

Complete complexes and spectral sequences

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Abstract: By analogy with the classical (Chasles-Schubert-Semple-Tyrell) spaces of complete quadrics and complete collineations, we introduce the variety of complete complexes. Its points can be seen as equivalence classes of spectral sequences of a certain type. We prove that the set of such equivalence classes has a structure of a smooth projective variety. We show that it provides a desingularization, with normal crossings boundary, of the Buchsbaum-Eisenbud variety of complexes, i.e., a compactification of the union of its maximal strata.

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0. Introduction

A. Background and motivation. The spaces of complete collineations and complete quadrics form a beautiful and very important chapter of algebraic

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geometry, going back to the classical works of Chasles and Schubert in the 19th century, see [Se], [Ty], [L3], [L2], [Va], [Th], [DGMP], [Kau] and references therein. They provide explicit examples of wonderful compactifications (i.e., of smooth compactifications with normal crossings boundary).

To recall the basic example, the group $\mathbb{P}GL_n(\mathbb{C}) = GL_n(\mathbb{C})/(\mathbb{C}^* \cdot \mathbf{1})$ has an obvious compactification by the projective space $\mathbb{P}(\text{Mat}_n(\mathbb{C}))$ but it is not wonderful since the complement, the determinantal variety, is highly singular. Now let V, W be two \mathbb{C} -vector spaces of the same dimension n . A *complete collineation* from V to W is a sequence of the following data (assumed to be nonzero and considered each up to a non-zero scalar factor):

- (0) A linear operator $A = A_0 : V \rightarrow W$, possibly degenerate (not an isomorphism). Note that $\text{Ker}(A)$ and $\text{Coker}(A)$ have the same dimension.
- (1) A linear operator $A_1 : \text{Ker}(A_0) \rightarrow \text{Coker}(A_0)$, possibly degenerate.
- (2) A linear operator $A_2 : \text{Ker}(A_1) \rightarrow \text{Coker}(A_1)$, possibly degenerate, and so on, until we obtain a non-degenerate linear operator.

One of the main results of the classical theory says that the set of complete collineations has a natural structure of a smooth projective variety over \mathbb{C} , containing $\mathbb{P}GL_n(\mathbb{C})$ as an open subset (A_0 non-degenerate) so that the complement is a divisor with normal crossings.

We now want to look at this classical construction from a more modern perspective. We can view a linear operator $A : V \rightarrow W$ as a 2-term cochain complex. Then the sequence (A_ν) as above is nothing but a *spectral sequence*: a sequence of complexes (E_ν^\bullet, D^ν) such that each $E_{\nu+1}^\bullet$ is identified with the cohomology $H_{D^\nu}^\bullet(E_\nu^\bullet)$.

This suggests a generalization of the construction of complete collineations involving more full-fledged (simply graded) spectral sequences. In this paper we develop such a generalization. The role of the group $GL_n(\mathbb{C})$ (or its projectivization $\mathbb{P}GL_n(\mathbb{C})$) is played by appropriate strata in the *Buchsbaum-Eisenbud variety of complexes* $C(V^\bullet)$ and its projectivization $\mathbb{P}C(V^\bullet)$. Here V^\bullet is a graded vector space and $C(V^\bullet)$ consists of all ways of making V^\bullet into a cochain complex, see §2 and [Ke], [DS] for more background. The varieties $C(V^\bullet)$ are known to share many important properties of determinantal varieties, in particular, they are *spherical varieties*: the action of the group $GL(V^\bullet) = \prod GL(V^i)$ on the coordinate ring has simple spectrum, i.e., each irreducible representation enters at most once.

B. Summary of results. Our results can be summarized as follows. For simplicity, consider the projective variety of complexes $\mathbb{P}C(V^\bullet)$. Let $\mathbb{P}C^\circ(V^\bullet)$

be the union of its maximal $GL(V^\bullet)$ -orbits, a smooth open dense subvariety in $\mathbb{P}C(V^\bullet)$, see (6.7).

At the same time let $\mathbb{P}SS(V^\bullet)$ be the set of equivalence classes of spectral sequences (E_ν^\bullet, D^ν) , $\nu = 0, \dots, N$, of variable (finite) length N , see §7A, in which:

- $E_0^\bullet = V^\bullet$.
- Each D^ν , $\nu = 0, \dots, N - 1$, is not entirely zero and considered up to an overall scalar.
- The “abutment” E_N^\bullet does not admit any two consecutive nonzero spaces (so the spectral sequence must degenerate at E_N^\bullet).

Then:

- (1) The set $\mathbb{P}SS(V^\bullet)$ admits the structure of a smooth projective variety $\overline{\mathbb{P}C}(V^\bullet)$ over \mathbb{C} .
- (2) $\overline{\mathbb{P}C}(V^\bullet)$ contains $\mathbb{P}C^\circ(V^\bullet)$ as an open dense part, and the complement $\mathbb{P}SS(V^\bullet) - \mathbb{P}C^\circ(V^\bullet)$ is a divisor with normal crossings.
- (3) One can obtain $\overline{\mathbb{P}C}(V^\bullet)$ as the successive blowup of the closures of the natural strata in $\mathbb{P}C(V^\bullet)$.

These results are obtained by combining Theorems 6.10 and 7.3. The realization of $\overline{\mathbb{P}C}(V^\bullet)$ as an iterated blowup generalizes the approach of Vainsencher [Va] to complete collineations. In the main body of the paper we work over any algebraically closed field \mathbf{k} of characteristic 0 and consider the varieties $C(V^\bullet)$ as well. Also, more generally, for any graded locally free sheaf of finite rank V^\bullet over an arbitrary normal variety X over \mathbf{k} , we introduce relative versions of varieties of complexes $C_X(V^\bullet)$ and $\mathbb{P}C_X(V^\bullet)$ (cf. Section 4). When X is smooth we obtain the analogs of the results (1)-(3) above.

C. Phenomena behind the results. The main phenomenon that makes our theory work, is the remarkable *self-similarity of the variety of complexes*. More precisely, $C(V^\bullet)$ is subdivided into strata (loci of complexes with prescribed ranks of the differentials). The transverse slice to a stratum passing through a point $D \in C(V^\bullet)$ (i.e., a differential in V^\bullet), is itself a variety of complexes but corresponding to the graded vector space $H_D^\bullet(V^\bullet)$ of cohomology of D (cf. Propositions 2.9 and 4.4). This generalizes the familiar self-similarity of the determinantal varieties: the transverse slice to the stratum formed by matrices of fixed rank inside a determinantal variety, is itself a determinantal variety of smaller size. In particular, our analysis implies that our stratification is conical in the sense of MacPherson and Procesi [MP].

Further, the classical intuitive reason behind the appearance of complete collineations has a transparent homological meaning. To recall this reason,

consider a 1-parameter family $A(t)$ of linear operators $V \rightarrow W$ (depending, say, analytically on a complex number t near 0). If $A(t)$ is nondegenerate for $t \neq 0$ but $A_0 = A(0)$ is degenerate, then the “next Taylor coefficient” of $A(t)$ gives $A_1 : \text{Ker}(A_0) \rightarrow \text{Coker}(A_0)$, the further Taylor coefficients give A_2 and so on. This gives the limit $\lim_{t \rightarrow 0} A(t)$ in the space of complete collineations in the classical theory.

If we now have an analytic 1-parameter family $D(t)$ of differentials in the same graded \mathbb{C} -vector space V^\bullet , we can view the Taylor expansion of $D(t)$ as a single differential \mathbf{D} in the graded $\mathbb{C}((t))$ -vector space $V^\bullet \otimes_{\mathbb{C}} \mathbb{C}((t))$. The fact that $D(t)$ is analytic at 0 (so we are talking about Taylor, not Laurent expansions), means that \mathbf{D} preserves the t -adic filtration in $V^\bullet \otimes_{\mathbb{C}} \mathbb{C}((t))$. The associated spectral sequence of the filtered complex is essentially simply graded, and it represents the limit of $(V^\bullet, D(t))$, as $t \rightarrow 0$, in our compactification.

D. Future directions. We expect our varieties of complete complexes to have interesting enumerative invariants, generalizing the many remarkable properties of complete collineations.

Historically, the first example of a “complete” variety of geometric objects was the Chasles-Schubert space \overline{Q}_n of complete quadrics, which gives a wonderful compactification of the variety Q_n of smooth quadric hypersurfaces in \mathbb{P}^n , see [DGMP]. From our point of view, Q_n can be seen as a particular case of the *variety of self-dual complexes*. That is, we start with a graded (by \mathbb{Z} or $\mathbb{Z} + \frac{1}{2}$) vector space V^\bullet which is identified with its graded dual by a graded symmetric bilinear form and consider all ways of making V^\bullet into a self-dual complex. The corresponding analog of \overline{Q}_n is then formed by the variety of *self-dual spectral sequences*. We leave its study to a future work.

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1. Categories of complexes

Let \mathbf{k} be an algebraically closed field and Λ be a finitely generated commutative \mathbf{k} -algebra. We denote by gMod_Λ the category of finitely generated graded Λ -modules V^\bullet . That is, $V^\bullet = \bigoplus_{i \in \mathbb{Z}} V^i$, with all V^i finitely generated, and $V^i = 0$ for $|i| \gg 0$. For $j \in \mathbb{Z}$ the shifted graded module $V^\bullet[j]$ is defined by $(V^\bullet[j])^i = V^{j+i}$.

We denote by $\mathcal{C}om_\Lambda$ the category of cochain complexes over Λ , i.e., of graded modules $V^\bullet = \bigoplus V^i \in \mathbf{gMod}_\Lambda$ equipped with a differential D , a collection of Λ -linear maps $D_i : V^i \rightarrow V^{i+1}$ satisfying $D_{i+1} \circ D_i = 0$. We will consider $D = (D_i)$ as a morphism $V^\bullet \rightarrow V^\bullet[1]$ in \mathbf{gMod}_Λ . For a complex (V^\bullet, D) we have the graded module $H_D^\bullet(V^\bullet)$ of cohomology.

We define the shifted complex $(V^\bullet, D)[j]$ to have the underlying graded module $V^\bullet[j]$ as above and the differential $D^{V^\bullet[j]}$ having components given by

$$(1.1) \quad D_i^{V^\bullet[j]} = (-1)^j D_{j+i}^{V^\bullet}.$$

One way to explain this formula is to represent $V^\bullet[j] = \Lambda[j] \otimes_\Lambda V^\bullet$ (here $\Lambda[j]$ is the ring Λ put in degree $(-j)$). Then (1.1) corresponds to defining the differential via the graded Leibniz rule.

We recall that a morphism of complexes $f : (V^\bullet, D) \rightarrow (V'^\bullet, D')$ is called *null-homotopic*, if it is of the form $f = D's + sD$, where $s : V^\bullet \rightarrow V'^\bullet[-1]$ is any morphism in \mathbf{gMod}_Λ . In this case we write $f \sim 0$. For two morphisms of complexes $f, g : (V^\bullet, D) \rightarrow (V'^\bullet, D')$ we say that f is *homotopic to g* and write $f \sim g$, if $f - g \sim 0$. Null-homotopic morphisms form an ideal in $\mathbf{Mor}(\mathcal{C}om_\Lambda)$, and the quotient category is called the *homotopy category of complexes* and will be denoted $\mathcal{H}ot_\Lambda$.

Definition 1.2. (a) A complex $(V^\bullet, D) \in \mathcal{C}om_\Lambda$ will be called *admissible*, if:

- (a1) Each V^i is a projective Λ -module.
- (a2) Each image $\text{Im}(D_i) \subset V^{i+1}$ is, locally, on the Zariski topology of $\text{Spec}(\Lambda)$, a direct summand in V^{i+1} .

(b) Let X be a \mathbf{k} -scheme of finite type. A complex V^\bullet of coherent sheaves on X will be called *admissible*, if, for any affine open $U \subset X$, the complex $\Gamma(U, V^\bullet)$ of modules over $\Lambda = \mathcal{O}(U)$, is admissible.

Proposition 1.3. (a) For an admissible complex (V^\bullet, D) , each $H_D^i(V^\bullet)$ is a projective Λ -module.

(b) Assume that Λ is a local ring. Then any admissible complex $V^\bullet \in \mathcal{C}om_\Lambda$ can be written as a direct sum $V^\bullet = A^\bullet \oplus H^\bullet$, where A^\bullet is an admissible acyclic complex ($H^\bullet(A^\bullet) = 0$), and H^\bullet is an admissible complex with zero differential (so $H^\bullet \simeq H^\bullet(V^\bullet)$).

(c) Further, for a local Λ , any acyclic admissible complex A^\bullet is contractible (i.e., $\text{Id}_{A^\bullet} \sim 0$). Therefore, for any two admissible complexes (V^\bullet, D)

and (V'^{\bullet}, D') over Λ we have

$$\mathrm{Hom}_{\mathcal{H}ot_{\Lambda}}((V^{\bullet}, D), (V'^{\bullet}, D')) \simeq \mathrm{Hom}_{\mathfrak{g}Mod_{\Lambda}}(H_D^{\bullet}(V^{\bullet}), H_{D'}^{\bullet}(V'^{\bullet}))$$

Proof: (a) Since $\mathrm{Im}(D_i)$ is locally a direct summand of a projective Λ -module V^{i+1} , it is projective. Therefore $\mathrm{Ker}(D_i)$ which is the kernel of the surjective morphism $D_i : V^i \rightarrow \mathrm{Im}(D_i)$ is itself projective and, moreover, locally a direct summand in V^i . So $\mathrm{Im}(D_{i-1})$ being, locally, a direct summand in V^i , is in fact locally a direct summand in $\mathrm{Ker}(D_i)$, and so $H_D^i(V) = \mathrm{Ker}(D_i)/\mathrm{Im}(D_{i-1})$ is projective.

