \frac{1}{2}\Big(\sum_{s=1}^{n} \lambda^{(s)} + \lambda\Big) + 1.$$

Since X has no base points, p and q do not have common roots. Combining with the equality $\mathrm{Wr}(p,q) = 2T_1$, one has that the space spanned by p and q has singular points at $z_1, \dots, z_n$ and ∞ only. Moreover, the exponents at z_s, $s = 1, \dots, n$, are equal to $0, \lambda^{(s)} + 1$, and the exponents at ∞ are equal to $-\deg p, -\deg q$.

By Theorem 3.2, the space $\mathrm{span}\{p,q\}$ corresponds to a common eigenvector of the $\mathfrak{gl}_2$ Bethe subalgebra in the subspace $\big(\bigotimes_{s=1}^{n} V_{(\lambda^{(s)},0)}(z_s) \otimes V_{(d-2-\deg p, d-1-\deg q)}(\infty)\big)^{\mathfrak{sl}_2}$.

Conversely, given a common eigenvector of the $\mathfrak{gl}_2$ Bethe algebra in $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_2}$, by Theorem 3.2, it corresponds to a space $\tilde{X}$ of polynomials in $\mathrm{Gr}(2,d)$ without base points. Let $\{p,q\}$ be a basis of $\tilde{X}$, define a space of polynomials $\mathrm{span}\{p^2, pq, q^2\}$ in $\mathrm{Gr}(3,d)$. It is easy to see that $\mathrm{span}\{p^2, pq, q^2\} \in \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ is a pure self-dual space. □

Let $X \in \mathrm{Gr}(2,d)$, denote by X^2 the space spanned by f^2 for all polynomials $f \in X$. It is clear that $X^2 \in \mathrm{sGr}(3, 2d-1)$. Define

$$\pi : \mathrm{Gr}(2,d) \to \mathrm{sGr}(3, 2d-1) \qquad (4.8)$$

by sending X to X^2. The map π is an injective algebraic map.

Corollary 4.20. *The map π defines a bijection between the subset of spaces of polynomials without base points in* $\mathrm{Gr}(2,d)$ *and the subset of pure self-dual spaces in* $\mathrm{sGr}(3, 2d-1)$. □

Note that not all self-dual spaces in $\mathrm{sGr}(3, 2d-1)$ can be expressed as X^2 for some $X \in \mathrm{Gr}(2,d)$ since the greatest common divisor of a self-dual space does not have to be a square of a polynomial.

4.7. Geometry and topology

It would be very interesting to determine the topology and geometry of the strata and cycles of $\mathrm{Gr}(N,d)$ and of $\mathrm{sGr}(N,d)$. In particular, it would be interesting to understand the geometry and topology of the self-dual Grassmannian $\mathrm{sGr}(N,d)$. Here are some simple examples of small dimension.

Of course, $\mathrm{sGr}(N,N) = \mathrm{Gr}(N,N)$ is just one point. Also, $\mathrm{sGr}(2r+1, 2r+2)$ is just $\mathbb{P}^1$.

Consider $\mathrm{sGr}(2r, 2r+1)$, $r \geqslant 1$. It has only two strata: $\mathrm{s}\Omega_{(\omega_r,\omega_r),(0,0)}$ and $\mathrm{s}\Omega_{(0),(1)}$. Moreover, the reduced Wronski map has degree 1 and defines a bijection: $\overline{\mathrm{Wr}} : \mathrm{sGr}(2r, 2r+1) \to \mathrm{Gr}(1,3)$. In particular, the $\mathfrak{so}_{2r+1}$-stratification in this case is identified with the swallowtail $\mathfrak{gl}_1$-stratification of quadratics. There are two strata: polynomials with two distinct roots and polynomials with one double root. Therefore, through the reduced Wronski map, the self-dual Grassmannian $\mathrm{sGr}(2r, 2r+1)$ can be identified with $\mathbb{P}^2$ with coordinates $(a_0 : a_1 : a_2)$ and the stratum $\mathrm{s}\Omega_{(0),(1)}$ is a nonsingular curve of degree 2 given by the equation $a_1^2 - 4a_0a_2 = 0$.

Consider $\mathrm{sGr}(2r+1, 2r+3)$, $r \geqslant 1$. In this case, we have $r+2$ strata: $\mathrm{s}\Omega_{(\omega_i,\omega_i),(0,0)}$, $i = 1, \dots, r$, $\mathrm{s}\Omega_{(0,0),(1,1)}$, and $\mathrm{s}\Omega_{(0),(2)}$. The reduced Wronski map $\overline{\mathrm{Wr}} : \mathrm{sGr}(2r+1, 2r+3) \to \mathrm{Gr}(1,3)$ restricted to any strata again has degree 1. Therefore, through the reduced Wronski map, the self-dual Grassmannian $\mathrm{sGr}(2r+1, 2r+3)$ can be identified with $r+1$ copies of $\mathbb{P}^2$ all intersecting in the same nonsingular degree 2 curve corresponding to the stratum $\mathrm{s}\Omega_{(0),(2)}$. In particular, every 2-dimensional $\mathfrak{sp}_{2r}$-cycle is just $\mathbb{P}^2$.

Consider $\mathrm{sGr}(2r+1, 2r+4)$, $r \geqslant 1$. We have $\dim \mathrm{sGr}(2r+1, 2r+4) = 3$. This is the last case when for all strata the coverings of Proposition 4.9 have degree one. There are already many strata. For example, consider $\mathrm{sGr}(5,8)$, that is $r = 2$. There are four strata of dimension 3 corresponding to the following sequences of $\mathfrak{sp}_4$-weights and 3-tuples of nonnegative integers:

$$\boldsymbol{\Lambda}_1 = (\omega_1, \omega_1, 0), \quad \boldsymbol{k}_1 = (0,0,1); \qquad \boldsymbol{\Lambda}_2 = (\omega_1, \omega_1, \omega_2), \quad \boldsymbol{k}_2 = (0,0,0);$$
$$\boldsymbol{\Lambda}_3 = (\omega_2, \omega_2, 0), \quad \boldsymbol{k}_3 = (0,0,1); \qquad \boldsymbol{\Lambda}_4 = (0,0,0), \quad \boldsymbol{k}_4 = (1,1,1).$$

By the reduced Wronski map, the stratum $\Omega_{\boldsymbol{\Lambda}_4,\boldsymbol{k}_4}$ is identified with the subset of $\mathrm{Gr}(1,4)$ represented by cubic polynomials without multiple roots and the cycle $\overline{\Omega}_{\boldsymbol{\Lambda}_4,\boldsymbol{k}_4}$ with $\mathrm{Gr}(1,4) = \mathbb{P}^3$. The stratification of $\overline{\Omega}_{\boldsymbol{\Lambda}_4,\boldsymbol{k}_4}$ is just the swallowtail of cubic polynomials. However, for other three strata the reduced Wronski map has degree 3. Using instead the map in Proposition 4.9, we identify each of these strata with $\mathring{\mathbb{P}}_3/(\mathbb{Z}/2\mathbb{Z})$ or with the subset of $\mathrm{Gr}(1,3) \times \mathrm{Gr}(1,2)$ represented by a pair of polynomials (p_1, p_2), such that $\deg(p_1) \leqslant 2$, $\deg(p_2) \leqslant 1$ and such that all three roots (including infinity) of p_1p_2 are distinct. Then the corresponding $\mathfrak{sp}_4$-cycles $\overline{\Omega}_{\boldsymbol{\Lambda}_i,\boldsymbol{k}_i}$, $i = 1,2,3$, are identified with $\mathrm{Gr}(1,3) \times \mathrm{Gr}(1,2) = \mathbb{P}^2 \times \mathbb{P}^1$.

