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Wall-crossing functors for quantized symplectic resolutions: perversity and partial Ringel dualities

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To Yuri Ivanovich Manin on the occasion of his 80th birthday

Abstract: In this paper we study wall-crossing functors between categories of modules over quantizations of symplectic resolutions. We prove that wall-crossing functors through faces are perverse equivalences and use this to verify an Etingof type conjecture for quantizations of Nakajima quiver varieties associated to affine quivers. In the case when there is a Hamiltonian torus action on the resolution with finitely many fixed points so that it makes sense to speak about categories $\mathcal{O}$ over quantizations, we introduce new standardly stratified structures on these categories $\mathcal{O}$ and relate the wall-crossing functors to the Ringel duality functors associated to these standardly stratified structures.

1. Introduction

In this paper we study the wall-crossing (a.k.a. twisting) functors between categories of modules over quantizations of symplectic resolutions. These functors are derived equivalences introduced in this generality in [BPW, Section 6.4] and further studied in [BL].

More precisely, let X^0 denote a normal affine Poisson variety admitting a symplectic resolution of singularities. We also assume that X^0 is conical in the sense that it comes with a $\mathbb{C}^\times$-action that contracts X^0 to a single point and rescales the Poisson bracket. The symplectic resolutions of X^0 are parameterized by cones (to be called *chambers*) of a certain rational hyperplane arrangement in $H^2(X,\mathbb{R})$, where X is any of these resolutions. By X^θ we denote a resolution corresponding to the open cone containing a generic element $\theta \in H^2(X,\mathbb{R})$. We can speak about filtered quantizations $\mathcal{A}^\theta_\lambda$ of (the structure sheaf of) X^θ, where $\lambda \in H^2(X,\mathbb{C})$ is a quantization parameter. Further, it

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makes sense to speak about the category of coherent $\mathcal{A}^\theta_\lambda$-modules to be denoted by $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$. For θ, θ' lying in two different chambers we have a derived equivalence $\mathfrak{WC}_{\theta' \leftarrow \theta} : D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda)) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{A}^{\theta'}_\lambda))$, introduced in [BPW, Section 6.4], to be called the *wall-crossing functor*. We will be interested in two special situations:

- θ, θ' lie in two chambers that are opposite to one another with respect to their common face,
- the same condition but for θ and $-\theta'$.

The two extremes here is when these chambers share a common codimension 1 face and when one is the negative of the other.

There are two important results about the wall-crossing functors in this situation that will be obtained in this paper. First, we will show that the wall-crossing functor for the cones opposite with respect to a common face is perverse in the sense of Chuang and Rouquier and give some description of the corresponding filtration. This result was obtained in some special cases in [BL, Sections 8,10] and in the closely related setting of rational Cherednik algebras in [L5]. The proof in the present situation closely follows [L5] but we need to replace some missing ingredients such as the restriction functors for Harish-Chandra bimodules.

The perversity of the wall-crossing functors is used to prove a generalization of the main result of [BL], Etingof's conjecture on the number of finite dimensional irreducible modules for symplectic reflection algebras. In [BL], Bezrukavnikov and the author interpreted the conjecture in terms of quantized Nakajima quiver varieties. The conjecture was proved for quivers of finite type and also of affine type with special framing. In this paper we give a proof for affine quivers with an arbitrary framing[1].

Our second main result concerns the situation when there is a Hamiltonian torus T acting on X with finitely many fixed points. In this case one can fix a generic one-parameter subgroup $\nu : \mathbb{C}^\times \to T$ and consider the corresponding category $\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda) \subset \mathrm{Coh}(\mathcal{A}^\theta_\lambda)$, introduced in this generality in [BLPW]. This is a highest weight category whose simple objects are labelled by X^T. In [L7] we have introduced *compatible standardly stratified structures* on $\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda)$ that roughly speaking come from degenerating ν. In this paper we will produce a new kind of compatible standardly stratified structures that come from deforming λ. There is such a structure associated to each face of the chamber containing θ. Namely, let Γ be a face of the chamber containing θ and χ be

[1]The proof of this generalization was previously obtained in [L4], by now a retired preprint, but the proof in the present paper is much simpler and more straightforward.

an integral point in the interior of Γ. Set $\theta' := \theta - N\chi$ for $N \gg 0$. We show that the wall-crossing functor $\mathfrak{WC}^{-1}_{\theta \leftarrow \theta'}$ is the Ringel duality functor coming from the standardly stratified structure given by θ.

The paper is organized as follows. Section 2 recalls various preliminaries on symplectic resolutions and their quantizations following mostly [BPW] and [BL]. These preliminaries include the structure of resolutions and their quantizations, Harish-Chandra bimodules, localization theorems and wall-crossing functors. In Section 3 we will state and prove a theorem on a perversity of wall-crossing functors through faces. For this, we will need to study classical and quantum slices to symplectic leaves and restriction functors for Harish-Chandra bimodules. We use the perversity to prove an Etingof-type conjecture for quantized quiver varieties associated to quivers of affine types. Finally, in Section 4 we deal with standardly stratified categories and their Ringel dualities. The most nontrivial part is to introduce the standardly stratified structures on $\mathcal{O}(\mathcal{A}^\theta_\lambda)$ coming from deforming λ. Then we introduce Ringel duality for standardly stratified categories and show that wall-crossing functors as in the previous paragraph give Ringel duality functors.

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2. Preliminaries

2.1. Symplectic resolutions

Let X be a smooth symplectic variety (with form ω) equipped also with an action of $\mathbb{C}^\times$ subject to the following conditions:

(a) There is a positive integer d such that $t.\omega = t^d\omega$.
(b) The algebra $\mathbb{C}[X]$ is finitely generated and is positively graded: there are no negative components and the zero component consists of scalars.
(c) The natural morphism $\rho : X \to X^0 := \mathrm{Spec}(\mathbb{C}[X])$ is a projective resolution of singularities.

We will say that X is a conical symplectic resolution. Thanks to (b), we can talk about the point $0 \in X^0$. Conditions (b) and (c) imply that

$\lim_{t\to 0} t.x$ exists and lies in $\rho^{-1}(0)$. So we will call the $\mathbb{C}^\times$-actions on X and X^0 *contracting*.

By the Grauert-Riemenschneider theorem, we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. By results of Kaledin, [K1, Theorem 2.3], X^0 has finitely many symplectic leaves.

We will be interested in *conical deformations* $X_{\mathfrak{P}}/\mathfrak{P}$, where $\mathfrak{P}$ is a finite dimensional vector space, and $X_{\mathfrak{P}}$ is a symplectic scheme over $\mathfrak{P}$ with symplectic form $\widehat{\omega} \in \Omega^2(X_{\mathfrak{P}}/\mathfrak{P})$ that specializes to ω and also with a $\mathbb{C}^\times$-action on $X_{\mathfrak{P}}$ having the following properties:

- the morphism $X_{\mathfrak{P}} \to \mathfrak{P}$ is $\mathbb{C}^\times$-equivariant, where we consider the action of $\mathbb{C}^\times$ on $\mathfrak{P}$ given by $t.p = t^{-d}p$,
- the restriction of the action to X coincides with the contracting action,
- $t.\widehat{\omega} := t^d\widehat{\omega}$.

It turns out that there is a universal conical deformation $\tilde{X}$ over $\tilde{\mathfrak{P}} := H^2(X, \mathbb{C})$ (any other deformation is obtained via the pull-back with respect to a linear map $\mathfrak{P} \to \tilde{\mathfrak{P}}$). We will often write $X_{\tilde{\mathfrak{P}}}$ instead of $\tilde{X}$.

For $\lambda \in \tilde{\mathfrak{P}}$, let us write X_λ for the corresponding fiber of $X_{\tilde{\mathfrak{P}}} \to \tilde{\mathfrak{P}}$. If X, X' are two conical symplectic resolutions of X, then there are open subvarieties $\breve{X} \subset X, \breve{X}' \subset X'$ with $\operatorname{codim}_X X \setminus \breve{X}, \operatorname{codim}_X X \setminus \breve{X} \geqslant 2$ and $\breve{X} \xrightarrow{\sim} \breve{X}'$, see, e.g., [BPW, Proposition 2.19]. This allows to identify the Picard groups $\operatorname{Pic}(X) = \operatorname{Pic}(X')$. Moreover, the Chern class map defines an isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Pic}(X) \xrightarrow{\sim} H^2(X, \mathbb{C})$. See, e.g., [BPW, Section 2.3]. Let $\tilde{\mathfrak{P}}_{\mathbb{Z}}$ be the image of $\operatorname{Pic}(X)$ in $H^2(X, \mathbb{C})$.

Set $\tilde{\mathfrak{P}}_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \tilde{\mathfrak{P}}_{\mathbb{Z}}$. There is a finite group W acting on $\tilde{\mathfrak{P}}_{\mathbb{R}}$ as a reflection group, such that the movable cone C of X (that does not depend on the choice of a resolution) is a fundamental chamber for W.

Now consider the locus of $\lambda \in \tilde{\mathfrak{P}}$ such that the fiber X_λ of $X_{\tilde{\mathfrak{P}}} \to \tilde{\mathfrak{P}}$ over λ is not affine. As Namikawa checked in [Nam], this locus is the union of hyperplanes defined over $\mathbb{R}$. Let $H_1, \dots, H_k \subset \tilde{\mathfrak{P}}_{\mathbb{R}}$ be the real forms of these hyperplanes. It turns out that the collection $H_1, \dots, H_k$ is W-stable and includes all the walls for W. The hyperplanes H_i that intersect C split it into the union of polyhedral cones. These cones are in bijection with the conical symplectic resolutions of X^0, where to a resolution we assign its ample cone. Let $C_1, \dots, C_m$ denote all the ample cones.

Definition 2.1. We say that an element $\theta \in \tilde{\mathfrak{P}}_{\mathbb{Q}}$ is *generic* if it does not lie in $\bigcup_{i=1}^k H_i$. The cones wC_i (for $w \in W$) will be called *chambers*.

For a generic element θ, we will write X^θ for the resolution corresponding to the ample cone containing $W\theta \cap C$. Further, if $w\theta \in C$, we will choose a

different identification of $H^2(X^\theta, \mathbb{C})$ with $\tilde{\mathfrak{P}}$, one twisted by w (so that the ample cone actually contains θ).

2.2. Quantizations

We start by introducing the various versions of quantizations that we are going to consider.

Let X be a Poisson scheme (of finite type over $\mathbb{C}$) and $d \in \mathbb{Z}_{>0}$. By a formal quantization of X (relative to d) we mean a sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{D}_\hbar$ in Zariski topology on X together with an isomorphism $\iota : \mathcal{D}_\hbar/(\hbar) \xrightarrow{\sim} \mathcal{O}_X$ of sheaves of algebras that satisfy the following conditions.

- $\mathcal{D}_\hbar$ is flat over $\mathbb{C}[[\hbar]]$.
- The $\hbar$-adic filtration on $\mathcal{D}_\hbar$ is complete and separated.
- $[\mathcal{D}_\hbar, \mathcal{D}_\hbar] \subset \hbar^d \mathcal{D}_\hbar$. This gives a Poisson bracket on $\mathcal{D}_\hbar/(\hbar)$.
- ι is a Poisson isomorphism.

It is a standard fact that to give such a quantization in the case when X is affine is the same thing as to give a single algebra that is a formal quantization of $\mathbb{C}[X]$.

Now suppose that $\mathbb{C}^\times$ acts on X in such a way that $t \in \mathbb{C}^\times$ rescales the Poisson bracket on $\mathcal{O}_X$ by t^d. Then we can speak about graded formal quantization. By definition, this is a formal quantization $(\mathcal{D}_\hbar, \iota)$ such that $\mathcal{D}_\hbar$ is equipped with a $\mathbb{C}^\times$-action by algebra isomorphisms with $t.\hbar = t\hbar$ and such that ι is $\mathbb{C}^\times$-equivariant.

Let us now recall the notion of a filtered quantization. Suppose that $\mathbb{C}^\times$ acts on X as in the previous paragraph. Assume, in addition, that every point in X has a $\mathbb{C}^\times$-stable open affine neighborhood. This is the case when X is quasi-projective or by a theorem of Sumihiro, [S], when X is normal. By the conical topology on X we mean the topology, where "open" means Zariski open and $\mathbb{C}^\times$-stable. Note that $\mathcal{O}_X$ is a sheaf of graded algebras in the conical topology. By a filtered quantization of $\mathcal{O}_X$ we mean a pair $(\mathcal{D}, \iota)$, where $\mathcal{D}$ is a sheaf of $\mathbb{Z}$-filtered algebras (the filtration is ascending) in the conical topology on X and ι is an isomorphism $\operatorname{gr}\mathcal{D} \xrightarrow{\sim} \mathcal{O}_X$ of sheaves of graded algebras. These data are supposed to satisfy the following axioms.

- The topology on $\mathcal{D}$ induced by the filtration is complete and separated.
- If $\mathcal{D}_{\leqslant i}$ denotes the ith filtration component, then $[\mathcal{D}_{\leqslant i}, \mathcal{D}_{\leqslant j}] \subset \mathcal{D}_{\leqslant i+j-d}$ for all $i, j \in \mathbb{Z}$.
- The isomorphism $\iota : \operatorname{gr}\mathcal{D} \xrightarrow{\sim} \mathcal{O}_X$ is Poisson.

Let us explain a connection between graded formal and filtered quantizations. Let $\mathcal{D}_\hbar$ be a graded formal quantization. Then we can consider the subsheaf of $\mathbb{C}^\times$-finite sections $\mathcal{D}_{\hbar,fin}$ of $\mathcal{D}_\hbar$ restricted to conical topology. Then $\mathcal{D}_{\hbar,fin}/(\hbar-1)$ is a filtered quantization. Conversely, let $\mathcal{D}$ be a filtered quantization. Then we can consider the Rees sheaf $R_\hbar(\mathcal{D}) := \bigoplus_i \mathcal{D}_{\leqslant i}\hbar^i$. The $\hbar$-adic completion of $R_\hbar(\mathcal{D})$ uniquely extends to a sheaf in the Zariski topology that is a graded formal quantization. It is easy to see that these two procedures give mutually inverse bijections between the set of filtered quantizations and the set of graded formal quantizations.

Now let us discuss the classification of quantizations obtained in [BeKa, L3]. Let $\mathcal{Q}_\hbar(X)$ denote the set of isomorphism classes of formal quantizations of X. Bezrukavnikov and Kaledin, [BeKa, Section 4], defined a natural (in particular, compatible with pull-backs under open embeddings) map $\mathsf{Per} : \mathcal{Q}_\hbar(X) \to H^2(X,\mathbb{C})[[\hbar]]$. They have shown in [BeKa, Theorem 1.8] that if $H^i(X,\mathcal{O}_X) = 0$ for $i = 1,2$, then this map is an isomorphism. Now let $\mathcal{Q}_{\hbar,\mathbb{C}^\times}(X)$ denote the set of isomorphism classes of graded formal quantizations. It was shown in [L3, Section 2.3] that the composition $\mathcal{Q}_{\hbar,\mathbb{C}^\times}(X) \to \mathcal{Q}_\hbar(X) \to H^2(X,\mathbb{C})[[\hbar]]$ restricts to a bijection $\mathcal{Q}_{\hbar,\mathbb{C}^\times}(X) \xrightarrow{\sim} H^2(X,\mathbb{C})$. We conclude that the filtered quantizations of X are parameterized by $H^2(X,\mathbb{C})$. For $\lambda \in H^2(X,\mathbb{C})$, we write $\mathcal{D}_\lambda$ for the corresponding filtered quantization.

The definitions of a quantization admit several ramifications. First, instead of X we can consider a Poisson scheme $\widehat{X}$ over a vector space $\mathfrak{P}$ and talk about quantizations of $\widehat{X}/\mathfrak{P}$ that are now required to be sheaves of $\mathbb{C}[\mathfrak{P}]$-algebras. When we speak about graded quantizations, we will always assume that $\mathbb{C}^\times$ acts on $\mathfrak{P}$ by $t.p = t^{-d}p$. Also we can talk about quantizations of formal schemes. The classification results quoted in the previous paragraph still hold for formal quantizations: for a symplectic formal scheme X with $H^i(X,\mathcal{O}_X) = 0, i = 1,2$, the formal quantizations are classified by elements of $\mathbb{C}[[H^2_{DR}(X),\hbar]]$.

Now let us discuss quantizations of a conical symplectic resolution $X = X^\theta$ (see [BPW, Section 3]). The equalities $H^i(X,\mathcal{O}_X) = 0$ hold so the filtered quantizations are classified by $\tilde{\mathfrak{P}} = H^2(X,\mathbb{C})$. The quantization corresponding to $\lambda \in \tilde{\mathfrak{P}}$ will be denoted by $\mathcal{A}^\theta_\lambda$. Moreover, recall that we have the universal conical deformation $X_{\tilde{\mathfrak{P}}}$ of X. It was shown in [BeKa, Section 6.2] that $X_{\tilde{\mathfrak{P}}}$ admits a canonical quantization to be denoted by $\mathcal{A}^\theta_{\tilde{\mathfrak{P}}}$. It satisfies the following property: its specialization to $\lambda \in \tilde{\mathfrak{P}}$ coincides with $\mathcal{A}^\theta_\lambda$.

Since $H^i(X,\mathcal{O}_X) = 0$ for all $i > 0$, we see that $\mathcal{A}_\lambda := \Gamma(\mathcal{A}^\theta_\lambda)$ is a quantization of $\mathbb{C}[X]$, while $H^i(X,\mathcal{A}^\theta_\lambda) = 0$. It was shown in [BPW, Section 3.3] that $\mathcal{A}_\lambda$ is independent of the choice of θ.

Note that $\mathcal{A}^{\theta}_{-\lambda}$ is identified with $(\mathcal{A}^{\theta}_{\lambda})^{opp}$, [L3, Section 2.3], and hence $\mathcal{A}_{\lambda} \cong \mathcal{A}^{opp}_{-\lambda}$. Also we have $\mathcal{A}_{\lambda} \cong \mathcal{A}_{w\lambda}$ for all $\lambda \in \tilde{\mathfrak{P}}, w \in W$, see [BPW, Proposition 3.10].

2.3. Example: quiver varieties

Let us recall a special class of symplectic resolutions that will be of importance for us later. This class is the Nakajima quiver varieties.

Let Q be a quiver (=oriented graph, we allow loops and multiple edges). We can formally represent Q as a quadruple (Q_0, Q_1, t, h), where Q_0 is a finite set of vertices, Q_1 is a finite set of arrows, $t, h : Q_1 \to Q_0$ are maps that to an arrow a assign its tail and head. In this paper we are interested in the case when Q is of affine type, i.e., Q is an extended Dynkin quiver of type A, D, E.

Pick vectors $v = (v_i)_{i\in Q_0}, w = (w_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}_{\geqslant 0}$ and vector spaces V_i, W_i with $\dim V_i = v_i, \dim W_i = w_i$. Consider the (co)framed representation space

$$R = R(v,w) := \bigoplus_{a\in Q_1} \mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i\in Q_0} \mathrm{Hom}(V_i, W_i).$$

We will also consider its cotangent bundle $T^*R = R \oplus R^*$, this is a symplectic vector space that can be identified with

$$\bigoplus_{a\in Q_1} \Big(\mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \mathrm{Hom}(V_{h(a)}, V_{t(a)})\Big) \\ \oplus \bigoplus_{i\in Q_0} \left(\mathrm{Hom}(V_i, W_i) \oplus \mathrm{Hom}(W_i, V_i)\right).$$

The group $G := \prod_{k\in Q_0} \mathrm{GL}(V_k)$ naturally acts on T^*R and this action is Hamiltonian. Its moment map $\mu : T^*R \to \mathfrak{g}^*$ is dual to $x \mapsto x_R : \mathfrak{g} \to \mathbb{C}[T^*R]$, where x_R stands for the vector field on R induced by $x \in \mathfrak{g}$.

Fix a $\theta \in \mathbb{Z}^{Q_0}$ (to be called a *stability condition* later on) that is thought as a character of G via $\theta((g_k)_{k\in Q_0}) = \prod_{k\in Q_0} \det(g_k)^{\theta_k}$. Then, by definition, the quiver variety $\mathcal{M}^{\theta}(v,w)$ is the GIT Hamiltonian reduction $\mu^{-1}(0)^{\theta-ss}//G$. We are interested in two extreme cases: when θ is generic (and so $\mathcal{M}^{\theta}(v,w)$ is smooth and symplectic) and when $\theta = 0$ (and so $\mathcal{M}^{\theta}(v,w)$ is affine). We will write $\mathcal{M}(v,w)$ for $\mathrm{Spec}(\mathbb{C}[\mathcal{M}^{\theta}(v,w)])$. This is an affine variety independent of θ. A natural projective morphism $\rho : \mathcal{M}^{\theta}(v,w) \to \mathcal{M}(v,w)$ is a resolution of singularities; see, for example, [BL, Section 2.1]. Note also that we have compatible $\mathbb{C}^{\times}$-actions on $\mathcal{M}^{\theta}(v,w), \mathcal{M}(v,w)$ induced from the action on T^*R given by $t.(r,\alpha) := (t^{-1}r, t^{-1}\alpha), r \in R, \alpha \in R^*$. So $\mathcal{M}^{\theta}(v,w) \to \mathcal{M}(v,w)$

is a conical symplectic resolution. Note that if μ is flat, then $\mathcal{M}(v,w) \xrightarrow{\sim} \mathcal{M}^0(v,w)$. There is a combinatorial necessary and sufficient condition on μ to be flat due to Crawley-Boevey, [CB], but we do not need that.