(b) Since Λ is local, all locally direct summands discussed above are in fact direct summands of Λ -modules. Let $H^i \subset \mathrm{Ker}(D_i)$ be a direct complement to $\mathrm{Im}(D_{i-1})$, and let $W^i \subset V^i$ be a direct complement to $\mathrm{Ker}(D_i)$, so that $V^i = \mathrm{Im}(D_{i-1}) \oplus H^i \oplus W^i$. Then $H^{\bullet} \subset V^{\bullet}$ is a subcomplex with zero differential. Putting $A^i = \mathrm{Im}(D_{i-1}) \oplus W^i$, we get a subcomplex $A^{\bullet} \subset V^{\bullet}$ which is acyclic, and $V^{\bullet} = A^{\bullet} \oplus H^{\bullet}$.

(c) If A^{\bullet} is admissible and acyclic, we write, as before, $A^i = \mathrm{Im}(D_{i-1}) \oplus W^i$. Since $\mathrm{Im}(D_{i-1}) = \mathrm{Ker}(D_i)$, the restricted morphism $D_i|_{W^i}$ is an isomorphism $W^i \rightarrow \mathrm{Im}(D_i)$. Denote $s_{i+1} : A^{i+1} \rightarrow A^i$ to be the composite map

$$A^{i+1} \xrightarrow{\mathrm{pr}} \mathrm{Im}(D_i) \xrightarrow{D_i|_{W^i}^{-1}} W^i \hookrightarrow A^i.$$

Then $s = (s_i : A^i \rightarrow A^{i-1})$ satisfies $Ds + sD = \mathrm{Id}_{A^{\bullet}}$. □

When $\Lambda = \mathbf{k}$, all complexes are admissible, and we obtain:

Proposition 1.4. *Indecomposable objects in the abelian category $Com_{\mathbf{k}}$ are the following:*

- (1) $\mathbf{k}[j]$, $j \in \mathbb{Z}$;
- (2) $\{\mathbf{k} \xrightarrow{\mathrm{Id}} \mathbf{k}\}[j]$, $j \in \mathbb{Z}$ (the 2-term complex with differential being the identity).

Proof: A complex with trivial differential is a direct sum of summands of type (1). Further, an acyclic complex A^{\bullet} is a direct sum of summands of type (2). Indeed, in the notation of the proof of Proposition 1.3(c), A^{\bullet} splits into a direct sum of 2-term complexes $\{W^i \xrightarrow{D_i|_{W^i}} \mathrm{Im}(D_i)\}$, which are thus direct sums of summands of type (2). So our statement follows from Proposition 1.3(b). □

2. The affine variety of complexes, its strata and normal cones

From now on we assume that the characteristic of \mathbf{k} is 0. Let $V^\bullet = \bigoplus V^i$ be a finite-dimensional graded \mathbf{k} -vector space.

Definition 2.1. *The affine variety of complexes associated to V^\bullet is the closed subscheme*

$$C(V^\bullet) = \left\{ D = (D_i) \in \prod_i \text{Hom}(V^i, V^{i+1}) \mid D_{i+1} \circ D_i = 0 \text{ for all } i \right\}$$

in the affine space $\prod_i \text{Hom}(V^i, V^{i+1})$.

In other words $C(V^\bullet)$ is the subscheme of $\prod_i \text{Hom}(V^i, V^{i+1})$ formed by all ways of making V^\bullet into a complex. See [DS] for background. In particular, we note:

Proposition 2.2. *$C(V^\bullet)$ is a reduced scheme (affine algebraic variety). Further, each irreducible component of $C(V^\bullet)$ is a normal variety.* □

The group $GL(V^\bullet) = \prod_i GL(V^i)$ acts naturally on $C(V^\bullet)$. An element $g = (g_i)$, $g_i \in GL(V^i)$, sends $D = (D_i)$ to

$$(2.3) \quad (gD)_i = g_{i+1} D_i g_i^{-1}.$$

Orbits of $GL(V^\bullet)$ on $C(V^\bullet)$ are nothing but isomorphism classes of complexes (V^\bullet, D) with all possible D . We will call these orbits the *strata* of $C(V^\bullet)$ and denote by $[D]$ the stratum passing through D . By the Krull-Schmidt theorem, strata (isomorphism classes) are labelled by the multiplicities of the indecomposable summands of (V^\bullet, D) in the category $\text{Com}_{\mathbf{k}}$. Using the description of indecomposables given by Proposition 1.4, one obtains an explicit combinatorial description of the strata. Let us recall this description, together with some further properties of strata and their closures that have been established in [Go].

Without changing the essence of the problem, we can (and will) assume that $V^\bullet = \bigoplus_{i=0}^m V^i$ is concentrated in degrees $[0, m]$, and denote $n_i = \dim(V^i)$. Let R be the set of sequences $\mathbf{r} = (r_1, \dots, r_m)$, $r_i \in \mathbb{Z}_{\geq 0}$, satisfying the conditions

$$r_i + r_{i-1} \leq n_i, \quad i = 0, \dots, m + 1.$$

Here we put $r_0 = r_{m+1} = n_{m+1} = 0$. The set R is partially ordered by

$$\mathbf{r} \leq \mathbf{r}' \quad \Leftrightarrow \quad r_i \leq r'_i, \quad \forall i.$$

For any $\mathbf{r} \in R$ we denote

$$\begin{aligned} C_{\mathbf{r}}^{\circ}(V^{\bullet}) &= \{D \in C(V^{\bullet}) \mid \text{rk}(D_i) = r_{i+1}, i = 1, \dots, m\}, \\ C_{\mathbf{r}}(V^{\bullet}) &= \{D \in C(V^{\bullet}) \mid \text{rk}(D_i) \leq r_{i+1}, i = 1, \dots, m\}. \end{aligned}$$

Proposition 2.4 ([Go]). (a) *The strata of $C(V^{\bullet})$ are precisely the $C_{\mathbf{r}}^{\circ}(V^{\bullet})$, $\mathbf{r} \in R$. They are non-empty, locally closed, smooth subvarieties.*

(b) *The closure of $C_{\mathbf{r}}^{\circ}(V^{\bullet})$ is $C_{\mathbf{r}}(V^{\bullet})$. In particular, each $C_{\mathbf{r}}(V^{\bullet})$ is irreducible.*

(c) *We have $C_{\mathbf{r}}(V^{\bullet}) \subset C_{\mathbf{r}'}(V^{\bullet})$, if and only if $\mathbf{r} \leq \mathbf{r}'$.*

(d) *The irreducible components of $C(V^{\bullet})$ are precisely the $C_{\mathbf{r}}(V^{\bullet})$ where \mathbf{r} runs over maximal elements of the poset R . \square*

Proposition 2.5. (a) *The subvariety $C_{\mathbf{r}}(V^{\bullet})$ coincides with the subscheme in $C(V^{\bullet})$ given by the vanishing of the minors of size $r_{i+1} \times r_{i+1}$ of the differentials D_i for all i . In other words, the subscheme thus defined, is reduced.*

(b) *The scheme-theoretic intersection $C_{\mathbf{r}}(V^{\bullet}) \cap C_{\mathbf{s}}(V^{\bullet})$ coincides with the variety $C_{\min(\mathbf{r}, \mathbf{s})}(V^{\bullet})$, where*

$$\min(\mathbf{r}, \mathbf{s}) = (\min(r_1, s_1), \dots, \min(r_m, s_m)).$$

Proof: Part (a) is one of the main results of De Concini-Strickland [DS]. Part (b) follows from the following well known property of the determinantal ideals in the ring $\mathbf{k}[a_{ij}]$ of polynomials in the entries of an indeterminate $p \times q$ matrix $\|a_{ij}\|$. The ideal generated by all $r \times r$ minors contains the ideal generated by all $(r + 1) \times (r + 1)$ minors. \square

Let Y be a closed subscheme of a \mathbf{k} -scheme Z of finite type, and $I_Y \subset \mathcal{O}_Z$ the sheaf of ideals of Y . We denote by

$$\mathcal{N}_{Y/Z}^* = I_Y/I_Y^2, \quad \mathcal{N}_{Y/X} = \underline{\text{Hom}}_{\mathcal{O}_Y}(\mathcal{N}_{Y/Z}, \mathcal{O}_Y)$$

the conormal and normal sheaves to Y in Z . We will be particularly interested in the case when $\mathcal{N}_{Y/Z}^*$ (and therefore $\mathcal{N}_{Y/Z}$) is locally free, i.e., represents a vector bundle on Y . The total space of this vector bundle is then a scheme which we call the *normal bundle* to Y in Z and denote

$$N_{Y/Z} = \text{Spec}_{\mathcal{O}_Y} S_{\mathcal{O}_Y}^{\bullet}(I_Y/I_Y^2).$$

We further denote by

$$\text{NC}_{Y/Z} = \text{Spec}_{\mathcal{O}_Y} \left(\bigoplus_{n=0}^{\infty} I_Y^n/I_Y^{n+1} \right)$$

the *normal cone* to Y in Z . Because of the surjection of sheaves of \mathcal{O}_Y -algebras

$$S_{\mathcal{O}_Y}^\bullet(I_Y/I_Y^2) \longrightarrow \bigoplus_{n=0}^{\infty} I_Y^n/I_Y^{n+1},$$

$\mathrm{NC}_{Y/Z}$ is a closed subscheme in $N_{Y/Z}$. In particular, $\mathrm{NC}_{Y/Z}$ is a “cone bundle” over Y : it is equipped with an affine morphism $\mathrm{NC}_{Y/Z} \rightarrow Y$ whose fiber over a \mathbf{k} -point $y \in Y$ is a cone $(\mathrm{NC}_{Y/Z})_y$ in the linear space $(N_{Y/Z})_y$.

The above constructions extend easily to the case when Y is locally closed (instead of closed) subscheme in Z . In this case, we define

$$\mathcal{N}_{Y/Z}^* = \mathcal{N}_{Y/Z^\circ}^*, \quad N_{Y/Z} = N_{Y/Z^\circ} \quad \text{and} \quad \mathrm{NC}_{Y/Z} = \mathrm{NC}_{Y/Z^\circ},$$

where $Z^\circ \subset Z$ is any open subset containing Y and such that Y is closed in Z° . See [F] for more background.

We now specialize to the case when $Z = C(V^\bullet)$ and $Y = [D]$ is the stratum through a \mathbf{k} -point D . Let $\mathbf{r} = (r_1, \dots, r_m) \in R$ and let $*$ stand for any of the categories $\mathrm{Com}_{\mathbf{k}}$, $\mathrm{Hot}_{\mathbf{k}}$ or $\mathrm{gMod}_{\mathbf{k}}$. We denote by

$$\mathrm{Hom}_*^{\leq \mathbf{r}}((V^\bullet, D), (V^\bullet, D)[1]) \subset \mathrm{Hom}_*((V^\bullet, D), (V^\bullet, D)[1])$$

the closed subvariety formed by morphisms $f = (f_i : V^i \rightarrow V^{i+1})$ such that $\mathrm{rk}(f_i) \leq r_i$ for all i .

Proposition 2.6. (a) *The Zariski tangent space to $C(V^\bullet)$ at D is found by:*

$$T_D C(V^\bullet) = \mathrm{Hom}_{\mathrm{Com}_{\mathbf{k}}}((V^\bullet, D), (V^\bullet, D)[1]).$$

(b) *Suppose D is contained in some $C_{\mathbf{r}}(V^\bullet)$. Then, the Zariski tangent space to $C_{\mathbf{r}}(V^\bullet)$ at D is found by:*

$$T_D C_{\mathbf{r}}(V^\bullet) = \mathrm{Hom}_{\mathrm{Com}_{\mathbf{k}}}^{\leq \mathbf{r}}((V^\bullet, D), (V^\bullet, D)[1]).$$

Proof: (a) By definition, $T_D C(V^\bullet)$ is the set of points D_ε of $C(V^\bullet)$ with values in $\mathbf{k}[\varepsilon]/\varepsilon^2$ which extend the \mathbf{k} -point D . Since $C(V^\bullet)$ is embedded into the affine space $\mathrm{Hom}_{\mathrm{gVect}_{\mathbf{k}}}(V^\bullet, V^\bullet[1])$, we can write $D_\varepsilon = D + \varepsilon f$ where $f \in \mathrm{Hom}_{\mathrm{gVect}_{\mathbf{k}}}(V^\bullet, V^\bullet[1])$. The condition for D_ε to be a point of $C(V^\bullet)$ is the vanishing of $D_\varepsilon^2 = D^2 + \varepsilon(Df + fD)$. Since $D^2 = 0$ by assumption, we are left with $Df + fD = 0$ which, in virtue of the convention (1.1), means that $f : (V^\bullet, D) \rightarrow (V^\bullet, D)[1]$ is a morphism of complexes. Part (b) is similar. \square

Proposition 2.7. *The Zariski tangent space $T_D[D] \subset T_DC(V^\bullet)$ consists of those morphisms of complexes $(V^\bullet, D) \rightarrow (V^\bullet, D)[1]$, which are homotopic to 0.*

Proof: By definition, $[D] = GL(V^\bullet) \cdot D$ is the orbit of D under the action (2.3). Therefore

$$T_D[D] = \text{Im}\{\mathfrak{gl}(V^\bullet) \xrightarrow{h_D} T_DC(V^\bullet)\}$$

is the image of the Lie algebra $\mathfrak{gl}(V^\bullet)$ under the infinitesimal action h_D induced by (2.3). To differentiate (2.3), we take

$$g = (g_i), \quad g_i = 1 + \varepsilon s_i, \quad s_i \in \mathfrak{gl}(V^i), \quad \varepsilon^2 = 0.$$