A similar picture is observed for 3-dimensional strata in the case of $\mathrm{sGr}(2r, 2r+2)$. Consider, for example, $\mathrm{Gr}(2,4)$; see Example 3.9. Then the 4-dimensional stratum $\Omega_{(1,0),(1,0),(1,0),(1,0)}$ is dense and (relatively) complicated, as the corresponding covering in Proposition 3.4 has degree 2. But for the 3-dimensional strata the degrees are 1. Therefore, $\Omega_{(2,0),(1,0),(1,0)}$ and $\Omega_{(1,1),(1,0),(1,0)}$ are identified with $\mathring{\mathbb{P}}_3/(\mathbb{Z}/2\mathbb{Z})$ and the corresponding cycles are just $\mathrm{Gr}(1,3) \times \mathrm{Gr}(1,2) = \mathbb{P}^2 \times \mathbb{P}^1$.

5. More notation

5.1. Lie algebras

Let $\mathfrak{g}$ and $\mathfrak{h}$ be as in Section 2.2. One has the Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Introduce also the positive and negative Borel subalgebras $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$.

Let $\mathscr{G}$ be a simple Lie group, $\mathscr{B}$ a Borel subgroup, and $\mathscr{N} = [\mathscr{B}, \mathscr{B}]$ its unipotent radical, with the corresponding Lie algebras $\mathfrak{n}_+ \subset \mathfrak{b} \subset \mathfrak{g}$. Let $\mathscr{G}$ act on $\mathfrak{g}$ by adjoint action.

Let $E_1, \dots, E_r \in \mathfrak{n}_+$, $\check{\alpha}_1, \dots, \check{\alpha}_r \in \mathfrak{h}$, $F_1, \dots, F_r \in \mathfrak{n}_-$ be the Chevalley generators of $\mathfrak{g}$. Let p_{-1} be the regular nilpotent element $\sum_{i=1}^r F_i$. The set $p_{-1} + \mathfrak{b} = \{p_{-1} + b \mid b \in \mathfrak{b}\}$ is invariant under conjugation by elements of $\mathscr{N}$. Consider the quotient space $(p_{-1} + \mathfrak{b})/\mathscr{N}$ and denote the $\mathscr{N}$-conjugacy class of $g \in p_{-1} + \mathfrak{b}$ by $[g]_{\mathfrak{g}}$.

Let $\check{\mathcal{P}} = \{\check{\lambda} \in \mathfrak{h} | \langle \alpha_i, \check{\lambda} \rangle \in \mathbb{Z},\ i = 1, \dots, r\}$ and $\check{\mathcal{P}}^+ = \{\check{\lambda} \in \mathfrak{h} | \langle \alpha_i, \check{\lambda} \rangle \in \mathbb{Z}_{\geqslant 0},\ i = 1, \dots, r\}$ be the coweight lattice and the cone of dominant integral coweights. Let $\rho \in \mathfrak{h}^*$ and $\check{\rho} \in \mathfrak{h}$ be the Weyl vector and covector such that $\langle \rho, \check{\alpha}_i \rangle = 1$ and $\langle \alpha_i, \check{\rho} \rangle = 1$, $i = 1, \dots, r$.

The Weyl group $\mathcal{W} \subset \mathrm{Aut}(\mathfrak{h}^*)$ is generated by simple reflections s_i, $i = 1, \dots, r$,

$$s_i(\lambda) = \lambda - \langle \lambda, \check{\alpha}_i \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The restriction of the bilinear form $(\cdot, \cdot)$ to $\mathfrak{h}$ is nondegenerate and induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. The action of $\mathcal{W}$ on $\mathfrak{h}$ is given by $s_i(\check{\mu}) = \check{\mu} - \langle \alpha_i, \check{\mu} \rangle \check{\alpha}_i$ for $\check{\mu} \in \mathfrak{h}$. We use the notation

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \cdot \check{\lambda} = w(\check{\lambda} + \check{\rho}) - \check{\rho}, \quad w \in \mathcal{W},\ \lambda \in \mathfrak{h}^*,\ \check{\lambda} \in \mathfrak{h},$$

for the shifted action of the Weyl group on $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively.

Let ${}^t\mathfrak{g} = \mathfrak{g}({}^tA)$ be the Langlands dual Lie algebra of $\mathfrak{g}$, then ${}^t(\mathfrak{so}_{2r+1}) = \mathfrak{sp}_{2r}$ and ${}^t(\mathfrak{sp}_{2r}) = \mathfrak{so}_{2r+1}$. A system of simple roots of ${}^t\mathfrak{g}$ is $\check{\alpha}_1, \dots, \check{\alpha}_r$ with the corresponding coroots $\alpha_1, \ldots, \alpha_r$. A coweight $\check{\lambda} \in \mathfrak{h}$ of $\mathfrak{g}$ can be identified with a weight of ${}^t\mathfrak{g}$.

For a vector space X we denote by $\mathcal{M}(X)$ the space of X-valued meromorphic functions on $\mathbb{P}^1$. For a group R we denote by $R(\mathcal{M})$ the group of R-valued meromorphic functions on $\mathbb{P}^1$.

5.2. $\mathfrak{sp}_{2r}$ as a subalgebra of $\mathfrak{sl}_{2r}$

Let $v_1, \ldots, v_{2r}$ be a basis of $\mathbb{C}^{2r}$. Define a nondegenerate skew-symmetric form χ on $\mathbb{C}^{2r}$ by

$$\chi(v_i, v_j) = (-1)^{i+1}\delta_{i,2r+1-j}, \quad i, j = 1, \ldots, 2r.$$

The special symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2r}$ by definition consists of all endomorphisms K of $\mathbb{C}^{2r}$ such that $\chi(Kv, v') + \chi(v, Kv') = 0$ for all $v, v' \in \mathbb{C}^{2r}$. This identifies $\mathfrak{sp}_{2r}$ with a Lie subalgebra of $\mathfrak{sl}_{2r}$.

Denote E_{ij} the matrix with zero entries except 1 at the intersection of the i-th row and j-th column.

The Chevalley generators of $\mathfrak{g} = \mathfrak{sp}_{2r}$ are given by

$$\begin{aligned}
&E_i = E_{i,i+1} + E_{2r-i,2r+1-i}, \quad F_i = E_{i+1,i} + E_{2r+1-i,2r-i}, \quad i = 1, \ldots, r-1,\\
&E_r = E_{r,r+1}, \qquad F_r = E_{r+1,r},\\
&\check{\alpha}_j = E_{jj} - E_{j+1,j+1} + E_{2r-j,2r-j} - E_{2r+1-j,2r+1-j}, \quad \check{\alpha}_r = E_{rr} - E_{r+1,r+1},\\
&\quad j = 1, \ldots, r-1.
\end{aligned}$$

Moreover, a coweight $\check{\lambda} \in \mathfrak{h}$ can be written as

$$\check{\lambda} = \sum_{i=1}^{r} \Big(\langle \alpha_i, \check{\lambda} \rangle + \cdots + \langle \alpha_{r-1}, \check{\lambda} \rangle + \langle \alpha_r, \check{\lambda} \rangle / 2 \Big) (E_{ii} - E_{2r+1-i,2r+1-i}). \tag{5.1}$$

In particular,

$$\check{\rho} = \sum_{i=1}^{r} \frac{2r-2i+1}{2} (E_{ii} - E_{2r+1-i,2r+1-i}).$$

For convenience, we denote the coefficient of E_{ii} in the right hand side of (5.1) by $(\check{\lambda})_{ii}$, for $i = 1, \ldots, 2r$.

5.3. $\mathfrak{so}_{2r+1}$ as a subalgebra of $\mathfrak{sl}_{2r+1}$

Let $v_1, \dots, v_{2r+1}$ be a basis of $\mathbb{C}^{2r+1}$. Define a nondegenerate symmetric form χ on $\mathbb{C}^{2r+1}$ by

$$\chi(v_i, v_j) = (-1)^{i+1}\delta_{i,2r+2-j}, \quad i, j = 1, \dots, 2r+1.$$

The special orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}_{2r+1}$ by definition consists of all endomorphisms K of $\mathbb{C}^{2r+1}$ such that $\chi(Kv, v') + \chi(v, Kv') = 0$ for all $v, v' \in \mathbb{C}^{2r+1}$. This identifies $\mathfrak{so}_{2r+1}$ with a Lie subalgebra of $\mathfrak{sl}_{2r+1}$.

Denote E_{ij} the matrix with zero entries except 1 at the intersection of the i-th row and j-th column.