Now let us proceed to the quantum setting. We will work with quantizations of $\mathcal{M}^\theta(v,w), \mathcal{M}(v,w)$. Consider the algebra $D(R)$ of differential operators on R. The group G naturally acts on $D(R)$ with a quantum comoment map $\Phi : \mathfrak{g} \to D(R), x \mapsto x_R$. We can consider the quantum Hamiltonian reduction $\mathcal{A}^0_\lambda(v,w) = [D(R)/D(R)\{x_R - \langle \lambda, x\rangle | x \in \mathfrak{g}\}]^G$. It is a quantization of $\mathcal{M}^0(v,w) = \mathcal{M}(v,w)$ when the moment map μ is flat. We can also define a quantization $\mathcal{A}^\theta_\lambda(v,w)$ of $\mathcal{M}^\theta(v,w)$ by quantum Hamiltonian reduction. Namely, we can microlocalize $D(R)$ to a sheaf in the conical topology so that we can consider the restriction of $D(R)$ to $(T^*R)^{\theta-ss}$, let $\mathcal{D}^{\theta-ss}$ denote the restriction. Let π stand for the quotient morphism $\mu^{-1}(0)^{\theta-ss} \to \mu^{-1}(0)^{\theta-ss}/G = \mathcal{M}^\theta(v,w)$. Let us notice that $\mathcal{D}^{\theta-ss}/\mathcal{D}^{\theta-ss}\{x_R - \langle \lambda, x\rangle | x \in \mathfrak{g}\}$ is scheme-theoretically supported on $\mu^{-1}(0)^{\theta-ss}$ and so can be regarded as a sheaf in conical topology on that variety. Set

$$\mathcal{A}^\theta_\lambda(v,w) := [\pi_* \left(\mathcal{D}^{\theta-ss}/\mathcal{D}^{\theta-ss}\{x_R - \langle \lambda, x\rangle\}\right)]^G,$$

this is a quantization of $\mathcal{M}^\theta(v,w)$. We note that the period of $\mathcal{A}^\theta_\lambda(v,w)$ equals to the cohomology class in $H^2(\mathcal{M}^\theta(v,w),\mathbb{C})$ defined by λ up to a shift by a fixed element in $H^2(\mathcal{M}^\theta(v,w),\mathbb{C})$. We will write $\mathcal{A}_\lambda(v,w)$ for $\Gamma(\mathcal{A}^\theta_\lambda(v,w))$. We have $\mathcal{A}_\lambda(v,w) = \mathcal{A}^0_\lambda(v,w)$ for a Zariski generic λ, see [BL, Section 2.2].

Below we will need a standard and well-known result about symplectic leaves in quiver varieties.

Lemma 2.2. *Let $v' \leqslant v$ be a root of Q. Pick a Zariski generic parameter λ with $v' \cdot \lambda = 0$. If the variety $\mathcal{M}_\lambda(v,w)$ has a single symplectic leaf that is a point, then v' is a real root.*

Proof. Recall that from Q and w we can form a new quiver Q^w with one additional vertex ∞ and w_i arrow from i to ∞. Then we can form the double quiver $\bar{Q}^w$ so that T^*R is the representation space of $\bar{Q}^w$ of dimension $(v,1)$.

Consider a representation r of $\bar{Q}^w$ lying in $\mu^{-1}(\lambda)$ and having closed orbit. Let H denote the stabilizer of r in G and U be the symplectic part of the slice representation in X. Then the point corresponding to x in the quotient $\mu^{-1}(\lambda)//G$ is a symplectic leaf if and only if $U^H = \{0\}$. Recall, see e.g. [BL, Section 2.1], that the space U is recovered as follows. We decompose r into the sum of the irreducible $\bar{Q}^w$-modules: $r = r^0 \oplus (r^1)^{\oplus n_1} \oplus \ldots \oplus (r_k)^{\oplus n_k}$, where r_0 is an irreducible representation with dimension vector of the form $(v^0,1)$ and r_i are pairwise non-isomorphic irreducible representations of $\bar{Q}$ of dimension

vector v^i. By the choice of λ, we see that all $v^i, i > 0$, are proportional to v'. We note that U is the representation of another quiver that has vertices $1, \dots, k$, with $1 - \frac{1}{2}(v^i, v^i)$ loops at the vertex i. If $U^H = \{0\}$, then there are no loops and so each v^i is a real root and hence v' is a real root. □

2.4. Harish-Chandra bimodules

Now let $\mathcal{A}$ be a $\mathbb{Z}$-filtered algebra with a complete and separated filtration $\mathcal{A} = \bigcup_{i\in\mathbb{Z}} \mathcal{A}_{\leqslant i}$. We assume that $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-d}$ for some fixed $d \in \mathbb{Z}_{>0}$ and that $\operatorname{gr}\mathcal{A}$ is finitely generated.

By a Harish-Chandra $\mathcal{A}$-bimodule we mean a bimodule $\mathcal{B}$ that can be equipped with a complete and separated filtration $\mathcal{B} = \bigcup_{i\in\mathbb{Z}} \mathcal{B}_{\leqslant i}$ such that

- The filtration is compatible with the filtration on $\mathcal{A}$.
- $[\mathcal{A}_{\leqslant i}, \mathcal{B}_{\leqslant j}] \subset \mathcal{B}_{\leqslant i+j-d}$.
- $\operatorname{gr}\mathcal{B}$ is a finitely generated $\operatorname{gr}\mathcal{A}$-module.

For a HC bimodule $\mathcal{B}$ we can define its associated variety $\mathrm{V}(\mathcal{B})$ inside of the reduced scheme associated to $\operatorname{gr}\mathcal{A}$ in the usual way. Note that $\mathrm{V}(\mathcal{B})$ is a Poisson subvariety.

When $\mathcal{A}^1, \mathcal{A}^2$ are quotients of $\mathcal{A}$ we can speak about HC $\mathcal{A}^1$-$\mathcal{A}^2$-bimodules.

The following lemma is proved analogously to [L5, Proposition 3.8].

Lemma 2.3. *Let $\mathcal{B}^1, \mathcal{B}^2$ be HC $\mathcal{A}$-bimodules. Then $\operatorname{Tor}_i^{\mathcal{A}}(\mathcal{B}^1, \mathcal{B}^2)$, $\operatorname{Ext}^i_{\mathcal{A}}(\mathcal{B}^1, \mathcal{B}^2), \operatorname{Ext}^i_{\mathcal{A}^{opp}}(\mathcal{B}^1, \mathcal{B}^2)$ are HC.*

Now let us give an example of a HC bimodule. Let $\mathcal{A} = \mathcal{A}_{\tilde{\mathfrak{P}}}$. Pick $\chi \in \tilde{\mathfrak{P}}_{\mathbb{Z}}$. Consider a line bundle $\mathcal{O}(\chi)$ on $X^\theta_{\tilde{\mathfrak{P}}}$ (recall that $\tilde{\mathfrak{P}}_{\mathbb{Z}}$ is the image of $\operatorname{Pic}(X)$ in $H^2(X, \mathbb{C})$, for $\mathcal{O}(\chi)$ we take any line bundle corresponding to a lift of χ to $\operatorname{Pic}(X)$). Since $H^i(X^\theta_{\tilde{\mathfrak{P}}}, \mathcal{O}_{X^\theta_{\tilde{\mathfrak{P}}}}) = 0$ for $i > 0$, we see that $\mathcal{O}(\chi)$ admits a unique quantization $\mathcal{A}^\theta_{\tilde{\mathfrak{P}},\chi}$ to an $\mathcal{A}^\theta_{\tilde{\mathfrak{P}}}$-bimodule, where for $a \in \tilde{\mathfrak{P}}^*$ we have $[a, m] = \langle \chi, a\rangle m$ for any local section m of $\mathcal{B}$, see [BPW, Section 5.1] for details.

Now pick an affine subspace $\mathfrak{P} \subset \tilde{\mathfrak{P}}$. Set $\mathcal{A}^\theta_{\mathfrak{P},\chi} := \mathcal{A}^\theta_{\tilde{\mathfrak{P}},\chi} \otimes_{\mathbb{C}[\tilde{\mathfrak{P}}]} \mathbb{C}[\mathfrak{P}]$. Then the global section bimodule $\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi} := \Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi})$ is HC over $\mathcal{A}_{\tilde{\mathfrak{P}}}$.

Under some conditions, the bimodule $\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}$ is independent of θ.

Lemma 2.4. *Suppose that the vector subspace of $\tilde{\mathfrak{P}}$ associated to $\mathfrak{P}$ is not contained in the singular locus. Then $\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}$ is independent of θ.*

This is [BPW, Proposition 6.24]. The following lemma is [BPW, Proposition 6.26].

Lemma 2.5. *Suppose that $H^1(X, \mathcal{O}(\chi)) = 0$. Then the specialization of $\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}$ to $\lambda \in \mathfrak{P}$ coincides with $\mathcal{A}^{(\theta)}_{\lambda,\chi}$.*

Now let us discuss $\tilde{\mathfrak{P}}$-supports of HC bimodules. Let $\mathcal{B}$ be a HC $\mathcal{A}_{\tilde{\mathfrak{P}}}$-bimodule. By its right $\tilde{\mathfrak{P}}$-support we mean the set $\mathrm{Supp}^r_{\tilde{\mathfrak{P}}}(\mathcal{B})$ consisting of all $\lambda \in \tilde{\mathfrak{P}}$ such that $\mathcal{B}_\lambda := \mathcal{B} \otimes_{\mathbb{C}[\tilde{\mathfrak{P}}]} \mathbb{C}_\lambda$ is nonzero.

The following result was obtained in [L7, Proposition 2.6].

Lemma 2.6. *The subset $\mathrm{Supp}^r_{\tilde{\mathfrak{P}}}(\mathcal{B}) \subset \tilde{\mathfrak{P}}$ is Zariski closed. Its asymptotic cone is $\mathrm{Supp}_{\tilde{\mathfrak{P}}}(\mathrm{gr}\,\mathcal{B})$, where the associated graded is taken with respect to any good filtration.*

For a HC $\mathcal{A}_{\tilde{\mathfrak{P}}}$-bimodule $\mathcal{B}$, by $\mathrm{V}_0(\mathcal{B})$ we will denote $\mathrm{V}(\mathcal{B}) \cap X^0_0$.

2.5. Localization theorems

Let $X = X^\theta$ be a conical symplectic resolution. Let $\lambda \in \tilde{\mathfrak{P}}$ and $\mathcal{A}^\theta_\lambda$ be the corresponding filtered quantization of X with global sections $\mathcal{A}_\lambda$. We can consider the category $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ of all coherent sheaves of $\mathcal{A}^\theta_\lambda$-modules and its derived category $D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda))$, see [BL, Section 2.3]. We have the global section functor $\Gamma_\lambda = \mathrm{Hom}_{\mathcal{A}^\theta_\lambda}(\mathcal{A}^\theta_\lambda, \bullet) : \mathrm{Coh}(\mathcal{A}^\theta_\lambda) \to \mathcal{A}_\lambda\text{-mod}$ and its left adjoint functor $\mathrm{Loc}_\lambda := \mathcal{A}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} \bullet : \mathcal{A}_\lambda\text{-mod} \to \mathcal{A}^\theta_\lambda\text{-mod}$.

We say that abelian localization holds for (λ, X) (or (λ, θ)) if $\Gamma_\lambda, \mathrm{Loc}_\lambda$ are mutually inverse equivalences. The following result (proved in [BPW, Section 5.3]) gives a necessary and sufficient condition for the abelian localization to hold. Let χ be an ample element in $\tilde{\mathfrak{P}}_\mathbb{Z}$.

Lemma 2.7. *The following two conditions on $\lambda \in \tilde{\mathfrak{P}}$ are equivalent:*

- *Abelian localization holds for (λ, θ).*
- *There is $n > 0$ such that the bimodules $\mathcal{A}^{(\theta)}_{\lambda+mn\chi,n\chi}$ and $\mathcal{A}^{(\theta)}_{\tilde{\mathfrak{P}},-n\chi}|_{\lambda+m(n+1)\chi}$ are mutually inverse Morita equivalences for all $m \in \mathbb{Z}_{\geqslant 0}$.*

Recall that an open subset $U \subset \tilde{\mathfrak{P}}$ is said to be *asymptotically generic* (a terminology from [BL]) if the asymptotic cone of its complement is contained in the singular locus.

Corollary 2.8. *Let $\chi \in \tilde{\mathfrak{P}}_\mathbb{Z}$ be ample and m be such that the line bundle $\mathcal{O}_X(n\chi)$ has no higher cohomology. Then there is an asymptotically generic open subset $U_{n,\chi} \subset \tilde{\mathfrak{P}}$ such that abelian localization holds for (λ, θ) provided $\lambda + nm\chi \in U_{n,\chi}$ for all $m \in \mathbb{Z}_{\geqslant 0}$.*

Proof. The locus of λ, where the bimodules $\mathcal{A}^{(\theta)}_{\lambda,m\chi}$ and $\mathcal{A}^{(\theta)}_{\mathfrak{P},-m\chi}|_{\lambda+m\chi}$ are inverse Morita equivalences is Zariski open and asymptotically generic. This is proved using Lemma 2.6, compare to the proof of (2) of [BL, Proposition 4.5]. The claim of our corollary follows from Lemma 2.7. □

A weaker version of this result was obtained in [BPW, Section 5.3].

We consider the derived functors $R\Gamma_\lambda : D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda)) \to D^b(\mathcal{A}_\lambda\text{-mod})$ and $L\,\mathrm{Loc}_\lambda : D^-(\mathcal{A}_\lambda\text{-mod}) \to D^-(\mathrm{Coh}(\mathcal{A}^\theta_\lambda))$ (if $\mathcal{A}_\lambda$ has finite homological dimension, then $L\,\mathrm{Loc}_\lambda$ restricts to a functor between the bounded derived categories).

We say that *derived localization* holds for (λ, X) (or (λ,θ)) if $R\Gamma_\lambda$ and $L\,\mathrm{Loc}_\lambda$ are mutually inverse equivalences. In this case, $\mathcal{A}_\lambda$ has finite homological dimension. In all known examples, the converse is also true, however, this fact is not proved in general.

2.6. Wall-crossing functors

Let θ, θ' be two generic elements of $H^2(X,\mathbb{Q})$ and $\lambda \in \tilde{\mathfrak{P}}$. Following [BPW, Section 6.3], we are going to produce a derived equivalence $\mathfrak{WC}_{\theta'\leftarrow\theta} : D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda)) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{A}^{\theta'}_\lambda))$ assuming abelian localization holds for (λ,θ) and derived localization holds for (λ,θ'). Then we set $\mathfrak{WC}_{\theta'\leftarrow\theta} := L\,\mathrm{Loc}^{\theta'}_\lambda \circ \Gamma^\theta_\lambda$. Note that this functor is right t-exact.

We can give a different realization of $\mathfrak{WC}_{\theta'\leftarrow\theta}$. Namely, pick $\lambda' \in \lambda + \tilde{\mathfrak{P}}_\mathbb{Z}$ such that abelian localization holds for (λ',θ'). We identify $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ with $\mathcal{A}_\lambda$-mod by means of Γ^θ_λ and $\mathrm{Coh}(\mathcal{A}^{\theta'}_\lambda)$ with $\mathcal{A}_{\lambda'}$-mod by means of $\Gamma^{\theta'}_{\lambda'}(\mathcal{A}^{\theta'}_{\lambda',\lambda'-\lambda} \otimes_{\mathcal{A}^{\theta'}_\lambda} \bullet)$. Under these identifications, the functor $\mathfrak{WC}_{\theta'\leftarrow\theta}$ becomes $\mathfrak{WC}_{\lambda'\leftarrow\lambda} := \mathcal{A}^{(\theta)}_{\lambda,\lambda'-\lambda} \otimes^L_{\mathcal{A}_\lambda} \bullet$, see [BPW, Proposition 6.31].

We will need to study the functor $\mathfrak{WC}_{\lambda,\lambda'-\lambda}$ as $\lambda'-\lambda$ is fixed and λ varies along a suitable affine subspace of $\tilde{\mathfrak{P}}$. Namely, we take any face Γ of any chamber C and consider the chamber C' opposite to C with respect Γ. For example, if $\Gamma = \{0\}$, then $C' = -C$, while for Γ of codimension 1, we get the unique chamber C' sharing the face Γ with C. Now pick a parameter λ_0 such that abelian localization holds for (λ_0,θ) with θ in the interior of C. Let $\mathfrak{P}_0$ be the vector subspace in $\tilde{\mathfrak{P}}$ spanned by Γ and $\mathfrak{P} := \lambda_0 + \mathfrak{P}_0$. Further, fix $\chi \in \tilde{\mathfrak{P}}_\mathbb{Z}$ such that abelian localization holds for $\lambda'_0 := \lambda_0 + \chi$ and θ', an element in the interior of C'.

Proposition 2.9. *Possibly after replacing λ_0 with an element $\lambda_0+\psi$ (and χ with $\chi-\psi$), where $\psi \in \tilde{\mathfrak{P}}_\mathbb{Z}$ and abelian localization holds for $(\lambda_0+\psi,\theta)$, we have the following: for a Zariski generic $\lambda \in \mathfrak{P}$, the functor $\mathcal{A}^{(\theta')}_{\mathfrak{P},\chi}|_\lambda \otimes^L_{\mathcal{A}_\lambda} \bullet$ is an equivalence $D^b(\mathcal{A}_\lambda\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_{\lambda+\chi}\text{-mod})$.*

In the proof we will need a connection between derived localization and global sections functors and homological duality functors. Let us recall the latter. Assume that the algebra $\mathcal{A}_\lambda$ has finite homological dimension. First, we have a functor $D_\lambda : D^b(\mathcal{A}_\lambda\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_{-\lambda}\text{-mod})^{opp}$ given by $R\operatorname{Hom}_{\mathcal{A}_\lambda}(\bullet, \mathcal{A}_\lambda)$ (here we use the identification $\mathcal{A}_\lambda^{opp} \cong \mathcal{A}_{-\lambda}$ mentioned in the end of Section 2.2). Second, for a generic element $\vartheta \in \tilde{\mathfrak{P}}_{\mathbb{Q}}$ we have a functor $D_\lambda^\vartheta : D^b(\operatorname{Coh}(\mathcal{A}_\lambda^\vartheta)) \xrightarrow{\sim} D^b(\operatorname{Coh}(\mathcal{A}_{-\lambda}^\vartheta))^{opp}$ given by $R\mathcal{H}om_{\mathcal{A}_\lambda^\vartheta}(\bullet, \mathcal{A}_\lambda^\vartheta)$.

Lemma 2.10. *We have* $D_\lambda \cong R\Gamma_{-\lambda}^\vartheta \circ D_\lambda^\vartheta \circ L\operatorname{Loc}_\lambda^\vartheta$.

Proof. Note that $R\Gamma_{-\lambda}^\vartheta \circ D_\lambda^\vartheta(\bullet) = R\operatorname{Hom}_{\mathcal{A}_\lambda^\vartheta}(\bullet, \mathcal{A}_\lambda^\vartheta)$. Now

$$R\Gamma_{-\lambda}^\vartheta \circ D_\lambda^\vartheta \circ L\operatorname{Loc}_\lambda^\vartheta(M) = R\operatorname{Hom}_{\mathcal{A}_\lambda^\vartheta}(\mathcal{A}_\lambda^\vartheta \otimes^L_{\mathcal{A}_\lambda} M, \mathcal{A}_\lambda^\vartheta)$$
$$= R\operatorname{Hom}_{\mathcal{A}_\lambda}(M, \mathcal{A}_\lambda^\vartheta) = R\operatorname{Hom}_{\mathcal{A}_\lambda}(M, \mathcal{A}_\lambda) = D_\lambda(M).$$

Here $R\operatorname{Hom}_{\mathcal{A}_\lambda}(M, \mathcal{A}_\lambda) \to R\operatorname{Hom}_{\mathcal{A}_\lambda}(M, \mathcal{A}_\lambda^\vartheta)$ is induced by $\mathcal{A}_\lambda \to \mathcal{A}_\lambda^\vartheta$, it is an isomorphism for any M because it is an isomorphism for $M = \mathcal{A}_\lambda$. □

Corollary 2.11. *Suppose that* $\mathcal{A}_\lambda, \mathcal{A}_{-\lambda}$ *have finite homological dimension. The functor* $R\Gamma_{-\lambda}^\vartheta$ *is an equivalence if and only if* $L\operatorname{Loc}_\lambda^\vartheta$ *is. Equivalently, derived localization holds for* (λ, ϑ) *if and only if it holds for* $(-\lambda, \vartheta)$.

Proof of Proposition 2.9. It is enough to prove this claim for a Weil generic element of $\mathfrak{P}$, compare to [L5, Section 5.3]. We only need to check that derived localization holds for (λ, θ'). By Corollary 2.11, this is equivalent to the claim that derived localization holds for $(-\lambda, \theta')$. We note that (perhaps, after replacing λ_0 with $\lambda_0 + \psi$ as in the statement of the proposition) abelian localization holds for $(-\lambda, \theta')$ (this follows from the choice of λ (Weil generic) and Corollary 2.8). The claim of the proposition follows. □

Now let us discuss a special class of wall-crossing functors, the long wall-crossing functors.

Pick a generic $\theta \in \tilde{\mathfrak{P}}_{\mathbb{Q}}$ and $\lambda, \lambda^- \in \tilde{\mathfrak{P}}$ subject to the following conditions:

1. Abelian localization holds for $(\lambda, \theta), (-\lambda, -\theta), (\lambda^-, \theta)$.
2. $\lambda^- - \lambda \in \tilde{\mathfrak{P}}_{\mathbb{Z}}$.

So we get the wall-crossing functor $\mathfrak{WC}_{\lambda^- \leftarrow \lambda} : D^b(\mathcal{A}_\lambda\text{-mod}) \to D^b(\mathcal{A}_{\lambda^-}\text{-mod})$.