Then

$$(gD)_i = (1 + \varepsilon s_{i+1})D_i(1 - \varepsilon s_i) = D_i + \varepsilon(s_{i+1}D_i - D_i s_i)$$

which is precisely a perturbation of D by a morphism homotopic to 0. Since s_i can be arbitrary, the statement follows. \square

Since $[D]$ is an orbit of $GL(V^\bullet)$, the conormal sheaf $\mathcal{N}_{[D]/C(V^\bullet)}^*$ is locally free and so we can speak about the normal bundle $N_{[D]/C(V^\bullet)}$. Proposition 1.3(c) together with the above implies:

Corollary 2.8. *The fiber at D of the normal bundle $N_{[D]/C(V^\bullet)}$ is found by:*

$$(N_{[D]/C(V^\bullet)})_D \simeq \text{Hom}_{\mathfrak{g}\text{Mod}_{\mathbf{k}}}(H_D^\bullet(V^\bullet), H_D^\bullet(V^\bullet)[1]). \quad \square$$

Proposition 2.9. (a) *The fiber of $\text{NC}_{[D]/C(V^\bullet)}$ at D is found by:*

$$(\text{NC}_{[D]/C(V^\bullet)})_D \simeq C(H_D^\bullet(V^\bullet)) \subset \text{Hom}_{\mathfrak{g}\text{Mod}_{\mathbf{k}}}(H_D^\bullet(V^\bullet), H_D^\bullet(V^\bullet)[1]).$$

(b) *Suppose D is contained in some $C_{\mathbf{r}}(V^\bullet)$. Then, the fiber of $\text{NC}_{[D]/C_{\mathbf{r}}(V^\bullet)}$ at D is found by:*

$$\begin{aligned} (\text{NC}_{[D]/C_{\mathbf{r}}(V^\bullet)})_D &\simeq C_{\mathbf{r}-\mathbf{r}_D}(H_D^\bullet(V^\bullet)) \\ &\subset \text{Hom}_{\mathfrak{g}\text{Mod}_{\mathbf{k}}}^{\leq \mathbf{r}-\mathbf{r}_D}(H_D^\bullet(V^\bullet), H_D^\bullet(V^\bullet)[1]), \end{aligned}$$

where \mathbf{r}_D is the sequence of ranks of the differential D .

Proof: (a) We note first that

$$(2.10) \quad (\mathrm{NC}_{[D]/C(V^\bullet)})_D \simeq \mathrm{NC}_{D/C(V^\bullet)}/T_D[D],$$

the quotient of the tangent cone at the point D by the action of the vector space $T_D[D]$ (we consider this space as an algebraic group acting on the tangent cone). Now, the normal cone to any subvariety Y in any X at a \mathbf{k} -point y is found, inside $T_y Y$, by:

- (1) Considering all $\mathbf{k}[[\varepsilon]]$ -points x_ε of X which are (1st order) tangent to Y
- (2) Restricting the equations of Y in X to such x_ε .
- (3) Equating to 0 the next (after the linear) lowest nonvanishing terms in ε in these equations.

In our case $X = \mathrm{Hom}_{\mathbf{gMod}_\mathbf{k}}(V^\bullet, V^\bullet[1])$ is an affine space, so in Step (1) above it is enough to take the $\mathbf{k}[[\varepsilon]]$ -points of the form $D_\varepsilon = D + \varepsilon f$ with $f \in \mathrm{Hom}_{\mathbf{gVect}_\mathbf{k}}(V^\bullet, V^\bullet[1])$. By Proposition 2.6, for D_ε to be 1st order tangent to $C(V^\bullet)$, it is necessary and sufficient that f be a morphism in $\mathrm{Com}_\mathbf{k}$, not just in $\mathbf{gMod}_\mathbf{k}$.

Further, in Step(2), the equations of $C(V^\bullet)$ after restricting to D_ε are the matrix elements of

$$D_\varepsilon^2 = D^2 + \varepsilon(Df + fD) + \varepsilon^2 f^2,$$

so the next nonvanishing coefficient in Step (3) is f^2 . This means that

$$\mathrm{NC}_{D/C(V^\bullet)} = \{f \in \mathrm{Hom}_{\mathrm{Com}_\mathbf{k}}((V^\bullet, D), (V^\bullet, D)[1]) \mid f^2 = 0\}.$$

Our statement now follows from (2.10), the identification of $T_D[D]$ in Proposition 2.7 and Proposition 1.3(c).

(b) The isomorphism of Proposition 1.3(c) restricts to an isomorphism of the subspace $\mathrm{Hom}_{\mathcal{H}\mathrm{ot}_\mathbf{k}}^{\leq r}((V^\bullet, D), (V^\bullet, D)[1])$ with $\mathrm{Hom}_{\mathbf{gMod}_\mathbf{k}}^{\leq r-rD}(H_D^\bullet(V^\bullet), H_D^\bullet(V^\bullet)[1])$. So it suffices to repeat the argument of (a). \square

3. The projective variety of complexes

It follows from Definition 2.1 that $C(V^\bullet)$ is given by homogeneous (quadratic) equations in the linear space $\mathrm{Hom}_{\mathbf{gMod}_\mathbf{k}}(V^\bullet, V^\bullet[1])$; those are the matrix elements of all the maps $D_{i+1} \circ D_i : V^i \rightarrow V^{i+2}$, where $D_i \in \mathrm{Hom}(V^i, V^{i+1})$ for all i . We can therefore give the following definition:

Definition 3.1. *The projective variety of complexes associated to V^\bullet is the projectivization*

$$\mathbb{P}C(V^\bullet) \subset \mathbb{P}(\text{Hom}_{\mathbf{gMod}_k}(V^\bullet, V^\bullet[1]))$$

of $C(V^\bullet)$.

It follows that $\mathbb{P}C(V^\bullet)$ is a reduced scheme (projective algebraic variety), and each of its irreducible components is normal.

For any nonzero differential $D = (D_i : V^i \rightarrow V^{i+1})$ in $C(V^\bullet)$, we denote its image in $\mathbb{P}C(V^\bullet)$ by $\mathbb{P}D$. Further, for any $\mathbf{0} \neq \mathbf{r} \in R$, we denote by $\mathbb{P}C_{\mathbf{r}}^\circ(V^\bullet)$ and $\mathbb{P}C_{\mathbf{r}}(V^\bullet)$ the images of the stratum $C_{\mathbf{r}}^\circ(V^\bullet)$ and of its closure in $\mathbb{P}C(V^\bullet)$ respectively. We call the $\mathbb{P}C_{\mathbf{r}}^\circ(V^\bullet)$ for $\mathbf{r} \neq \mathbf{0}$, the *strata* of $\mathbb{P}\text{Com}(V^\bullet)$ and denote by $[\mathbb{P}D]$ the stratum passing through $\mathbb{P}D$.

The properties of the affine varieties of complexes and their strata imply at once:

- Proposition 3.2.** (a) *The strata of $\mathbb{P}C(V^\bullet)$ are precisely the $GL(V^\bullet)$ -orbits. They are non-empty, locally closed, smooth subvarieties.*
 (b) *The closure of $\mathbb{P}C_{\mathbf{r}}^\circ(V^\bullet)$ is $\mathbb{P}C_{\mathbf{r}}(V^\bullet)$. In particular, each $\mathbb{P}C_{\mathbf{r}}(V^\bullet)$ is irreducible.*
 (c) *We have $\mathbb{P}C_{\mathbf{r}}(V^\bullet) \subset \mathbb{P}C_{\mathbf{r}'}(V^\bullet)$ if and only if $\mathbf{r} \leq \mathbf{r}'$.*
 (d) *The irreducible components of $\mathbb{P}C(V^\bullet)$ are precisely the $\mathbb{P}C_{\mathbf{r}}(V^\bullet)$ where \mathbf{r} runs over maximal elements of the poset R . \square*

Proposition 3.3. (a) *The fiber of $\text{NC}_{[\mathbb{P}D]/\mathbb{P}C(V^\bullet)}$ at $\mathbb{P}D$ is found by:*

$$(\text{NC}_{[\mathbb{P}D]/\mathbb{P}C(V^\bullet)})_{\mathbb{P}D} \simeq C(H_D^\bullet(V^\bullet)).$$

- (b) *Suppose $\mathbb{P}D$ is contained in some $\mathbb{P}C_{\mathbf{r}}(V^\bullet)$. The fiber of $\text{NC}_{[\mathbb{P}D]/\mathbb{P}C_{\mathbf{r}}(V^\bullet)}$ at $\mathbb{P}D$ is found by:*

$$(\text{NC}_{[\mathbb{P}D]/\mathbb{P}C_{\mathbf{r}}(V^\bullet)})_{\mathbb{P}D} \simeq C_{\mathbf{r}-\mathbf{r}_D}(H_D^\bullet(V^\bullet)). \quad \square$$

4. The relative varieties of complexes and their normal cones

Let X be a normal algebraic variety over \mathbf{k} . We denote by \mathbf{gCoh}_X the category of graded coherent sheaves $\mathcal{F}^\bullet = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^i$ and morphisms preserving the grading. For any two $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{gCoh}_X$ we have a coherent sheaf $\underline{\text{Hom}}_{\mathbf{gCoh}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ on X (local homomorphisms).

Let $V^\bullet = \bigoplus_{i=0}^m V^i$ be a graded vector bundle (locally free sheaf of finite rank) on X concentrated in degrees $[0, m]$. For any \mathbf{k} -point $x \in X$ we denote by V_x^\bullet the graded vector space obtained as the fiber of V^\bullet at x . The constructions of §2 admit obvious relative versions.

Definition 4.1. *The relative affine variety of complexes associated to V^\bullet is the X -scheme $C_X(V^\bullet) \xrightarrow{\pi} X$ that represents the functor on schemes given by $T \mapsto \{(\phi, D) | \phi : T \rightarrow X \text{ a morphism of schemes, } D \text{ a differential in } \phi^*V^\bullet\}$.*

By definition, $C_X(V^\bullet)$ carries the *universal complex* of vector bundles

$$(\underline{V}^\bullet, \underline{D}), \quad \underline{V}^\bullet = \pi^*V^\bullet, \\ \underline{D} = \text{the universal differential coming from Definition 4.1.}$$

In particular, the fiber of π at a \mathbf{k} -point $x \in X$ is the variety of complexes $C(V_x^\bullet)$.

We denote by

$$M = \mathcal{H}om_{\text{gCoh}_X}(V^\bullet, V^\bullet[1]) = \text{Spec}_{\mathcal{O}_X} S^\bullet \left(\bigoplus_{i=0}^{m-1} V^i \otimes (V^{i+1})^* \right)$$

the total space of the vector bundle $\underline{\text{Hom}}_{\text{gCoh}_X}(V^\bullet, V^\bullet[1])$ (“relative space of matrices”). Then $C_X(V^\bullet)$ is a closed conic subvariety in M . Let $\mathbb{P}M \rightarrow X$ be the projectivization of M over X .

Definition 4.2. *The relative projective variety of complexes associated to V^\bullet is the projectivization $\mathbb{P}C_X(V^\bullet) \subset \mathbb{P}M$ of $C_X(V^\bullet)$.*

As before, for any $\mathbf{r} \in R$ we denote by $C_{X,\mathbf{r}}^\circ(V^\bullet)$, (resp. by $C_{X,\mathbf{r}}(V^\bullet)$) the locally closed, (resp. closed) subvariety in $C_X(V^\bullet)$ formed by differentials $D = (D_i)$ with the rank of D_i equal to r_i everywhere (resp. $\leq r_i$ everywhere). We refer to the $C_{X,\mathbf{r}}^\circ(V^\bullet)$ as the *strata* of $C_X(V^\bullet)$.

If $\mathbf{r} \neq \mathbf{0}$, we denote by $\mathbb{P}C_{X,\mathbf{r}}^\circ(V^\bullet)$, resp. $\mathbb{P}C_{X,\mathbf{r}}(V^\bullet)$, the image of $C_{X,\mathbf{r}}^\circ(V^\bullet)$, resp. $C_{X,\mathbf{r}}(V^\bullet)$ in $\mathbb{P}C_X(V^\bullet)$. We call the $\mathbb{P}C_{X,\mathbf{r}}^\circ(V^\bullet)$ the *strata* of $\mathbb{P}C_X(V^\bullet)$.

Proposition 4.3. (a) $C_X(V^\bullet)$ and $\mathbb{P}C_X(V^\bullet)$ are reduced schemes; each irreducible component of these schemes is a normal variety.

(b) If X is smooth, then each stratum $C_{X,\mathbf{r}}^\circ(V^\bullet)$, $\mathbb{P}C_{X,\mathbf{r}}^\circ(V^\bullet)$ is smooth.

(c) The subvariety $C_{X,\mathbf{r}}(V^\bullet)$ coincides with the subscheme in $C_X(V^\bullet)$ given by the vanishing of the minors of size $r_{i+1} \times r_{i+1}$ of the differentials D_i for all i .