The Chevalley generators of $\mathfrak{g} = \mathfrak{so}_{2r+1}$ are given by

$$\begin{aligned} &E_i = E_{i,i+1} + E_{2r+1-i,2r+2-i}, \quad F_i = E_{i+1,i} + E_{2r+2-i,2r+1-i}, \\ &\quad i = 1, \dots, r, \\ &\check{\alpha}_j = E_{jj} - E_{j+1,j+1} + E_{2r+1-j,2r+1-j} - E_{2r+2-j,2r+2-j}, \quad j = 1, \dots, r. \end{aligned}$$

Moreover, a coweight $\check{\lambda} \in \mathfrak{h}$ can be written as

$$\check{\lambda} = \sum_{i=1}^{r} \Big(\langle \alpha_i, \check{\lambda}\rangle + \dots + \langle \alpha_r, \check{\lambda}\rangle\Big)(E_{ii} - E_{2r+2-i,2r+2-i}). \tag{5.2}$$

In particular,

$$\check{\rho} = \sum_{i=1}^{r} (r+1-i)(E_{ii} - E_{2r+2-i,2r+2-i}).$$

For convenience, we denote the coefficient of E_{ii} in the right hand side of (5.2) by $(\check{\lambda})_{ii}$, for $i = 1, \dots, 2r+1$.

5.4. Lemmas on spaces of polynomials

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)}, \lambda)$ be a sequence of partitions with at most N parts such that $|\boldsymbol{\Lambda}| = N(d-N)$ and let $\boldsymbol{z} = (z_1, \dots, z_n, \infty) \in \mathring{\mathbb{P}}_{n+1}$.

Given an N-dimensional space of polynomials X, denote by $\mathcal{D}_X$ the monic scalar differential operator of order N with kernel X. The operator $\mathcal{D}_X$ is a monodromy-free Fuchsian differential operator with rational coefficients.

Lemma 5.1. *A subspace $X \subset \mathbb{C}_d[x]$ is a point of $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ if and only if the operator $\mathcal{D}_X$ is Fuchsian, regular in $\mathbb{C} \setminus \{z_1, \dots, z_n\}$, the exponents at z_s, $s = 1, \dots, n$, being equal to $\lambda_N^{(s)}, \lambda_{N-1}^{(s)} + 1, \dots, \lambda_1^{(s)} + N - 1$, and the exponents at ∞ being equal to $1 + \lambda_N - d, 2 + \lambda_{N-1} - d, \dots, N + \lambda_1 - d$.* □

Let $\boldsymbol{T} = (T_1, \dots, T_N)$ be associated with $\boldsymbol{\Lambda}, \boldsymbol{z}$, see (4.1). Let $\Gamma = \{u_1, \dots, u_N\}$ be a basis of $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$, define a sequence of polynomials

$$y_{N-i} = \mathrm{Wr}^{\dagger}(u_1, \dots, u_i), \quad i = 1, \dots, N-1. \tag{5.3}$$

Denote $(y_1, \dots, y_{N-1})$ by $\boldsymbol{y}_\Gamma$. We say that $\boldsymbol{y}_\Gamma$ is *constructed from the basis* Γ.

Lemma 5.2 ([MV1]). *Suppose $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and let $\Gamma = \{u_1, \dots, u_N\}$ be a basis of X. If $\boldsymbol{y}_\Gamma = (y_1, \dots, y_{N-1})$ is constructed from Γ, then*

$$\mathcal{D}_X = \Big(\partial_x - \ln'\Big(\frac{T_1 \cdots T_N}{y_1}\Big)\Big)\Big(\partial_x - \ln'\Big(\frac{y_1 T_2 \cdots T_N}{y_2}\Big)\Big) \times \dots$$
$$\times \Big(\partial_x - \ln'\Big(\frac{y_{N-2} T_{N-1} T_N}{y_{N-1}}\Big)\Big)\Big(\partial_x - \ln'(y_{N-1} T_N)\Big).$$

□

Let $\mathcal{D} = \partial_x^N + \sum_{i=1}^N h_i(x)\partial_x^{N-i}$ be a differential operator with meromorphic coefficients. The operator $\mathcal{D}^* = \partial_x^N + \sum_{i=1}^N (-1)^i \partial_x^{N-i} h_i(x)$ is called the *formal conjugate to* $\mathcal{D}$.

Lemma 5.3. *Let $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and let $\{u_1, \dots, u_N\}$ be a basis of X, then*

$$\frac{\mathrm{Wr}(u_1, \dots, \widehat{u_i}, \dots, u_N)}{\mathrm{Wr}(u_1, \dots, u_N)}, \quad i = 1, \dots, N,$$

form a basis of $\mathrm{Ker}((\mathcal{D}_X)^*)$. *The symbol $\widehat{u_i}$ means that u_i is skipped. Moreover, given an arbitrary factorization of $\mathcal{D}_X$ to linear factors, $\mathcal{D}_X = (\partial_x + f_1)(\partial_x + f_2)\dots(\partial_x + f_N)$, we have $(\mathcal{D}_X)^* = (\partial_x - f_N)(\partial_x - f_{N-1})\dots(\partial_x - f_1)$.*

Proof. The first statement follows from Theorem 3.14 of [MTV2]. The second statement follows from the first statement and Lemma A.5 of [MV1]. □

Lemma 5.4. *Let $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$. Then*

$$\mathcal{D}_{X^\dagger} = (T_1 \cdots T_N) \cdot (\mathcal{D}_X)^* \cdot (T_1 \cdots T_N)^{-1}.$$

Proof. The statement follows from Lemma 5.3 and the definition of $X^\dagger$. □

Lemma 5.5. *Suppose $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is a pure self-dual space and z is an arbitrary complex number, then there exists a basis $\Gamma = \{u_1, \dots, u_N\}$ of X such that for $\boldsymbol{y}_\Gamma = (y_1, \dots, y_{N-1})$ given by (5.3), we have $y_i = y_{N-i}$ and $y_i(z) \neq 0$ for every $i = 1, \dots, N-1$.*

Proof. The lemma follows from the proofs of Theorem 8.2 and Theorem 8.3 of [MV1]. □

6. $\mathfrak{g}$-oper

We fix N, $N \geqslant 4$, and set $\mathfrak{g}$ to be the Langlands dual of $\mathfrak{g}_N$. Explicitly, $\mathfrak{g} = \mathfrak{sp}_{2r}$ if $N = 2r$ and $\mathfrak{g} = \mathfrak{so}_{2r+1}$ if $N = 2r + 1$.

6.1. Miura $\mathfrak{g}$-oper

Fix a global coordinate x on $\mathbb{C} \subset \mathbb{P}^1$. Consider the following subset of differential operators

$$\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1) = \{\partial_x + p_{-1} + \boldsymbol{v} \mid \boldsymbol{v} \in \mathcal{M}(\mathfrak{b})\}.$$

This set is stable under the gauge action of the unipotent subgroup $\mathscr{N}(\mathcal{M}) \subset \mathscr{G}(\mathcal{M})$. The space of $\mathfrak{g}$-*opers* is defined as the quotient space $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1) := \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)/\mathscr{N}(\mathcal{M})$. We denote by $[\nabla]$ the class of $\nabla \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$.

We say that $\nabla = \partial_x + p_{-1} + \boldsymbol{v} \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ is *regular* at $z \in \mathbb{P}^1$ if $\boldsymbol{v}$ has no pole at z. A $\mathfrak{g}$-oper $[\nabla]$ is said to be *regular* at z if there exists $f \in \mathscr{N}(\mathcal{M})$ such that $f^{-1} \cdot \nabla \cdot f$ is regular at z.

Let $\nabla = \partial_x + p_{-1} + \boldsymbol{v}$ be a representative of a $\mathfrak{g}$-oper $[\nabla]$. Consider ∇ as a $\mathscr{G}$-connection on the trivial principal bundle $p : \mathscr{G} \times \mathbb{P}^1 \to \mathbb{P}^1$. The connection has singularities at the set $\mathrm{Sing} \subset \mathbb{C}$ where the function $\boldsymbol{v}$ has poles (and maybe at infinity). Parallel translations with respect to the connection define the monodromy representation $\pi_1(\mathbb{C} \setminus \mathrm{Sing}) \to \mathscr{G}$. Its image is called the *monodromy group* of ∇. If the monodromy group of one of the representatives of $[\nabla]$ is contained in the center of $\mathscr{G}$, we say that $[\nabla]$ is a *monodromy-free* $\mathfrak{g}$-oper.