We want to study the behavior of this functor on the subcategory $D^b_{hol}(\mathcal{A}_\lambda) \subset D^b(\mathcal{A}_\lambda\text{-mod})$ of all objects with holonomic homology. Recall that we say that an $\mathcal{A}_\lambda$-module M is *holonomic* if its associated variety $\mathrm{V}(M)$ intersects all leaves of X^0 at isotropic subvarieties, equivalently, if $\pi^{-1}(\mathrm{V}(M))$ is an isotropic subvariety of X, see [L6, Section 5].

It is easy to see that $\mathrm{V}(H^i(DM)) \subset \mathrm{V}(M)$ for all i. From here and $D^2 = \mathrm{id}$, it follows that D restricts to an equivalence $D^b_{hol}(\mathcal{A}_\lambda) \xrightarrow{\sim} D^b_{hol}(\mathcal{A}_{-\lambda})^{opp}$.

Similarly, we can define the full subcategory $D^b_{hol}(\mathcal{A}^\vartheta_\lambda) \subset D^b(\mathrm{Coh}(\mathcal{A}^\vartheta_\lambda))$. It was checked in [BL, Section 4], that $D^\vartheta_\lambda[\frac{1}{2}\dim X]$ is a t-exact equivalence $D^b_{hol}(\mathcal{A}^\vartheta_\lambda) \xrightarrow{\sim} D^b_{hol}(\mathcal{A}^\vartheta_{-\lambda})$.

The functor $\mathfrak{WC}_{\lambda^-\leftarrow\lambda}$ also restricts to an equivalence $D^b_{hol}(\mathcal{A}_\lambda) \xrightarrow{\sim} D^b_{hol}(\mathcal{A}_{\lambda^-})$.

The following result was obtained (in a special case but the proof in the general case is the same) in [BL, Proposition 5.13] (proved in [BL, Section 8]).

Proposition 2.12. *Under the assumptions (1),(2) above, there is a t-exact equivalence $D^b_{hol}(\mathcal{A}_{\lambda^-}) \xrightarrow{\sim} D^b_{hol}(\mathcal{A}_{-\lambda})^{opp}$ that intertwines $\mathfrak{WC}_{\lambda^-\leftarrow\lambda}$ with D.*

Here is a corollary of this proposition also proved in [BL, Section 8].

Corollary 2.13. *For $M \in \mathcal{A}_\lambda$-mod, the following two conditions are equivalent:*

(1) $H_i(\mathfrak{WC}_{\lambda^-\leftarrow\lambda}M) = 0$ *for* $i < \frac{1}{2}\dim X$.
(2) $\dim M < \infty$.

3. Perversity of wall-crossing

3.1. Main result

Let $(\lambda^1,\theta^1),(\lambda^2,\theta^2) \in \tilde{\mathfrak{P}} \times \tilde{\mathfrak{P}}_{\mathbb{Q}}$, where θ^1,θ^2 are generic, be such that abelian localization holds for these pairs, and $\lambda^2 - \lambda^1 \in \tilde{\mathfrak{P}}_{\mathbb{Z}}$. Let C^1, C^2 denote the chambers of θ^1,θ^2, respectively. We assume that C^1 and C^2 are opposite to each other with respect to their common face, say Γ. In other words, there is an interval whose midpoint is generic in Γ, while the end points are generic in C^1, C^2. We are going to prove that the functor $\mathfrak{WC}_{\lambda^2\leftarrow\lambda^1}$ is a *perverse equivalence* in the sense of Chuang and Rouquier, [R1, Section 2.6].

Let us recall the general definition. Let $\mathcal{T}^1,\mathcal{T}^2$ be triangulated categories equipped with t-structures that are homologically finite (each object in $\mathcal{T}^i$ has only finitely many nonzero homology groups). Let $\mathcal{C}^1,\mathcal{C}^2$ denote the hearts of $\mathcal{T}^1,\mathcal{T}^2$, respectively.

We are going to recall the definition of a perverse equivalence with respect to filtrations $\mathcal{C}^i = \mathcal{C}^i_0 \supset \mathcal{C}^i_1 \supset \ldots \supset \mathcal{C}^i_k = \{0\}$ by Serre subcategories. By definition, this is a triangulated equivalence $\mathcal{T}^1 \to \mathcal{T}^2$ subject to the following conditions:

(P1) For any j, the equivalence $\mathcal{F}$ restricts to an equivalence $\mathcal{T}^1_{\mathcal{C}^1_j} \to \mathcal{T}^2_{\mathcal{C}^2_j}$, where we write $\mathcal{T}^i_{\mathcal{C}^i_j}, i = 1, 2$, for the category of all objects in $\mathcal{T}^i$ with homology (computed with respect to the t-structures of interest) in $\mathcal{C}^i_j$.

(P2) For $M \in \mathcal{C}^1_j$, we have $H_\ell(\mathcal{F}M) = 0$ for $\ell < j$ and $H_\ell(\mathcal{F}M) \in \mathcal{C}^2_{j+1}$ for $\ell > j$.

(P3) The functor $M \mapsto H_j(\mathcal{F}M)$ induces an equivalence $\mathcal{C}^1_j/\mathcal{C}^1_{j+1} \xrightarrow{\sim} \mathcal{C}^2_j/\mathcal{C}^2_{j+1}$ of abelian categories.

Now let $\mathfrak{P}^1, \mathfrak{P}^2 \subset \tilde{\mathfrak{P}}$ denote the affine subspaces $\mathfrak{P}^i := \lambda^i + \mathrm{Span}_{\mathbb{C}}(\Gamma)$. We can shift the space $\mathfrak{P}^1$ (see Proposition 2.9) such that the derived localization holds for a Weil generic point of this space and the stability condition θ^2. We will produce chains of two-sided ideals

$$\mathcal{A}_{\mathfrak{P}^i} := \mathcal{I}^0_{\mathfrak{P}^i} \supset \mathcal{I}^1_{\mathfrak{P}^i} \supset \ldots \mathcal{I}^q_{\mathfrak{P}^i} \supset \mathcal{I}^{q+1}_{\mathfrak{P}^i} = \{0\},$$

where $q = \frac{1}{2}\dim X$, having the following property:

(*) For a Weil generic parameter $\widehat{\lambda}^i \in \mathfrak{P}^i$, the specialization $\mathcal{I}^j_{\widehat{\lambda}^i}$ is the minimal two-sided ideal $\mathcal{I} \subset \mathcal{A}_{\widehat{\lambda}^i}$ such that GK-dim $\mathcal{A}_{\widehat{\lambda}^i}/\mathcal{I} < 2j$.

(*) implies that $(\mathcal{I}^j_{\lambda^i})^2 = \mathcal{I}^j_{\lambda^i}$ for a Weil generic $\lambda^i \in \mathfrak{P}^i$ and hence also for a Zariski generic λ^i. Note that the ideal $\mathcal{I}^j_{\lambda^i}$ is well-defined for a Zariski generic λ^i as in the proof of [L9, Lemma 2.9]. We set $\mathcal{C}^i_j = (\mathcal{A}_{\lambda^i}/\mathcal{I}^j_{\lambda^i})$-mod. This is a Serre subcategory of $\mathcal{C}^i$.

Theorem 3.1. *We assume that X^0 has conical slices (see Definition 3.2 below). Suppose that derived localization holds for a Weil generic point $\widehat{\lambda}^1 \in \mathfrak{P}^1$ and θ^2. Pick χ in the chamber of θ^2 such that $H^1(X^{\theta_2}, \mathcal{O}(\chi)) = 0$. For a Zariski generic $\lambda^1 \in \mathfrak{P}^1$ and $\chi \in \tilde{\mathfrak{P}}_{\mathbb{Z}}$ such that $\mathfrak{P}^2 = \mathfrak{P}^1 + \chi$ and abelian localization holds for (λ^i, θ^i), where $\lambda^2 = \lambda^1 + \chi$, the functor $\mathfrak{WC}_{\lambda^1 \to \lambda^2}$ is a perverse equivalence with respect to the filtrations $\mathcal{C}^i_j \subset \mathcal{C}^i, i = 1, 2$.*

3.2. Slices

Here we are going to impose an additional assumption. Pick a point $x \in X^0$. Let $\mathcal{L} = \mathcal{L}_x$ denote the symplectic leaf through x. Consider the completion $\mathbb{C}[X^0]^{\wedge_x}$ of $\mathbb{C}[X^0]$. Then we can embed $\mathbb{C}[\mathcal{L}_x]^{\wedge_x}$ into $\mathbb{C}[X^0]^{\wedge_x}$ and this embedding is unique up to a twist with a Hamiltonian automorphism of $\mathbb{C}[X^0]^{\wedge_x}$, see [K1, Section 3]. Moreover, $\mathbb{C}[X^0]^{\wedge_x}$ splits into the completed tensor product $\mathbb{C}[X^0]^{\wedge_x} = \mathbb{C}[\mathcal{L}]^{\wedge_x} \widehat{\otimes} \underline{\widehat{A}}_x$, where $\underline{\widehat{A}}_x$ is the centralizer of $\mathbb{C}[\mathcal{L}]^{\wedge_x}$ in $\mathbb{C}[X^0]^{\wedge_x}$, [K1, Section 3]. Below we will often omit the subscript x.

Let $\underline{\widehat{X}}^0 (= \underline{\widehat{X}}^0_x)$ denote the formal spectrum of $\underline{\widehat{A}}$. We can view $\underline{\widehat{X}}_0$ as a formal subscheme of X^0 so that $X^{0\wedge_x} = \mathcal{L}^{\wedge_x} \times \underline{\widehat{X}}^0$. We call $\underline{\widehat{X}}^0$ a slice to x in X^0.

Definition 3.2. We say that $\underline{\widehat{X}}^0$ is *conical* if there is a pro-rational $\mathbb{C}^\times$-action on $\underline{\widehat{A}}$ that

- rescales the Poisson bracket on $\underline{\widehat{A}}$ by t^{-d},
- whose weights on the maximal ideal of $\underline{\widehat{A}}$ are positive,
- and that lifts to the preimage $\underline{\widehat{X}}$ of $\underline{\widehat{X}}^0$ in X.

Further, we say that X^0 *has conical slices* if $\underline{\widehat{X}}^0$ is conical for all $x \in X^0$. This holds for all examples of X^0 that we know.

Let $\underline{A}$ denote the $\mathbb{C}^\times$-finite part of $\underline{\widehat{A}}$. This is a Poisson subalgebra. Then $\underline{X}^0 := \operatorname{Spec}(\underline{A})$ is a Poisson variety with a contracting action of $\mathbb{C}^\times$. Moreover, $\underline{X}^0$ admits a symplectic resolution: the formal neighborhood of the zero fiber in the resolution of $\underline{X}^0$ coincides with the formal neighborhood of $\rho^{-1}(x)$. So $\underline{\tilde{\mathfrak{P}}} := H^2(\underline{X}, \mathbb{C}) = H^2(\rho^{-1}(x), \mathbb{C})$.

Now let $\mathcal{A}$ be a quantization of $\mathbb{C}[X^0]$. We are going to produce a slice quantization of $\mathbb{C}[\underline{X}^0]$, a construction that first appeared in [L1] with some refinements given in [L2]. In what follows we assume that d is even (we can always replace d with its multiple by replacing the contracting torus with its cover). Let $V := T_x\mathcal{L}$, this is a symplectic vector space (let ω_V denote the form). Consider the homogenized Weyl algebra $\mathbb{A}_\hbar(V)$ with the relations $uv - vu = \hbar^d \omega_V(u, v)$. We can consider the completion $\mathbb{A}_\hbar(V)^{\wedge_0}$ of $\mathbb{A}_\hbar(V)$ at $0 \in V$. Note that this is an algebra flat over $\mathbb{C}[[\hbar]]$ and $\mathbb{A}_\hbar(V)^{\wedge_0}/(\hbar) = \mathbb{C}[\mathcal{L}]^{\wedge_x}$.

Now consider the Rees algebra $\mathcal{A}_\hbar$ of $\mathcal{A}$ and its completion $\mathcal{A}_\hbar^{\wedge_x}$. We can lift the embedding $\mathbb{C}[\mathcal{L}]^{\wedge_x} \hookrightarrow \mathbb{C}[X^0]^{\wedge_x}$ to an embedding $\mathbb{A}_\hbar(V)^{\wedge_0} \hookrightarrow \mathcal{A}_\hbar^{\wedge_x}$ (that is unique up to a twist with an automorphism of the form $\exp(\hbar^{1-d} f)$ for $f \in \mathcal{A}_\hbar^{\wedge_x}$), see [L2, Section 2.1]. Moreover, the centralizer $\underline{\widehat{A}}_\hbar$ of $\mathbb{A}_\hbar(V)^{\wedge_0}$ in $\mathcal{A}_\hbar^{\wedge_x}$ satisfies $\underline{\widehat{A}}_\hbar/(\hbar) = \underline{\widehat{A}}$ so that we have $\mathcal{A}_\hbar^{\wedge_x} = \mathbb{A}_\hbar(V)^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \underline{\widehat{A}}_\hbar$.

Assume now that $\mathcal{A} = \mathcal{A}_\lambda$, where $\lambda \in \tilde{\mathfrak{P}}$.

Lemma 3.3. *The following statements are true:*

(1) *The action of* $\mathbb{C}^\times$ *on* $\underline{\widehat{A}}$ *lifts to a pro-rational* $\mathbb{C}^\times$*-action on* $\underline{\widehat{\mathcal{A}}}_\hbar$ *by algebra automorphisms with* $\hbar$ *of degree* 1.

(2) *Let* $\underline{\mathcal{A}}_\hbar$ *denote the* $\mathbb{C}^\times$*-finite part of* $\underline{\widehat{\mathcal{A}}}_\hbar$*, then* $\underline{\mathcal{A}} := \underline{\mathcal{A}}_\hbar/(\hbar - 1)$ *is the algebra of global sections of the filtered quantization of* $\underline{X}$*, whose period is the pull-back of* λ *to* $\rho^{-1}(x)$.

Proof. Let us prove (1). It is enough to lift the $\mathbb{C}^\times$-action on $\underline{\widehat{A}}\widehat{\otimes}\mathbb{C}[[V]]$ to $\underline{\widehat{\mathcal{A}}}_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbb{A}_\hbar(V)^{\wedge_0}$ (by changing the embedding of V into the latter tensor product we may achieve that V is $\mathbb{C}^\times$-stable so that the $\mathbb{C}^\times$-action will restrict to $\underline{\widehat{\mathcal{A}}}_\hbar$).

The action of $\mathbb{C}^\times$ on $\mathcal{A}_\hbar$ gives rise to the Euler derivation that we denote by eu. The derivation extends to the completion $\mathcal{A}_\hbar^{\wedge_x}$ that we have identified with $\underline{\widehat{\mathcal{A}}}_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbb{A}_\hbar(V)^{\wedge_0}$. Now on $\underline{\widehat{A}}\widehat{\otimes}\mathbb{C}[[V]]$ we have two derivations eu and $\underline{\mathsf{eu}}$, the latter comes from the $\mathbb{C}^\times$-action on $\underline{\widehat{A}}\widehat{\otimes}\mathbb{C}[[V]]$, where the action on V is by dilations. The difference $\delta := \mathsf{eu} - \underline{\mathsf{eu}}$ is a Poisson derivation of $\underline{\widehat{A}}\widehat{\otimes}\mathbb{C}[[V]]$. It is enough to show that it is Hamiltonian, then we can lift $\underline{\mathsf{eu}}$ to a derivation $\underline{\mathsf{eu}}$ of $\underline{\widehat{\mathcal{A}}}_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbb{A}_\hbar(V)^{\wedge_0}$, which is easily seen to integrate to a $\mathbb{C}^\times$-action.

To prove that δ is Hamiltonian, we note that both eu and $\underline{\mathsf{eu}}$ lift to $\underline{\widehat{X}}$. So δ also lifts to a symplectic vector field on $\underline{\widehat{X}}$. On the other hand, by a result of Kaledin, [K2, Corollary 1.5], $H^1(\rho^{-1}(x),\mathbb{C}) = 0$. It follows that $\underline{\widehat{X}}$ has no 1st de Rham cohomology. This shows that δ is Hamiltonian and finishes the proof of (1).

Let us prove (2). We can still form the completion $(\mathcal{A}_\hbar^\theta)^{\wedge_{\pi^{-1}(x)}}$ that will be a formal quantization of the formal neighborhood $X^{\wedge_{\pi^{-1}(x)}}$ of $\pi^{-1}(x)$ in X. The notion of period still makes sense. It follows from the construction in [BeKa, Section 4] that the period of $(\mathcal{A}_\hbar^\theta)^{\wedge_x}$ is the pull-back $\underline{\lambda}$ of λ to $\pi^{-1}(x)$. On the other hand, the centralizer of $\mathbb{A}_\hbar(V)^{\wedge_0}$ in $(\mathcal{A}_\hbar^\theta)^{\wedge_x}$ (to be denoted by $\underline{\widehat{\mathcal{A}}}_\hbar^\theta$) is a quantization of the slice in $X^{\wedge_{\pi^{-1}(x)}}$. Note that the global sections of $\underline{\widehat{\mathcal{A}}}_\hbar^\theta$ are $\underline{\widehat{\mathcal{A}}}_\hbar$. By the proof of (1), $\mathbb{C}^\times$ acts by automorphisms of $(\mathcal{A}_\hbar^\theta)^{\wedge_{\pi^{-1}(x)}}$ preserving V so that, modulo $\hbar$, we get the contracting action on $X^{\wedge_{\pi^{-1}(x)}}$. So $\mathbb{C}^\times$ acts on $\underline{\widehat{\mathcal{A}}}_\hbar^\theta$. From here we get a filtered quantization $\underline{\mathcal{A}}^\theta$ of $\underline{X}$ with period $\underline{\lambda}$. By the construction, its global sections are $\underline{\mathcal{A}}$. This finishes the proof of (2). □

Remark 3.4. We can also consider slices for the varieties $X^0_{\tilde{\mathfrak{P}}}, X^\theta_{\tilde{\mathfrak{P}}}$ and the algebra $\mathcal{A}_{\tilde{\mathfrak{P}}}$. For the same reasons as before, we get the deformations $\underline{X}^0_{\tilde{\mathfrak{P}}}, \underline{X}^\theta_{\tilde{\mathfrak{P}}}$ of $\underline{X}^0, \underline{X}$. Further, we get the quantization $\underline{\mathcal{A}}^\theta_{\tilde{\mathfrak{P}}}$ of $\underline{X}_{\tilde{\mathfrak{P}}}$ and its algebra of global sections $\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}}$. Part (2) of Lemma 3.3 shows that $\underline{\mathcal{A}}^\theta_{\tilde{\mathfrak{P}}}$ is obtained from the canonical quantization of $\underline{X}^\theta_{\underline{\tilde{\mathfrak{P}}}}$ under the pull-back with respect to the natural map $\tilde{\mathfrak{P}} \to \underline{\tilde{\mathfrak{P}}}$. Recall that we write $\underline{\tilde{\mathfrak{P}}}$ for $H^2(\underline{X},\mathbb{C})$.

3.3. Restriction functors for HC bimodules

Now we are going to define restriction functors between the categories of HC bimodules. Namely, we pick a point $x \in X^0$. This allows us to define the slice

algebras $\underline{\mathcal{A}}_\lambda$ for $\mathcal{A}_\lambda$ and $\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}}$ for $\mathcal{A}_{\tilde{\mathfrak{P}}}$. We are going to produce an exact functor $\bullet_{\dagger,x} : \mathrm{HC}(\mathcal{A}_{\tilde{\mathfrak{P}}}) \to \mathrm{HC}(\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}})$.

Pick $\mathcal{B} \in \mathrm{HC}(\mathcal{A}_{\tilde{\mathfrak{P}}})$. Choose a good filtration on $\mathcal{B}$ and form the corresponding Rees $\mathcal{A}_{\tilde{\mathfrak{P}},\hbar}$-bimodule $\mathcal{B}_\hbar$. This bimodule comes with the Euler derivation eu. The derivation extends to the completion $\mathcal{B}_\hbar^{\wedge_x}$ that is an $\mathcal{A}_{\tilde{\mathfrak{P}},\hbar}^{\wedge_x}$-bimodule. Similarly to [L1, Section 3.3], we see that $\mathcal{B}_\hbar^{\wedge_x}$ decomposes into the product $\mathbb{A}_\hbar(V)^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\widehat{\underline{\mathcal{B}}}_\hbar$, where $\widehat{\underline{\mathcal{B}}}_\hbar$ stands for the centralizer of V in $\mathcal{B}_\hbar^{\wedge_x}$. Note that we can equip $\widehat{\underline{\mathcal{B}}}_\hbar$ with an Euler derivation that is compatible with the derivation $\underline{\mathsf{eu}}$ on $\widehat{\underline{\mathcal{A}}}_{\tilde{\mathfrak{P}},\hbar}$. Namely, recall the proof of Lemma 3.3; there is an element $a \in \mathbb{A}_\hbar(V)^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\widehat{\underline{\mathcal{A}}}_{\tilde{\mathfrak{P}},\hbar}$ such that $\mathsf{eu} - \underline{\mathsf{eu}} = \hbar^{-d}[a,\cdot]$. Now we can *define* the derivation $\underline{\mathsf{eu}}$ of $\mathcal{B}_\hbar^{\wedge_x}$ as $\mathsf{eu} - \hbar^{-d}[a,\cdot]$. It restricts to $\widehat{\underline{\mathcal{B}}}_\hbar$. Then we define the $\underline{\mathcal{A}}_{\tilde{\mathfrak{P}},\hbar}$-subbimodule $\underline{\mathcal{B}}_\hbar$ of $\widehat{\underline{\mathcal{B}}}_\hbar$ as the the $\underline{\mathsf{eu}}$-finite part of $\widehat{\underline{\mathcal{B}}}_\hbar$. The bimodule $\underline{\mathcal{B}}_\hbar$ is gradeable and is finitely generated over $\underline{\mathcal{A}}_\hbar$. We set $\mathcal{B}_{\dagger,x} := \underline{\mathcal{B}}_\hbar/(\hbar-1)\underline{\mathcal{B}}_\hbar$. The assignment $\mathcal{B} \mapsto \mathcal{B}_{\dagger,x}$ is indeed a functor. It follows easily from the construction that this functor is exact, compare to [L1, Section 3.4].