(d) The scheme-theoretic intersection $C_{X,\mathbf{r}}(V^\bullet) \cap C_{X,\mathbf{s}}(V^\bullet)$ coincides with the variety $C_{X,\min(\mathbf{r},\mathbf{s})}(V^\bullet)$, where

$$\min(\mathbf{r}, \mathbf{s}) = (\min(r_1, s_1), \dots, \min(r_m, s_m)).$$

Proof: Parts (a)-(c) follow from the absolute case, cf. Proposition 2.5. The proof of (d) is also similar to the proof of Proposition 2.5(b). \square

Let S be a stratum of $C_X(V^\bullet)$. We denote by $\underline{V}^\bullet_S = \underline{V}^\bullet|_S$ the restriction to S of the universal complex \underline{V}^\bullet . By the definition of the strata, \underline{V}^\bullet_S is an admissible complex (Def. 1.2), and therefore its graded sheaf of cohomology with respect to the restriction of the differential \underline{D} , is locally free (a graded vector bundle). We denote this graded vector bundle by $H^\bullet_S := H^\bullet_{\underline{D}}(\underline{V}^\bullet_S)$. Note that H^\bullet_S descends canonically to a graded vector bundle on the projectivization $\mathbb{P}S$, which we will denote by $H^\bullet_{\mathbb{P}S}$.

Propositions 2.9 and 3.3, describing the normal cones to the strata fiberwise, can be formulated in a neater, global way, using relative varieties of complexes. The proofs are identical and we omit them.

Proposition 4.4. (a) *Let $S = C^\circ_{X,r'}(X)$ be a stratum in $C_X(V^\bullet)$ and $r' \leq r$. Then*

$$\text{NC}_S(C_X(V^\bullet)) = C_S(H^\bullet_S), \quad \text{NC}_S(C_{X,r}(V^\bullet)) = C_{S,r-r'}(H^\bullet_S).$$

(b) *Let $\mathbb{P}S = \mathbb{P}C^\circ_{X,r'}(X)$ be a stratum in $\mathbb{P}C_X(V^\bullet)$ and $r' \leq r$. Then*

$$\text{NC}_{\mathbb{P}S}(\mathbb{P}C_X(V^\bullet)) = C_{\mathbb{P}S}(H^\bullet_{\mathbb{P}S}), \quad \text{NC}_{\mathbb{P}S}(\mathbb{P}C_{X,r}(V^\bullet)) = C_{\mathbb{P}S,r-r'}(H^\bullet_{\mathbb{P}S}). \quad \square$$

5. Charts in varieties of complexes

We keep the notation of §4. The goal of this section is to prove:

Proposition 5.1. (a) *Let c be a \mathbf{k} -point of $C_X(V^\bullet)$ (resp. $\mathbb{P}C_X(V^\bullet)$) belonging to a stratum S . There exists an isomorphism of an étale neighborhood of c in $C_X(V^\bullet)$ (resp. in $\mathbb{P}C_X(V^\bullet)$) with an étale neighborhood of c in $\text{NC}_{S/C_X(V^\bullet)}$ (resp. in $\text{NC}_{S/\mathbb{P}C_X(V^\bullet)}$).*

(b) *Suppose that c (and therefore S) is contained in some $C_{X,r}(V^\bullet)$ (resp. $\mathbb{P}C_{X,r}(V^\bullet)$). Then the isomorphism of part (a) restricts to an isomorphism of an étale neighborhood of c in $C_{X,r}(V^\bullet)$ (resp. in $\mathbb{P}C_{X,r}(V^\bullet)$) with an étale neighborhood of c in $\text{NC}_{S/C_{X,r}(V^\bullet)}$ (resp. $\text{NC}_{S/\mathbb{P}C_{X,r}(V^\bullet)}$).*

Remark 5.2. (a) Combining Propositions 5.1 and 4.4, we obtain the following conclusion (“self-similarity of varieties of complexes”). Any relative affine or projective variety of complexes is modeled, near any point c , by another affine variety of complexes, which is typically simpler than the original one (depending on the singular nature of c).

(b) The proposition implies, in particular, that the natural stratifications of $C_X(V^\bullet)$ and $\mathbb{P}C_X(V^\bullet)$ are conical in the sense of MacPherson and Procesi [MP].

Proof of Proposition 5.1: We begin by a series of reductions. First, we only need to prove the statement for the affine variety of complexes: the projective case follows immediately from that by descent.

Second, we need only to give the proof for part (a): part (b) will follow by identical arguments.

Third, it is enough to consider only the absolute case when X is a point. Indeed, by restricting, if necessary, to an open subset of X , we can assume that the graded vector bundle V^\bullet is trivial, identified with $X \times V_x^\bullet$, where V_x^\bullet is the fiber at some $x \in X$. In this case $C_X(V^\bullet) = X \times C(V_x^\bullet)$, and the stratum S has the form $X \times S_x$, where S_x is a stratum in $C(V_x^\bullet)$. A point c has then the form $c = (x, D)$ where $x \in X$ and D lies in S_x . A chart for $C_X(V^\bullet)$ near c will follow from a chart for $C(V_x^\bullet)$ near D .

So we assume that $X = \text{Spec}(\mathbf{k})$ and V^\bullet is a graded vector space. Our point c is therefore just a differential D in V^\bullet and $S = [D]$. The remainder of the proof is subdivided into three steps.

Step 1: We write $V^\bullet = H^\bullet \oplus A^\bullet$, where H^\bullet is a complex with zero differential and A^\bullet is an acyclic complex (see Proposition 1.3). Let D_A be the differential of A^\bullet . Then we can write D in matrix form as

$$(5.3) \quad D = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_A \end{array} \right)$$

where the zero upper left part corresponds to H^\bullet . Now, we have an embedding of $C(H^\bullet)$ into $C(V^\bullet)$ defined by

$$(5.4) \quad \delta \mapsto D_\delta := \left(\begin{array}{c|c} \delta & 0 \\ \hline 0 & D_A \end{array} \right), \quad \delta \in C(H^\bullet).$$

Further, H^\bullet is isomorphic to $H_D^\bullet(V^\bullet)$, the cohomology of V^\bullet . Therefore, by Proposition 2.9, we deduce that $C(H^\bullet)$ is isomorphic to the fiber of the normal cone $\text{NC}_{[D]/C(V^\bullet)}$ over D . In other words, we have embedded the fiber of the normal cone over D back into $C(V^\bullet)$. Our goal in the steps to follow is to extend this embedding to an étale map from an open neighborhood of the fiber to an open neighborhood of D in $C(V^\bullet)$.

Step 2: Let St_D be the stabilizer of D for the action of $GL(V^\bullet)$ on $C(V^\bullet)$. We write $g \in GL(V^\bullet)$ as $g = (g_i : V^i \rightarrow V^i)_{i=0}^m$, and then write each g_i in the matrix form with respect to $V^i = H^i \oplus A^i$:

$$g_i = \left(\begin{array}{c|c} P_i & Q_i \\ \hline R_i & S_i \end{array} \right), \quad P_i : H^i \rightarrow H^i, \quad R_i : H^i \rightarrow A^i, \text{ etc.}$$

The condition that $g \in \text{St}_D$ means, in virtue of (5.3) and the action law (2.3):

$$(5.5) \quad Q_{i+1}D_i = D_iR_i = 0, \quad S_{i+1}D_i = D_iS_i,$$

which is a set of linear homogeneous equations on the matrix elements of the g_i . In other words, (5.5) defines a linear subspace $W \subset \text{End}(V^\bullet) = \bigoplus_i \text{End}_{\mathbf{k}}(V^i)$, and $\text{St}_D = W \cap GL(V^\bullet)$. Choose a complementary affine subspace L to W passing through $\mathbf{1} \in \text{End}(V^\bullet)$ (which is the unit element of $GL(V^\bullet)$). Then set $U = L \cap GL(V^\bullet)$.

Lemma 5.6. *The action map $\phi : U \rightarrow [D]$, $g \mapsto g \cdot D$, is birational, and its differential at $\mathbf{1} \in U$ is an isomorphism.*

Proof: It is clear from the construction (the tangent space $T_{\mathbf{1}}U$ is a complement to $T_{\mathbf{1}}\text{St}_D$), that $\dim(U) = \dim[D]$, and $d_{\mathbf{1}}\phi$ is an isomorphism. To see that ϕ is birational, fix a generic $g_0 \in U$ and see how many $g \in U$ are there such that $\phi(g) = \phi(g_0)$. The latter condition means $g \cdot D = g_0 \cdot D$, i.e., $g_0^{-1}g \in \text{St}_D$, or, in other words, $g \in g_0 \cdot \text{St}_D$. Since St_D is the intersection of $GL(V^\bullet)$ with a linear subspace in $\text{End}(V^\bullet)$, the coset $g_0 \cdot \text{St}_D$ also has this property, so $L \cap (g_0 \cdot \text{St}_D)$ typically consists of one point, i.e. ϕ is generically bijective onto its image. Since $[D]$ is normal and the ground field \mathbf{k} has characteristic 0, Zariski's Main Theorem for quasifinite morphisms implies that ϕ is generically an open immersion, i.e. birational. \square

Let us now extend (5.4) to an embedding

$$(5.7) \quad \eta : \text{Hom}(H^\bullet, H^\bullet[1]) \hookrightarrow \text{Hom}(V^\bullet, V^\bullet[1]), \quad \delta \mapsto D_\delta$$

by allowing δ to be an arbitrary morphism of graded vector spaces $H^\bullet \rightarrow H^\bullet[1]$ (not necessarily satisfying $\delta^2 = 0$).

Lemma 5.8. *Inside the tangent space $T_D \text{Hom}(V^\bullet, V^\bullet[1])$ we have $\text{Im}(d_0\eta) \cap T_D[D] = 0$.*

Proof: Identifying the tangent space in question with the vector space $\text{Hom}(V^\bullet, V^\bullet[1])$, we have

$$\text{Im}(d_0\eta) = \text{Hom}(H^\bullet, H^\bullet[1]) \subset \text{Hom}(H^\bullet \oplus A^\bullet, H^\bullet[1] \oplus A^\bullet[1]) = \text{Hom}(V^\bullet, V^\bullet[1]).$$

Note that $\text{Im}(d_0\eta)$ consists of morphisms of complexes, not just of graded vector spaces, since H^\bullet has zero differential and is a direct summand in V^\bullet as a complex. On the other hand, $T_D[D]$ consists, by Proposition 2.7, of morphisms of complexes $V^\bullet \rightarrow V^\bullet[1]$ which are homotopic to 0. Such morphisms induce

the zero map on the cohomology. On the other hand, morphisms from $\text{Im}(d_0\eta)$ are faithfully represented by their action on the cohomology, since $H^\bullet \simeq H_D^\bullet(V^\bullet)$. Therefore the intersection of the two subspaces is 0. \square

Lemma 5.9. *The action map*

$$\Psi : U \times \text{Hom}(H^\bullet, H^\bullet[1]) \longrightarrow \text{Hom}(V^\bullet, V^\bullet[1]), \quad (g, \delta) \mapsto g \cdot D_\delta,$$

is, at the point $(\mathbf{1}, 0)$, étale onto its image.

Proof: As both the source and target of Ψ are smooth, it is enough to show that the differential $d_{(\mathbf{1}, 0)}\Psi$ is an injective linear map. Now,

$$T_{(\mathbf{1}, 0)}(U \times \text{Hom}(H^\bullet, H^\bullet[1])) = T_{\mathbf{1}}U \oplus \text{Hom}(H^\bullet, H^\bullet[1]).$$

The restriction of $d_{(\mathbf{1}, 0)}\Psi$ to the summand $T_{\mathbf{1}}U$ is the map $d_{\mathbf{1}}\phi$ which, by Lemma 5.6, maps it isomorphically to $T_D[D]$. The restriction of $d_{(\mathbf{1}, 0)}\Psi$ to the summand $\text{Hom}(H^\bullet, H^\bullet[1])$ is the embedding $d_0\eta$, see (5.7). So, by Lemma 5.8, its image does not intersect the image of the summand $T_{\mathbf{1}}U$, which is $T_D[D]$. This means that the map from the direct sum of the two summands is injective. \square

Corollary 5.10. *The action map*

$$\Phi : U \times C(H^\bullet) \longrightarrow C(V^\bullet), \quad (g, \delta) \mapsto g \cdot D_\delta,$$

(the restriction of Ψ) is étale at the point $(\mathbf{1}, 0)$.

Proof: By Lemma 5.9, Φ is, at $(\mathbf{1}, 0)$, étale onto its image. This image is contained in $C(V^\bullet)$. Now, we look at the irreducible components K of $C(V^\bullet)$ through D . Applying Proposition 2.4(d), we see that they are in bijection with irreducible components of $C(H^\bullet)$ through 0 and therefore with irreducible components K' of $U \times C(H^\bullet)$ through $(\mathbf{1}, 0)$. The dimension of each component K' is equal to the dimension of the corresponding K , and we see that K is covered by K' near D (and therefore $\Phi(K') = K$). So $\text{Im}(\Phi) = C(V^\bullet)$. \square

Step 3: The isomorphism $H^\bullet \simeq H_D^\bullet(V^\bullet)$ induces, by Proposition 2.9, an identification

$$\xi : C(H^\bullet) \rightarrow C(H_D^\bullet(V^\bullet)) = (\text{NC}_{[D]/C(V^\bullet)})_D.$$

We extend it to a map

$$\Xi : U \times C(H^\bullet) \longrightarrow \text{NC}_{[D]/C(V^\bullet)}, \quad (g, \delta) \mapsto g \cdot \xi(D_\delta),$$

using the action of $GL(V^\bullet)$ on $NC_{[D]/C(V^\bullet)}$. This morphism is birational and, moreover, biregular near $\{1\} \times C(H^\bullet)$ by Lemma 5.6, since $g \in U$ takes the fiber of $NC_{[D]/C(V^\bullet)}$ over D to the fiber over $g \cdot D$.

Now, the composition $\Phi \circ \Xi^{-1} : NC_{[D]/C(V^\bullet)} \rightarrow C(V^\bullet)$ is a rational map, regular and étale at the point $D \in NC_{[D]/C(V^\bullet)}$. Proposition 5.1 is proved.