A *Miura* $\mathfrak{g}$-*oper* is a differential operator of the form $\nabla = \partial_x + p_{-1} + \boldsymbol{v}$, where $\boldsymbol{v} \in \mathcal{M}(\mathfrak{h})$.

A $\mathfrak{g}$-oper $[\nabla]$ has *regular singularity* at $z \in \mathbb{P}^1 \setminus \{\infty\}$, if there exists a representative ∇ of $[\nabla]$ such that

$$(x-z)^{\check{\rho}} \cdot \nabla \cdot (x-z)^{-\check{\rho}} = \partial_x + \frac{p_{-1} + \boldsymbol{w}}{x - z},$$

where $\boldsymbol{w} \in \mathcal{M}(\mathfrak{b})$ is regular at z. The residue of $[\nabla]$ at z is $[p_{-1} + \boldsymbol{w}(z)]_{\mathfrak{g}}$. We denote the residue of $[\nabla]$ at z by $\mathrm{res}_z[\nabla]$.

Similarly, a $\mathfrak{g}$-oper $[\nabla]$ has *regular singularity* at $\infty \in \mathbb{P}^1$, if there exists a representative ∇ of $[\nabla]$ such that

$$x^{\check{\rho}} \cdot \nabla \cdot x^{-\check{\rho}} = \partial_x + \frac{p_{-1} + \tilde{\boldsymbol{w}}}{x},$$

where $\tilde{\boldsymbol{w}} \in \mathcal{M}(\mathfrak{b})$ is regular at ∞. The residue of $[\nabla]$ at ∞ is $-[p_{-1}+\tilde{\boldsymbol{w}}(\infty)]_{\mathfrak{g}}$. We denote the residue of $[\nabla]$ at ∞ by $\mathrm{res}_\infty[\nabla]$.

Lemma 6.1. *For any $\check{\lambda}, \check{\mu} \in \mathfrak{h}$, we have $[p_{-1} - \check{\rho} - \check{\lambda}]_{\mathfrak{g}} = [p_{-1} - \check{\rho} - \check{\mu}]_{\mathfrak{g}}$ if and only if there exists $w \in \mathcal{W}$ such that $\check{\lambda} = w \cdot \check{\mu}$.* □

Hence we can write $[\check{\lambda}]_{\mathcal{W}}$ for $[p_{-1} - \check{\rho} - \check{\lambda}]_{\mathfrak{g}}$. In particular, if $[\nabla]$ is regular at z, then $\mathrm{res}_z[\nabla] = [0]_{\mathcal{W}}$.

Let $\check{\boldsymbol{\Lambda}} = (\check{\lambda}^{(1)}, \dots, \check{\lambda}^{(n)}, \check{\lambda})$ be a sequence of $n+1$ dominant integral $\mathfrak{g}$-coweights and let $\boldsymbol{z} = (z_1, \dots, z_n, \infty) \in \mathring{\mathbb{P}}_{n+1}$. Let $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\check{\boldsymbol{\Lambda}}, \boldsymbol{z}}$ denote the set of all $\mathfrak{g}$-opers with at most regular singularities at points z_s and ∞ whose residues are given by

$$\mathrm{res}_{z_s}[\nabla] = [\check{\lambda}^{(s)}]_{\mathcal{W}}, \quad \mathrm{res}_\infty[\nabla] = -[\check{\lambda}]_{\mathcal{W}}, \quad s = 1, \dots, n,$$

and which are regular elsewhere. Let $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}}, \boldsymbol{z}} \subset \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\check{\boldsymbol{\Lambda}}, \boldsymbol{z}}$ denote the subset consisting of those $\mathfrak{g}$-opers which are also monodromy-free.

Lemma 6.2 ([F]). *For every $\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}}, \boldsymbol{z}}$, there exists a Miura $\mathfrak{g}$-oper as one of its representatives.* □

Lemma 6.3 ([F]). *Let ∇ be a Miura $\mathfrak{g}$-oper, then $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\check{\boldsymbol{\Lambda}}, \boldsymbol{z}}$ if and only if the following conditions hold:*

(i) ∇ is of the form

$$\nabla = \partial_x + p_{-1} - \sum_{s=1}^{n} \frac{w_s \cdot \check{\lambda}^{(s)}}{x - z_s} - \sum_{j=1}^{m} \frac{\tilde{w}_j \cdot 0}{x - t_j} \tag{6.1}$$

for some $m \in \mathbb{Z}_{\geqslant 0}$, $w_s \in \mathcal{W}$ for $s = 1, \dots, n$ and $\tilde{w}_j \in \mathcal{W}$, $t_j \in \mathbb{P}^1 \setminus \boldsymbol{z}$ for $j = 1, \dots, m$,

(ii) there exists $w_\infty \in \mathcal{W}$ such that

$$\sum_{s=1}^{n} w_s \cdot \check{\lambda}^{(s)} + \sum_{j=1}^{m} \tilde{w}_j \cdot 0 = w_\infty \cdot \check{\lambda}, \tag{6.2}$$

(iii) $[\nabla]$ is regular at t_j for $j = 1, \dots, m$.

□

Remark 6.4. The condition (6.2) implies that $\sum_{s=1}^{n} \langle \alpha_r, \check{\lambda}^{(s)} \rangle + \langle \alpha_r, \check{\lambda} \rangle$ is even if $N = 2r$.

6.2. Miura transformation

Following [DS], one can associate a linear differential operator L_∇ to each Miura $\mathfrak{g}$-oper $\nabla = \partial_x + p_{-1} + \boldsymbol{v}(x)$, $\boldsymbol{v}(x) \in \mathcal{M}(\mathfrak{h})$.

In the case of $\mathfrak{sl}_{r+1}$, $\boldsymbol{v}(x) \in \mathcal{M}(\mathfrak{h})$ can be viewed as an $(r+1)$-tuple $(v_1(x), \dots, v_{r+1}(x))$ such that $\sum_{i=1}^{r+1} v_i(x) = 0$. The *Miura transformation* sends $\nabla = \partial_x + p_{-1} + \boldsymbol{v}(x)$ to the operator

$$L_\nabla = (\partial_x + v_1(x)) \dots (\partial_x + v_{r+1}(x)).$$

Similarly, the Miura transformation takes the form

$$L_\nabla = (\partial_x + v_1(x)) \dots (\partial_x + v_r(x))(\partial_x - v_r(x)) \dots (\partial_x - v_1(x))$$

for $\mathfrak{g} = \mathfrak{sp}_{2r}$ and

$$L_\nabla = (\partial_x + v_1(x)) \dots (\partial_x + v_r(x))\partial_x(\partial_x - v_r(x)) \dots (\partial_x - v_1(x))$$

for $\mathfrak{g} = \mathfrak{so}_{2r+1}$. The formulas of the corresponding linear differential operators for the cases of $\mathfrak{sp}_{2r}$ and $\mathfrak{so}_{2r+1}$ can be understood with the embeddings described in Sections 5.2 and 5.3.

It is easy to see that different representatives of $[\nabla]$ give the same differential operator, we can write this map as $[\nabla] \mapsto L_{[\nabla]}$.

Recall the definition of $(\check{\lambda})_{ii}$ for $\check{\lambda} \in \mathfrak{h}$ from Sections 5.2 and 5.3.

Lemma 6.5. *Suppose ∇ is a Miura $\mathfrak{g}$-oper with $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$, then $L_{[\nabla]}$ is a monic Fuchsian differential operator with singularities at points in $\boldsymbol{z}$ only. The exponents of $L_{[\nabla]}$ at z_s, $s = 1, \dots, n$, are $(\check{\lambda}^{(s)})_{ii} + N - i$, and the exponents at ∞ are $-(\check{\lambda})_{ii} - N + i$, $i = 1, \dots, N$.*

Proof. Note that ∇ satisfies the conditions (i)-(iii) in Lemma 6.3. By Theorem 5.11 in [F] and Lemma 6.1, we can assume $w_s = 1$ for given s. The lemma follows directly. □

Denote by $Z(\mathscr{G})$ the center of $\mathscr{G}$, then

$$Z(\mathscr{G}) = \begin{cases} \{I_{2r+1}\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ \{\pm I_{2r}\} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}. \end{cases}$$

We have the following lemma.