The following two properties of $\bullet_{\dagger,x}$ are established as in [BL, Section 5.5].

Lemma 3.5. *The associated variety* $\mathrm{V}(\mathcal{B}_{\dagger,x})$ *is the unique conical subvariety in* $\underline{X}_{0,\tilde{\mathfrak{P}}}$ *such that* $\mathrm{V}(\mathcal{B}_{\dagger,x}) \times \mathcal{L}^{\wedge_x} = \mathrm{V}(\mathcal{B})^{\wedge_x}$.

Lemma 3.6. *The functor* $\bullet_{\dagger,x}$ *intertwines the Tor and the Ext functors. More precisely, we get the following:*

$$\mathrm{Tor}_i^{\mathcal{A}_{\tilde{\mathfrak{P}}}}(\mathcal{B}^1,\mathcal{B}^2)_{\dagger,x} \cong \mathrm{Tor}_i^{\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}}}(\mathcal{B}^1_{\dagger,x},\mathcal{B}^2_{\dagger,x}),$$
$$\mathrm{Ext}^i_{\mathcal{A}_{\tilde{\mathfrak{P}}}}(\mathcal{B}^1,\mathcal{B}^2)_{\dagger,x} \cong \mathrm{Ext}^i_{\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}}}(\mathcal{B}^1_{\dagger,x},\mathcal{B}^2_{\dagger,x}).$$

The latter equality holds for Ext's of left $\mathcal{A}_{\tilde{\mathfrak{P}}}$*-modules and of right modules.*

Let $\mathcal{L}$ denote the leaf through x. Consider the full subcategory $\mathrm{HC}_{\overline{\mathcal{L}}}(\mathcal{A}_{\tilde{\mathfrak{P}}}) \subset \mathrm{HC}(\mathcal{A}_{\tilde{\mathfrak{P}}})$ consisting of all HC bimodules $\mathcal{B}$ such that $\mathrm{V}_0(\mathcal{B}) \subset \overline{\mathcal{L}}$. Also consider the full subcategory $\mathrm{HC}_{fin}(\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}}) \subset \mathrm{HC}(\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}})$ of all bimodules that are finitely generated over $\mathbb{C}[\tilde{\mathfrak{P}}]$. By Lemma 3.5, the functor $\bullet_{\dagger,x}$ restricts to $\mathrm{HC}_{\overline{\mathcal{L}}}(\mathcal{A}_{\tilde{\mathfrak{P}}}) \to \mathrm{HC}_{fin}(\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}})$.

Lemma 3.7. *The functor* $\bullet_{\dagger,x} : \mathrm{HC}_{\overline{\mathcal{L}}}(\mathcal{A}_{\tilde{\mathfrak{P}}}) \to \mathrm{HC}_{fin}(\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}})$ *admits a right adjoint functor, to be denoted by* $\bullet^{\dagger,x}$.

Proof. This is proved as an analogous statement in [L1] (see Proposition 3.3.4 and 3.4.1 in *loc.cit.*) using [L6, Lemma 3.9] instead of [L1, Section 3.2]. □

Now let us study the behavior of $\bullet_{\dagger,x}$ on wall-crossing bimodules.

Proposition 3.8. *Suppose that $\lambda \in \tilde{\mathfrak{P}}, \chi \in \tilde{\mathfrak{P}}_{\mathbb{Z}}$, that abelian localization holds for $(\lambda+\chi,\theta)$ and that $H^1(X^\theta, \mathcal{O}(\chi)) = 0$. Then $(\mathcal{A}^{(\theta)}_{\lambda,\chi})_{\dagger,x} = \underline{\mathcal{A}}^{(\theta)}_{\lambda,\chi}$.*

Here and below we write $\underline{\mathcal{A}}^{(\theta)}_{\lambda,\chi}$ for the $\underline{\mathcal{A}}_{\tilde{\mathfrak{P}}}$-bimodule defined similarly to $\mathcal{A}^{(\theta)}_{\lambda,\chi}$.

Proof. From $H^1(X^\theta, \mathcal{O}(\chi)) = 0$ we deduce that $\operatorname{gr} \mathcal{A}^{(\theta)}_{\lambda,\chi} = \Gamma(\mathcal{O}(\chi))$. By the formal function theorem, we have that $\Gamma(\mathcal{O}(\chi))^{\wedge_x}$ coincides with the global sections of $\mathcal{O}(\chi)^{\wedge_{\pi^{-1}(x)}}$. It follows that

$$(\mathcal{A}^{(\theta)}_{\lambda,\chi,\hbar})^{\wedge_x}/\hbar(\mathcal{A}^{(\theta)}_{\lambda,\chi,\hbar})^{\wedge_x} = \Gamma(\mathcal{O}(\chi)^{\wedge_{\pi^{-1}(x)}}) \tag{3.1}$$

Now let $\underline{\mathcal{O}}(\chi)$ denote the line bundle on $\underline{X}$ obtained by restricting $\mathcal{O}(\chi)$. From (3.1) and the construction of the functor $\bullet_{\dagger,x}$, we conclude that

$$\operatorname{gr}\left((\mathcal{A}^{(\theta)}_{\lambda,\chi})_{\dagger,x}\right) = \Gamma(\underline{\mathcal{O}}(\chi)). \tag{3.2}$$

On the other hand, we have a natural homomorphism

$$(\mathcal{A}^{(\theta)}_{\lambda,\chi,\hbar})^{\wedge_x} \to \Gamma\left((\mathcal{A}^{\theta}_{\lambda,\chi,\hbar})^{\wedge_{\pi^{-1}(x)}}\right).$$

Note that

$$\Gamma\left((\mathcal{A}^{\theta}_{\lambda,\chi,\hbar})^{\wedge_{\pi^{-1}(x)}}\right) = \mathbb{A}_\hbar^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \Gamma\left((\underline{\mathcal{A}}^{\theta}_{\lambda,\chi,\hbar})^{\wedge_{\underline{\pi}^{-1}(0)}}\right).$$

This yields a filtered bimodule homomorphism $(\mathcal{A}^{(\theta)}_{\lambda,\chi})_{\dagger,x} \to \underline{\mathcal{A}}^{(\theta)}_{\lambda,\chi}$. The corresponding homomorphism $\operatorname{gr}\left((\mathcal{A}^{(\theta)}_{\lambda,\chi})_{\dagger,x}\right) \to \operatorname{gr} \underline{\mathcal{A}}^{(\theta)}_{\lambda,\chi}$ intertwines the isomorphism $\operatorname{gr}\left((\mathcal{A}^{(\theta)}_{\lambda,\chi})_{\dagger,x}\right) \xrightarrow{\sim} \Gamma(\underline{\mathcal{O}}(\chi))$ and the inclusion $\operatorname{gr} \underline{\mathcal{A}}^{(\theta)}_{\lambda,\chi} \hookrightarrow \Gamma(\underline{\mathcal{O}}(\chi))$. It follows that $\operatorname{gr}\left((\mathcal{A}^{(\theta)}_{\lambda,\chi})_{\dagger,x}\right) \xrightarrow{\sim} \operatorname{gr} \underline{\mathcal{A}}^{(\theta)}_{\lambda,\chi}$ and hence $(\mathcal{A}^{(\theta)}_{\lambda,\chi})_{\dagger,x} \xrightarrow{\sim} \underline{\mathcal{A}}^{(\theta)}_{\lambda,\chi}$. □

3.4. Proof of Theorem 3.1

The proof follows the strategy of [L5, Section 6] using besides Corollary 2.13 and Proposition 3.8.

First, let us produce the ideals $\mathcal{I}^j_{\mathfrak{P}^i}, i = 1, 2$. We start with $\mathcal{I}^1_{\mathfrak{P}^i}$.

Lemma 3.9. *There is an ideal $\mathcal{I}^1_{\mathfrak{P}^i} \subset \mathcal{A}_{\mathfrak{P}^i}$ that specializes to the minimal ideal of finite codimension for a Weil generic parameter in $\mathfrak{P}^i$ and such that $\mathcal{A}^{\mathfrak{P}^i}/\mathcal{I}^{\mathfrak{P}^i}_1$ is finitely generated as a module over $\mathbb{C}[\mathfrak{P}^i]$.*

Proof. The proof repeats that of [L5, Lemma 5.1] (using the fact that the algebra $\mathcal{A}_\lambda$ has a unique minimal ideal of finite codimension, see [L6, Section 4.3], instead of appealing to the category $\mathcal{O}$ as in [L5]). □

Now let us construct the ideals $\mathcal{I}^j_{\mathfrak{P}^i}$ for arbitrary j as in [L5, Section 5.2]. We set

$$\mathcal{I}^j_{\mathfrak{P}^i} = \left(\bigcap_{\mathcal{L}} (\mathcal{I}^{1,\mathcal{L}}_{\mathfrak{P}^i})^{\dagger,x}\right)^j,$$

where the union is taken over all symplectic leaves $\mathcal{L} \subset X_0$ with $\dim \mathcal{L} < 2j$ and $x \in \mathcal{L}$. Here $\mathcal{I}^{1,\mathcal{L}}_{\mathfrak{P}^i} \subset \mathcal{A}^{\mathcal{L}}_{\mathfrak{P}^i}$ (the slice algebra corresponding to the leaf $\mathcal{L}$) is the ideal constructed similarly to $\mathcal{I}^1_{\mathfrak{P}^i} \subset \mathcal{A}_{\mathfrak{P}^i}$. Similarly to [L5, Lemma 5.2], we see that, for a Weil generic $\widehat{\lambda}^i \subset \mathfrak{P}^i$, the ideal $\mathcal{I}^j_{\widehat{\lambda}^i}$ is the minimal ideal $\mathcal{I} \subset \mathcal{A}_{\widehat{\lambda}^i}$ with GK-$\dim \mathcal{A}_{\widehat{\lambda}^i}/\mathcal{I} < 2j$.

Now, similarly to the proof of [L5, Theorem 6.1], Theorem 3.1 follows from the next proposition. Here we pick a Zariski generic $\lambda^1 \in \mathfrak{P}^1$ and set $\mathcal{A}_1 := \mathcal{A}_{\lambda^1}, \mathcal{A}_2 := \mathcal{A}_{\lambda^1+\chi}, \mathcal{B} := \mathcal{A}^{(\theta^2)}_{\lambda^1,\chi}, \mathcal{I}^j_1 := \mathcal{I}^j_{\lambda^1}, \mathcal{I}^j_2 := \mathcal{I}^j_{\lambda^1+\chi}$.

Proposition 3.10. *The following is true.*

(a) For all i, j, we have $\mathcal{I}^j_2 \operatorname{Tor}^{\mathcal{A}_1}_i(\mathcal{B}, \mathcal{A}_1/\mathcal{I}^j_1) = 0$.
(b) For all i, j, we have $\operatorname{Tor}^{\mathcal{A}_2}_i(\mathcal{A}_2/\mathcal{I}^j_2, \mathcal{B})\mathcal{I}^j_1 = 0$.
(c) We have $\operatorname{Tor}^{\mathcal{A}_1}_i(\mathcal{B}, \mathcal{A}_1/\mathcal{I}^j_1) = 0$ for $i < n + 1 - j$.
(d) We have $\mathcal{I}^{j-1}_2 \operatorname{Tor}^{\mathcal{A}_1}_i(\mathcal{B}, \mathcal{A}_1/\mathcal{I}^j_1) = \operatorname{Tor}^{\mathcal{A}_2}_i(\mathcal{A}_2/\mathcal{I}^j_2, \mathcal{B})\mathcal{I}^{j-1}_1 = 0$ for $i > n + 1 - j$.
(e) Set $\mathcal{B}_j := \operatorname{Tor}^{\mathcal{A}_1}_{n+1-j}(\mathcal{B}, \mathcal{A}_1/\mathcal{I}^j_1)$. The kernel and the cokernel of the natural homomorphism

$$\mathcal{B}_j \otimes_{\mathcal{A}_1} \operatorname{Hom}_{\mathcal{A}_2}(\mathcal{B}_j, \mathcal{A}_2/\mathcal{I}^j_2) \to \mathcal{A}_2/\mathcal{I}^j_2$$

are annihilated by $\mathcal{I}^{j-1}_2$ on the left and on the right.
(f) The kernel and the cokernel of the natural homomorphism

$$\operatorname{Hom}_{\mathcal{A}_1}(\mathcal{B}_j, \mathcal{A}_1/\mathcal{I}^j_1) \otimes_{\mathcal{A}_2} \mathcal{B}_j \to \mathcal{A}_1/\mathcal{I}^j_1.$$

are annihilated on the left and on the right by $\mathcal{I}^{j-1}_1$.

Proof. The proof of this proposition closely follows that of [L5, Proposition 6.3]. As in that proof (see Step 4 there), it is enough to prove (a)-(f) in the case when λ^1 is Weil generic in $\mathfrak{P}^1$. The proof that (a),(b) hold is the same as in Step 1 of the proof of [L5, Proposition 6.3]. To prove (c)-(f), we start with the case of $j = 1$. Here these claims follow from Corollary 2.13. Now the

proof for arbitrary j repeats that of Step 3 of the proof of [L5, Proposition 6.3], where we use Proposition 3.8 to show that the restriction of the wall-crossing bimodule is still a wall-crossing bimodule (note that we can take sufficiently ample χ in the definition of a wall-crossing bimodule and hence the cohomology vanishing required in Proposition 3.8 holds). □

Remark 3.11. All varieties X^0 we know have conical slices. One can prove Theorem 3.1 even without this assumption, but the proof is considerably more technical.

3.5. Application to Etingof type conjecture

Here we consider a quiver Q of affine type. We use the notation from Section 2.3.

We consider the category $\mathcal{A}^\theta_\lambda(v,w)\text{-mod}_{\rho^{-1}(0)}$ of all coherent $\mathcal{A}^\theta_\lambda(v,w)$-modules supported at $\rho^{-1}(0)$. We are going to describe $K_0(\mathcal{A}^\theta_\lambda(v,w)\text{-mod}_{\rho^{-1}(0)})$ (we always consider complexified K_0) confirming [BL, Conjecture 1.1] when Q is affine. The dimension of this K_0 coincides with the number of finite dimensional irreducible representations of $\mathcal{A}_\lambda(v,w)$ provided the homological dimension of $\mathcal{A}_\lambda(v,w)$ is finite, see, e.g., [BL, Section 1.5].

Let us write ω for the dominant weight of $\mathfrak{g}(Q)$ with labels w_i. Further, we set $\nu=\omega-\sum_{i\in Q_0} v_i\alpha_i$, where we write α_i for the simple root of Q corresponding to $i\in Q_0$. Recall that, by [Nak], the homology group $H_{mid}(\mathcal{M}^\theta(v,w))$ (where "mid" stands for $\dim_{\mathbb{C}}\mathcal{M}^\theta(v,w)$) is identified with the weight space $L_\omega[\nu]$ of weight ν in the irreducible integrable $\mathfrak{g}(Q)$-module L_ω with highest weight ω. Further, by [BaGi] (see also [BL, Section 11]), we have a natural inclusion $K_0(\mathcal{A}_\lambda(v,w)\text{-mod}_{\rho^{-1}(0)})\hookrightarrow H_{mid}(\mathcal{M}^\theta(v,w))$ given by the characteristic cycle map CC_λ. We want to describe the image of CC_λ.

Following [BL, Section 3], we define a subalgebra $\mathfrak{a}(=\mathfrak{a}_\lambda)\subset\mathfrak{g}(Q)$ and an $\mathfrak{a}$-submodule $L^{\mathfrak{a}}_\omega\subset L_\omega$. By definition, $\mathfrak{a}$ is spanned by the Cartan subalgebra $\mathfrak{t}\subset\mathfrak{g}(Q)$ and all root spaces $\mathfrak{g}_\beta(Q)$ where $\beta=\sum_{i\in Q_0}b_i\alpha^i$ is a real root with $\sum_{i\in Q_0}b_i\lambda_i\in\mathbb{Z}$. For $L^{\mathfrak{a}}_\omega$ we take the $\mathfrak{a}$-submodule of L_ω generated by the extremal weight spaces (those, where the weight is conjugate to the highest one under the action of the Weyl group).

Theorem 3.12. *Let Q be of affine type. Then the map CC_λ is injective and the image of $K_0(\mathcal{A}^\theta_\lambda(v,w)\text{-mod}_{\rho^{-1}(0)})$ in $L_\omega[\nu]$ under CC_λ coincides with $L^{\mathfrak{a}}_\omega\cap L_\omega[\nu]$.*

Proof of Theorem 3.12. It was checked in [BL, Section 7] that $L^{\mathfrak{a}}_\omega\cap L_\omega[\nu]$ is contained in the image of CC_λ. According to [BL, Section 11], to finish

the proof one needs to check that there are no notrivial *extremal* finite dimensional modules (defined in [BL, Section 11.1]). This, in turn, follows if one proves that the wall-crossing through the wall $\ker\delta$ (where δ stands for the indecomposable imaginary root of Q) cannot have a homological shift of $\frac{1}{2}\dim\mathcal{M}^\theta(v,w)$. This reduction was obtained in [BL, Section 11.2].

So let us check that the homological shift for the wall-crossing to $\ker\delta$ is less than $\dim\mathcal{M}^\theta(v,w)/2$. Thanks to Theorem 3.1, it is enough to prove the following. Let $\mathfrak{P}^1$ be an affine subspace in $\tilde{\mathfrak{P}}$ with associated vector space $\ker\delta$. We need to show that the ideal $\mathcal{I}^1_{\mathfrak{P}^1}$ coincides with the algebra $\mathcal{A}_{\mathfrak{P}^1}(v,w)$ (at least after localization to a Zariski generic locus). The quotient $\mathcal{A}_{\mathfrak{P}^1}(v,w)/\mathcal{I}^1_{\mathfrak{P}^1}$ is finitely generated over $\mathbb{C}[\mathfrak{P}^1]$ and so is $\operatorname{gr}\left(\mathcal{A}_{\mathfrak{P}^1}(v,w)/\mathcal{I}^1_{\mathfrak{P}^1}\right)$. But $\operatorname{gr}\mathcal{I}^1_{\mathfrak{P}^1}$ is a Poisson ideal. What remains to prove is that the variety $\mathcal{M}_p(v,w)$ has no symplectic leaves that are single points as long as $p\in\ker\delta$ is Zariski generic (provided $\mathcal{M}_p(v,w)$ is not a point itself). This follows from Lemma 2.2. □

3.6. Wall-crossing bijections and annihilators

We use the notation of Theorem 3.1. Being perverse, the wall-crossing functor $\mathfrak{WC}_{\lambda^2\leftarrow\lambda^1}$ induces a bijection $\mathfrak{wc}_{\lambda^2\leftarrow\lambda^1}:\operatorname{Irr}(\mathcal{A}_{\lambda^1})\to\operatorname{Irr}(\mathcal{A}_{\lambda^2})$ between the sets of irreducible modules (to be called the *wall-crossing bijection*). In this section, we are going to investigate a compatibility of these bijections with the annihilators.

The following proposition generalizes the left cell part of [L8, Theorem 1.1(i)].

Proposition 3.13. *Let* $N_1,N_2\subset\operatorname{Irr}(\mathcal{A}_{\lambda^1})$ *be such that* $\operatorname{Ann}_{\mathcal{A}_{\lambda^1}}(N_1)=\operatorname{Ann}_{\mathcal{A}_{\lambda^1}}(N_2)$. *Let* $M_i:=\mathfrak{wc}_{\lambda^2\leftarrow\lambda^1}(N_i), i=1,2$. *Then* $\operatorname{Ann}_{\mathcal{A}_{\lambda^2}}(M_1)=\operatorname{Ann}_{\mathcal{A}_{\lambda^2}}(M_2)$.

Proof. Let us write $\mathcal{A}^j$ for $\mathcal{A}_{\lambda^j}$ and $\mathcal{B}$ for the wall-crossing $\mathcal{A}^2$-$\mathcal{A}^1$-bimodule. Let $\mathcal{J}:=\operatorname{Ann}_{\mathcal{A}^1}(N_i), i=1,2$.

Note that $\mathcal{B}\otimes^L_{\mathcal{A}^1}\bullet$ is a perverse equivalence between $\mathrm{HC}(\mathcal{A}^1)$ and $\mathrm{HC}(\mathcal{A}^2\text{-}\mathcal{A}^1)$ (viewed as hearts of standard t-structures on the full subcategories of $D^b(\mathcal{A}^1\text{-bimod}), D^b(\mathcal{A}^2\text{-}\mathcal{A}^1\text{-bimod})$ of all complexes with HC homology). The filtrations are again defined by the annihilation by the ideals $\mathcal{I}^j_{\lambda^1},\mathcal{I}^j_{\lambda^2}$ from the left.