6. Complete complexes via blowups

Let X be a smooth irreducible variety over \mathbf{k} , and V^\bullet be a graded vector bundle over X with grading situated in degrees $[0, m]$, and $\text{rk}(V^i) = n_i$. We keep all the other notations of §4. In this section we construct the *variety of complete complexes*, resp. *projective variety of complete complexes* by successively blowing up closures of strata in $C_X(V^\bullet)$ and $\mathbb{P}C_X(V^\bullet)$ respectively.

A. Reminder on blowups. Let Z be a scheme of finite type over \mathbf{k} and $Y \subset Z$ a closed subscheme with sheaf of ideal I_Y . The *blowup of Y in Z* is the scheme

$$\text{Bl}_Y(Z) = \text{Proj} \left(\bigoplus_{n=0}^{\infty} I_Y^n \right).$$

See [F] and [H] for general background. In particular, we have a natural projection

$$p : \text{Bl}_Y(Z) \rightarrow Z, \quad p^{-1}(Y) = \mathbb{P}NC_{Y/Z} := \text{Proj} \left(\bigoplus_{n=0}^{\infty} I_Y^n / I_Y^{n+1} \right),$$

which restricts to an isomorphism $\text{Bl}_Y(Z) - p^{-1}(Y) \rightarrow Z - Y$. We will be especially interested in the case when Y is a smooth algebraic variety and the conormal sheaf $\mathcal{N}_{Y/Z}^*$ is locally free, in which case $\mathbb{P}NC_{Y/Z} \subset \mathbb{P}N_{Y/Z}$ is a closed subscheme in the projectivization of the normal bundle $N_{Y/Z}$. Compare with §2.

If $W \subset Z$ is a closed subscheme, the *strict transform* of W is defined as $W^{\text{st}} = \text{Bl}_{W \cap Y}(W)$ where $W \cap Y$ is the scheme-theoretic intersection. It is a closed subscheme in $\text{Bl}_Y(Z)$.

If Z is an algebraic variety and W is an irreducible subvariety, then W^{st} is equal to the closure in $\text{Bl}_Y(Z)$ of $p^{-1}(W - Y)$ (in particular, it is empty, if $W \subset Y$). More generally, if $W \subset Z$ is any subvariety, its *dominant transform* \widetilde{W} is defined as [L4]:

$$\widetilde{W} = \begin{cases} W^{\text{st}}, & \text{if } W \not\subset Y; \\ \text{total inverse image } p^{-1}(W), & \text{if } W \subset Y. \end{cases}$$

Proposition 6.1. *Let Z be a scheme of finite type over \mathbf{k} and $W_1, W_2 \subset Z$ be closed subschemes. Let $Y = W_1 \cap W_2$ (scheme-theoretic intersection). Then the strict transforms of W_1 and W_2 in $\text{Bl}_Y(Z)$ are disjoint.*

Proof: This is Exercise 7.12 in [H]. □

B. Details on the poset of strata. Recall the poset R of integer vectors $\mathbf{r} = (r_1, \dots, r_m)$ labelling the strata (as well as the closures of the strata) of $C_X(V^\bullet)$, see §2 for the absolute case, extended in §4 to the relative case. Thus the zero vector $\mathbf{0}$ is the minimal element of R . For $\mathbf{r} \in R$ we denote $|\mathbf{r}| = \sum r_i$, and call this number the *length* of \mathbf{r} . We denote $R_l = \{\mathbf{r} \in R : |\mathbf{r}| = l\}$, and similarly for $R_{\geq l}, R_{< l}$ etc.

Proposition 6.2. (a) *The poset R is ranked with rank function $|\mathbf{r}|$. That is, for any $\mathbf{r} \leq \mathbf{r}'$ all maximal chains of strict inequalities $\mathbf{r} = \mathbf{r}^{(0)} < \dots < \mathbf{r}^{(l)} = \mathbf{r}'$ have the same cardinality $l + 1$, where $l = |\mathbf{r}'| - |\mathbf{r}|$.*

(b) *The set of minimal elements of $R_{\geq l}$ coincides with R_l .*

Proof: Both statements follow from the next property which is obvious from the definition of R by inequalities.

Lemma 6.3. *If $\mathbf{r} \in R$ and $r_i \neq 0$, then $\mathbf{r} - e_i \in R$, where e_i is the i th basis vector (1 at the position i , zeroes everywhere else). □*

We now introduce the notation for some subsets of R :

R^{\max} denotes the set of maximal elements of R (which label irreducible components of $C_X(V^\bullet)$ as well as of $\mathbb{P}C_X(V^\bullet)$).

$R^b = R - R^{\max}$ (labels closures of non-maximal strata in $C_X(V^\bullet)$, to be blown up).

$R^b_{>l} = R^b \cap R_{>l}$ etc. In particular, $R^b_{>0}$ labels closures of non-maximal strata in $\mathbb{P}C_X(V^\bullet)$.

Definition 6.4. *A graded vector space V^\bullet is sparse, if the numbers $n_i = \dim(V^i)$ are such that $n_i n_{i+1} = 0$ for each i .*

Remark 6.5. V^\bullet is sparse, iff $C(V^\bullet) = \{0\}$.

Proposition 6.6. *A vector $\mathbf{r} \in R$ lies in R^{\max} if and only if for each $D \in C_{\mathbf{r}}^\circ(V^\bullet)$ the graded vector space $H_D^\bullet(V^\bullet)$ is sparse.*

Proof: Indeed, the fiber of the normal cone to the stratum $[D] = C_{\mathbf{r}}^\circ(V^\bullet)$ over D is, by Proposition 2.9, identified with $C(H_D^\bullet(V^\bullet))$. Saying that $[D]$ is a maximal stratum is equivalent to saying that its normal cone consists of just the zero section. □

We denote by

$$(6.7) \quad C^\circ(V^\bullet) = \bigcup_{\mathbf{r} \in R^{\max}} C_{\mathbf{r}}^\circ(V^\bullet), \quad \mathbb{P}C^\circ(V^\bullet) = \bigcup_{\mathbf{r} \in R^{\max}} \mathbb{P}C_{\mathbf{r}}^\circ(V^\bullet)$$

the union of the maximal strata. It is a smooth open dense subvariety in $C(V^\bullet)$, resp. $\mathbb{P}C(V^\bullet)$. We will refer to it as the *generic part* of $C(V^\bullet)$, resp. $\mathbb{P}C(V^\bullet)$.

C. Main constructions and results. Let d be the maximal value of $|\mathbf{r}|$ for $\mathbf{r} \in R$. Our first result gives a construction of a series of blowups of the varieties of complexes with good properties (at each step we perform a blowup with a smooth center). For convenience of inductive arguments, we formulate the results for both affine and projective varieties of complexes.

Theorem 6.8. *There exist towers of blowups*

$$\begin{aligned} \overline{C}_X(V^\bullet) &= C_X^{(d)}(V^\bullet) \rightarrow \cdots \rightarrow C_X^{(1)}(V^\bullet) \rightarrow C_X^{(0)}(V^\bullet) = C_X(V^\bullet), \\ \overline{\mathbb{P}C}_X(V^\bullet) &= \mathbb{P}C_X^{(d)}(V^\bullet) \rightarrow \cdots \rightarrow \mathbb{P}C_X^{(2)}(V^\bullet) \rightarrow \mathbb{P}C_X^{(1)}(V^\bullet) = \mathbb{P}C_X(V^\bullet) \end{aligned}$$

with the following properties (which define them uniquely). For $\mathbf{r} \in R$ let $C_{X,\mathbf{r}}^{(l)}(V^\bullet) \subset C_X^{(l)}(V^\bullet)$ be the iterated dominant transform of $C_{X,\mathbf{r}}(V^\bullet) \subset C_X(V^\bullet)$, and for $\mathbf{r} \in R_{>0}$ let $\mathbb{P}C_{X,\mathbf{r}}^{(l)}(V^\bullet) \subset \mathbb{P}C_X^{(l)}(V^\bullet)$ be the iterated dominant transform of $\mathbb{P}C_{X,\mathbf{r}}(V^\bullet) \subset \mathbb{P}C_X(V^\bullet)$.

- (a) For any given l and for $|\mathbf{r}| = l$, the subvarieties $C_{X,\mathbf{r}}^{(l)}$ (resp. $\mathbb{P}C_{X,\mathbf{r}}^{(l)}$) are smooth and disjoint.
- (b) We have

$$\begin{aligned} C_X^{(l+1)}(V^\bullet) &= \text{Bl} \prod_{|\mathbf{r}|=l} C_{X,\mathbf{r}}^{(l)}(V^\bullet) C_X^{(l)}(V^\bullet), \\ \mathbb{P}C_X^{(l+1)}(V^\bullet) &= \text{Bl} \prod_{|\mathbf{r}|=l} \mathbb{P}C_{X,\mathbf{r}}^{(l)}(V^\bullet) \mathbb{P}C_X^{(l)}(V^\bullet). \end{aligned}$$

The theorem will allow us to make the following definition.

Definition 6.9. *The relative variety of complete complexes associated to V^\bullet , is the variety $\overline{C}_X(V^\bullet)$. The relative projective variety of complete complexes associated to V^\bullet , is the variety $\overline{\mathbb{P}C}_X(V^\bullet)$.*

Our second result says that the iterated blowup we construct, provides wonderful compactifications of the open strata in the irreducible components of the varieties of complexes.

- Theorem 6.10.** (a) *The variety $\overline{C}_X(V^\bullet)$ (resp. $\overline{\mathbb{P}C}_X(V^\bullet)$) is smooth and equal to the disjoint union of the $C_{X,\mathbf{r}}^{(d)}(V^\bullet)$ (resp. of the $\mathbb{P}C_{X,\mathbf{r}}(V^\bullet)$) for $\mathbf{r} \in R^{\max}$.*
- (b) *The varieties $\Delta_{\mathbf{r}} = C_{X,\mathbf{r}}^{(d)}(V^\bullet)$, $\mathbf{r} \in R^b$, are smooth and form a divisor with normal crossings in $\overline{C}_X(V^\bullet)$ which we denote $\partial\overline{C}_X(V^\bullet)$. Similarly, the varieties $\mathbb{P}\Delta_{\mathbf{r}} = \mathbb{P}C_{X,\mathbf{r}}^{(d)}(V^\bullet)$, $\mathbf{r} \in R_{>0}^b$, are smooth and form a divisor with normal crossings in $\overline{\mathbb{P}C}_X(V^\bullet)$, which we denote $\partial\overline{\mathbb{P}C}_X(V^\bullet)$,*
- (c) *The complement $\overline{C}_X(V^\bullet) - \partial\overline{C}_X(V^\bullet)$ is identified with the disjoint union of the strata $C_{X,\mathbf{r}}^\circ(V^\bullet)$ for $\mathbf{r} \in R^{\max}$ (i.e., with the open strata in the irreducible components of $C_X(V^\bullet)$). Similarly, $\overline{\mathbb{P}C}_X(V^\bullet) - \partial\overline{\mathbb{P}C}_X(V^\bullet)$ is identified with the union of the strata $\mathbb{P}C_{X,\mathbf{r}}^\circ(V^\bullet)$ for $\mathbf{r} \in R^{\max}$.*

In fact, our construction provides, along the way, a wonderful compactification of any stratum in any variety of complexes.

- Theorem 6.11.** (a) *The projection $C_{X,\mathbf{r}}^{(\mathbf{r}|)}(V^\bullet) \rightarrow C_{X,\mathbf{r}}(V^\bullet)$ is birational and biregular over the open stratum $C_{X,\mathbf{r}}^\circ(V^\bullet)$. The subvarieties $C_{X,\mathbf{r}}^{(\mathbf{r}|)}(V^\bullet) \cap C_{X,\mathbf{r}'}^{(\mathbf{r}|)}(V^\bullet)$, $\mathbf{r}' < \mathbf{r}$, are smooth and form a divisor with normal crossings in $C_{X,\mathbf{r}}^{(\mathbf{r}|)}(V^\bullet)$.*
- (b) *Similarly, the projection $\mathbb{P}C_{X,\mathbf{r}}^{(\mathbf{r}|)}(V^\bullet) \rightarrow \mathbb{P}C_{X,\mathbf{r}}(V^\bullet)$ is birational and biregular over the open stratum $\mathbb{P}C_{X,\mathbf{r}}^\circ(V^\bullet)$. The subvarieties $\mathbb{P}C_{X,\mathbf{r}}^{(\mathbf{r}|)}(V^\bullet) \cap \mathbb{P}C_{X,\mathbf{r}'}^{(\mathbf{r}|)}(V^\bullet)$, $0 < \mathbf{r}' < \mathbf{r}$, are smooth and form a divisor with normal crossings in $\mathbb{P}C_{X,\mathbf{r}}^{(\mathbf{r}|)}(V^\bullet)$.*

Our strategy for proving Theorems 6.8 - 6.11 consists in reducing the analysis of each blowup, by means of étale local charts, to the simplest case: the blowup of the zero section in the relative affine variety of complexes. This strategy agrees with the general approach to conical stratifications of singular varieties, sketched in §3.3 of [MP]. Our analysis also provides additional information that will be used in §7 to identify \mathbf{k} -points of the blowup with spectral sequences.

We start by analyzing this simplest case.