Lemma 6.6. *Suppose ∇ is a Miura $\mathfrak{g}$-oper with $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\mathbf{\Lambda}},\boldsymbol{z}}$. If $\mathfrak{g} = \mathfrak{so}_{2r+1}$, then $L_{[\nabla]}$ is a monodromy-free differential operator. If $\mathfrak{g} = \mathfrak{sp}_{2r}$, then the monodromy of $L_{[\nabla]}$ around z_s is $-I_{2r}$ if and only if $\langle \alpha_r, \check{\lambda}^{(s)} \rangle$ is odd for given $s \in \{1, \ldots, n\}$.* □

6.3. Relations with pure self-dual spaces

Let $\check{\mathbf{\Lambda}} = (\check{\lambda}^{(1)}, \ldots, \check{\lambda}^{(n)}, \check{\lambda})$ be a sequence of $n+1$ dominant integral $\mathfrak{g}$-coweights and let $\boldsymbol{z} = (z_1, \ldots, z_n, \infty) \in \mathring{\mathbb{P}}_{n+1}$.

Consider $\check{\mathbf{\Lambda}}$ as a sequence of dominant integral $\mathfrak{g}_N$-weights. Choose d large enough so that $k := d - N - \sum_{s=1}^{n} (\check{\lambda}^{(s)})_{11} - (\check{\lambda})_{11} \geqslant 0$. (We only need to consider the case that $\sum_{s=1}^{n} (\check{\lambda}^{(s)})_{11} + (\check{\lambda})_{11}$ is an integer for $N = 2r$, see Lemma 4.4 and Remark 6.4.) Let $\boldsymbol{k} = (0, \ldots, 0, k)$. Note that we always have $|\check{\mathbf{\Lambda}}_{A,\boldsymbol{k}}| = N(d-N)$ and spaces of polynomials in $\mathrm{s}\Omega_{\check{\mathbf{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$ ($= \mathrm{s}\Omega_{\check{\mathbf{\Lambda}}_{A,\boldsymbol{k}},\boldsymbol{z}}$) are pure self-dual spaces.

Theorem 6.7. *There exists a bijection between $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\mathbf{\Lambda}},\boldsymbol{z}}$ and $\mathrm{s}\Omega_{\check{\mathbf{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$ given by the map $[\nabla] \mapsto \mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f)$, where $\boldsymbol{T} = (T_1, \ldots, T_N)$ is associated with $\check{\mathbf{\Lambda}}_{A,\boldsymbol{k}}, \boldsymbol{z}$ and $f = (T_1 \ldots T_N)^{-1/2}$.*

Proof. We only prove it for the case of $\mathfrak{g} = \mathfrak{sp}_{2r}$. Suppose $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\mathbf{\Lambda}},\boldsymbol{z}}$, by Lemmas 6.2 and 6.3, we can assume ∇ has the form (6.1) satisfying the conditions (i), (ii), and (iii) in Lemma 6.3.

Note that if $\langle \alpha_r, \check{\lambda}^{(s)} \rangle$ is odd, f has monodromy $-I_{2r}$ around the point z_s. By Lemma 6.6, one has that $f^{-1} \cdot L_{[\nabla]} \cdot f$ is monodromy-free around the point z_s for $s = 1, \ldots, n$. Note also that $\sum_{s=1}^{n} \langle \alpha_r, \check{\lambda}^{(s)} \rangle + \langle \alpha_r, \check{\lambda} \rangle$ is even, it follows that $f^{-1} \cdot L_{[\nabla]} \cdot f$ is also monodromy-free around the point ∞. Hence $f^{-1} \cdot L_{[\nabla]} \cdot f$ is a monodromy-free differential operator.

It follows from Lemmas 5.1 and 6.5 that $\mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f) \in \Omega_{\check{\mathbf{\Lambda}}_{A,\boldsymbol{k}},\boldsymbol{z}}$. Since $L_{[\nabla]}$ takes the form

$$(\partial_x + v_1(x)) \ldots (\partial_x + v_r(x))(\partial_x - v_r(x)) \ldots (\partial_x - v_1(x)),$$

it follows that $\mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f)$ is a pure self-dual space by Lemma 5.4.

If there exist $[\nabla_1], [\nabla_2] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\mathbf{\Lambda}},\boldsymbol{z}}$ such that $f^{-1} \cdot L_{[\nabla_1]} \cdot f = f^{-1} \cdot L_{[\nabla_2]} \cdot f$, then they are the same differential operator constructed from different bases of $\mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f)$ as described in Lemma 5.2. Therefore they correspond to the same $\mathfrak{so}_{2r+1}$-population by Theorem 7.5 of [MV1]. It follows from Theorem 4.2 and remarks in Section 4.3 of [MV2] that $[\nabla_1] = [\nabla_2]$.

Conversely, give a self-dual space $X \in \mathrm{s}\Omega_{\check{\boldsymbol{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$. By Lemma 5.5, there exists a basis Γ of X such that for $\boldsymbol{y}_\Gamma = (y_1, \dots, y_{N-1})$ we have $y_i = y_{N-i}$, $i = 1, \dots, N-1$. Following [MV2], define $\boldsymbol{v} \in \mathcal{M}(\mathfrak{h})$ by

$$\langle \alpha_i, \boldsymbol{v} \rangle = -\ln' \Big(T_i \prod_{j=1}^{r} y_j^{-a_{i,j}} \Big),$$

then we introduce the Miura $\mathfrak{g}$-oper $\nabla_\Gamma = \partial_x + p_{-1} + \boldsymbol{v}$, which only has regular singularities. It is easy to see from Lemma 5.2 that $f^{-1} \cdot L_{[\nabla_\Gamma]} \cdot f = \mathcal{D}_X$. It follows from the same argument as the previous paragraph that $[\nabla_\Gamma] = [\nabla_{\Gamma'}]$ for any other basis Γ' of X and hence $[\nabla_\Gamma]$ is independent of the choice of Γ. Again by Lemma 5.5, for any $x_0 \in \mathbb{C} \setminus \boldsymbol{z}$ we can choose Γ such that $y_i(x_0) \neq 0$ for all $i = 1, \dots, N-1$, it follows that $[\nabla_\Gamma]$ is regular at x_0. By exponents reasons (see Lemma 6.5), we have

$$\mathrm{res}_{z_s}[\nabla_\Gamma] = [\check{\lambda}^{(s)}]_{\mathcal{W}}, \quad \mathrm{res}_\infty[\nabla_\Gamma] = -[\check{\lambda}]_{\mathcal{W}}, \quad s = 1, \dots, n.$$

On the other hand, $[\nabla_\Gamma]$ is monodromy-free by Theorem 4.1 of [MV2]. It follows that $[\nabla_\Gamma] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$, which completes the proof. □

7. Proof of main theorems

7.1. Proof of Theorems 3.6 and 3.8

We prove Theorem 3.6 first.

By assumption, $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(n-1)})$ is a simple degeneration of $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$. Without loss of generality, we assume that $\xi^{(i)} = \lambda^{(i)}$ for $i = 1, \dots, n-2$ and

$$\dim(V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})^{\mathrm{sing}}_{\xi^{(n-1)}} > 0.$$

Recall the strata $\Omega_{\boldsymbol{\Lambda}}$ is a union of intersections of Schubert cells $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$, see (3.6). Taking the closure of $\Omega_{\boldsymbol{\Lambda}}$ is equivalent to allowing coordinates of $\boldsymbol{z} \in \mathring{\mathbb{P}}_n$ coincide.