Consider the HC $\mathcal{A}^1$-bimodule $\mathcal{A}^1/\mathcal{J}$. The ideal $\mathcal{J}$ is primitive hence prime and so, by classical results of Borho and Kraft, [BoKr, Corollary 3.6], the inclusion $\mathcal{J}\subsetneq\tilde{\mathcal{J}}$ implies $\text{GK-}\dim(\mathcal{A}^1/\mathcal{J})>\text{GK-}\dim(\mathcal{A}^1/\tilde{\mathcal{J}})$. It follows

that the HC bimodule $\mathcal{A}^1/\mathcal{J}$ has simple socle, say S. Let T be the corresponding simple $\mathcal{A}^2$-$\mathcal{A}^1$-bimodule. We claim that $\operatorname{Ann}_{\mathcal{A}^2}(M_j)$ coincides with the left annihilator, $\operatorname{LAnn}_{\mathcal{A}^2}(T)$, of T.

First of all, note that $\operatorname{Ann}_{\mathcal{A}^2}(M_j), \operatorname{LAnn}_{\mathcal{A}^2}(T)$ are primitive ideals. Moreover,

$$\operatorname{V}(\mathcal{A}^2/\operatorname{Ann}_{\mathcal{A}^2}(M_j)) = \operatorname{V}(\mathcal{A}^2/\operatorname{LAnn}_{\mathcal{A}^2}(T)) = \operatorname{V}(\mathcal{A}^1/\mathcal{J}).$$

Indeed, as in [BL, Theorem 10.2],

$$\operatorname{V}(\mathcal{A}^2/\operatorname{Ann}_{\mathcal{A}^2}(M_j)) = \operatorname{V}(\mathcal{A}^1/\operatorname{Ann}_{\mathcal{A}^1}(N_j)),$$
$$\operatorname{V}(\mathcal{A}^2/\operatorname{LAnn}_{\mathcal{A}^2}(T)) = \operatorname{V}(\mathcal{A}^1/\operatorname{LAnn}_{\mathcal{A}^1}(S)).$$

As in the proof of [BL, Theorem 10.2], we see that M_j is the head of $\mathcal{B}_\ell \otimes_{\mathcal{A}^1} N_j$ and T is the head of $\mathcal{B}_\ell \otimes_{\mathcal{A}^1} S$, where $\mathcal{B}_\ell$ is the Tor bimodule introduced in Proposition 3.10. Note that $S \otimes_{\mathcal{A}^1} N_j \twoheadrightarrow N_j$. By axiom (P3) in the definition of a perverse equivalence, the kernels of $\mathcal{B}_\ell \otimes_{\mathcal{A}^1} N_j \twoheadrightarrow M_j$ are annihilated by $\mathcal{I}^2_{\ell+1}$ and the same is true for the kernel of $\mathcal{B}_\ell \otimes_{\mathcal{A}^1} S \twoheadrightarrow T$. So we get epimorphisms $T \otimes_{\mathcal{A}^1} N_j \twoheadrightarrow M_j$. From here we see that $\operatorname{LAnn}_{\mathcal{A}^2}(T) \subset \operatorname{Ann}_{\mathcal{A}^2}(M_j)$. Since the associated varieties of these primitive ideals coincide, we apply the result of Borho and Kraft again, and get $\operatorname{LAnn}_{\mathcal{A}^2}(T) = \operatorname{Ann}_{\mathcal{A}^2}(M_j)$. □

4. Wall-crossing functors as partial Ringel dualities

4.1. Highest weight categories

Let us start by recalling the standard notion of a highest weight category.

Basic assumptions. Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category equivalent to the category of finite dimensional modules over a unital associative finite dimensional $\mathbb{C}$-algebra. Let $\mathcal{T}$ be an indexing set for the simples in $\mathcal{C}$, we write $L(\tau)$ for the simple object indexed by τ and $P(\tau)$ for its projective cover.

By a highest weight structure on $\mathcal{C}$ we mean a partial order $\leqslant$ on $\mathcal{T}$ that satisfies the axioms (HW1) and (HW2) below. For $\tau \in \mathcal{T}$, let $\mathcal{C}_{\leqslant\tau}$ (resp., $\mathcal{C}_{<\tau}$) denote the Serre span of $L(\tau')$ with $\tau' \leqslant \tau$ (resp., $\tau' < \tau$). Here is our first axiom:

(HW1) The quotient category $\mathcal{C}_{\leqslant\tau}/\mathcal{C}_{<\tau}$ is equivalent to the category of vector spaces.

Let $\Delta(\tau)$ denote the projective cover of $L(\tau)$ in $\mathcal{C}_{\leqslant\tau}$, by definition, this is the standard object corresponding to τ. Note that we have a natural epimorphism $P(\tau) \twoheadrightarrow \Delta(\tau)$. Here is our second axiom:

(HW2) The kernel of $P(\tau) \twoheadrightarrow \Delta(\tau)$ is filtered with $\Delta(\tau')$, where $\tau' > \tau$.

Recall that in any highest weight category one has costandard objects $\nabla(\tau)$, $\tau \in \mathcal{T}$, with $\dim \mathrm{Ext}^i(\Delta(\tau), \nabla(\tau')) = \delta_{i,0}\delta_{\tau,\tau'}$. By a *tilting* in $\mathcal{C}$ we mean an object that is standardly filtered (admits a filtration by Δ's) and also costandardly filtered. The indecomposable tilting objects are indexed by $\mathcal{T}$: we have a unique indecomposable tilting $T(\tau)$ that admits an embedding $\Delta(\tau) \hookrightarrow T(\tau)$ with standardly filtered cokernel.

Now let us recall the notion of Ringel duality that we will be generalizing below. Let $\mathcal{C}_1, \mathcal{C}_2$ be two highest weight categories. Let $\mathcal{C}_2^{\Delta}, \mathcal{C}_1^{\nabla}$ denote the full subcategories of standardly and costandardly filtered objects in $\mathcal{C}_2, \mathcal{C}_1$, respectively. Let R be an equivalence $\mathcal{C}_1^{\nabla} \xrightarrow{\sim} \mathcal{C}_2^{\Delta}$ of exact categories. Let T denote the tilting generator of $\mathcal{C}_1$, i.e., the sum of all indecomposable tilting objects. Then $\mathcal{C}_2$ gets identified with $\mathrm{End}(T)^{opp}$-mod and the equivalence R above becomes $\mathrm{Hom}(T, \bullet)$. We also have a derived equivalence $R\,\mathrm{Hom}(T, \bullet) : D^b(\mathcal{C}_1) \to D^b(\mathcal{C}_2)$. This equivalence maps injectives to tiltings and, obviously, tiltings to projectives. We write $\mathcal{C}_1^{\vee}$ for $\mathcal{C}_2$. The functor R is called the (covariant) *Ringel duality*, and the category $\mathcal{C}_1^{\vee}$ is called the *Ringel dual* of $\mathcal{C}_1$.

4.2. Categories $\mathcal{O}$ for symplectic resolutions and cross-walling functors

Now let us recall an example of a highest weight category from [BLPW].

Suppose that we have a conical symplectic resolution X that comes equipped with a Hamiltonian action of a torus T that commutes with the contracting $\mathbb{C}^\times$-action. Let $\lambda \in \tilde{\mathfrak{P}}$. The action of T on $\mathcal{O}_X$ lifts to a Hamiltonian action of T on $\mathcal{A}_\lambda^\theta$. So we get a Hamiltonian action on $\mathcal{A}_\lambda$. By Φ we denote a quantum comoment map $\mathfrak{t} \to \mathcal{A}_\lambda$, recall that it is defined up to adding a character of $\mathfrak{t}$.

Let $\nu : \mathbb{C}^\times \to T$ be a one-parameter subgroup. The subgroup ν induces a grading $\mathcal{A}_\lambda = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_\lambda^{i,\nu}$. We set $\mathcal{A}_\lambda^{\geqslant 0,\nu} = \bigoplus_{i\geqslant 0} \mathcal{A}_\lambda^{i,\nu}$ and define $\mathcal{A}_\lambda^{>0,\nu}$ similarly. Further, set $\mathsf{C}_\nu(\mathcal{A}_\lambda) := \mathcal{A}_\lambda^{0,\nu}/\bigoplus_{i>0} \mathcal{A}_\lambda^{-i,\nu}\mathcal{A}_\lambda^{i,\nu}$. Note that $\mathcal{A}_\lambda/\mathcal{A}_\lambda\mathcal{A}_\lambda^{>0,\nu}$ is an $\mathcal{A}_\lambda$-$\mathsf{C}_\nu(\mathcal{A}_\lambda)$-bimodule, while $\mathcal{A}_\lambda/\mathcal{A}_\lambda^{<0,\nu}\mathcal{A}_\lambda$ is a $\mathsf{C}_\nu(\mathcal{A}_\lambda)$-$\mathcal{A}_\lambda$-bimodule.

Define the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ as the full subcategory of $\mathcal{A}_\lambda$-mod consisting of all modules, where the action of $\mathcal{A}_\lambda^{>0,\nu}$ is locally nilpotent. We get two functors $\Delta_\nu, \nabla_\nu : \mathsf{C}_\nu(\mathcal{A}_\lambda)$-mod $\to \mathcal{O}_\nu(\mathcal{A}_\lambda)$ given by

$$\Delta_\nu(N) := (\mathcal{A}_\lambda/\mathcal{A}_\lambda\mathcal{A}_\lambda^{>0,\nu}) \otimes_{\mathsf{C}_\nu(\mathcal{A}_\lambda)} N, \nabla_\nu(N) := \mathrm{Hom}_{\mathsf{C}_\nu(\mathcal{A}_\lambda)}(\mathcal{A}_\lambda/\mathcal{A}_\lambda^{<0,\nu}\mathcal{A}_\lambda, N).$$

Now suppose that T acts on X with finitely many fixed points. We say that a one-parameter group $\nu : \mathbb{C}^\times \to T$ is *generic* if $X^{\nu(\mathbb{C}^\times)} = X^T$. Equivalently, ν is generic if and only if it does not lie in $\ker \kappa$ for any character κ of the T-action on $\bigoplus_{p\in X^T} T_pX$. The hyperplanes $\ker \kappa$ split the lattice $\mathrm{Hom}(\mathbb{C}^\times, T)$ into the union of polyhedral regions to be called *chambers* (of one-parameter subgroups).

Suppose that ν is generic. Further, pick a generic (see Definition 2.1) $\theta \in \tilde{\mathfrak{P}}_{\mathbb{Q}}$ and $\lambda_0 \in \tilde{\mathfrak{P}}$. Let $\lambda := \lambda_0 + n\theta$ for $n \gg 0$.

Proposition 4.1. *The following is true:*

(1) *The category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ only depends on the chamber of ν.*
(2) *The natural functor $D^b(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \to D^b(\mathcal{A}_\lambda\text{-mod})$ is a full embedding.*
(3) $\mathsf{C}_\nu(\mathcal{A}_\lambda) = \mathbb{C}[X^T]$.
(4) *More generally, we have $\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda) = \bigoplus_Z \mathcal{A}^Z_{\iota_Z^*(\lambda)-\rho_Z}$, where the summation is taken over the irreducible components Z of $X^{\nu_0(\mathbb{C}^\times)}$, ι_Z is the embedding $Z \hookrightarrow X$, $\iota_Z^* : H^2(X,\mathbb{C}) \to H^2(Z,\mathbb{C})$ is the corresponding pull-back map, ρ_Z is a suitable element of $H^2(Z,\mathbb{C})$ and $\mathcal{A}^Z_{\iota_Z^*(\lambda)-\rho_Z}$ stands for the global sections of the filtered quantization of Z with period $\iota_Z^*(\lambda) - \rho_Z$.*
(5) *The category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is highest weight, where the standard objects are $\Delta_\nu(p)$, the costandard objects are $\nabla_\nu(p)$, where $p \in X^T$. For an order, which is a part of the definition of a highest weight structure, we take the contraction order on X^T defined by ν.*
(6) *Suppose ν_0 lies in the face of a chamber containing ν. Then $\Delta_{\nu_0}, \nabla_{\nu_0}$ restrict to exact functors $\mathcal{O}_\nu(\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda)) \to \mathcal{O}_\nu(\mathcal{A}_\lambda)$.*
(7) *The functor $\mathfrak{WC}^{-1}_{\lambda\leftarrow\lambda^-} : D^b(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \to D^b(\mathcal{O}_\nu(\mathcal{A}_{\lambda^-}))$ is a Ringel duality functor.*

Proof. (1) follows from [BLPW, Corollary 3.19]. (2) is [BLPW, Corollary 5.13]. (3) is [BLPW, Proposition 5.3]. (4) follows from [L7, Propositions 5.3,5.7]. (5) follows from [BLPW, Proposition 6.7]. (6) follows from [L7, Proposition 6.9, Section 6.5]. (7) is [L7, Proposition 7.7]. □

Let us recall the cross-walling (a.k.a. shuffling) functors introduced in [BLPW, Section 8] and studied in more detail in [L7]. Let ν, ν' be two generic one-parameter subgroups. Then there is a unique functor $\mathfrak{CW}_{\nu'\leftarrow\nu} : D^b(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \to D^b(\mathcal{O}_{\nu'}(\mathcal{A}_\lambda))$ with the property that

$$\mathrm{Hom}_{D^b(\mathcal{A}_\lambda\text{-mod})}(M,N) = \mathrm{Hom}_{D^b(\mathcal{O}_{\nu'}(\mathcal{A}_\lambda))}(\mathfrak{CW}_{\nu'\leftarrow\nu}M, N).$$

This was proved in [BLPW, Section 8.2]

The following results were obtained in [L7, Section 7]. We choose a parameter λ in the same way as for Proposition 4.1.

Proposition 4.2. *The functor $\mathfrak{CW}_{\nu'\leftarrow\nu}$ has the following properties.*

(1) *The functor is an equivalence for all ν, ν'.*
(2) *Suppose that a sequence ν, ν', ν'' is reduced (meaning that any wall that does not separate ν, ν'' does not separate ν, ν' either). Then $\mathfrak{CW}_{\nu''\leftarrow\nu} \xrightarrow{\sim} \mathfrak{CW}_{\nu''\leftarrow\nu'} \circ \mathfrak{CW}_{\nu'\leftarrow\nu}$.*
(3) *The functor $\mathfrak{CW}_{-\nu\leftarrow\nu}[\dim X/2] : D^b(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \xrightarrow{\sim} D^b(\mathcal{O}_{-\nu}(\mathcal{A}_\lambda))$ is a Ringel duality functor.*
(4) *Let ν_0 be a one-parameter subgroup lying in a common face of the chambers of ν, ν'. Then the functors $\mathfrak{CW}_{\nu'\leftarrow\nu} \circ \Delta_{\nu_0}$ and $\Delta_{\nu_0} \circ \underline{\mathfrak{CW}}_{\nu'\leftarrow\nu}$ from $D^b(\mathcal{O}_\nu(\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda)))$ to $D^b(\mathcal{O}_{\nu'}(\mathcal{A}_\lambda))$ are naturally isomorphic. Here $\underline{\mathfrak{CW}}_{\nu'\leftarrow\nu}$ stands for the cross-walling functor $D^b(\mathcal{O}_\nu(\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda))) \xrightarrow{\sim} D^b(\mathcal{O}_{\nu'}(\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda)))$.*

4.3. Standardly stratified categories

Here we are going to recall the definition of standardly stratified structures generalizing highest weight ones. We follow [LW]. The definition given there is more restrictive than in [CPS] but is less restrictive than in [ADL].

Let $\mathcal{C}, \mathcal{T}, L(\tau), P(\tau)$ have the same meaning as in Basic assumptions of Section 4.1. The additional structure of a standardly stratified category on $\mathcal{C}$ is a partial *pre-order* $\leqslant$ on $\mathcal{T}$ that should satisfy certain axioms to be explained below. Let us write Ξ for the set of equivalence classes of $\leqslant$, this is a poset (with partial order again denoted by $\leqslant$) that comes with a natural surjection $\varrho : \mathcal{T} \twoheadrightarrow \Xi$. The pre-order $\leqslant$ defines a filtration on $\mathcal{C}$ by Serre subcategories indexed by Ξ. Namely, to $\xi \in \Xi$ we assign the subcategories $\mathcal{C}_{\leqslant\xi}$ that is the Serre span of the simples $L(\tau)$ with $\varrho(\tau) \leqslant \xi$. Define $\mathcal{C}_{<\xi}$ analogously and let $\mathcal{C}_\xi$ denote the quotient $\mathcal{C}_{\leqslant\xi}/\mathcal{C}_{<\xi}$. Let π_ξ denote the quotient functor $\mathcal{C}_{\leqslant\xi} \twoheadrightarrow \mathcal{C}_\xi$. Let us write $\Delta_\xi : \mathcal{C}_\xi \to \mathcal{C}_{\leqslant\xi}$ for the left adjoint functor of π_ξ. Also we write $\operatorname{gr}\mathcal{C}$ for $\bigoplus_\xi \mathcal{C}_\xi$, Δ for $\bigoplus_\xi \Delta_\xi : \operatorname{gr}\mathcal{C} \to \mathcal{C}$. We call Δ the *standardization functor*. Finally, for $\tau \in \varrho^{-1}(\xi)$ we write $L_\xi(\tau)$ for $\pi_\xi(L(\tau))$, $P_\xi(\tau)$ for the projective cover of $L_\xi(\tau)$ in $\mathcal{C}_\xi$ and $\Delta(\tau)$ for $\Delta_\xi(P_\xi(\tau))$. The object $\Delta(\tau)$ is called *standard*. Note that there is a natural epimorphism $P(\tau) \twoheadrightarrow \Delta(\tau)$. The object $\overline{\Delta}(\tau) := \Delta_\xi(L_\xi(\tau))$ is called *proper standard*.

The axioms to be satisfied by $(\mathcal{C}, \leqslant)$ in order to give a standardly stratified structure are as follows.

(SS1) The functor $\Delta : \operatorname{gr}\mathcal{C} \to \mathcal{C}$ is exact.

(SS2) The projective $P(\tau)$ admits an epimorphism onto $\Delta(\tau)$ whose kernel is filtered by $\Delta(\tau')$'s, where $\tau' > \tau$.

We will also need the notion of a *weakly* standardly stratified category. Here we keep (SS1) but use a weaker version of (SS2):

(SS2′) The projective $P(\tau)$ admits an epimorphism onto $\Delta(\tau)$ whose kernel admits a filtration with successive quotients $\Delta_\xi(M_\xi)$, where $\xi > \varrho(\tau)$ and M_ξ is some object in $\mathcal{C}_\xi$.

Note that (SS1) allows to identify $K_0(\operatorname{gr}\mathcal{C})$ and $K_0(\mathcal{C})$ by means of Δ. If (SS2) also holds, then we also have the identification of $K_0(\operatorname{gr}\mathcal{C}\text{-proj})$ and $K_0(\mathcal{C}\text{-proj})$.

If all quotient categories $\mathcal{C}_\xi$ are equivalent to Vect, then a standardly stratified category is the same as a highest weight category. On the opposite end, if we take the trivial pre-order on $\mathcal{T}$, then there is no additional structure.

4.3.1. (Proper) standardly filtered objects We say that an object in $\mathcal{C}$ is standardly filtered if it admits a filtration whose successive quotients are standard. The notion of a proper standardly filtered object is introduced in a similar fashion. The categories of the standardly filtered and of the proper standardly filtered objects will be denoted by $\mathcal{C}^\Delta$ and $\mathcal{C}^{\overline{\Delta}}$. Note that (SS1) implies that $\mathcal{C}^\Delta \subset \mathcal{C}^{\overline{\Delta}}$.

Lemma 4.3. *Suppose (SS1) holds. Let M be an object in $\mathcal{C}^{\overline{\Delta}}$ such that all proper standard quotients are of the form $\overline{\Delta}(\tau)$ with $\varrho(\tau) = \xi$. Then $M = \Delta_\xi(\pi_\xi(M))$.*

This is [L7, Lemma 3.1].

Also note that in a weakly standardly stratified category the following hold:

$$\operatorname{Ext}^i_{\mathcal{C}}(\Delta_\xi(M), \Delta_{\xi'}(N)) \neq 0 \Rightarrow \xi' \leqslant \xi. \tag{4.1}$$

$$\operatorname{Ext}^i_{\mathcal{C}}(\Delta_\xi(M), \Delta_\xi(N)) = \operatorname{Ext}^i_{\mathcal{C}_\xi}(M, N). \tag{4.2}$$

4.3.2. Subcategories and quotients Suppose that $\mathcal{C}$ is weakly standardly stratified.

Let Ξ_0 be a poset ideal in Ξ. Let $\mathcal{C}_{\Xi_0}$ denote the Serre span of the simples $L(\tau)$ with $\varrho(\tau) \in \Xi_0$. Then $\mathcal{C}_{\Xi_0}$ is a standardly stratified category with pre-order on $\mathcal{T}_0 := \varrho^{-1}(\Xi_0)$ restricted from Ξ. Note that, for $\tau \in \mathcal{T}_0$, we have

$\overline{\Delta}_{\Xi_0}(\tau) = \overline{\Delta}(\tau), \Delta_{\Xi_0}(\tau) = \Delta(\tau)$, where the subscript Ξ_0 refers to the objects computed in $\mathcal{C}_{\Xi_0}$.

The embedding $\iota_{\Xi_0} : \mathcal{C}_{\Xi_0}^{\overline{\Delta}} \hookrightarrow \mathcal{C}^{\overline{\Delta}}$ admits a left adjoint functor $\iota_{\Xi_0}^!$, to an object $M \in \mathcal{C}^{\overline{\Delta}}$, this functor assigns the maximal quotient lying in $\mathcal{C}_{\Xi_0}^{\overline{\Delta}}$.

