Slopes of indecomposable F-isocrystals

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Abstract: We prove that for an indecomposable convergent or overconvergent F-isocrystal on a smooth irreducible variety over a perfect field of characteristic p, the gap between consecutive slopes at the generic point cannot exceed 1. (This may be thought of as a crystalline analogue of the following consequence of Griffiths transversality: for an indecomposable variation of complex Hodge structures, there cannot be a gap between non-zero Hodge numbers.) As an application, we deduce a refinement of a result of V. Lafforgue on the slopes of Frobenius of an ℓ-adic local system.

We also prove similar statements for G-local systems (crystalline and ℓ-adic ones), where G is a reductive group.

We translate our results on local systems into properties of the p-adic absolute values of the Hecke eigenvalues of a cuspidal automorphic representation of a reductive group over the adeles of a global field of characteristic $p > 0$.

Keywords: F-isocrystal, local system, slope, Newton polygon, Frobenius, hypergeometric sheaf.

1. Introduction and main results

Let k be a perfect field of characteristic $p > 0$. Let X be a smooth irreducible quasi-compact scheme over k. Let $|X|$ denote the set of closed points of X. Let $\eta \in X$ be the generic point.

1.1. The main theorem

1.1.1. Convergent and overconvergent F-isocrystals.   To varieties over k, one associates a family of Weil cohomology theories indexed by primes ℓ, consisting of ℓ-adic étale cohomology for $\ell \neq p$ and Berthelot's *rigid cohomology* [LeS] for $\ell = p$. For X as above, the category of $\mathbb{Q}_\ell$-local systems has not one but two p-adic analogues; these are the categories of *convergent F-isocrystals* and *overconvergent F-isocrystals*[1] on X, respectively denoted by $F\text{-Isoc}(X)$ and $F\text{-Isoc}^\dagger(X)$. Roughly speaking, convergent F-isocrystals are defined (locally) using the Raynaud generic fiber of a p-adic lift of X (e.g., for $X = \mathbb{A}_k^1$, take the closed unit disc over $\mathrm{Frac}\, W(k)$), whereas overconvergent

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[1]An overview of the theory of both types of F-isocrystals is given in [Ke6]. The precise definitions of $F\text{-Isoc}(X)$ and $F\text{-Isoc}^\dagger(X)$ can be found in [O, §2] and [Ber, §2.3], respectively. Let us note that in the word "F-isocrystal" the letter F stands for the Frobenius corresponding to $\mathbb{F}_p$ (i.e., raising to the power of p). Let us add that $F\text{-Isoc}(k) = F\text{-Isoc}(\mathrm{Spec}\, k) = F\text{-Isoc}^\dagger(\mathrm{Spec}\, k)$ is just the category of finite-dimensional vector spaces V over $\mathrm{Frac}\, W(k)$ equipped with a σ-linear isomorphism $F : V \xrightarrow{\sim} V$, where $\sigma \in \mathrm{Aut}\, W(k)$ is the unique automorphism such that $\sigma(x) \equiv x^p \bmod p$.

F-isocrystals are defined on some slightly larger region (e.g., a disc of radius greater than 1).

In particular, there is a canonical restriction functor $F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}(X)$. It is known to be *fully faithful* (the proof of this fact is not straightforward, see [Ke2, Theorem 1.1] or Theorem 2.2.2 herein); we thus view $F\text{-Isoc}^\dagger(X)$ as a full subcategory of $F\text{-Isoc}(X)$.

Remark 1.1.2. An object of $F\text{-Isoc}^\dagger(X)$ is indecomposable in $F\text{-Isoc}^\dagger(X)$ if and only if it is indecomposable in $F\text{-Isoc}(X)$. This is a particular case of the following lemma.