Let $\boldsymbol{z}_0 = (z_1, \dots, z_{n-1}) \in \mathring{\mathbb{P}}_{n-1}$. Let $X \in \Omega_{\boldsymbol{\Xi},\boldsymbol{z}_0}$. By Theorem 3.2, there exists a common eigenvector $v \in (V_{\boldsymbol{\Xi},\boldsymbol{z}_0})^{\mathfrak{sl}_N}$ of the Bethe algebra $\mathcal{B}$ such that $\mathcal{D}_v = \mathcal{D}_X$.

Let $\boldsymbol{z}_0' = (z_1, \dots, z_{n-1}, z_{n-1})$. Consider the $\mathcal{B}$-module $V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'}$, then we have

$$\begin{aligned} V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'} =& (\bigotimes_{s=1}^{n-2} V_{\lambda^{(s)}}(z_s)) \otimes (V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})(z_{n-1}) \\ =& \bigoplus_{\mu} c^{\mu}_{\lambda^{(n-1)},\lambda^{(n)}} (\bigotimes_{s=1}^{n-2} V_{\lambda^{(s)}}(z_s)) \otimes V_{\mu}(z_{n-1}), \end{aligned}$$

where $c^{\mu}_{\lambda^{(n-1)},\lambda^{(n)}} := \dim(V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})^{\mathrm{sing}}_{\mu}$ are the Littlewood-Richardson coefficients. Since $\dim(V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})^{\mathrm{sing}}_{\xi^{(n-1)}} > 0$, we have $V_{\boldsymbol{\Xi},\boldsymbol{z}_0} \subset V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'}$. In particular, $(V_{\boldsymbol{\Xi},\boldsymbol{z}_0})^{\mathfrak{sl}_N} \subset (V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'})^{\mathfrak{sl}_N}$. Hence v is a common eigenvector of the Bethe algebra $\mathcal{B}$ on $(V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'})^{\mathfrak{sl}_N}$ such that $\mathcal{D}_v = \mathcal{D}_X$.

It follows that X is a limit point of $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ as z_n approaches z_{n-1}. This completes the proof of Theorem 3.6.

Theorem 3.8 follows directly from Theorem 3.6.

7.2. Proof of Theorems 4.5, 4.12, and 4.13

We prove Theorem 4.5 first. We follow the convention of Section 6.

We can identify the sequence $\check{\boldsymbol{\Lambda}} = (\check\lambda^{(1)}, \dots, \check\lambda^{(n)}, \check\lambda)$ of dominant integral $\mathfrak{g}$-coweights as a sequence of dominant integral $\mathfrak{g}_N$-weights. Consider the $\mathfrak{g}_N$-module $V_{\check{\boldsymbol{\Lambda}}} = V_{\check\lambda^{(1)}} \otimes \cdots \otimes V_{\check\lambda^{(n)}} \otimes V_{\check\lambda}$. It follows from Theorem 3.2 and Corollary 3.3 of [R] that there exists a bijection between the joint eigenvalues of the $\mathfrak{g}_N$ Bethe algebra $\mathcal{B}$ acting on $(V_{\check\lambda^{(1)}}(z_1) \otimes \cdots \otimes V_{\check\lambda^{(n)}}(z_n))^{\mathrm{sing}}$ and the $\mathfrak{g}$-opers in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$ for all possible dominant integral $\mathfrak{g}$-coweight $\check\lambda$. In fact, one can show that Theorem 3.2 and Corollary 3.3 of [R] are also true for the subspaces of $(V_{\check\lambda^{(1)}}(z_1) \otimes \cdots \otimes V_{\check\lambda^{(n)}}(z_n))^{\mathrm{sing}}_{\check\lambda}$ with specific $\mathfrak{g}_N$-weight $\check\lambda$. Recall that $\boldsymbol{k} = (0, \dots, 0, k)$, where $k = d - N - \sum_{s=1}^{n} (\check\lambda^{(s)})_{11} - (\check\lambda)_{11} \geqslant 0$. Since one has the canonical isomorphism of $\mathcal{B}$-modules

$$(V_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N} \cong (V_{\check\lambda^{(1)}}(z_1) \otimes \cdots \otimes V_{\check\lambda^{(n)}}(z_n))^{\mathrm{sing}}_{\check\lambda},$$

by Theorem 6.7, we have the following theorem.

Theorem 7.1. *There exists a bijection between the joint eigenvalues of the* $\mathfrak{g}_N$ *Bethe algebra* $\mathcal{B}$ *acting on* $(V_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N}$ *and* $\mathrm{s}\Omega_{\check{\boldsymbol{\Lambda}},\boldsymbol{k},\boldsymbol{z}} \subset \mathrm{sGr}(N,d)$ *such that given a joint eigenvalue of* $\mathcal{B}$ *with a corresponding* $\mathcal{B}$*-eigenvector* v *in* $(V_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N}$ *we have* $\mathrm{Ker}\,((T_1 \dots T_N)^{1/2} \cdot \mathcal{D}_v \cdot (T_1 \dots T_N)^{-1/2}) \in \mathrm{s}\Omega_{\check{\boldsymbol{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$. □

The fact that Ker $((T_1 \dots T_N)^{1/2} \cdot \mathcal{D}_v \cdot (T_1 \dots T_N)^{-1/2}) \in \mathrm{s}\Omega_{\check{\mathbf{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$ for the eigenvector $v \in (V_{\check{\mathbf{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N}$ of the $\mathfrak{g}_N$ Bethe algebra (except for the case of even N when there exists $s \in \{1, 2, \dots, n\}$ such that $\langle \alpha_r, \check{\lambda}^{(s)} \rangle$ is odd) also follows from the results of [LMV] and [MM].

Note that by Proposition 2.10 in [R], the i-th coefficient of the scalar differential operator $L_{[\nabla]}$ in Theorem 6.7 is obtained by action of a universal series $G_i(x) \in \mathcal{U}(\mathfrak{g}_N[t][[x^{-1}]])$. Theorem 4.5 for the case of $N \geqslant 4$ is a direct corollary of Theorems 6.7 and 7.1.

Thanks to Theorem 4.5, Theorems 4.12 and 4.13 can be proved in a similar way as Theorems 3.6 and 3.8.

Appendix A. Self-dual spaces and ϖ-invariant vectors

A.1. Diagram automorphism ϖ

There is a diagram automorphism $\varpi : \mathfrak{sl}_N \to \mathfrak{sl}_N$ such that

$$\varpi(E_i) = E_{N-i}, \quad \varpi(F_i) = F_{N-i}, \quad \varpi^2 = 1, \quad \varpi(\mathfrak{h}_A) = \mathfrak{h}_A.$$

The automorphism ϖ is extended to the automorphism of $\mathfrak{gl}_N$ by

$$\mathfrak{gl}_N \to \mathfrak{gl}_N, \quad e_{ij} \mapsto (-1)^{i-j-1} e_{N+1-j,N+1-i}, \quad i, j = 1, \dots, N.$$

By abuse of notation, we denote this automorphism of $\mathfrak{gl}_N$ also by ϖ.

The restriction of ϖ to the Cartan subalgebra $\mathfrak{h}_A$ induces a dual map $\varpi^* : \mathfrak{h}_A^* \to \mathfrak{h}_A^*$, $\lambda \mapsto \lambda^\star$, by

$$\lambda^\star(h) = \varpi^*(\lambda)(h) = \lambda(\varpi(h)),$$

for all $\lambda \in \mathfrak{h}_A^*, h \in \mathfrak{h}_A$.

Let $(\mathfrak{h}_A^*)^0 = \{\lambda \in \mathfrak{h}_A^* \mid \lambda^\star = \lambda\} \subset \mathfrak{h}_A^*$. We call elements of $(\mathfrak{h}_A^*)^0$ *symmetric weights.*

Let $\mathfrak{h}_N$ be the Cartan subalgebra of $\mathfrak{g}_N$. Consider the root system of type A_{N-1} with simple roots $\alpha_1^A, \dots, \alpha_{N-1}^A$ and the root system of $\mathfrak{g}_N$ with simple roots $\alpha_1, \dots, \alpha_{[\frac{N}{2}]}$.