Now let $\mathcal{C}^{\Xi_0}$ be the quotient category $\mathcal{C}/\mathcal{C}_{\Xi_0}$. Let π_{Ξ_0} denote the quotient functor $\mathcal{C} \to \mathcal{C}^{\Xi_0}$ and let $\pi_{\Xi_0}^!$ be its left adjoint. The category $\mathcal{C}^{\Xi_0}$ is standardly stratified with pre-order on $\Lambda^0 := \Lambda \setminus \Lambda_0$ restricted from Λ. For $\xi \in \Xi^0 := \Xi \setminus \Xi_0$ we have $\Delta_\xi^0 = \pi_{\Xi_0} \circ \Delta_\xi$. Let us also point out that $\pi_{\Xi_0}^!$ defines a full embedding $(\mathcal{C}^{\Xi_0})^{\overline{\Delta}} \hookrightarrow \mathcal{C}^{\overline{\Delta}}$ whose image coincides with the full subcategory $\mathcal{C}^{\overline{\Delta},\Lambda^0}$ consisting of all objects that admit a filtration with successive quotients $\overline{\Delta}(\tau)$ with $\tau \in \mathcal{T}^0$. This embedding sends $\Delta^{\Xi_0}(\tau)$ to $\Delta(\tau)$, $P^{\Xi_0}(\tau)$ to $P(\tau)$.

The following lemma describes the derived categories of $\mathcal{C}^{\Xi_0}$ and $\mathcal{C}_{\Xi_0}$.

Lemma 4.4. *A natural functor $D^b(\mathcal{C}_{\Xi_0}) \hookrightarrow D^b(\mathcal{C})$ is a fully faithful embedding. Moreover, $D^b(\mathcal{C}^{\Xi_0}) = D^b(\mathcal{C})/D^b(\mathcal{C}_{\Xi_0})$.*

See [L7, Lemma 3.2] for a proof.

4.3.3. Opposite category Let $\mathcal{C}$ be a standardly stratified category. It turns out that the opposite category $\mathcal{C}^{opp}$ is also standardly stratified with the same pre-order $\leqslant$, see [LW, Section 1.2]. The standard and proper standard objects for $\mathcal{C}^{opp}$ are denoted by $\nabla(\tau)$ and $\overline{\nabla}(\tau)$, when viewed as objects of $\mathcal{C}$, they are called *costandard* and *proper costandard*. The right adjoint functor to π_ξ will be denoted by ∇_ξ and we write ∇ for $\bigoplus_\xi \nabla_\xi$ (this is the so called *costandardization functor*). So we have $\overline{\nabla}(\tau) = \nabla(L_\xi(\tau))$ and $\nabla(\tau) = \nabla(I_\xi(\tau))$, where $I_\xi(\tau)$ is the injective envelope of $L_\xi(\tau)$ in $\mathcal{C}_\xi$.

Let us write $\mathcal{C}^\nabla, \mathcal{C}^{\overline{\nabla}}$ for the subcategories of costandardly and of proper costandardly filtered objects. We have the following standard lemma (that was used in [LW] to verify the claims in the previous paragraph).

Lemma 4.5 (Lemma 2.4 in [LW])**.** *The following is true.*

(1) $\dim \mathrm{Ext}^i(\Delta(\tau), \overline{\nabla}(\tau')) = \dim \mathrm{Ext}^i(\overline{\Delta}(\tau), \nabla(\tau')) = \delta_{i,0}\delta_{\tau,\tau'}$.

(2) *For $N \in \mathcal{C}$, we have $N \in \mathcal{C}^\nabla$ (resp., $N \in \mathcal{C}^{\overline{\nabla}}$) if and only if* $\mathrm{Ext}^1(\overline{\Delta}(\tau), N) = 0$ *(resp.,* $\mathrm{Ext}^1(\Delta(\tau), N) = 0$*) for all τ. Similar characterizations are true for $\mathcal{C}^\Delta, \mathcal{C}^{\overline{\Delta}}$.*

Let us also note the following fact.

Lemma 4.6. *Let $\mathcal{C}$ be a weakly standardly stratified category. Then $\mathcal{C}$ is standardly stratified if and only if the right adjoint ∇_ξ of π_ξ is exact.*

This is [L7, Lemma 3.4].

4.3.4. Equivalences Let $\mathcal{C}_1, \mathcal{C}_2$ be two weakly standardly stratified categories and let Ξ_1, Ξ_2 be the corresponding posets. By an equivalence of $(\mathcal{C}_1, \Xi_1), (\mathcal{C}_2, \Xi_2)$ we mean a pair (Φ, ϕ) consisting of an equivalence $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ of abelian categories and a poset isomorphism $\phi : \Xi_1 \to \Xi_2$ such that the bijection between the simples induced by Φ is compatible with ϕ. Clearly, Φ induces an equivalence $\operatorname{gr}\Phi : \operatorname{gr}\mathcal{C}_1 \xrightarrow{\sim} \operatorname{gr}\mathcal{C}_2$ and $\Phi\circ\Delta \cong \Delta\circ\operatorname{gr}\Phi, \Phi\circ\nabla \cong \nabla\circ\operatorname{gr}\Phi$.

4.3.5. Example: categorical tensor products The formalism of standardly stratified categories was introduced in [LW] to treat tensor products of categorical representations of Kac-Moody algebras. Namely, let $\mathfrak{g}$ be a Kac-Moody algebra and $\mathcal{V}_1, \dots, \mathcal{V}_k$ be minimal $\mathfrak{g}$-categorifications (in the sense of Rouquier, [R2]) that categorify irreducible integrable highest weight representations. Webster in [W] constructed the categorical tensor product $\mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_k$ that was equipped in [LW] with a structure of a standardly stratified category. The reader is referred to [LW] for details.

4.3.6. Example: standardly stratified structure on $\mathcal{O}$ from degenerating ν Let us give another example of a standardly stratified category. Let X, T be as in Section 4.2. Pick a generic $\nu : \mathbb{C}^\times \to T$ and let ν_0 lie in the closure of the chamber containing ν. Then ν_0 defines an order on the set of irreducible components of $X^{\nu_0(\mathbb{C}^\times)}$ (by contraction, see [L7, Section 6.1] for details). So we get the pre-order $\leqslant_{\nu_0}$ on the set X^T. It is easy to see (and was checked in [L7, Section 6.1]) that the order $\leqslant_\nu$ refines $\leqslant_{\nu_0}$.

Now pick a sufficiently dominant quantization parameter λ (i.e., a parameter of the form $\lambda_0 + n\chi$, where $n \gg 0$) and consider the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$.

The following proposition is the main result of [L7, Section 6].

Proposition 4.7. *The pre-order $\leqslant_{\nu_0}$ defines a standardly stratified structure on $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. The associated graded category is $\mathcal{O}_\nu(\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda))$. The standardization functor is Δ_{ν_0}, while the costandardization functor is ∇_{ν_0}.*

4.4. Standardly stratified structure on $\mathcal{O}$ from deforming parameters

This is one of two central parts of this section. Here we introduce a new standardly stratified structure on $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ coming from deforming λ along an affine subspace parallel to a face in a suitable chamber.

4.4.1. Main result Pick a face Γ of a classical chamber C and an element $\lambda^\circ \in \tilde{\mathfrak{P}}$. Let $\mathfrak{P}_0$ denote the vector subspace of $\tilde{\mathfrak{P}}$ spanned by Γ. Set $\mathfrak{P}^1 := \lambda^\circ + \mathfrak{P}_0$.

Lemma 4.8. *There is an asymptotically generic Zariski open subset $\tilde{\mathfrak{P}}^0 \subset \tilde{\mathfrak{P}}$ with the following properties.*

(1) *We have an algebra isomorphism $\mathsf{C}_\nu(\mathcal{A}_\lambda) \cong \mathbb{C}[X^T]$ for any $\lambda \in \tilde{\mathfrak{P}}^0$.*
(2) *For any $\lambda \in \tilde{\mathfrak{P}}^0$, the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is highest weight with standard objects $\Delta_{\nu,\lambda}(p)$ and costandard objects $\nabla_{\nu,\lambda}(p)$.*

Proof. (1) follows from the proof of [L7, Proposition 5.3] (the statement of this proposition is slightly weaker than (1)).

Now let $\widehat{\mathfrak{P}}^0$ be an asymptotically generic open subset of $\mathfrak{P}$ satisfying the conditions of (1). The category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is highest weight with standards $\Delta_\nu(p)$ and costandards $\nabla_\nu(p)$ if and only if $\mathrm{Ext}^2(\Delta_{\nu,\lambda}(p), \nabla_{\nu,\lambda}(p')) = 0$, see the proof of [BLPW, Theorem 5.12]. That $\mathrm{Ext}^2(\Delta_{\nu,\lambda}(p), \nabla_{\nu,\lambda}(p')) = 0$ for λ in an asymptotically generic open subset follows from the proof in [BLPW, Appendix], where instead of $\mathfrak{P}$ we had a suitable line in there. Let us provide details for reader's convenience.

We observe that $\mathrm{Ext}^2_{\mathcal{A}_\lambda}(\Delta_{\nu,\lambda}(p), \nabla_{\nu,\lambda}(p')) = \mathrm{Tor}_2^{\mathcal{A}_\lambda}(\Delta_{\nu,\lambda}(p), \Delta^r_{-\nu,\lambda}(p'))^*$, where $\Delta^r_{-\nu,\lambda}(p')$ is the Verma module for the opposite algebra. So it is enough to show that

$$\mathrm{Tor}_2^{\mathcal{A}_\lambda}(\Delta_{\nu,\lambda}(p), \Delta^r_{-\nu,\lambda}(p')) = 0$$

for λ in an asymptotically generic open subset. Consider the universal modules $\Delta_{\nu,\hat{\mathfrak{P}}}(p), \Delta^r_{-\nu,\hat{\mathfrak{P}}}(p')$. As in [BLPW, Appendix], we see that $\mathrm{Tor}_2^{\mathcal{A}_\lambda}(\Delta_{\nu,\mathfrak{P}}(p), \Delta^r_{-\nu,\mathfrak{P}}(p'))$ is a finitely generated $\mathbb{C}[\mathfrak{P}]$-module that admits a filtration such that the associated graded $\mathbb{C}[\mathfrak{P}]$-module is finitely generated and is supported on the singular locus in $\mathfrak{P}$. This implies the the complement to the support of $\mathrm{Tor}_2(\Delta_{\nu,\mathfrak{P}}(p), \Delta^r_{-\nu,\mathfrak{P}}(p'))$ is an asymptotically generic open subset. For $\mathfrak{P}^1$ we can take the intersection of this open subset with $\mathfrak{P}^0$. □

It follows that, possibly after replacing λ°, with $\lambda^\circ + \chi$ for $\chi \in \tilde{\mathfrak{P}}_\mathbb{Z} \cap C$ we may assume that $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is highest weight for a Zariski generic $\lambda \in \mathfrak{P}^1$. Recall, [L7, Lemma 6.4], that the order is introduced as follows. Let $c_\lambda(p)$ denote the image of $h \in \mathcal{A}^T_\lambda$ under the composition

$$\mathcal{A}^T_\lambda \twoheadrightarrow \mathbb{C}_\nu(\mathcal{A}_\lambda) \twoheadrightarrow \mathbb{C}_p. \tag{4.3}$$

We set $p <_\lambda p'$ if $c_\lambda(p') - c_\lambda(p) \in \mathbb{Z}_{>0}$. Recall that $c_\lambda(p) - c_\lambda(p')$ is a linear function whose value at $\chi \in \tilde{\mathfrak{P}}_\mathbb{Z}$ coincides with $\alpha_p(\chi) - \alpha_{p'}(\chi)$, where we write

$\alpha_p(\chi)$ for the character of the action of ν in the fiber of $\mathcal{O}(\chi)$, see [L7, Lemma 6.4].

Now pick a sufficiently general integral point χ in the interior of Γ. For $\lambda = \lambda^\circ + N\chi$ for $N \gg 0$, the order $<_\lambda$ can be described in the following way. Set $p \prec_\chi p'$ if $\alpha_{p'}(\chi) - \alpha_p(\chi) > 0$ and $p \sim_\chi p'$ if $\alpha_p(\chi) = \alpha_{p'}(\chi)$. For $p \sim_\chi p'$, the difference $c_{p'}(\lambda) - c_p(\lambda)$ is independent of $\lambda \in \mathfrak{P}^1$. In particular, if $p \sim_\chi p'$, then we have $p <_\lambda p'$ if and only if $p <_{\widehat{\lambda}} p'$ for a Weil generic $\widehat{\lambda} \in \mathfrak{P}^1$. So we see that the order $<_\lambda$ is refined by the following order $\leqslant$: we have $p < p'$ if $p \prec_\chi p'$ or $p <_{\widehat{\lambda}} p'$ (note that the latter automatically implies $p \sim_\chi p'$). We also would like to point out that $\preceq_\chi$ is a pre-order on X^T (refined by $\leqslant_\lambda$).

Here is the main result of the present section.

Proposition 4.9. *Let λ be as above. The category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is standardly stratified with respect to the pre-order $\preceq_\chi$. We have a labeling preserving equivalence $\operatorname{gr}\mathcal{O}_\nu(\mathcal{A}_\lambda) \cong \mathcal{O}_\nu(\mathcal{A}_{\widehat{\lambda}})$ for a Weil generic $\widehat{\lambda} \in \mathfrak{P}^1$; in particular, the right hand side is independent of $\widehat{\lambda}$.*

We prove this result in the rest of the section.

4.4.2. Objects $\overline{\Delta}_{\mathfrak{P}^1}(p), \overline{\nabla}_{\mathfrak{P}^1}(p)$ Note that every $p \in X^T$ defines an algebra homomorphism $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}^1}) \to \mathbb{C}[\mathfrak{P}^1]$ (similar to Section 4.3) that is an isomorphism over an asymptotically generic open subset in $\mathfrak{P}^1$. This allows to define the $\mathcal{A}_{\mathfrak{P}^1}$-modules $\Delta_{\mathfrak{P}^1}(p), \nabla_{\mathfrak{P}^1}(p)$ that specialize to $\Delta_\lambda(p), \nabla_\lambda(p)$ for a Zariski generic $\lambda \in \mathfrak{P}^1$.

We start by constructing $\mathcal{A}_{\mathfrak{P}^1}$-modules $\overline{\Delta}_{\mathfrak{P}^1}(p), \overline{\nabla}_{\mathfrak{P}^1}(p)$ that have the following properties:

(1) they are generically free over $\mathfrak{P}^1$,
(2) specialize to $L_{\widehat{\lambda}}(p)$ at $\widehat{\lambda}$,
(3) we have $\overline{\Delta}_\lambda(p) \twoheadrightarrow L_\lambda(p)$ (resp., $L_\lambda(p) \hookrightarrow \overline{\nabla}_\lambda(p)$) with kernel (resp., cokernel) filtered with $L_\lambda(p')$ for $p' <_\chi p$.

Later we will see that these objects specialized to λ become proper standard and proper costandard for the standardly stratified structure on $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ we are going to produce.

The objects $\overline{\Delta}_{\mathfrak{P}^1}(p)$ are constructed as quotients of $\Delta_{\mathfrak{P}^1}(p)$ similarly to the proof of [L9, Lemma 3.3]. Namely, let us consider all labels p' such that $p' \sim_\chi p, p' <_{\widehat{\lambda}} p$. Then we consider the object $M_{\mathfrak{P}^1} := \bigoplus_{p'} \Delta_{\mathfrak{P}^1}(p')$. Then we define the object $\Delta^i_{\mathfrak{P}^1}(p)$ recursively as follows: $\Delta^0_{\mathfrak{P}^1}(p) = \Delta_{\mathfrak{P}^1}(p)$ and $\Delta^i_{\mathfrak{P}^1}(p)$ is the cokernel of the natural homomorphism

$$\operatorname{Hom}_{\mathcal{A}_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, \Delta^{i-1}_{\mathfrak{P}^1}(p)) \otimes_{\mathbb{C}[\mathfrak{P}_1]} M_{\mathfrak{P}^1} \to \Delta^{i-1}_{\mathfrak{P}_1}(p).$$

For i sufficiently large, $\Delta^j_{\mathfrak{P}^1}(p) \xrightarrow{\sim} \Delta^{j+1}_{\mathfrak{P}^1}(p)$ for all $j \geqslant i$ because $\mathcal{A}_{\mathfrak{P}}$ is Noetherian. We take $\Delta^i_{\mathfrak{P}^1}(p)$ for $\overline{\Delta}_{\mathfrak{P}^1}(p)$.

The objects $\overline{\Delta}_{\mathfrak{P}^1}(p)$ have properties (1)-(3) similarly to [L9, Lemma 3.3].

The modules $\overline{\nabla}_{\mathfrak{P}^1}(p)$ are produced as follows. We set $N_{\mathfrak{P}^1} := \bigoplus_{p'} \nabla_{\mathfrak{P}^1}(p')$, $\nabla^0_{\mathfrak{P}^1}(p) := \nabla_{\mathfrak{P}^1}(p)$. Take generators $\varphi_1, \dots, \varphi_\ell$ of the $\mathbb{C}[\mathfrak{P}^1]$-module

$$\mathrm{Hom}_{\mathcal{A}_{\mathfrak{P}^1}}(\nabla^{i-1}_{\mathfrak{P}^1}(p), N_{\mathfrak{P}^1}).$$

Then for $\nabla^i_{\mathfrak{P}^1}(p)$ we take the intersection of the kernels of $\varphi_i, i = 1, \dots, \ell$. So we have a descending chain $\nabla_{\mathfrak{P}^1}(p) \supset \nabla^1_{\mathfrak{P}^1}(p) \supset \nabla^2_{\mathfrak{P}^1}(p) \supset \dots$. This chain does not need to stabilize. However, it does stabilize after a specialization to a Weil generic point. Define $\overline{\nabla}_{\mathfrak{P}^1}(p)$ as $\overline{\nabla}^i_{\mathfrak{P}^1}(p)$ when $\overline{\nabla}^i_\lambda(p)$ stabilizes for a Weil generic $\lambda \in \mathfrak{P}^1$.

As in [L9, Lemma 3.3], one shows that the modules $\overline{\nabla}_{\mathfrak{P}^1}(p)$ have properties (1)-(3).

4.4.3. Filtrations Now let us establish an important property of the objects $\Delta_{\mathfrak{P}^1}(p), \nabla_{\mathfrak{P}^1}(p)$.

Lemma 4.10. *We have a principal Zariski open subset $\widehat{\mathfrak{P}}^1 \subset \mathfrak{P}^1$ and a filtration $\Delta_{\widehat{\mathfrak{P}}^1}(p)(:= \mathbb{C}[\widehat{\mathfrak{P}}^1] \otimes_{\mathbb{C}[\mathfrak{P}^1]} \Delta_{\mathfrak{P}^1}(p)) = F_0 \supsetneq F_1 \supsetneq F_2 \dots \supsetneq F_k \supsetneq F_{k+1} = \{0\}$ with the following properties:*

(i) *F_i/F_{i+1} is flat over $\widehat{\mathfrak{P}}^1$.*
(ii) *$(F_0/F_1)_{\lambda^1} \cong \overline{\Delta}_{\lambda^1}(p)$ for $\lambda^1 \in \widehat{\mathfrak{P}}^1$.*
(iii) *For $i \geqslant 0, \lambda^1 \in \widehat{\mathfrak{P}}^1$, we have $(F_i/F_{i+1})_{\lambda^1} \cong \overline{\Delta}_{\lambda^1}(p^i)^{\oplus m_i}$, where $p^i \sim_\chi p, p^i <_{\widehat{\lambda}} p$ and $m_i \in \mathbb{Z}_{>0}$ independent of λ^1.*

Proof. The construction of the filtration is based on the construction of $\overline{\Delta}_{\mathfrak{P}^1}(\lambda)$. Namely, let us order the labels satisfying $p' \sim_\chi p, p' <_{\widehat{\lambda}} p$ in a non-increasing way: $p_s, p_{s-1}, \dots, p_0 = p$ (p_s is the smallest). Then define F_k as the image of $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{P}^1}}(\Delta_{\mathfrak{P}^1}(p_s), \Delta_{\mathfrak{P}^1}(p)) \otimes_{\mathfrak{P}^1} \Delta_{\mathfrak{P}^1}(p_s) \to \Delta_{\mathfrak{P}^1}(p)$. It satisfies (i) and (iii) (with $p_i = p_s$). Shrinking $\mathfrak{P}^1$ to a principal Zariski open subset $\widehat{\mathfrak{P}}^1$, we see that $\mathrm{Hom}_{\mathcal{A}_{\widehat{\mathfrak{P}}^1}}(\Delta_{\widehat{\mathfrak{P}}^1}(p_s), \Delta_{\widehat{\mathfrak{P}}^1}(p)/F_k) = 0$. In particular, every homomorphism $\Delta_{\widehat{\mathfrak{P}}^1}(p_{s-1}) \to \Delta_{\widehat{\mathfrak{P}}^1}(p)/F_1$ factors through $\overline{\Delta}_{\widehat{\mathfrak{P}}^1}(p_{s-1})$. We define F_{k-1}/F_k as the image of $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{P}^1}}\left(\overline{\Delta}_{\mathfrak{P}^1}(p_{s-1}), \Delta_{\mathfrak{P}^1}(p)\right) \otimes_{\mathfrak{P}^1} \overline{\Delta}_{\mathfrak{P}^1}(p_{s-1}) \to \Delta_{\mathfrak{P}^1}(p)$. Then we shrink $\widehat{\mathfrak{P}}^1$. We continue considering homomorphisms from $\Delta_{\widehat{\mathfrak{P}}^1}(p_s), \overline{\Delta}_{\widehat{\mathfrak{P}}^1}(p_{s-1})$ until we arrive at the situation when both $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{P}^1}}\left((\overline{\Delta}_{\widehat{\mathfrak{P}}^1}(p_s), \Delta_{\mathfrak{P}^1}(p)/F_i\right), \mathrm{Hom}_{\mathcal{A}_{\widehat{\mathfrak{P}}^1}}\left(\overline{\Delta}_{\widehat{\mathfrak{P}}^1}(p_{s-1}), \Delta_{\widehat{\mathfrak{P}}^1}(p)/F_i\right)$ are zero

(we do arrive at this situation because the length of a Weil generic specialization of $\Delta_{\widehat{\mathfrak{P}}^1}(p)/F_i$ reduces after each step). In particular, possibly after shrinking $\widehat{\mathfrak{P}}^1$, every homomorphism $\Delta_{\widehat{\mathfrak{P}}^1}(p_{s-2}) \to \Delta_{\widehat{\mathfrak{P}}^1}(p)/F_i$ factors through $\overline{\Delta}_{\widehat{\mathfrak{P}}^1}(p_{s-2})$. Then we repeat the argument. □

The dual statement holds for $\nabla_{\mathfrak{P}^1}(p)$ (we consider the filtration by $\overline{\nabla}_{\widehat{\mathfrak{P}}^1}(p_i)$'s). Here we will take the descending filtration and use the fact that the lengthes of the filtration terms at Weil generic points are decreasing.