Lemma 1.1.3. *Let* $F : \mathcal{C} \to \mathcal{C}'$ *be a fully faithful functor between abelian categories. Then an object* $M \in \mathcal{C}$ *is indecomposable if and only if* $F(M)$ *is.*

Proof. Indecomposability of M (resp. $F(M)$) means that $\operatorname{End} M$ (resp. $\operatorname{End} F(M)$) has no non-trivial idempotents. On the other hand, $\operatorname{End} F(M) \simeq \operatorname{End} M$ by full faithfulness. $\square$

1.1.4. Slopes. Let $M \in F\text{-Isoc}(X)$ have rank n. Then for any $x \in X$ one has numbers $a_i^x(M) \in \mathbb{Q}$, $1 \le i \le n$, called the *slopes* of M at x (see [Ka1, §1.3]). We order them so that $a_i^x(M) \ge a_{i+1}^x(M)$. One can think of the collection of slopes at a fixed $x \in X$ as a dominant rational coweight of the group $GL(n)$.

Let us recall the definition of slopes. For any $x \in X$, let x_{perf} denote the spectrum of the perfection of the residue field of x. Then $F\text{-Isoc}(x_{\text{perf}})$ has a canonical $\mathbb{Q}$-grading. In particular, the pullback of $M \in F\text{-Isoc}(X)$ to x_{perf} is $\mathbb{Q}$-graded. Let d_r denote the rank of its component of degree $r \in \mathbb{Q}$. The numbers $a_i^x(M)$ are characterized by the following property: each $r \in \mathbb{Q}$ occurs among them d_r times.

Here is our main result.

Theorem 1.1.5. *Let* $\eta \in X$ *be the generic point. Let* $M \in F\text{-Isoc}(X)$ *be indecomposable and of rank* n. *Then* $a_i^\eta(M) - a_{i+1}^\eta(M) \le 1$ *for all* $i \in \{1, \dots, n-1\}$.

The proof will be given in §3. In fact, it will be shown that Theorem 1.1.5 easily follows from full faithfulness of the restriction functor $F\text{-Isoc}(X) \to F\text{-Isoc}(U)$, where $U \subset X$ is a dense open subset. The latter statement (Theorem 2.2.3) immediately follows from previously known results; however, the proofs of these results are difficult. (Hopefully, J. Kramer-Miller's theory of F-isocrystals with logarithmic decay [KM] will provide an easier proof of Theorem 1.1.5, which bypasses some of these difficult results.)

Remark 1.1.6. Theorem 1.1.5 may be viewed as an analogue for F-isocrystals of the following consequence of Griffiths transversality: for an indecomposable variation of complex Hodge structures, there cannot be a gap between non-zero Hodge numbers. The local version of this observation is an unpublished result from the second author's PhD thesis [Ke, §5].

The number $\sum_{i=1}^{n} a_i^x(M)$ is the slope of $\det M$ at x; it is well known that this number does not depend on $x \in X$. Set $A(M) := \frac{1}{n} \cdot \sum_{i=1}^{n} a_i^x(M)$.

Corollary 1.1.7. *In the situation of Theorem 1.1.5, for all $x \in X$ one has*

$$(1.1) \qquad \sum_{i=1}^{r} a_i^x(M) - rA(M) \le r(n-r)/2 \quad \text{for all } r \in \{1, \ldots, n-1\}.$$

Remark 1.1.8. The meaning of $r(n-r)/2$ is as follows: $r(n-r)/2 = \sum_{i=1}^{r} c_i$, where $c_1, \ldots, c_n$ are the numbers such that $\sum_{i=1}^{n} c_i = 0$ and $c_i - c_{i+1} = 1$ for $i \in \{1, \ldots, n-1\}$. In fact, $c_i = \frac{n+1}{2} - i$.

Proof of Corollary 1.1.7. The function $x \mapsto \sum_{i=1}^{r} a_i^x(M)$ is known to be lower semicontinuous (by semicontinuity of the Newton polygon, see [Ka1, Cor. 2.3.2]). So it suffices to check (1.1) for $x = \eta$.