There is a linear isomorphism $P_\varpi^* : \mathfrak{h}_N^* \to (\mathfrak{h}_A^*)^0$, $\lambda \mapsto \lambda_A$, where λ_A is defined by

$$\langle \lambda_A, \check{\alpha}_i^A \rangle = \langle \lambda_A, \check{\alpha}_{N-i}^A \rangle = \langle \lambda, \check{\alpha}_i \rangle, \quad i = 1, \dots, \left[\frac{N}{2}\right]. \tag{A.1}$$

Let $\lambda \in \mathfrak{h}_A^*$ and fix two nonzero highest weight vectors $v_\lambda \in (V_\lambda)_\lambda, v_{\lambda^\star} \in (V_{\lambda^\star})_{\lambda^\star}$. Then there exists a unique linear isomorphism $\mathcal{I}_\varpi : V_\lambda \to V_{\lambda^\star}$ such that

$$\mathcal{I}_\varpi(v_\lambda) = v_{\lambda^\star}, \quad \mathcal{I}_\varpi(gv) = \varpi(g)\mathcal{I}_\varpi(v), \tag{A.2}$$

for all $g \in \mathfrak{sl}_N, v \in V_\lambda$. In particular, if λ is a symmetric weight, $\mathcal{I}_\varpi$ is a linear automorphism of V_λ, where we always assume that $v_\lambda = v_{\lambda^\star}$.

Let M be a finite-dimensional $\mathfrak{sl}_N$-module with a weight space decomposition $M = \bigoplus_{\mu \in \mathfrak{h}_A^*}(M)_\mu$. Let $f : M \to M$ be a linear map such that $f(hv) = \varpi(h)f(v)$ for $h \in \mathfrak{h}_A, v \in M$. Then it follows that $f((M)_\mu) \subset (M)_{\mu^\star}$ for all $\mu \in \mathfrak{h}_A^*$. Define a formal sum

$$\mathrm{Tr}^\varpi_M f = \sum_{\mu \in (\mathfrak{h}_A^*)^0} \mathrm{Tr}(f|_{(M)_\mu})e(\mu),$$

where $\mathrm{Tr}(f|_{(M)_\mu})$ for $\mu \in (\mathfrak{h}_A^*)^0$ denotes the trace of the restriction of f to the weight space $(M)_\mu$.

Lemma A.1. *We have* $\mathrm{Tr}^\varpi_{M\otimes M'}(f \otimes f') = (\mathrm{Tr}^\varpi_M f) \cdot (\mathrm{Tr}^\varpi_{M'} f')$. □

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of dominant integral $\mathfrak{g}_N$-weights, then the tuple $\mathbf{\Lambda}^A = (\lambda_A^{(1)}, \dots, \lambda_A^{(n)})$ is a sequence of symmetric dominant integral $\mathfrak{sl}_N$-weights. Let $V_{\mathbf{\Lambda}^A} = \bigotimes_{s=1}^n V_{\lambda_A^{(s)}}$. The tensor product of maps $\mathcal{I}_\varpi$ in (A.2) with respect to $\lambda_A^{(s)}$, $s = 1, \dots, n$, gives a linear isomorphism

$$\mathcal{I}_\varpi : V_{\mathbf{\Lambda}^A} \to V_{\mathbf{\Lambda}^A}, \tag{A.3}$$

of $\mathfrak{sl}_N$-modules. Note that the map $\mathcal{I}_\varpi$ preserves the weight spaces with symmetric weights and the corresponding spaces of singular vectors. In particular, $(V_{\mathbf{\Lambda}^A})^{\mathfrak{sl}_N}$ is invariant under $\mathcal{I}_\varpi$.

Lemma A.2. *Let μ be a $\mathfrak{g}_N$-weight. Then we have*

$$\dim(V_{\mathbf{\Lambda}})^{\mathrm{sing}}_\mu = \mathrm{Tr}\big(\mathcal{I}_\varpi|_{(V_{\mathbf{\Lambda}^A})^{\mathrm{sing}}_{\mu_A}}\big), \qquad \dim(V_{\mathbf{\Lambda}})_\mu = \mathrm{Tr}\big(\mathcal{I}_\varpi|_{(V_{\mathbf{\Lambda}^A})_{\mu_A}}\big).$$

In particular, $\dim(V_{\mathbf{\Lambda}})^{\mathfrak{g}_N} = \mathrm{Tr}\big(\mathcal{I}_\varpi|_{(V_{\mathbf{\Lambda}^A})^{\mathfrak{sl}_N}}\big)$.

Proof. The statement follows from Lemma A.1 and Theorem 1 of Section 4.4 of [FSS]. □

A.2. Action of ϖ on the Bethe algebra

The automorphism ϖ is extended to the automorphism of current algebra $\mathfrak{gl}_N[t]$ by the formula $\varpi(g\otimes t^s) = \varpi(g)\otimes t^s$, where $g \in \mathfrak{gl}_N$ and $s = 0, 1, 2, \dots$. Recall the operator $\mathcal{D}^{\mathcal{B}}$, see (2.3).

Proposition A.3. *We have the following identity*

$$\varpi(\mathcal{D}^{\mathcal{B}}) = \partial_x^N + \sum_{i=1}^{N}(-1)^i \partial_x^{N-i} B_i(x).$$

Proof. It follows from the proof of Lemma 3.5 of [BHLW] that no nonzero elements of $\mathcal{U}(\mathfrak{gl}_N[t])$ kill all $\bigotimes_{s=1}^{n} L(z_s)$ for all $n \in \mathbb{Z}_{>0}$ and all $z_1, \dots, z_n$. It suffices to show the identity when it evaluates on $\bigotimes_{s=1}^{n} L(z_s)$.

Following the convention of [MTV6], define the $N \times N$ matrix $\mathcal{G}_h = \mathcal{G}_h(N, n, x, p_x, \boldsymbol{z}, \boldsymbol{\lambda}, X, P)$ by the formula

$$\mathcal{G}_h := \Big((p_x - \lambda_i)\, \delta_{ij} + \sum_{a=1}^{n} (-1)^{i-j} \frac{x_{N+1-i,a}\, p_{N+1-j,a}}{x - z_a} \Big)_{i,j=1}^{N}.$$

By Theorem 2.1 of [MTV6], it suffices to show that

$$\begin{aligned} &\operatorname{rdet}(\mathcal{G}_h) \prod_{a=1}^{n} (x - z_a) \\ &= \sum_{A,B,|A|=|B|} \prod_{b\notin A} (p_x - \lambda_b) \prod_{a\notin B} (x - z_a) \det(x_{ab})_{a\in A}^{b\in B} \det(p_{ab})_{a\in A}^{b\in B}. \end{aligned} \tag{A.4}$$

The proof of (A.4) is similar to the proof of Theorem 2.1 in [MTV6] with the following modifications.

Let m be a product whose factors are of the form $f(x)$, p_x, p_{ij}, x_{ij} where $f(x)$ is a rational function in x. Then the product m will be called *normally ordered* if all factors of the form p_x, x_{ij} are on the left from all factors of the form $f(x)$, p_{ij}.

Correspondingly, in Lemma 2.4 of [MTV6], we put the normal order for the first i factors of each summand. □

We have the following corollary of Proposition A.3.

Corollary A.4. *The $\mathfrak{gl}_N$ Bethe algebra $\mathcal{B}$ is invariant under ϖ, that is $\varpi(\mathcal{B}) = \mathcal{B}$.* □

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of partitions with at most N parts and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$.

Let $v \in (V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ be an eigenvector of the $\mathfrak{gl}_N$ Bethe algebra $\mathcal{B}$. Denote $\varpi(\mathcal{D}^{\mathcal{B}})_v$ the scalar differential operator obtained by acting by the formal operator $\varpi(\mathcal{D}^{\mathcal{B}})$ on v.

Corollary A.5. *Let $v \in (V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ be a common eigenvector of the $\mathfrak{gl}_N$ Bethe algebra; then the identity $\varpi(\mathcal{D}^{\mathcal{B}})_v = (\mathcal{D}_v)^*$ holds.* □

Let $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(n)})$ be a sequence of N-tuples of integers. Suppose

$$\xi^{(s)} - \lambda^{(s)} = m_s(1, \dots, 1), \quad s = 1, \dots, n.$$

Define the following rational functions depending on m_s, $s = 1, \dots, n$,

$$\varphi(x) = \prod_{s=1}^{n} (x - z_s)^{m_s}, \quad \psi(x) = \ln'(\varphi(x)) = \sum_{s=1}^{n} \frac{m_s}{x - z_s}.$$

Here we use the convention that $1/(x - z_s)$ is considered as the constant function 0 if $z_s = \infty$.