4.4.4. Objects $\tilde{\Delta}_{\mathfrak{P}^1}(p), \tilde{\nabla}_{\mathfrak{P}^1}(p)$ Now let us produce the objects $\tilde{\Delta}_{\mathfrak{P}^1}(p)$ whose specializations will later be shown to be standard for the standardly stratified structure. Namely, let us order the labels p' with $p' \sim_\chi p$ in a non-decreasing way: $p_1 > p_2 > \ldots > p_n$. Let us define the objects $\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_k, k \leqslant i$, inductively. We set $\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_i = \Delta_{\mathfrak{P}^1}(p_i)$. If $\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_k$ is already defined, then for $\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_{k-1}$ we take the universal extension

$$\begin{aligned} 0 \quad &\to \quad \mathrm{Ext}^1_{\mathcal{A}_{\mathfrak{P}^1}}(\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_k, \Delta_{\mathfrak{P}^1}(p_{k-1})) \otimes_{\mathbb{C}[\mathfrak{P}^1]} \Delta_{\mathfrak{P}^1}(p_{k-1}) \to \tilde{\Delta}_{\mathfrak{P}^1}(p_i)_{k-1} \\ &\to \quad \tilde{\Delta}_{\mathfrak{P}^1}(p_i)_k \to 0. \end{aligned}$$

We then set $\tilde{\Delta}_{\mathfrak{P}^1}(p_i) := \tilde{\Delta}_{\mathfrak{P}^1}(p_i)_1$. Note that this mirrors the construction of the projective objects in highest weight categories.

Lemma 4.11. *For λ' equal to either λ or to a Weil generic $\widehat{\lambda} \in \mathfrak{P}^1$, we have an epimorphism $P_{\lambda'}(p_i) \twoheadrightarrow \tilde{\Delta}_{\lambda'}(p_i)$. It is an isomorphism when $\lambda' = \widehat{\lambda}$ and its kernel is filtered with $\Delta_\lambda(p')$, where $p' >_\chi p$, when $\lambda' = \lambda$.*

Recall that here by λ we mean a point of $\mathfrak{P}^1$ of the form $\lambda^\circ + n\chi$, where λ° is some fixed point and $n \gg 0$.

Proof. First of all, let us recall a standard fact. Let $B_{\mathfrak{P}^1}$ be a $\mathbb{C}[\mathfrak{P}^1]$-algebra that is flat over $\mathbb{C}[\mathfrak{P}^1]$ and $M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1}$ be finitely generated $B_{\mathfrak{P}^1}$-modules that are flat in a neighborhood of a point $x \in \mathfrak{P}^1$. Then if we have $\mathrm{Ext}^i_{B_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1})_x \xrightarrow{\sim} \mathrm{Ext}^i_{B_x}(M_x, N_x)$ for all $i = 0, \ldots, k-1$, then $\mathrm{Ext}^k_{B_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1})_x \hookrightarrow \mathrm{Ext}^k_{B_x}(M_x, N_x)$.

The algebra $\mathcal{A}_{\mathfrak{P}^1}$ is countable dimensional. It follows that the spaces $\mathrm{Ext}^i_{\mathcal{A}_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1})$ is at most countable dimensional for any finitely generated $\mathcal{A}_{\mathfrak{P}^1}$-modules $M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1}$ and any i. In particular, the set of $\lambda^1 \in \mathfrak{P}^1$ such that the maximal ideal of λ^1 has a nonzero annihilator in $\mathrm{Ext}^i_{\mathcal{A}_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1})$, $i = 1, 2$, is countable. If the maximal ideal of λ^1 has zero annihilator in $\mathrm{Ext}^i_{\mathcal{A}_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1})$, then $\mathrm{Ext}^{i-1}_{\mathcal{A}_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1})_{\widehat{\lambda}} = \mathrm{Ext}^{i-1}_{\mathcal{A}_{\widehat{\lambda}}}(M_{\widehat{\lambda}}, N_{\widehat{\lambda}})$. Therefore,

for a Weil generic $\widehat{\lambda} \in \mathfrak{P}^1$, we have $\mathrm{Ext}^{i-1}_{\mathcal{A}_{\mathfrak{P}^1}}(M_{\mathfrak{P}^1}, N_{\mathfrak{P}^1})_{\widehat{\lambda}} = \mathrm{Ext}^{i-1}_{\mathcal{A}_{\widehat{\lambda}}}(M_{\widehat{\lambda}}, N_{\widehat{\lambda}}), i = 1, 2$. From here we deduce by induction that $\left(\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_k\right)_{\widehat{\lambda}} = \tilde{\Delta}_{\widehat{\lambda}}(p_i)_k$ (where the object on the right hand side is defined analogously to $\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_k$). This shows that $P_{\widehat{\lambda}}(p_i) = \tilde{\Delta}_{\widehat{\lambda}}(p_i)$.

Let us consider the case of $\lambda' = \lambda$ now. We prove by induction on k that

$$\mathrm{Ext}^j_{\mathcal{A}_{\mathfrak{P}^1}}(\tilde{\Delta}_{\mathfrak{P}^1}(p_i)_k, \Delta_{\mathfrak{P}^1}(p_{k-1}))_\lambda = \mathrm{Ext}^j_{\mathcal{A}_\lambda}(\tilde{\Delta}_\lambda(p_i)_k, \Delta_\lambda(p_{k-1})), \tag{4.4}$$

for $j = 0, 1$. Thanks to the inductive construction of the indecomposable projective objects in highest weight categories, this will imply the claim of the proposition.

Let us do the case of $j = 0$ first. Clearly, we have an inclusion of the left hand side into the right hand side. The dimension of the left hand side is the same for all Zariski generic $\lambda \in \mathfrak{P}^1$. So it equals to the multiplicity of $L_{\hat{\lambda}}(p_i)$ in $\Delta_{\hat{\lambda}}(p_k)$. That, in turn, equals to the multiplicity of $\bar{\Delta}_\lambda(p_i)$ in $\Delta_\lambda(p_k)$. Recall that the multiplicity of $L_\lambda(p_{i_1})$ in $\bar{\Delta}_\lambda(p_{i_2})$ equals δ_{i_1,i_2} by Section 4.4.2. It follows that the multiplicity of $\bar{\Delta}_\lambda(p_i)$ in $\Delta_\lambda(p_k)$ equals that of $L_\lambda(p_i)$ in $\Delta_\lambda(p_k)$. On the other hand, this multiplicity is bigger than or equal to the right hand side of (4.4). This finishes the proof of (4.4) in the case $j = 0$.

Let us proceed to the $j = 1$ case. So far, we know that, first, the left hand side of (4.4) is included into the right hand side and, second, (4.4) holds for λ replaced with a Weil generic $\widehat{\lambda} \in \mathfrak{P}^1$. It follows from the inductive construction of the indecomposable projectives that to show (4.4), we need to show that the multiplicity of $\Delta_\lambda(p_j)$ in $P_\lambda(p_i)$ (for $j < i$) coincides with that of $\Delta_{\widehat{\lambda}}(p_j)$ in $P_{\widehat{\lambda}}(p_i)$. By the BGG reciprocity, this is equivalent to $[\nabla_\lambda(p_j) : L_\lambda(p_i)] = [\nabla_{\widehat{\lambda}}(p_j) : L_{\widehat{\lambda}}(p_i)]$. By the ∇-analog of Lemma 4.10, the right hand side coincides with the multiplicity of $\overline{\nabla}_\lambda(p_i)$ in $\nabla_\lambda(p_j)$. By the properties (1)-(3) of $\overline{\nabla}_\lambda(p_i)$, that coincides with the multiplicity $[\nabla_\lambda(p_j) : L_\lambda(p_i)]$. That latter coincides with $[P_\lambda(p_i) : \Delta_\lambda(p_j)]$. So we see that $[P_\lambda(p_i) : \Delta_\lambda(p_j)] = [P_{\hat{\lambda}}(p_i) : \Delta_{\hat{\lambda}}(p_j)]$. □

Similarly, we define the objects $\tilde{\nabla}_{\mathfrak{P}^1}(p)$. A direct analog of Lemma 4.10 holds.

4.4.5. Standardly stratified structure We proceed to proving that the pre-order $\preceq_\chi$ defines a standardly stratified structure on $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. We start by proving two technical lemmas.

Lemma 4.12. *For a Zariski generic $\lambda \in \mathfrak{P}^1$, we have* $\dim \mathrm{Ext}^i_{\mathcal{A}_\lambda}(\tilde{\Delta}_\lambda(p), \overline{\nabla}_\lambda(p')) = \delta_{i0}\delta_{pp'}$.

Proof. Note that $\overline{\nabla}_\lambda(p')$ is quasi-isomorphic to a complex whose terms are filtered with $\nabla_\lambda(p'')$ with $p'' \sim_\chi p'$. Using the ∇-analog of Lemma 4.10, we see that $\mathrm{Ext}^i_{\mathcal{A}_\lambda}(\widetilde{\Delta}_\lambda(p), \overline{\nabla}_\lambda(p')) = 0$ for all i unless $p \sim_\chi p'$.

Now consider the case $p \sim_\chi p'$. By Lemma 4.11, we have an exact sequence $0 \to K \to P_\lambda(p) \to \widetilde{\Delta}_\lambda(p) \to 0$, where K is filtered by $\Delta_\lambda(p'')$'s for $p'' \prec_\chi p$. By the previous paragraph, $\mathrm{Ext}^i_{\mathcal{A}_\lambda}(K, \overline{\nabla}_\lambda(p')) = 0$ for all i. So

$$\mathrm{Ext}^i_{\mathcal{A}_\lambda}(\tilde{\Delta}_\lambda(p), \overline{\nabla}_\lambda(p')) = \mathrm{Ext}^i_{\mathcal{A}_\lambda}(P_\lambda(p), \overline{\nabla}_\lambda(p')). \tag{4.5}$$

Now recall, Section 4.4.2, that $L_\lambda(p)$ appears in $\overline{\nabla}_\lambda(p')$ with multiplicity $\delta_{p,p'}$. We apply (4.5) to finish the proof. □

Similarly, we get $\dim \mathrm{Ext}^i_{\mathcal{A}_\lambda}(\overline{\Delta}_\lambda(p), \tilde{\nabla}_\lambda(p')) = \delta_{i0}\delta_{pp'}$.

Lemma 4.13. *Let $M \in \mathcal{O}_\nu(\mathcal{A}_\lambda)$ be such that $\mathrm{Ext}^i_{\mathcal{A}_\lambda}(M, \overline{\nabla}_\lambda(p)) = 0$ for all p. Then M is filtered by $\tilde{\Delta}$'s.*

Proof. Thanks to the ∇-analog of Lemma 4.10, we see that M is Δ-filtered. Let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be the equivalence classes for $\sim_\chi$ ordered so that $\mathcal{T}_i \prec_\chi \mathcal{T}_j \Rightarrow i < j$. Then we have a filtration on M by objects of the form M_j, where M_j is filtered by $\Delta_\lambda(p')$'s, where $p' \in \mathcal{T}_j$. What we need to check is that each M_i is the direct sum of $\widetilde{\Delta}_\lambda(p')$'s.

Similarly to the proof of Lemma 4.12, $\mathrm{Ext}^i(M_j, \overline{\nabla}_\lambda(p)) = 0$ if $p \notin \mathcal{T}_j$. So in the proof of the lemma it is enough to assume that $M = M_j$ for some j.

The equality $\mathrm{Ext}^i_{\mathcal{A}_\lambda}(M, \overline{\nabla}_\lambda(p)) = 0$ is equivalent to $\mathrm{Ext}^i_{\mathcal{O}}(M, \overline{\nabla}_\lambda(p)) = 0$, which, in turn, is equivalent to $\mathrm{Ext}^i_{\mathcal{O}_p}(\pi_p(M), \pi_p(\overline{\nabla}_\lambda(p))) = 0$, where we write $\mathcal{O}_p$ for the subquotient highest weight category corresponding to the interval $\{p' | p' \sim_\chi p\}$ and π_p for the quotient $\mathcal{O}_{\preceq_\chi p} \twoheadrightarrow \mathcal{O}_p$. Since the objects $\pi_p(\overline{\nabla}_\lambda(p'))$ are simple, we deduce that $\pi_p(M)$ is projective. Note that the natural homomorphism $\pi_p^!(\pi_p(M)) \to M$ is an epimorphism as π_p does not kill any simple in the head of M. Also then classes of $\pi_p^!(\pi_p(M))$ and M in the category of standardly filtered objects coincide. So $\pi_p^!(\pi_p(M)) \xrightarrow{\sim} M$, which finishes the proof. □

Now we can complete the proof that $(\mathcal{O}_\nu(\mathcal{A}_\lambda), <_\chi)$ is a standardly stratified category. By Lemma 4.13, any projective in $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is $\tilde{\Delta}$-filtered, which is (SS2). Similarly, every injective is $\tilde{\nabla}$-filtered. It follows that the quotient functor π_ξ maps the injective objects in the subcategory $\mathcal{O}_{\preceq_\chi p} \subset \mathcal{O}_\nu(\mathcal{A}_\lambda)$ to injective objects in the quotient $\mathcal{O}_p$. Equivalently, by [L7, Lemma 3.4], the functor $\pi_\xi^!$ is exact, which is (SS1).

4.4.6. Associated graded category To finish the proof of Proposition 4.9 we need to check that $\operatorname{gr}\mathcal{O}_\nu(\mathcal{A}_\lambda) \cong \mathcal{O}_\nu(\mathcal{A}_{\widehat{\lambda}})$. Note that the subquotient category $\mathcal{O}_p$ of $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ corresponding to the equivalence class of p with respect to $\sim_\chi$ is equivalent to the category of right modules over the algebra $B_\lambda := \operatorname{End}_{\mathcal{A}_\lambda}(\bigoplus_{p'} \tilde{\Delta}_\lambda(p'))$, where the summation is taken over all $p' \sim_\chi p$. We can also consider the algebra $B_{\mathfrak{P}^1} := \operatorname{End}_{\mathcal{A}_{\mathfrak{P}^1}}(\bigoplus_{p'} \tilde{\Delta}_{\mathfrak{P}^1}(p'))$, the algebra B_λ is the specialization of $B_{\mathfrak{P}^1}$ to λ (and the same is true for a Weil generic element $\widehat{\lambda}$). What remains to prove is the following lemma.

Lemma 4.14. *We have an algebra isomorphism $B_\lambda \cong B_{\widehat{\lambda}}$ that respects the primitive idempotents $e_{p'}$, where $p' \sim_\chi p$.*

Proof. After passing to a principal Zariski open subset $\widehat{\mathfrak{P}}^1 \subset \mathfrak{P}^1$, we achieve that the algebra $\mathcal{B}_{\widehat{\mathfrak{P}}^1}$ is a free $\mathbb{C}[\widehat{\mathfrak{P}}^1]$-module with a basis including the idempotents $e_{p'}$ and compatible with the decomposition $\bigoplus_{p',p''} e_{p'}\mathcal{B}_{\widehat{\mathfrak{P}}^1}e_{p''} = \mathcal{B}_{\widehat{\mathfrak{P}}^1}$. This gives rise to a morphism $\widehat{\mathfrak{P}}^1 \to \mathfrak{X}$, where $\mathfrak{X}$ denotes the variety of associative products such that the elements $e_{p'}$ are idempotents. Isomorphisms correspond to a suitable algebraic group action. What we need to prove is that a Zariski open subset of $\widehat{\mathfrak{P}}^1$ maps to a single orbit. For this we note that we have a labeling preserving isomorphism $\mathcal{B}_\lambda \cong \mathcal{B}_{\lambda+\chi'}$, where χ' is an integral element of Γ. It follows that the elements $\lambda, \lambda+\chi'$ map to the same orbit. But the set $\{\lambda+\chi'\}$ is Zariski dense in $\widehat{\mathfrak{P}}^1$, which implies our claim. □

4.5. Ringel dualities for standardly stratified structures

In this section we discuss Ringel duality for standardly stratified categories[2]. The most important example of Ringel duality functors comes from wall-crossing functors.

4.5.1. Tilting and cotilting objects An object in $\mathcal{C}$-tilt $:= \mathcal{C}^\Delta \cap \mathcal{C}^{\overline{\nabla}}$ is called *tilting*. We can construct an indecomposable tilting $T(\tau)$ similarly to the highest weight case. Namely, let us order elements of $\mathcal{T}$, $\tau_1, \dots, \tau_n$ in such a way that $\tau_i \succeq \tau_j$ implies $i \leqslant j$. Define the objects $T_j(\tau_i)$, where $j \leqslant i$, inductively as follows:

- $T_i(\tau_i) := \Delta(\tau_i)$.
- Once $T_j(\tau_i)$ is constructed, for $T_{j-1}(\tau)$ we take the universal extension of $\operatorname{Ext}^1(\Delta(\tau_{j-1}), T_j(\tau_i)) \otimes \Delta(\tau_{j-1})$ by $T_j(\tau_i)$. So we have an exact sequence

$$0 \to T_j(\tau_i) \to T_{j-1}(\tau_i) \to \operatorname{Ext}^1(\Delta(\tau_{j-1}), T_j(\tau_i)) \otimes \Delta(\tau_{j-1}) \to 0.$$

[2]Preliminary versions of sections 4.5.1-4.5.4 appeared in one of the first drafts of our joint paper [LW] with Ben Webster and were later removed.

We set $T(\tau_i) := T_1(\tau_i)$. From the construction of $T(\tau)$ and (4.1) it is easy to see that $\mathrm{Ext}^1(\Delta(\tau'), T(\tau)) = 0$ for any τ'. By (2) of Lemma 4.5, $T(\tau) \in \mathcal{C}^{\overline{\nabla}}$, so $T(\tau)$ is indeed tilting.

Lemma 4.15. *The object $T(\tau)$ is indecomposable. Moreover, any indecomposable tilting object in $\mathcal{C}$ is isomorphic to precisely one $T(\tau)$.*

Proof. Note that by the construction of $T(\tau)$, the label τ is uniquely determined by the following property: there is an embedding $\Delta(\tau) \hookrightarrow T(\tau)$ such that the cokernel is standardly filtered. Moreover, Lemma 4.5(2) implies that a direct summand of a standardly filtered object is standardly filtered. These two observations imply both claims of the lemma. □

Applying this construction to $\mathcal{C}^{opp}$ we get *cotilting objects*.

Below we will need some further easy properties of tilting objects.

Lemma 4.16. *We have an epimorphism $T(\tau) \twoheadrightarrow \nabla(P_\xi(\tau))$, where $\xi = \varrho(\tau)$, whose kernel lies in $\mathcal{C}^{\overline{\nabla}}_{<\xi}$.*

Proof. In the proof we can assume that ξ is the largest element of Ξ (if not, we pass to the standardly stratified subcategory $\mathcal{C}_{\preceq\xi}$). Note that $\mathrm{Ext}^1(\overline{\nabla}(\tau'), \overline{\nabla}(\tau)) = 0$ if $\tau' \prec \tau$. It follows that we have a canonical exact sequence

$$0 \to K \to T(\tau) \to C \to 0,$$

where $K \in \mathcal{C}_{<\xi} \cap \mathcal{C}^{\overline{\nabla}}$ and C is filtered with successive quotients of the form $\overline{\nabla}(\tau)$ with $\varrho(\tau) = \xi$. It remains to prove that $C = \nabla(P_\xi(\tau))$. We have $\pi_\xi(C) = \pi_\xi(T(\tau)) = \pi_\xi(\Delta(\tau)) = P_\xi(\tau)$. Then we can apply Lemma 4.3 to $\mathcal{C}^{opp}$. □

Lemma 4.17. *Let $M \in \mathcal{C}^{\Delta}$ and $N \in \mathcal{C}^{\overline{\nabla}}$. Then there is a tilting object T_N with an epimorphism $T_N \twoheadrightarrow N$ whose kernel lies in $\mathcal{C}^{\overline{\nabla}}$. Furthermore, any morphism $M \to N$ factors through T_N.*

Proof. We can construct T_N by taking the consecutive universal extensions of N by $\overline{\nabla}(\tau_i) \otimes \mathrm{Ext}^1(N, \overline{\nabla}(\tau_i))$ for i ranging from 1 to $n = |\mathcal{T}|$ (we order labels in non-decreasing way with respect to $\leqslant$). The claim about morphisms follows from Lemma 4.5(1) as the kernel of $T_N \twoheadrightarrow N$ is proper costandardly filtered. □

4.5.2. Definition of Ringel duality Let $\mathcal{C}_1$ be a standardly stratified category and $\mathcal{C}_2$ be a weakly standardly stratified category. By a Ringel duality data, we mean a pair $(\mathcal{R}, \theta)$ of

- A poset isomorphism $\theta : \Xi_1 \xrightarrow{\sim} \Xi_2^{opp}$.
- an equivalence $\mathcal{R} : D^b(\mathcal{C}_1) \xrightarrow{\sim} D^b(\mathcal{C}_2)$ of triangulated categories that restricts to an equivalence $\mathcal{C}_1^{\overline{\nabla}} \xrightarrow{\sim} \mathcal{C}_2^{\overline{\Delta}}$ of exact categories.
- If $\mathcal{R}(\overline{\nabla}(\tau)) = \overline{\Delta}(\tau')$, then $\varrho(\tau') = \theta \circ \varrho(\tau)$.