Set $b_i := a_i^\eta(M) - A(M) - c_i$, where $c_1, \ldots, c_n$ are as in Remark 1.1.8. We have to check that $\sum_{i=1}^{r} b_i \le 0$. It is clear that $\sum_{i=1}^{n} b_i = 0$, and Theorem 1.1.5 tells us that $b_i \le b_{i+1}$. So $n \sum_{i=1}^{r} b_i = n \sum_{i=1}^{r} b_i - r \sum_{i=1}^{n} b_i = \sum_{i=1}^{r} \sum_{j=r+1}^{n} (b_i - b_j) \le 0.$ $\quad\square$

Remark 1.1.9. Let $\check{\omega}^x(M)$ denote the dominant rational coweight of $SL(n)$ corresponding to the numbers $a_i^x(M) - A(M)$, $1 \le i \le n$. Let $\check{\rho}$ denote the sum of the fundamental coweights of $SL(n)$. Corollary 1.1.7 says that $\check{\rho} - \check{\omega}^x(M)$ belongs to the "positive cone" (i.e., the convex cone generated by the simple coroots). Theorem 1.1.5 says that $\check{\rho} - \check{\omega}^\eta(M)$ belongs to the dominant cone (which is *strictly* contained in the positive cone if $n \ge 3$).

1.2. Counterexamples

One can ask whether in the situation of Theorem 1.1.5 the inequality $a_i^x(M) - a_{i+1}^x(M) \le 1$ holds for *all* $x \in X$ and $i \in \{1, \ldots, n-1\}$. If $n = 2$ the

answer is "yes" by Corollary 1.1.7 because in this case $a_1^x(M) - a_2^x(M) = 2(a_1^x(M) - A(M))$. In general, the answer is *no*. In Appendix A we construct counterexamples for $n \in \{3, 4\}$ using hypergeometric local systems in the sense of N. Katz.

1.3. An application to ℓ-adic local systems

Now assume that the ground field k is *finite*. Let $|X|$ denote the set of closed points of X. For $x \in |X|$ let $\deg x$ denote the degree over $\mathbb{F}_p$ of the residue field of x.

Fix an algebraic closure $\overline{\mathbb{Q}}_\ell \supset \mathbb{Q}_\ell$, and let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$.

1.3.1. Algebraicity for $\overline{\mathbb{Q}}_\ell$-sheaves. If $\mathcal{E}$ is a $\overline{\mathbb{Q}}_\ell$-sheaf on X, $x \in |X|$, and $\bar{x}$ is a geometric point of X with image x, then one can consider the eigenvalues of the geometric Frobenius acting on the stalk $\mathcal{E}_{\bar{x}}$; for brevity, we will call them "Frobenius eigenvalues of $\mathcal{E}_x$" (or "Frobenius eigenvalues of $\mathcal{E}$ at x").

We say that $\mathcal{E}$ is *algebraic* if the Frobenius eigenvalues of $\mathcal{E}_x$ are in $\overline{\mathbb{Q}}$ for every x. It is known that any indecomposable $\overline{\mathbb{Q}}_\ell$-sheaf becomes algebraic after tensoring by the pullback of some local system on $\operatorname{Spec} \mathbb{F}_p$ (for lisse sheaves this is [Laf, Cor. VII.8]; in general see, e.g., [Dr, Cor. B.8]). So algebraicity is a mild assumption.

1.3.2. Slopes for algebraic lisse $\overline{\mathbb{Q}}_\ell$-sheaves. Fix a valuation $v : \overline{\mathbb{Q}}^\times \to \mathbb{Q}$ such that $v(p) = 1$. Let $\mathcal{E}$ be an algebraic lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X of rank n.

By assumption, for each $x \in |X|$ the Frobenius eigenvalues of $\mathcal{E}_x$ are in $\overline{\mathbb{Q}}^\times$. Applying to them the map $v : \overline{\mathbb{Q}}^\times \to \mathbb{Q}$ and dividing by $\deg x$, one gets n rational numbers. We call them the *slopes* of $\mathcal{E}$ at x. We denote them by $a_i^x(\mathcal{E})$; as before, we order them so that $a_i^x(\mathcal{E}) \geq a_{i+1}^x(\mathcal{E})$.

The number $A(\mathcal{E}) := \frac{1}{n} \cdot \sum\limits_{i=1}^{n} a_i^x(\mathcal{E})$ does not depend on $x \in |X|$: indeed, by [Del, Prop. 1.3.4(i)] there exists $m \in \mathbb{N}$ such that $(\det \mathcal{E})^{\otimes m}$ is a pullback of a rank 1 local system on $\operatorname{Spec} \mathbb{F}_p$.