D. Inductive step: structure of the first blowup. Consider $C_X^{(1)}(V^\bullet) = \text{Bl}_X C_X(V^\bullet)$, where $X = C_{X,0}(V^\bullet)$ is the zero section. Since $C_X(V^\bullet) \rightarrow X$ is conic over X , we have the projection $q : C_X^{(1)}(V^\bullet) \rightarrow \mathbb{P}C_X(V^\bullet)$ realizing $C_X^{(1)}(V^\bullet)$ as the total space of the relative line bundle $\mathcal{O}(-1)$. Therefore the strict transforms of the varieties $C_{X,\mathbf{r}}(V^\bullet)$ (i.e., of the closures of strata) in

$C_X^{(1)}(V^\bullet)$ are:

$$C_{X,\mathbf{r}}^{(1)}(V^\bullet) = q^{-1}(\mathbb{P}C_{X,\mathbf{r}}(V^\bullet)), \quad |\mathbf{r}| \geq 1.$$

The lowest closures of the strata (coinciding with the corresponding strata) in $\mathbb{P}C_X(V^\bullet)$ are the $\mathbb{P}C_{X,\mathbf{r}}(V^\bullet)$ for $|\mathbf{r}| = 1$, i.e., for $\mathbf{r} = e_i$ for some i . Being the strata, they are smooth and disjoint and carry vector bundles $H_{\mathbb{P}C_{X,\mathbf{r}}(V^\bullet)}^\bullet$, see §4. Let

$$H_{\mathbf{r}}^\bullet = q^*(H_{\mathbb{P}C_{X,\mathbf{r}}(V^\bullet)}^\bullet), \quad |\mathbf{r}| = 1.$$

This is a vector bundle on $C_{X,\mathbf{r}}^{(1)}(V^\bullet)$. Proposition 5.1 implies, now, by pullback:

Proposition 6.12. *Let $|\mathbf{r}| = 1$. Then:*

(a) *Let c be any \mathbf{k} -point of $C_{X,\mathbf{r}}^{(1)}(V^\bullet)$. There exists an isomorphism of an étale neighborhood of c in $C_{X,\mathbf{r}}^{(1)}(V^\bullet)$ with an étale neighborhood of c in the relative variety of complexes $C_{C_{X,\mathbf{r}}^{(1)}(V^\bullet)}(H_{\mathbf{r}}^\bullet)$. Here, in the second variety, c is understood as lying in the zero section.*

(b) *Further, the isomorphism in (a) can be chosen so that it takes, for any $\mathbf{r}' \geq \mathbf{r}$, an étale neighborhood of c in $C_{X,\mathbf{r}'}^{(1)}(V^\bullet)$ to an étale neighborhood of c in $C_{C_{X,\mathbf{r}}^{(1)}(V^\bullet), \mathbf{r}' - \mathbf{r}}(H_{\mathbf{r}}^\bullet)$. \square*

E. Proof of Theorem 6.8 and a “disjointness lemma”. We prove, by induction in l , the compound statement consisting of Theorem 6.8 and the following claim.

Proposition 6.13. *Let $l = 0, \dots, d$. Let $|\mathbf{r}| = l$ and c be any \mathbf{k} -point of $C_{X,\mathbf{r}}^{(l)}(V^\bullet)$ (resp. of $\mathbb{P}C_{X,\mathbf{r}}^{(l)}(V^\bullet)$).*

(a) *There exist:*

- *A Zariski open neighborhood U of c in $C_{X,\mathbf{r}}^{(l)}(V^\bullet)$ (resp. in $\mathbb{P}C_{X,\mathbf{r}}^{(l)}(V^\bullet)$).*
- *A graded vector bundle H^\bullet on U .*
- *An isomorphism $\Xi = \Xi_{\mathbf{r},l}$ of an étale neighborhood of c in $C_X^{(l)}(V^\bullet)$ (resp. in $\mathbb{P}C_X^{(l)}(V^\bullet)$) with an étale neighborhood of c in $C_U(H^\bullet)$.*

(b) *Further, for any $\mathbf{r}' \geq \mathbf{r}$, Ξ restricts to an isomorphism of an étale neighborhood of c in $C_{X,\mathbf{r}'}^{(l)}(V^\bullet)$ with an étale neighborhood of c in $C_{U, \mathbf{r}' - \mathbf{r}}(H^\bullet)$.*

We assume the statements proven for a given value of l (as well as for all the previous values). In particular, we define $C_X^{(l+1)}(V^\bullet)$ and $\mathbb{P}C_X^{(l+1)}(V^\bullet)$ by the formulas in Theorem 6.8(b). After this, we prove Theorem 6.8 for $l + 1$.

The statement of Theorem 6.8(a) for $l + 1$ reads as follows. *For different \mathbf{r} with $|\mathbf{r}| = l + 1$, the subvarieties $C_{X,\mathbf{r}}^{(l+1)}(V^\bullet)$, resp. $\mathbb{P}C_{X,\mathbf{r}}^{(l+1)}(V^\bullet)$, are smooth and disjoint.*

To prove this, we note the following. By Proposition 6.13(b) for l , we know that for any \mathbf{r}' with $|\mathbf{r}'| = l$, the variety $C_X^{(l)}(V^\bullet)$ (resp. $\mathbb{P}C_X^{(l)}(V^\bullet)$) behaves near $C_{X,\mathbf{r}'}^{(l)}(V^\bullet)$ (resp. near $\mathbb{P}C_{X,\mathbf{r}'}^{(l)}(V^\bullet)$), like the variety $C_U(H^\bullet)$ behaves near its zero section U (étale local identification). So $\text{Bl}_{C_{X,\mathbf{r}'}^{(l)}(V^\bullet)}C_X^{(l)}(V^\bullet)$ (resp. $\text{Bl}_{\mathbb{P}C_{X,\mathbf{r}'}^{(l)}(V^\bullet)}\mathbb{P}C_X^{(l)}(V^\bullet)$) together with its closures of the strata, will be étale locally identified with $\text{Bl}_U(C_U(H^\bullet))$ together with its closures of the strata. This latter blowup was studied in Proposition 6.12. In particular, as pointed out in the discussion just above that proposition, its lowest closures of the strata, $C_{U,\mathbf{r}''}^{(1)}(H^\bullet)$, $|\mathbf{r}''| = 1$, are smooth and disjoint. But Proposition 6.13(b) (for l) implies that these lowest closures of the strata are étale locally identified with the strict transforms of the $C_{X,\mathbf{r}}^{(l)}(V^\bullet)$, $|\mathbf{r}| = l + 1$ in $C_X^{(l+1)}(V^\bullet)$, i.e. with $C_{X,\mathbf{r}}^{(l+1)}(V^\bullet)$. This proves part (a) of Theorem 6.8 for $l + 1$.

Part (b) of Theorem 6.8 for $l + 1$ now constitutes the definition of $C_X^{(l+1)}(V^\bullet)$ and $\mathbb{P}C_X^{(l+1)}(V^\bullet)$.

We now prove Proposition 6.13 for $l + 1$. For this we just need to combine two charts:

- (1) The étale chart given by Proposition 6.12 for $\text{Bl}_U(C_U(H^\bullet))$
- (2) The (source- and target-wise) blowup of the already constructed étale chart $\Xi_{\mathbf{r}',t}$, $|\mathbf{r}'| = l$, from Proposition 6.13 for l .

This concludes the inductive proof of Theorem 6.8 and Proposition 6.13. \square

Let us conclude this part with the following “disjointness lemma”, to be used later.

Lemma 6.14. *Let $\mathbf{r}, \mathbf{s} \in R$. Suppose that $\mathbf{r} \not\leq \mathbf{s}$ and $\mathbf{s} \not\leq \mathbf{r}$. Let $\mathbf{r}' = \min(\mathbf{r}, \mathbf{s})$ (see Proposition 2.5) and $l' = |\mathbf{r}'|$. Then the varieties*

$$C_{X,\mathbf{r}}^{(l'+1)}(V^\bullet), C_{X,\mathbf{s}}^{(l'+1)}(V^\bullet) \subset C_X^{(l'+1)}(V^\bullet)$$

are disjoint.

Proof: By Proposition 4.3(d), $C_{X,\mathbf{r}'}(V^\bullet) = C_{X,\mathbf{r}}(V^\bullet) \cap C_{X,\mathbf{s}}(V^\bullet)$ (scheme-theoretic intersection). Therefore the strict transforms of $C_{X,\mathbf{r}}(V^\bullet)$ and $C_{X,\mathbf{s}}(V^\bullet)$ in the blowup of $C_X(V^\bullet)$ along $C_{X,\mathbf{r}'}(V^\bullet)$ are disjoint by Proposition 6.1. Now, over the open stratum $C_{X,\mathbf{r}'}^\circ(V^\bullet)$, this blowup coincides with

$C_X^{(l'+1)}(V^\bullet)$ so we conclude that the image of the intersection $C_{X,r}^{(l'+1)}(V^\bullet) \cap C_{X,s}^{(l'+1)}(V^\bullet)$ in $C_X(V^\bullet)$ does not meet $C_{X,r'}^\circ(V^\bullet)$.

It remains to eliminate the possibility of a point $p \in C_{X,r'}^{(l'+1)}(V^\bullet)$ belonging to $C_{X,r}^{(l'+1)}(V^\bullet) \cap C_{X,s}^{(l'+1)}(V^\bullet)$ and projecting to a point in some smaller stratum S inside $C_{X,r'}(V^\bullet)$. Such a stratum has the form $S = C_{X,t}^\circ(V^\bullet)$ with $\mathbf{t} < \mathbf{r}'$.

Let q be the image of p in $C_X^{(|t|)}(V^\bullet)$. By Proposition 6.13, near q , the variety $C_X^{(|t|)}(V^\bullet)$ is étale locally identified with some $C_S(H^\bullet)$ so that $C_{X,r}^{(|t|)}(V^\bullet)$, resp. $C_{X,s}^{(|t|)}(V^\bullet)$, resp. $C_{X,r'}^{(|t|)}(V^\bullet)$ is identified with $C_{S,r-t}(H^\bullet)$, resp. $C_{S,s-t}(H^\bullet)$ resp. $C_{S,r'-t}(H^\bullet)$. Under this identification, the relevant part of $C_{X,r}^{(|t|+1)}(V^\bullet)$ is just the blowup of (the relevant part of) $C_{S,r-t}(H^\bullet)$ along $C_{S,r'-t}(H^\bullet)$, and similarly for $C_{X,s}^{(|t|+1)}(V^\bullet)$. Now observe that $\min(\mathbf{r} - \mathbf{t}, \mathbf{s} - \mathbf{t}) = \min(\mathbf{r}, \mathbf{s}) - \mathbf{t}$. So, as before, the center of the blowup is the scheme-theoretic intersection of two subvarieties and so their strict transforms in the blowup are disjoint. We have thus shown that $C_{X,r}^{(|t|+1)}(V^\bullet)$ and $C_{X,s}^{(|t|+1)}(V^\bullet)$ do not intersect over S . Since $|t| < l'$, their subsequent iterated dominant transforms $C_{X,r}^{(l'+1)}(V^\bullet)$ and $C_{X,s}^{(l'+1)}(V^\bullet)$, respectively, do not intersect over S as well. \square

F. Proof of Theorems 6.10 and 6.11. We start with some reductions.

First, we will treat only the blowups $C_X^{(l)}(V^\bullet)$ of the affine varieties of complexes. The treatment of the $\mathbb{P}C_X^{(l)}(V^\bullet)$ is completely parallel.

Second, note that Theorem 6.10 is a particular case of Theorem 6.11. Indeed, each $C_{X,r}^{(d)}(V^\bullet)$, $\mathbf{r} \in R^{\max}$, will first appear as $C_{X,r}^{(|r|)}(V^\bullet)$ and will not change in the subsequent blowups. So we concentrate on the proof of Theorem 6.11.

Let us write $C_r^{(l)} = C_{X,r}^{(l)}(V^\bullet)$. For any $\mathbf{r} \in R$ and any $l \leq |\mathbf{r}|$ put

$$(6.15) \quad W_r^{(l)} = C_r^{(l)} - \bigcup_{\mathbf{s} < \mathbf{r}, l \leq |\mathbf{s}|} (C_r^{(l)} \cap C_s^{(l)}).$$

This is an open subvariety in $C_r^{(l)}$. Put also

$$D_r^{(l)} = W_r^{(l)} \cap \bigcup_{\mathbf{s} < \mathbf{r}, l > |\mathbf{s}|} C_s^{(l)}.$$

When $l = 0$, we have that $W_r^{(0)} = C_r^\circ = C_{X,r}^\circ(V^\bullet)$ is the open stratum corresponding to \mathbf{r} in the variety of complexes, while $D_r^{(0)} = \emptyset$.

When $l = |\mathbf{r}|$, we have that $W_{\mathbf{r}}^{(|\mathbf{r}|)} = C_{\mathbf{r}}^{(|\mathbf{r}|)}$, and

$$D_{\mathbf{r}}^{(|\mathbf{r}|)} = C_{\mathbf{r}}^{(|\mathbf{r}|)} \cap \bigcup_{\mathbf{s} < \mathbf{r}} C_{\mathbf{s}}^{(|\mathbf{r}|)}$$

is the divisor which is claimed in the theorem to be a divisor with normal crossings. So it suffices to prove the following more general statement. In this statement and its proof we will use the following terminology. A pair (Z, D) will be called *wonderful*, if Z is a smooth variety and D is a divisor with normal crossings in Z .

Proposition 6.16. *For any $\mathbf{r} \in R$ and any $l \leq |\mathbf{r}|$, the pair $(W_{\mathbf{r}}^{(l)}, D_{\mathbf{r}}^{(l)})$ is wonderful.*

Proof: We proceed by induction in l . The case $l = 0$ is clear from the above. Suppose the statement is proved for a given value of l , and suppose that $(l + 1) \leq |\mathbf{r}|$, so that the next statement is a part of the proposition. Look at the blowup map $p : C^{(l+1)} \rightarrow C^{(l)}$ with the smooth center $\coprod_{|\mathbf{s}|=l} C_{\mathbf{s}}^{(l)}$, as in Theorem 6.8(b).