Note that an equivalence $\mathcal{C}_1^{\overline{\nabla}} \to \mathcal{C}_2^{\overline{\Delta}}$ of exact categories automatically maps a proper costandard object to a proper standard one, and this induces a bijection between the labelling sets of simples.

We say that $(\mathcal{C}_2, \Xi_2)$ is a Ringel dual of $(\mathcal{C}_1, \Xi_1)$. We call $\mathcal{R}$ the *Ringel duality functor*.

4.5.3. Existence Let $\mathcal{C}$ be a standardly stratified category. Let $T := \bigoplus_{\tau \in \mathcal{T}} T(\tau)$ be the tilting generator. Consider the category $\mathcal{C}^\vee := \mathrm{End}_{\mathcal{C}}(T)^{opp}$-mod and the functor $\mathcal{R} := \mathrm{RHom}_{\mathcal{C}}(T, \bullet) : D^b(\mathcal{C}) \to D^b(\mathcal{C}^\vee)$.

The functor $\mathcal{R}$ is an equivalence. Indeed, it is easy to see that the higher self-extensions of T vanish. This shows that $\mathcal{R}$ is a quotient functor. The functor is an equivalence because the objects $T(\tau)$ generate the triangulated category $D^b(\mathcal{C})$. The equivalence $\mathcal{R}$ is exact on $\mathcal{C}^{\overline{\nabla}}$.

Proposition 4.18. *The category $\mathcal{C}^\vee$ is weakly standardly stratified with poset Ξ^{opp} and the pair $(\mathcal{R}, \mathrm{id})$ is a Ringel duality data. Moreover, for any other Ringel dual category $\mathcal{C}'$ and Ringel duality data $(\mathcal{R}', \theta')$, there is an equivalence $(\Phi, \phi) : \mathcal{C}^\vee \to \mathcal{C}'$ of weakly standardly stratified categories such that $\mathcal{R}'$ is isomorphic to $\Phi \circ \mathcal{R}$ and $\theta' = \phi$.*

We will start by proving that $\mathcal{C}^\vee$ is indeed a weakly standardly stratified category (a harder part) and then prove a uniqueness statement. After that we will briefly discuss conditions under which $\mathcal{C}^\vee$ is standardly stratified and not just weakly standardly stratified.

4.5.4. Weakly standardly stratified structure on $\mathcal{C}^\vee$ The following proposition shows that $\mathcal{C}^\vee$ is Ringel dual to $\mathcal{C}$.

Proposition 4.19. *The category $\mathcal{C}^\vee$ has a weakly standardly stratified structure for the opposite preorder on $\mathcal{T}$. We have an identification $\mathrm{gr}\,\mathcal{C}^\vee \cong \mathrm{gr}\,\mathcal{C}$ and the functor $\Delta^\vee(\bullet) : \mathrm{gr}\,\mathcal{C}^\vee \to \mathcal{C}^\vee$ coincides with $\mathrm{Hom}_{\mathcal{C}}(T, \nabla(\bullet))$.*

Proof. The proof is in several steps.

Step 1. For an ideal $\Xi_0 \subset \Xi$, the category $(\mathcal{C}_{\Xi_0})^\vee$ is naturally identified with a quotient of $\mathcal{C}^\vee$. The quotient functor is $M \mapsto M e_{\Xi_0}$, where e_{Ξ_0} denotes the central idempotent that is the projection from T to $\bigoplus_{\tau \in \Lambda_0} T(\tau)$. So for a coideal $\Xi^0 \subset \Xi$ (=ideal $\Xi^0 \subset \Xi^{opp}$) we can define the subcategory $(\mathcal{C}^\vee)_{\Xi^0} \subset \mathcal{C}^\vee$

as the kernel of the projection $\mathcal{C}^\vee \twoheadrightarrow (\mathcal{C}_{\Xi\setminus\Xi^0})^\vee$. This gives rise to a filtration on $\mathcal{C}^\vee$ by Serre subcategories. In particular, we can talk about the subquotients $\mathcal{C}^\vee_\xi$.

Step 2. Now we claim that $(\mathcal{C}^\vee)_\xi$ is naturally identified with $\mathcal{C}_\xi$. By the definition of our filtration, we can represent $(\mathcal{C}^\vee)_\xi$ as the kernel of the quotient $(\mathcal{C}_{\leqslant\xi})^\vee \twoheadrightarrow (\mathcal{C}_{<\xi})^\vee$. So, in order to prove the claim in the beginning of the paragraph, we can assume that ξ is the largest element of Ξ. We claim that the required equivalence $\mathcal{C}_\xi \xrightarrow{\sim} \mathcal{C}^\vee_\xi$ is given by the functor $\iota : \mathrm{Hom}(T, \nabla_\xi(\bullet)) : \mathcal{C}_\xi \to \mathcal{C}^\vee$. The image of ι is contained in $(\mathcal{C}^\vee)_\xi$ because $\mathrm{Hom}(T(\tau'), \nabla(\tau)) = 0$ for $\tau' < \tau$. So ι is a functor $\mathcal{C}_\xi \to (\mathcal{C}^\vee)_\xi$.

Let us show that ι is an equivalence. Consider the quotient A_1 of $\mathrm{End}(T)^{opp}$ by the 2-sided ideal of all morphisms that factor as $T \to T_{<\xi} \to T$ for $T_{<\xi} \in \mathcal{C}_{<\xi}$-tilt. So $(\mathcal{C}^\vee)_\xi$ is just A_1-mod. Consider the object $R := \bigoplus_{\tau\in\varrho^{-1}(\xi)} \nabla(P_\xi(\tau))$ that is a quotient of T in such a way that the kernel lies in $\mathcal{C}_{<\xi}$, see Lemma 4.16. Clearly a morphism $T \to T$ induces a morphism $R \to R$ and so we get a homomorphism $\mathrm{End}(T)^{opp} \to A_2$, where $A_2 := \mathrm{End}(R)^{opp}$ so that $\mathcal{C}_\xi = A_2$-mod. Note that the homomorphism $\mathrm{End}(T)^{opp} \to A_2$ factors through A_1 because we have $\mathrm{Hom}(\bigoplus_{\varrho(\tau)<\xi} T(\tau), R) = 0$. Let φ be the resulting homomorphism $A_1 \to A_2$. It is straightforward from the construction of $\iota : \mathcal{C}_\xi \to (\mathcal{C}^\vee)_\xi$ is just φ^*.

So we need to check that φ is an isomorphism. By Lemma 4.17, any homomorphism $T \to \nabla(P_\xi(\tau))$ lifts to an endomorphism of T. So φ is surjective. Now let ψ be an endomorphism of T whose image lies in the kernel K of $T \twoheadrightarrow \nabla(P_\xi(\tau))$. Recall that $K \in \mathcal{C}_{<\xi} \cap \mathcal{C}^{\overline{\nabla}}$. So, by Lemma 4.17, there is a tilting object $T_K \in \mathcal{C}_{<\xi}$ with $T_K \twoheadrightarrow K$ and every morphism $T \to K$ factors through T_K. It follows that any morphism $T \to K$ lies in the kernel of $\mathrm{End}(T)^{opp} \twoheadrightarrow A_1$. Hence φ is injective. This finishes the proof of the claim that ι is an equivalence and establishes an equivalence $\mathrm{gr}\,\mathcal{C} \cong \mathrm{gr}\,\mathcal{C}^\vee$ that we will be using from now on.

Step 3. Fix $\xi \in \Xi$ and set $\Xi^0 := \{\xi' \in \Xi | \xi' \geqslant \xi\}, \Xi_0 := \Xi \setminus \Xi^0$. Let us show that, under the identification $\mathcal{C}_\xi \cong \mathcal{C}^\vee_\xi$, the quotient functor $(\pi^\vee)_\xi : (\mathcal{C}^\vee)_{\geqslant\xi} \twoheadrightarrow \mathcal{C}_\xi$ gets identified with $\pi_{\Xi_0,\xi}(N \otimes_{\mathrm{End}(T)} T)$, where we write $\pi_{\Xi_0,\xi}$ for the quotient functor $\mathcal{C}_{\Xi_0} \twoheadrightarrow \mathcal{C}_\xi$. Note that, for $N \in (\mathcal{C}^\vee)_{\geqslant\xi}$, the tensor product $N \otimes_{\mathrm{End}(T)} T$ lies in $\mathcal{C}_{\Xi_0}$ because $e_{\Xi_0} N = 0$. So the functor $\pi_{\Xi_0,\xi}(\bullet \otimes_{\mathrm{End}(T)} T)$ does make sense. To show the coincidence of the functors we can replace $\mathcal{C}$ by $\mathcal{C}_{\leqslant\xi}$ (and so $\mathcal{C}^\vee$ will be replaced with a suitable quotient). Then $\pi_\xi(T \otimes_{\mathrm{End}(T)^{opp}} \bullet)$ is just a quasi-inverse of $\mathrm{Hom}(T, \nabla_\xi(\bullet))$ and our claim follows.

Step 4. An isomorphism $(\pi^\vee)_\xi(\bullet) \cong \pi_{\Xi_0,\xi}(\bullet \otimes_{\mathrm{End}(T)} T)$ shows that the functor $\Delta^\vee_\xi : \mathcal{C}_\xi \to \mathcal{C}^\vee_{\geqslant\xi}$ defined by $\Delta^\vee_\xi(\bullet) := \mathrm{Hom}_{\mathcal{C}}(T, \nabla_\xi(\bullet))$ is left adjoint

to the projection $\mathcal{C}^\vee_{\geqslant\xi} \twoheadrightarrow \mathcal{C}^\vee_\xi$. The functor $\Delta^\vee_\xi$ is exact. Moreover, the functor $\mathrm{Hom}(T, \bullet)$ identifies $\mathcal{C}^{\overline{\nabla}}$ with $(\mathcal{C}^\vee)^{\overline{\Delta}}$. Under this identification, we have

$$\overline{\Delta}^\vee(\tau) = \overline{\nabla}(\tau), \quad P^\vee(\tau) = T(\tau). \qquad \square$$

4.5.5. Uniqueness Now let $\mathcal{C}'$ be another Ringel dual category of $\mathcal{C}$, let $(\mathcal{R}', \theta')$ be the corresponding Ringel duality data. It follows that $\mathcal{R} \circ \mathcal{R}'^{-1}$ is an equivalence $D^b(\mathcal{C}') \xrightarrow{\sim} D^b(\mathcal{C}^\vee)$. Every projective P' in $\mathcal{C}'$ satisfies $\mathrm{Ext}^1_{\mathcal{C}'}(P', \overline{\Delta}(\tau')) = 0$ for all labels τ'. It follows that $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{R}'^{-1}(P'), \overline{\nabla}(\tau)) = 0$ for all labels τ. So $\mathcal{R}'^{-1}(P')$ is Δ-filtered. It is also $\overline{\nabla}$-filtered by the definition of a Ringel duality functor. So $\mathcal{R}'^{-1}(P')$ is tilting and therefore $\mathcal{R} \circ \mathcal{R}'^{-1}$ is projective. It follows that $\mathcal{R} \circ \mathcal{R}'^{-1}$ restricts to $\mathcal{C}'$-proj $\to \mathcal{C}$-proj and so comes from an equivalence of abelian categories. This equivalence is automatically an equivalence of weakly standardly stratified categories (and θ' is the corresponding bijection of posets).

4.5.6. When $\mathcal{C}^\vee$ is standardly stratified In general, it seems that $\mathcal{C}^\vee$ is only weakly standardly stratified. Clearly, the claim that $\mathcal{C}^\vee$ is standardly stratified is equivalent to the following claim:

(*) All tilting objects in $\mathcal{C}$ admit a filtration with successive quotients of the form $\nabla_\xi(P_\xi)$, where P_ξ is a projective object in $\mathcal{C}_\xi$ (instead of just some object that is guaranteed by the condition of being tilting).

When all projectives in $\mathrm{gr}\,\mathcal{C}$ are injective, (*) becomes

(**) Tilting and co-tilting objects in $\mathcal{C}$ are the same.

For example, (**) is satisfied for tensor products of minimal categorifications studied in [LW]. In order to see that one applies an analog of the inductive construction of projective objects used in the proof of [LW, Theorem 6.1] to construct tilting objects using the dual splitting procedure from [LW, Section 4.4]. The inductive construction shows that all tilting objects are co-tilting.

4.5.7. Wall-crossing functors Let X, ν, λ, χ be as in Section 4.4.1. Set $\lambda' := \lambda - N\chi$ for $N \gg 0$. We are going to describe the Ringel dual of $(\mathcal{O}_\nu(\mathcal{A}_\lambda), \leqslant_\chi)$ and the corresponding Ringel duality functor.

Theorem 4.20. *The standardly stratified category $(\mathcal{O}_\nu(\mathcal{A}_{\lambda'}), \leqslant_{-\chi})$ is the Ringel dual of $(\mathcal{O}_\nu(\mathcal{A}_\lambda), \leqslant_\chi)$. The functor $\mathfrak{WC}^{-1}_{\lambda \leftarrow \lambda'}$ is the corresponding Ringel duality functor.*

Proof. We need to prove that there is a highest weight equivalence $\iota : \operatorname{gr}\mathcal{O}_\nu(\mathcal{A}_\lambda) \xrightarrow{\sim} \mathcal{O}_\nu(\mathcal{A}_{\widehat{\lambda}}) \xleftarrow{\sim} \operatorname{gr}\mathcal{O}_\nu(\mathcal{A}_{\lambda'})$ such that we have a functorial isomorphism $\mathfrak{WC}_{\lambda'\leftarrow\lambda}\Delta_\chi(M) \cong \nabla_{-\chi}(\iota(M))$. Let $\lambda^- := \lambda - N\theta$ for $N \gg 0$ so that λ', λ^- lie in chambers that are opposite with respect to the face containing $-\chi$. It follows that

$$\mathfrak{WC}_{\lambda\leftarrow\lambda^-} = \mathfrak{WC}_{\lambda\leftarrow\lambda'} \circ \mathfrak{WC}_{\lambda'\leftarrow\lambda^-}. \tag{4.6}$$

Since $\mathfrak{WC}^{-1}_{\lambda\leftarrow\lambda^-}$ is the Ringel duality functor for the highest weight structure on $\mathcal{O}_\nu(\mathcal{A}_\lambda)$, see Proposition 4.1, we only need to show that $\mathfrak{WC}_{\lambda'\leftarrow\lambda^-}$ maps $\Delta_{\lambda^-}(p)$ to $\Delta_{-\chi} \circ \nabla_{\widehat{\lambda}}(p)$. For the same reason as in [L9, Proposition 3.2], the object $\mathfrak{WC}_{\lambda'\leftarrow\lambda^-}\Delta_{\lambda^-}(p)$ has no higher homology and its class in K_0 coincides with $[\Delta_{\lambda'}(p)] = [\Delta_{-\chi}\circ\nabla_{\widehat{\lambda}}(p)]$. Because of this the functor $\mathfrak{WC}_{\lambda'\leftarrow\lambda^-}$ restricts to an equivalence

$$D^b(\mathcal{O}_\nu(\mathcal{A}_{\lambda'})_{\preceq_\chi p}) \xrightarrow{\sim} D^b(\mathcal{O}_\nu(\mathcal{A}_{\lambda^-})_{\preceq_\chi p}).$$

Hence there is an equivalence $\underline{\mathfrak{WC}} : D^b(\mathcal{O}_\nu(\mathcal{A}_{\widehat{\lambda^-}})) \xrightarrow{\sim} D^b(\mathcal{O}_\nu(\mathcal{A}_{\widehat{\lambda'}}))$ such that $\pi_p\circ\mathfrak{WC}^{-1}_{\lambda'\leftarrow\lambda^-} \cong \underline{\mathfrak{WC}}\circ\pi_p$. By adjunction, we get $\mathfrak{WC}_{\lambda'\leftarrow\lambda^-}\circ\Delta_{-\chi} \cong \Delta_{-\chi}\circ\underline{\mathfrak{WC}}$. Note that $\underline{\mathfrak{WC}}$ induces the trivial map between the K_0-groups and sends $\Delta_{\widehat{\lambda^-}}(p)$ to an object. The highest weight orders for $\mathcal{O}_\nu(\mathcal{A}_{\widehat{\lambda^-}}), \mathcal{O}_\nu(\mathcal{A}_{\widehat{\lambda'}})$ are opposite. It follows that $\underline{\mathfrak{WC}}$ sends $\Delta_{\widehat{\lambda^-}}(p)$ to $\nabla_{\widehat{\lambda'}}(p)$. A required equivalence ι is given by $\underline{\mathfrak{WC}} \circ \mathfrak{WC}^{-1}_{\widehat{\lambda'}\leftarrow\widehat{\lambda^-}}$. □

Remark 4.21. Proposition 4.9 and Theorem 4.20 are still true for categories $\mathcal{O}$ over Rational Cherednik algebras (since an analog of (4.6) is not known in that context, a priori, the Ringel duality functor will be given by $\mathfrak{WC}_{\lambda'\leftarrow\lambda^-} \circ \mathfrak{WC}^{-1}_{\lambda\leftarrow\lambda^-}$). We need to take the chamber structure defined by the c-order as in [L5, Section 2.6] and consider wall-crossing functors from *loc.cit.* The proofs carry over to the RCA situation more or less verbatim (in the proof of Lemma 4.14 we need to use equivalences from [L5, Proposition 4.2] rather than equivalences coming from localization theorem).

Remark 4.22. Note that the functor $\underline{\mathfrak{WC}}$ in the proof of Theorem 4.20 coincides with the wall-crossing functor $\mathfrak{WC}_{\widehat{\lambda'}\leftarrow\widehat{\lambda^-}}$ (where $\widehat{\bullet}$ means a Weil generic deformation of a parameter along the face Γ) up to post-composing with an abelian equivalence that is the identity on the level of K_0. The isomorphism $\Delta\circ\underline{\mathfrak{WC}} \cong \mathfrak{WC}_{\lambda'\leftarrow\lambda^-}\circ\Delta$ implies that the wall-crossing bijections (see Section 3.6) $\mathfrak{wc}_{\widehat{\lambda'}\leftarrow\widehat{\lambda^-}}$ and $\mathfrak{wc}_{\lambda'\leftarrow\lambda^-}$ coincide. This result was established in [L8, Theorem 1.1(iii)] and [L9, Proposition 3.1] for special varieties X.

Remark 4.23. The isomorphism $\Delta \circ \underline{\mathfrak{WC}} \cong \mathfrak{WC}_{\lambda' \leftarrow \lambda^-} \circ \Delta$ may be viewed as a categorical analog of a result predicted by Maulik and Okounkov and proved by Negut in the special case of affine type A quiver varieties. This result, [Ne, (3.21)], reduces the computation of a K-theoretic R-matrix $\mathbf{R}^{\pm}_{\mathsf{m}+r\theta,\mathsf{m}+(r+\varepsilon)\theta}$ to root quantum subalgebras.

4.5.8. Example of partial Ringel duality: cross-walling functors Let X, ν, ν_0, λ be as in 4.3.6. Assume, in addition, that all components of $X^{\nu_0(\mathbb{C}^\times)}$ have the same dimension. Let $d := (\dim X - \dim X^{\nu_0(\mathbb{C}^\times)})/2$. Define a one-parameter subgroup ν' as follows. Let ν_1 be such that $N\nu_0 + \nu_1$ lies in the same open chamber as ν. Then we set $\nu' := -N\nu_0 + \nu_1$.

The following is a corollary of Proposition 4.2 (compare to the proof of Theorem 4.20).

Proposition 4.24. *There is an abelian equivalence of the category* $(\mathcal{O}_{\nu'}(\mathcal{A}_\lambda), \leqslant_{-\nu_0})$ *with the Ringel dual of* $(\mathcal{O}_\nu(\mathcal{A}_\lambda), \leqslant_{\nu_0})$ *and is trivial on* K_0 *that intertwines the Ringel duality functor with* $\mathfrak{CW}^{-1}_{\nu \leftarrow \nu'}[-d]$.

4.5.9. Prospective example: Webster's functors Let $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be as in Section 4.3.5. Fix a composition $k = k_1 + \ldots + k_r$. Let w_0 be the longest element of S_k and w_0' be the longest element of $S_{k_1} \times S_{k_2} \times \ldots \times S_{k_r}$. Set $\sigma = w_0' w_0^{-1}$. Let $\mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_k$ denote the categorical tensor product of the categories $\mathcal{V}_1, \ldots, \mathcal{V}_k$.

For $\sigma \in S_k$, Webster in [W] defined right t-exact equivalences $\mathcal{T}_\sigma : D^b(\mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_k) \xrightarrow{\sim} D^b(\mathcal{V}_{s(1)} \otimes \ldots \otimes \mathcal{V}_{s(k)})$. He showed that these equivalences give rise to a braid group action. The category $\mathcal{V}_{\sigma(1)} \otimes \ldots \otimes \mathcal{V}_{\sigma(k)}$ is expected to be the Ringel dual of $\mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_k$ with respect to a suitable standardly stratified structure provided $\sigma = w_0' w_0^{-1}$. The functor $\mathcal{T}_\sigma^{-1}$ is expected to be the Ringel duality functor.

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