Theorem 1.3.3. *Let $\mathcal{E}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X of rank n, which is algebraic in the sense of §1.3.1.*

(i) There exists a unique n-uple of rational numbers $a_1^\eta(\mathcal{E}) \geq a_2^\eta(\mathcal{E}) \geq \ldots \geq a_n^\eta(\mathcal{E})$ with the following property: let U denote the set of all $x \in |X|$ such that $a_i^x(\mathcal{E}) = a_i^\eta(\mathcal{E})$ for all i, then U is non-empty, and for any curve $C \subset X$ the subset $U \cap |C|$ is open in $|C|$.

(ii) For all $x \in |X|$ and $r \in \{1, \ldots, n\}$ one has

$$\sum_{i=1}^{r} a_i^x(\mathcal{E}) \leq \sum_{i=1}^{r} a_i^{\eta}(\mathcal{E}).$$

(iii) If $\mathcal{E}$ is indecomposable then $a_i^{\eta}(\mathcal{E}) - a_{i+1}^{\eta}(\mathcal{E}) \leq 1$ for all $i < n$.

We will prove Theorem 1.3.3 in §5 by combining Theorem 1.1.5 with the existence of crystalline companions (a.k.a. "petits camarades cristallins") proved by T. Abe [Ab].

Remark 1.3.4. In statement (i) uniqueness is easy (it follows from Lemma 5.3.4 below).

Remark 1.3.5. See [Ke7] for some stronger assertions about the set U. In particular, U is open.

Remark 1.3.6. Similarly to the proof of Corollary 1.1.7, statements (ii) and (iii) imply that if $\mathcal{E}$ is indecomposable then

$$(1.2) \qquad \sum_{i=1}^{r} a_i^x(\mathcal{E}) - rA(\mathcal{E}) \leq r(n-r)/2$$

for all $x \in |X|$ and $r \in \{1, \ldots, n-1\}$. At least for irreducible $\mathcal{E}$, this inequality was proved by V. Lafforgue without using isocrystals, see [Laf2, Cor. 2.2]; we recall his proof in §6.3.3. A weaker inequality had been conjectured by Deligne, see Conjecture 1.2.10(iv) of [De1].

Remark 1.3.7. As in §1.2, in the situation of Theorem 1.3.3(iii) it can happen that $a_i^x(\mathcal{E}) - a_{i+1}^x(\mathcal{E}) > 1$ for some $x \in |X|$ and $i \in \{1, \ldots, n-1\}$. Examples (with $\mathcal{E}$ irreducible and $n \in \{3, 4\}$) are given in Appendix A.

1.4. Generalization to arbitrary reductive groups

Theorems 1.1.5 and 1.3.3 are about $GL(n)$-local systems (crystalline and ℓ-adic ones). We deduce from them similar statements for G-local systems, where G is a reductive group (see Proposition 8.5.1, Theorem 8.6.3, and Theorem 9.2.8). We allow G to be disconnected; this is convenient for applications to automorphic representations in §10 (where G appears as the Langlands dual of a given connected reductive group).

1.5. An application to automorphic representations

V. Lafforgue [Laf2] used automorphic representations and the Langlands correspondence to prove (1.2). Similarly, we use the Langlands correspondence to translate Theorem 1.1.5 into properties of the p-adic absolute values of the Hecke eigenvalues of a cuspidal automorphic representation of $GL(n, \mathbb{A}_F)$, where $\mathbb{A}_F$ is the ring of adeles of a global field F of characteristic $p > 0$, see Theorem 6.2.1, §6.3.4, and Example 6.3.5. We do not know whether these properties can be proved directly (i.e., without passing to F-isocrystals).

A part of Theorem 6.2.1 generalizes to automorphic representations of $G(\mathbb{A}_F)$, where F is as above and G is any reductive group over F, see Theorem 10.7.1(i-ii). We are unable to generalize to arbitrary reductive groups the other part of Theorem 6.2.1 (namely, the estimate for the generic slope of automorphic representations). However, Theorem 10.7.1(iii) says that such a generalization would follow from Conjecture 12.7 of [Laf3] (which goes back to J. Arthur). The proof of Theorem 10.7.1 uses the main theorem of V. Lafforgue's article [Laf3].