Lemma 6.17. *p is biregular over $W_{\mathbf{r}}^{(l)} \subset C^{(l)}$, i.e., $W_{\mathbf{r}}^{(l)}$ does not meet the center of the blowup.*

Proof of the lemma: Indeed, each $C_{\mathbf{s}}^{(l)}$, $|\mathbf{s}| = l$, will either not meet $C_{\mathbf{r}}^{(l)}$ and hence $W_{\mathbf{r}}^{(l)}$ (this will happen if $\mathbf{s} \not\leq \mathbf{r}$ by Lemma 6.14), or will meet $C_{\mathbf{r}}^{(l)}$ but will be removed in forming $W_{\mathbf{r}}^{(l)}$ (this will happen if $\mathbf{s} < \mathbf{r}$). \square

Denote by E the exceptional divisor of p (the preimage of the center of the blowup). The lemma means that any “new” point $w \in W_{\mathbf{r}}^{(l+1)}$ (i.e., a point not lifted by a local biregular map from a point in $W_{\mathbf{r}}^{(l)}$), lies in E . So it is enough to prove that $(W_{\mathbf{r}}^{(l+1)}, D_{\mathbf{r}}^{(l+1)})$ is wonderful only near such new points w , belonging to E .

So we choose such w and denote $c = p(w)$. Then $c \in C_{\mathbf{s}}^{(l)}$ for some \mathbf{s} with $|\mathbf{s}| = l$. Since $w \in W_{\mathbf{r}}^{(l+1)}$, we have $\mathbf{s} < \mathbf{r}$. We now apply Proposition 6.13 to get an open neighborhood U of c in $C_{\mathbf{s}}^{(l)}$, a graded vector bundle H^{\bullet} on U and an identification Ξ of an étale neighborhood of c in $C_{\mathbf{r}}^{(l)}$ with an étale neighborhood of c in $C_{U, \mathbf{r}-\mathbf{s}}(H^{\bullet})$. Applying the blowup along the intersection of U with the étale neighborhoods in the source and target of Ξ , we identify an étale neighborhood of w in $C_{\mathbf{r}}^{(l+1)}$ with the étale neighborhood of a point w' in $\text{Bl}_U(C_{U, \mathbf{r}-\mathbf{s}}(H^{\bullet}))$, which is the total space of the line bundle $\mathcal{O}(-1) \rightarrow \mathbb{P}C_{U, \mathbf{r}-\mathbf{s}}(H^{\bullet})$.

Lemma 6.18. (a) *The point w' lies on the zero section of the bundle $\mathcal{O}(-1)$.*

(b) *Identifying this zero section with $\mathbb{P}C_{U, \mathbf{r}-s}(H^\bullet)$, we have that w' lies in the open stratum $\mathbb{P}C_{U, \mathbf{r}-s}^\circ(H^\bullet)$.*

Proof of the lemma: (a) follows because w lies in the exceptional divisor of p .

(b) Applying (6.15) in our case, we can write that

$$w \in W_{\mathbf{r}}^{(l+1)} = C_{\mathbf{r}}^{(l+1)} - \bigcup_{\substack{\mathbf{r}' < \mathbf{r} \\ l+1 \leq |\mathbf{r}'|}} (C_{\mathbf{r}}^{(l+1)} \cap C_{\mathbf{r}'}^{(l+1)}) \subset C_{\mathbf{r}}^{(l+1)}.$$

Since $w \in E$, it represents a point in the projectivization of the normal cone

$$\mathrm{NC}_{C_s^{(l)}} W_{\mathbf{r}}^{(l)} \subset \mathrm{NC}_{C_s^{(l)}} C_{\mathbf{r}}^{(l)}.$$

The variety $\mathrm{NC}_{C_s^{(l)}} C_{\mathbf{r}}^{(l)}$ is identified under Ξ (étale locally around $c \in C_{\mathbf{r}}^{(l)}$) with $C_{\mathbf{r}-s}(H^\bullet)$. Under this identification, the parts removed in forming $W_{\mathbf{r}}^{(l+1)}$, namely $C_{\mathbf{r}}^{(l+1)} \cap C_{\mathbf{r}'}^{(l+1)}$ match the subvarieties $\mathbb{P}C_{\mathbf{r}'-s}(H^\bullet)$. More precisely, the intersection $C_{\mathbf{r}}^{(l+1)} \cap C_{\mathbf{r}'}^{(l+1)} \cap E$, is identified with $\mathbb{P}C_{\mathbf{r}'-s}(H^\bullet)$. So if the statement of part (b) is not true, then w would lie in one of the removed parts. \square

Now Proposition 6.16 follows from the next obvious statement.

Lemma 6.19. *Let (Z, D) be a wonderful pair, and $q : L \rightarrow Z$ be a line bundle. Then $(L, q^{-1}(D) \cup Z)$ (with $Z \subset L$ being the zero section), is a wonderful pair. \square*

Therefore, Theorems 6.10 and 6.11 are proved.

7. Complete complexes and spectral sequences

A. Single-graded spectral sequences. By a *spectral sequence* of \mathbf{k} -vector spaces we mean a sequence of complexes $(E_\nu^\bullet, D^\nu) \in \mathrm{Com}_{\mathbf{k}}$, $\nu = 0, \dots, k+1$ such that $E_{\nu+1}^\bullet = H_{D^\nu}^\bullet(E_\nu^\bullet)$ for each $\nu < k+1$. Here $k+1$ can be either a finite number or ∞ . If $k+1$ is finite, then the differential D^{k+1} is considered to be zero (so that there is no additional E_{k+2}^\bullet to speak of).

Definition 7.1. *A spectral sequence (E_ν^\bullet, D^ν) , $\nu = 0, \dots, k+1$, will be called reduced, if:*

- (1) E_0^\bullet (and therefore each E_ν^\bullet) is finite-dimensional, i.e., the total dimension $\dim(E_0^\bullet) < \infty$.

- (2) Each D^ν , $\nu = 1, \dots, k$, is not entirely 0, i.e., at least one component $D_i^\nu : E_\nu^i \rightarrow E_\nu^{i+1}$ is nonzero.
- (3) The graded vector space E_{k+1}^\bullet is sparse, see §6B.

We say that (E_ν^\bullet, D^ν) is strongly reduced if, in addition, $D^0 \neq 0$.

For a reduced spectral sequence we have $\dim(E_{\nu+1}^\bullet) < \dim(E_\nu^\bullet)$, so the length of such a sequence is bounded by $\dim(E_0^\bullet)$.

Definition 7.2. Let V^\bullet be a finite-dimensional graded \mathbf{k} -vector space. A complete complex (affine version) on V^\bullet is an equivalence class of reduced spectral sequences (E_ν^\bullet, D^ν) with $E_0^\bullet = V^\bullet$, where each differential $D^\nu, \nu \geq 1$ is considered modulo scaling (the same scalar for all components D_i^ν).

A complete complex (projective version) on V^\bullet is an equivalence class of strongly reduced spectral sequences (E_ν^\bullet, D^ν) with $E_0^\bullet = V^\bullet$, where each differential $D^\nu, \nu \geq 0$ is considered modulo scaling (the same scalar for all components D_i^ν).

We denote by $\text{SS}(V^\bullet)$ and $\text{PSS}(V^\bullet)$ the sets of complete complexes on V^\bullet in the affine and projective version respectively.

B. \mathbf{k} -points of $\overline{C}(V^\bullet)$ and $\overline{\mathbb{P}C}(V^\bullet)$ as spectral sequences. Let $C^\circ(V^\bullet)$ be the generic part of $C(V^\bullet)$, i.e., the union of the maximal strata, see (6.7). We have an embedding $C^\circ(V^\bullet)(\mathbf{k}) \subset \text{SS}(V^\bullet)$: a differential D making V^\bullet into a complex, is identified with a spectral sequence of length 1, that is, consisting only of $E_0^\bullet = V^\bullet$ and $E_1^\bullet = H_D^\bullet(E_0^\bullet)$. The fact that D lies in a maximal stratum means, by Proposition 6.6, means that E_1^\bullet is sparse, so condition (3) of Definition 7.1 is satisfied. We have a similar embedding $\mathbb{P}C(V^\bullet)(\mathbf{k}) \subset \text{PSS}(V^\bullet)$.

Theorem 7.3. We have identifications

$$\overline{C}(V^\bullet)(\mathbf{k}) \simeq \text{SS}(V^\bullet), \quad \overline{\mathbb{P}C}(V^\bullet)(\mathbf{k}) \simeq \text{PSS}(V^\bullet),$$

extending the above embeddings.

C. Stratification of complete complexes. Before proving the theorem, we study the natural stratifications of $\overline{C}(V^\bullet)$ and $\overline{\mathbb{P}C}(V^\bullet)$ given by the generic parts of all possible intersections of the boundary divisors in each of these wonderful compactifications. It is convenient to work in the relative situation of the relative varieties of complete complexes $\overline{C}_X(V^\bullet)$ and $\overline{\mathbb{P}C}_X(V^\bullet)$ corresponding to a graded vector bundle V^\bullet on a smooth variety X . We recall the divisors $\Delta_{\mathbf{r}} \subset \overline{C}_X(V^\bullet)$, $\mathbf{r} \in R^b$ and $\mathbb{P}\Delta_{\mathbf{r}} \subset \overline{\mathbb{P}C}_X(V^\bullet)$, $\mathbf{r} \in R_{>0}^b$ from Theorem

6.10. To emphasize their dependence on X, V^\bullet we will write $\Delta_{\mathbf{r}}^{\overline{C}_X(V^\bullet)}$ and $\mathbb{P}\Delta_{\mathbf{r}}^{\overline{\mathbb{P}C}_X(V^\bullet)}$ respectively, if needed.

Proposition 7.4. *Let $\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(k)} \in R^b$ (resp. $\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(k)} \in R_{>0}^b$) be distinct. The intersection $\Delta_{\mathbf{r}^{(1)}} \cap \Delta_{\mathbf{r}^{(2)}} \cap \dots \cap \Delta_{\mathbf{r}^{(k)}} \subset \overline{C}_X(V^\bullet)$ (resp. $\mathbb{P}\Delta_{\mathbf{r}^{(1)}} \cap \mathbb{P}\Delta_{\mathbf{r}^{(2)}} \cap \dots \cap \mathbb{P}\Delta_{\mathbf{r}^{(k)}} \subset \overline{\mathbb{P}C}_X(V^\bullet)$) is nonempty if and only if, after a permutation of the $\mathbf{r}^{(i)}$, we have $\mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k)}$.*

Proof: We only prove the statement about the variety of complete complexes; the projective case follows by identical arguments.

“If”: We proceed by induction on k , the case $k = 1$ being trivial. So we assume the statement proved for all X, V^\bullet and $\mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k-1)}$.

Suppose now some X, V^\bullet and $\mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k)}$ are given. We consider the stratum $S = C_{X, \mathbf{r}^{(1)}}^\circ(V^\bullet)$. By Propositions 4.4 and 5.1, each point of S has an étale neighborhood $U \rightarrow C_X(V^\bullet)$ identified with a part (étale) of $C_S(H^\bullet)$ where H^\bullet is the vector bundle of the cohomology on S . Under this identification, each subvariety $C_{S, \mathbf{s}}(H^\bullet)$ corresponds to $C_{X, \mathbf{r}^{(1)} + \mathbf{s}}(V^\bullet)$.

Accordingly, the preimage of U in $\overline{C}_X(V^\bullet)$ is identified with a part of $\overline{C}_S(H^\bullet)$ in such a way that the divisors $\Delta_{\mathbf{s}}^{\overline{C}_S(H^\bullet)}$ in $\overline{C}_S(H^\bullet)$ correspond to the divisors $\Delta_{\mathbf{r}^{(1)} + \mathbf{s}}^{\overline{C}_X(V^\bullet)}$ in $\overline{C}_X(V^\bullet)$. In particular, $\Delta_{\mathbf{r}^{(1)}}^{\overline{C}_X(V^\bullet)}$ corresponds to the dominant transform of the zero section of $C_S(H^\bullet)$ which is nothing but $\overline{\mathbb{P}C}_S(H^\bullet)$, the projective variety of complete complexes.

Since, by the inductive assumption, the intersection $\Delta_{\mathbf{r}^{(2)} - \mathbf{r}^{(1)}} \cap \dots \cap \Delta_{\mathbf{r}^{(k)} - \mathbf{r}^{(1)}}$ in $\overline{C}_S(H^\bullet)$ is nonempty, the intersection of their images in $\overline{\mathbb{P}C}_S(H^\bullet)$ is also non-empty. But by the above argument, this intersection in $\overline{\mathbb{P}C}_S(H^\bullet)$ is étale locally identified with a part of the intersection $\Delta_{\mathbf{r}^{(1)}} \cap \Delta_{\mathbf{r}^{(2)}} \cap \dots \cap \Delta_{\mathbf{r}^{(k)}}$ in $\overline{C}_X(V^\bullet)$, which is therefore nonempty too.