Lemma A.6. *For any formal power series $a(x)$ in x^{-1} with complex coefficients, the linear map obtained by sending $e_{ij}(x)$ to $e_{ij}(x) + \delta_{ij} a(x)$ induces an automorphism of $\mathfrak{gl}_N[t]$.* □

We denote the automorphism in Lemma A.6 by $\eta_{a(x)}$.

Lemma A.7. *The $\mathcal{B}$-module obtained by pulling $V_{\boldsymbol{\Lambda},\boldsymbol{z}}$ via $\eta_{\psi(x)}$ is isomorphic to $V_{\boldsymbol{\Xi},\boldsymbol{z}}$.* □

By Lemma A.7, we can identify the $\mathcal{B}$-module $V_{\boldsymbol{\Xi},\boldsymbol{z}}$ with the $\mathcal{B}$-module $V_{\boldsymbol{\Lambda},\boldsymbol{z}}$ as vector spaces. This identification is an isomorphism of $\mathfrak{sl}_N$-modules. For $v \in (V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ we use $\eta_{\psi(x)}(v)$ to express the same vector in $(V_{\boldsymbol{\Xi},\boldsymbol{z}})^{\mathfrak{sl}_N}$ under this identification.

Lemma A.8. *The following identity for differential operators holds*

$$\eta_{\psi(x)}(\mathcal{D}^{\mathcal{B}}) = \varphi(x) \mathcal{D}^{\mathcal{B}} (\varphi(x))^{-1}.$$

Proof. The lemma follows from the simple computation:

$$\varphi(x)(\partial_x - e_{ii}(x))(\varphi(x))^{-1} = \partial_x - e_{ii}(x) - \psi(x).$$ □

Proposition A.9. *Let $v \in (V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ be an eigenvector of the Bethe algebra such that $\mathcal{D}_v = \mathcal{D}_X$ for some $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$, then $\mathcal{D}_{\eta_{\psi(x)}(v)} = \mathcal{D}_{\varphi(x) \cdot X}$.*

Proof. With the identification between the $\mathcal{B}$-modules $V_{\boldsymbol{\Xi},\boldsymbol{z}}$ and $V_{\boldsymbol{\Lambda},\boldsymbol{z}}$, we have

$$\mathcal{D}_{\eta_{\psi(x)}(v)} = \big(\eta_{\psi(x)}(\mathcal{D}^{\mathcal{B}})\big)_v = \varphi(x)\mathcal{D}_v(\varphi(x))^{-1} = \varphi(x)\mathcal{D}_X(\varphi(x))^{-1} = \mathcal{D}_{\varphi(x)\cdot X}.$$

The second equality follows from Lemma A.8. □

A.3. $\mathcal{I}_\varpi$-invariant Bethe vectors and self-dual spaces

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a tuple of dominant integral $\mathfrak{g}_N$-weights. Recall the map $\mathcal{I}_\varpi : V_{\boldsymbol{\Lambda}^A} \to V_{\boldsymbol{\Lambda}^A}$, from (A.3).

Note that an $\mathfrak{sl}_N$-weight can be lifted to a $\mathfrak{gl}_N$-weight such that the N-th coordinate of the corresponding $\mathfrak{gl}_N$-weight is zero. From now on, we consider $\lambda_A^{(s)}$ from (A.1) as $\mathfrak{gl}_N$-weights obtained from (4.2), that is as the partitions with at most $N-1$ parts.

Let $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(n)})$ be a sequence of N-tuples of integers such that

$$\xi^{(s)} - \lambda_A^{(s)} = -(\lambda_A^{(s)})_1(1, \dots, 1), \quad s = 1, \dots, n.$$

Consider the $\mathfrak{sl}_N$-module $V_{\boldsymbol{\Lambda}^A}$ as the $\mathfrak{gl}_N$-module $V_{\boldsymbol{\Lambda}_A}$, the image of $V_{\boldsymbol{\Lambda}_A}$ under $\mathcal{I}_\varpi$ in (A.3), considered as a $\mathfrak{gl}_N$-module, is $V_{\boldsymbol{\Xi}}$. Furthermore, the image of $(V_{\boldsymbol{\Lambda}_A})^{\mathfrak{sl}_N}$ under $\mathcal{I}_\varpi$ is $(V_{\boldsymbol{\Xi}})^{\mathfrak{sl}_N}$.

Let $\boldsymbol{T} = (T_1, \dots, T_N)$ be associated with $\boldsymbol{\Lambda}_A, \boldsymbol{z}$, we have

$$T_1 \cdots T_N = \prod_{s=1}^{n} (x - z_s)^{(\lambda_A^{(s)})_1}.$$

Let $\varphi(x) = T_1 \cdots T_N$ and let $\psi(x) = \varphi'(x)/\varphi(x)$. Hence by Lemma A.7, the pull-back of $V_{\boldsymbol{\Xi},\boldsymbol{z}}$ through $\eta_{\psi(x)}$ is isomorphic to $V_{\boldsymbol{\Lambda}_A,\boldsymbol{z}}$. Furthermore, the pull-back of $(V_{\boldsymbol{\Xi},\boldsymbol{z}})^{\mathfrak{sl}_N}$ through $\eta_{\psi(x)}$ is isomorphic to $(V_{\boldsymbol{\Lambda}_A,\boldsymbol{z}})^{\mathfrak{sl}_N}$.

Theorem A.10. *Let $v \in (V_{\boldsymbol{\Lambda}_A,\boldsymbol{z}})^{\mathfrak{sl}_N}$ be an eigenvector of the $\mathfrak{gl}_N$ Bethe algebra $\mathcal{B}$ such that $\mathcal{D}_v = \mathcal{D}_X$ for some $X \in \Omega_{\boldsymbol{\Lambda}_A,\boldsymbol{z}}$, then $\mathcal{D}_{\eta_{\psi(x)}\circ\mathcal{I}_\varpi(v)} = \mathcal{D}_{X^\dagger}$. Moreover, X is self-dual if and only if $\mathcal{I}_\varpi(v) = v$.*

Proof. It follows from Proposition A.9, Corollary A.5, and Lemma 5.4 that

$$\begin{aligned}\mathcal{D}_{\eta_{\psi(x)}\circ\mathcal{I}_\varpi(v)} =& \varphi(x)\mathcal{D}_{\mathcal{I}_\varpi(v)}(\varphi(x))^{-1} = \varphi(x)\varpi(\mathcal{D}^{\mathcal{B}})_v(\varphi(x))^{-1} \\ =& (T_1 \dots T_N)(\mathcal{D}_X)^*(T_1 \dots T_N)^{-1} = \mathcal{D}_{X^\dagger}.\end{aligned}$$

Since $(\lambda_A^{(s)})_N = 0$ for all $s = 1, \dots, n$, X has no base points. Therefore X is self-dual if and only if $\mathcal{D}_X = \mathcal{D}_{X^\dagger}$. Suppose X is self-dual, it follows from

Theorem 3.2 that $\eta_{\psi(x)} \circ \mathcal{I}_\varpi(v)$ is a scalar multiple of v. By our identification, in terms of an $\mathfrak{sl}_N$-module homomorphism, $\eta_{\psi(x)}$ is the identity map. Moreover, since $\mathcal{I}_\varpi$ is an involution, we have $\mathcal{I}_\varpi(v) = \pm v$.

Finally, generically, we have an eigenbasis of the action of $\mathcal{B}$ in $(V_{\mathbf{\Lambda}_A,\boldsymbol{z}})^{\mathfrak{sl}_N}$ (for example for all $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$). In such a case, by the equality of dimensions using Lemma A.2, we have $\mathcal{I}_\varpi(v) = v$. Then the general case is obtained by taking the limit. □

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