“Only if”: The statement reduces to the following: if $\Delta_{\mathbf{r}} \cap \Delta_{\mathbf{s}} \neq \emptyset$, then $\mathbf{r} < \mathbf{s}$ or $\mathbf{s} < \mathbf{r}$. To prove this, suppose that $\mathbf{r} \not< \mathbf{s}$ and $\mathbf{s} \not< \mathbf{r}$. Let $\mathbf{r}' = \min(\mathbf{r}, \mathbf{s})$, see Proposition 2.5. and $l = |\mathbf{r}'|$. Then $\mathbf{r}' < \mathbf{r}, \mathbf{s}$. Our statement now follows from Lemma 6.14. \square

We now make precise the natural stratification of the varieties of complete complexes associated to their boundary. To this end, we give the following Definition:

Definition 7.5. *Let $T \subset R$ be any subset of the form $T = \{\mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k)}\}$, where $\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(k)} \in R^b$. We allow the case $k = 0$, i.e., $T = \emptyset$.*

1. The stratum of $\overline{C}_X(V^\bullet)$ associated to T is defined to be the locally closed subvariety

$$\Delta_T^\circ(X, V^\bullet) = \Delta_{\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(k)}}(X, V^\bullet) := \bigcap_{\mathbf{r} \in T} \Delta_{\mathbf{r}} - \bigcup_{s \notin T} \Delta_s \subset \overline{C}_X(V^\bullet).$$

2. Let $\mathbf{r}^{(1)} \neq \mathbf{0}$. The stratum of $\overline{\mathbb{P}C}_X(V^\bullet)$ associated to T is defined to be the locally closed subvariety

$$\begin{aligned} \mathbb{P}\Delta_T^\circ(X, V^\bullet) &= \mathbb{P}\Delta_{\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(k)}}(X, V^\bullet) \\ &:= \bigcap_{\mathbf{r} \in T} \mathbb{P}\Delta_{\mathbf{r}} - \bigcup_{s \notin T} \mathbb{P}\Delta_s \subset \overline{\mathbb{P}C}_X(V^\bullet). \end{aligned}$$

Remark 7.6. If $T = \emptyset$, the stratum of $\overline{C}_X(V^\bullet)$ (resp. $\overline{\mathbb{P}C}_X(V^\bullet)$) associated to T is the generic part $C_X^\circ(V^\bullet)$ (resp. $\mathbb{P}C_X^\circ(V^\bullet)$) of $\overline{C}_X(V^\bullet)$ (resp. $\overline{\mathbb{P}C}_X(V^\bullet)$) that is, the union of $C_{X, \mathbf{r}}^\circ(V^\bullet)$ (resp. $\mathbb{P}C_{X, \mathbf{r}}^\circ(V^\bullet)$) for all $\mathbf{r} \in R^{\max}$.

If $X = \text{Spec}(\mathbf{k})$, i.e., V^\bullet is just a graded \mathbf{k} -vector space, we abbreviate the notation for the above varieties to $\Delta_T^\circ(V^\bullet)$, resp. $\mathbb{P}\Delta_T^\circ(V^\bullet)$.

D. Proof of Theorem 7.3. Let V^\bullet be a graded \mathbf{k} -vector space, as in the theorem. We will identify each stratum in $\overline{C}(V^\bullet)$, resp. $\overline{\mathbb{P}C}(V^\bullet)$, with the set of spectral sequences with fixed numerical invariants. Let $T = \{\mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k)}\}$ be as above. Define the set $SS_T^\circ(V^\bullet) \subset SS(V^\bullet)$ to consist of equivalence classes of spectral sequences (E_ν^\bullet, D^ν) (see discussion after Definition 7.1) such that:

- (0) $E_0^\bullet = V^\bullet$ and $D^0 \in C_{\mathbf{r}^{(1)}}^\circ(E_0^\bullet)$;
- (1) $D^1 \in C_{\mathbf{r}^{(2)} - \mathbf{r}^{(1)}}^\circ(E_1)$, where $E_1^\bullet := H_{D^0}^\bullet(E_0^\bullet)$,
- (2) $D^2 \in C_{\mathbf{r}^{(3)} - \mathbf{r}^{(2)}}^\circ(E_2)$, and so on.

If $\mathbf{r}^{(1)} \neq \mathbf{0}$, we denote by $\mathbb{P}SS_T^\circ(V^\bullet)$ the subset of $\mathbb{P}SS(V^\bullet)$ corresponding to $SS_T^\circ(V^\bullet)$. It is clear that we have disjoint decompositions

$$SS(V^\bullet) = \bigsqcup_{T = \{\mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k)}\}} SS_T^\circ(V^\bullet), \quad \mathbb{P}SS(V^\bullet) = \bigsqcup_{T = \{0 < \mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k)}\}} \mathbb{P}SS_T^\circ(V^\bullet).$$

Theorem 7.3 is a consequence of the following refined statement.

Proposition 7.7. For any $T = \{\mathbf{r}^{(1)} < \dots < \mathbf{r}^{(k)}\}$ as above we have identifications

$$SS_T^\circ(V^\bullet) \simeq \Delta_T^\circ(V^\bullet)(\mathbf{k}), \quad \mathbb{P}SS_T^\circ(V^\bullet) \simeq \mathbb{P}\Delta_T^\circ(V^\bullet)(\mathbf{k}).$$

Proof of the proposition: For the proof, we work with relative complete varieties of complexes and analyze their strata, introduced in part C, in an inductive fashion.

Lemma 7.8. *Let X be a smooth variety over \mathbf{k} and V^\bullet a graded vector bundle on X . Then we have an isomorphism*

$$\Delta_{\mathbf{r}(1), \dots, \mathbf{r}(k)}^\circ(X, V^\bullet) \simeq \mathbb{P}\Delta_{\mathbf{r}(2)-\mathbf{r}(1), \mathbf{r}(3)-\mathbf{r}(1), \dots, \mathbf{r}(k)-\mathbf{r}(1)}^\circ(C_{X, \mathbf{r}(1)}^\circ(V^\bullet), H^\bullet),$$

where H^\bullet is the vector bundle of cohomology on the stratum $C_{X, \mathbf{r}(1)}^\circ(V^\bullet)$. We further have an isomorphism

$$\mathbb{P}\Delta_{\mathbf{r}(1), \dots, \mathbf{r}(k)}^\circ(X, V^\bullet) \simeq \mathbb{P}\Delta_{\mathbf{r}(2)-\mathbf{r}(1), \mathbf{r}(3)-\mathbf{r}(1), \dots, \mathbf{r}(k)-\mathbf{r}(1)}^\circ(\mathbb{P}C_{X, \mathbf{r}(1)}^\circ(V^\bullet), H^\bullet).$$

Knowing the lemma, the proof of Proposition 7.7 (and thus of Theorem 7.3) for strata in $\overline{C}_X(V^\bullet)$ is finished as follows. We construct inductively the following varieties X_ν together with graded vector bundles E_ν^\bullet on them:

- (0) $X_0 = \text{Spec}(\mathbf{k})$, and $E_0^\bullet = V^\bullet$.
- (1) X_1 is the stratum $\mathbb{P}C_{\mathbf{r}(1), X_0}^\circ(E_0^\bullet)$, and E_1^\bullet is the bundle of cohomology on this stratum.
- (2) X_2 is the stratum $\mathbb{P}C_{\mathbf{r}(2)-\mathbf{r}(1), X_1}^\circ(E_1^\bullet)$, and E_2^\bullet is the bundle of cohomology on this stratum,
-
- (k) X_k is the stratum $\mathbb{P}C_{\mathbf{r}(k)-\mathbf{r}(k-1), X_{k-1}}^\circ(E_{k-1}^\bullet)$, and E_k^\bullet is the bundle of cohomology on this stratum.

Lemma 7.8 implies, by induction, the following:

Corollary 7.9. *The variety X_k is identified with $\Delta_T^\circ(V^\bullet)$. □*

Proposition 7.7 for strata in $\overline{C}_X(V^\bullet)$ now follows because points of X_k are manifestly identified with equivalence classes of spectral sequences, as in Definition 7.1. The case of strata in $\overline{\mathbb{P}C}_X(V^\bullet)$ is treated similarly.

E. Proof of Lemma 7.8. By definition, $\Delta_{\mathbf{r}(1)} = \Delta_{\mathbf{r}(1)}(X, V^\bullet)$ is the iterated dominant transform of the closed subvariety $C_{X, \mathbf{r}(1)}(V^\bullet)$ in the first tower of blowups in Theorem 6.8. It follows that

$$\Delta_{\mathbf{r}(1)} - \bigcup_{\mathbf{s} < \mathbf{r}(1)} \Delta_{\mathbf{s}} = \widetilde{C}_{X, \mathbf{r}(1)}^\circ(V^\bullet)$$

is the iterated dominant transform of the open part (stratum) $C_{X, \mathbf{r}(1)}^\circ(V^\bullet)$. Let us analyze this iterated transform and the tower of blowups in more detail.

The first blowup nontrivial over $C_{X,\mathbf{r}^{(1)}}^\circ(V^\bullet)$, will appear at the stage $l = |\mathbf{r}^{(1)}|$. It will be the blowup along the dominant transform of $C_{X,\mathbf{r}^{(1)}}(V^\bullet)$ which, on our part, reduces to $C_{X,\mathbf{r}^{(1)}}^\circ(V^\bullet)$ itself. The corresponding dominant transform is, therefore, the total inverse image, i.e., the projectivization of the normal cone to $C_{X,\mathbf{r}^{(1)}}^\circ(V^\bullet)$ in $C_X(V^\bullet)$. This projectivization is the projective variety of complexes $\mathbb{P}C_{C_{X,\mathbf{r}^{(1)}}^\circ(V^\bullet)}(H^\bullet)$.

If we continue the construction of $\overline{C}_X(V^\bullet)$ in the tower of blowups of Theorem 6.8, then subsequent blowups along dominant transforms of the $C_{X,\mathbf{t}}(V^\bullet)$, $\mathbf{t} > \mathbf{r}^{(1)}$ will induce blowups of $\mathbb{P}C_{C_{X,\mathbf{r}^{(1)}}^\circ(V^\bullet)}(H^\bullet)$ along (dominant transforms of) the $\mathbb{P}C_{C_{X,\mathbf{r}^{(1)}}^\circ(V^\bullet),\mathbf{t}-\mathbf{r}^{(1)}}(H^\bullet)$. This will produce $\overline{\mathbb{P}C}_{C_{X,\mathbf{r}^{(1)}}^\circ(V^\bullet)}(H^\bullet)$, the relative projective variety of complete complexes. In other words, we have established an identification

$$\Delta_{\mathbf{r}^{(1)}} - \bigcup_{\mathbf{s} < \mathbf{r}^{(1)}} \Delta_{\mathbf{s}} \simeq \overline{\mathbb{P}C}_{C_{\mathbf{r}^{(1)},X}^\circ(V^\bullet)}(H^\bullet).$$

Under this identification the intersection of the LHS with each divisor $\Delta_{\mathbf{t}}^{\overline{C}_X(V^\bullet)}$, $\mathbf{t} > \mathbf{r}^{(1)}$, corresponds to the divisor $\mathbb{P}\Delta_{\mathbf{t}-\mathbf{r}^{(1)}}$ in $\overline{\mathbb{P}C}_{C_{\mathbf{r}^{(1)},X}^\circ(V^\bullet)}(H^\bullet)$. The lemma is immediate from this. \square

References

- [GM] S. I. GELFAND, Y. I. MANIN. *Methods of Homological Algebra*. Springer-Verlag (1993).
- [DGMP] C. DE CONCINI, M. GORESKY, R. MacPherson, C. Procesi. The geometry of quadrics and their degenerations. *Comm. Math. Helv.* **63** (1988) 337–413.
- [DS] C. DE CONCINI, E. STRICKLAND. On the variety of complexes. *Adv. Math.* **41** (1981), 57–77.
- [F] W. FULTON. *Intersection Theory*. Springer-Verlag (1984).
- [Go] N. GONCIULEA. Singular loci of varieties of complexes. II. *J. Algebra* **235** (2001), 547–558.
- [H] R. HARTSHORNE. *Algebraic Geometry*. Springer-Verlag (2006).
- [KP] M. KAPRANOV, S. PIMENOV. Derived varieties of complexes and Kostant’s theorem for $\mathfrak{gl}(m|n)$. arXiv:1504.00339.

- [Kau] I. KAUSZ. A modular compactification of the general linear group. *Documenta Math.* **5** (2000) 553–594.
- [Ke] G. R. KEMPF. Images of homogeneous vector bundles and varieties of complexes. *Bull. Amer. Math. Soc.* **81** (1975), 900–901.
- [L1] V. LAKSHMIBAI. Singular loci of varieties of complexes. *J. Pure Appl. Algebra* **152** (2000), 217–230.
- [L2] D. LAKSOV. The geometry of complete linear maps. *Ark. Math.* **26** (1988) 231–263.
- [L3] D. LAKSOV. Notes on the origin of complete collineations, in: Enumerative geometry and classical algebraic geometry (Nice, 1981), pp. 107–132. P. LE BARZ, Y. HERVIER Eds. Birkhäuser (1982).
- [L4] LI LI. Wonderful compactification of an arrangement of subvarieties. *Michigan Math. J.* **58** (2009) 535–563.
- [MP] R. MACPHERSON, C. PROCESI. Making conical compactifications wonderful. *Selecta Math.* **4** (1998) 125–139.
- [Se] J. G. SEMPLE. The variety whose points represent complete collineations of S_r on S'_r . *Rend. di Math. Univ. Rome* **10** (1951) 201–208.
- [Th] M. THADDEUS. Complete collineations revisited. *Math. Ann.* **315** (1999) 469–495.
- [Ty] J. A. TYRRELL. Complete quadrics and collineations in S_n . *Mathematica* **3** (1956) 69–79.
- [Va] I. VAINSENER. Complete collineations and blowing up determinantal ideals. *Math. Ann.* **267** (1984) 417–432.

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