1.6. Organization of the article

In §2 we combine some statements from the literature to show that for any open dense $U \subset X$, the restriction functor F-Isoc$(X) \to F$-Isoc(U) is fully faithful. This result plays a crucial role in the proof of our main Theorem 1.1.5.

In §3 we prove Theorem 1.1.5. In §4 we discuss some equivalent reformulations of Theorem 1.1.5. In §5 we prove Theorem 1.3.3. In §6 we discuss the application to automorphic representations of $GL(n)$ mentioned in §1.5.

In §7 we prove some lemmas on algebraic groups. In §8-9 they are used to prove the generalizations of Theorems 1.1.5 and 1.3.3 to arbitrary reductive groups. In §10 we treat the slopes of automorphic representations of arbitrary reductive groups by combining the results of §8 with the main theorem of [Laf3].

In Appendix A we provide the counterexamples promised in §1.2.

In Appendix B we recall R. Crew's results on F-Isoc(X) and F-Isoc$^\dagger(X)$ as Tannakian categories.

Acknowledgements

2. Full faithfulness of restriction functors

We reprise part of the discussion in [Ke6, §5] around the full faithfulness of various restriction functors.

2.1. Partial overconvergence

In addition to the two categories of isocrystals considered so far, we will need a third one: for U an open dense subset of X, let $F\text{-Isoc}(U, X)$ denote the category of F-isocrystals on U overconvergent within X. In particular, we have $F\text{-Isoc}(U, X) = F\text{-Isoc}(U)$ if $U = X$ and $F\text{-Isoc}(U, X) = F\text{-Isoc}^\dagger(U)$ if X is proper over k.

2.2. Full faithfulness

Theorem 2.2.1. *For any open dense $U \subset X$, the restriction functor $F\text{-Isoc}(X) \to F\text{-Isoc}(U, X)$ is fully faithful.*

Proof. See [Ke4, Theorem 5.2.1]. $\square$

Theorem 2.2.2. *For any open dense $U \subset X$, the restriction functor $F\text{-Isoc}(U, X) \to F\text{-Isoc}(U)$ is fully faithful. (This remains true even if X is not required to be smooth.)*

Proof. In the case where X is proper over k, this becomes the statement that $F\text{-Isoc}^\dagger(U) \to F\text{-Isoc}(U)$ is fully faithful, which is [Ke2, Theorem 1.1]. For the general case, see [Ke5, Theorem 4.2.1]. $\square$

By combining the preceding results, we obtain the following.

Theorem 2.2.3. *For any open dense $U \subset X$, the restriction functor $F\text{-Isoc}(X) \to F\text{-Isoc}(U)$ is fully faithful.*

Proof. Write the functor as a composition $F\text{-Isoc}(X) \to F\text{-Isoc}(U, X) \to F\text{-Isoc}(U)$. These functors are fully faithful by Theorem 2.2.1 and Theorem 2.2.2, respectively. This completes the proof. $\square$

3. Proof of Theorem 1.1.5

3.1. Reduction of Theorem 1.1.5 to Proposition 3.1.4(a)

By Theorem 2.2.3 and Lemma 1.1.3, if $M \in F\text{-Isoc}(X)$ is indecomposable, then so is its restriction to any non-empty open subset of X. By semicontinuity

of the Newton polygon, we may reduce Theorem 1.1.5 to the case where the Newton polygon of M is the same at all $x \in X$. In this case M admits a slope filtration[2], so Theorem 1.1.5 reduces to the following statement in the spirit of [Ke, Theorem 5.2.1].

Proposition 3.1.1. *Let $M_1, M_2 \in F\text{-Isoc}(X)$ and $s_1, s_2 \in \mathbb{Q}$. Suppose that M_i is isoclinic of slope s_i at each point of X. If $s_1 - s_2 > 1$, then $\mathrm{Ext}^1(M_1, M_2) = 0$.*

A proof of Proposition 3.1.1 is given below. A slightly different proof, more directly based on [Ke], is given in [Ke6, Appendix A].

It is known[3] that $F\text{-Isoc}(X)$ identifies with the category of Frobenius-equivariant objects in the category $\mathrm{Isoc}(X) := \mathrm{Crys}(X) \otimes \mathbb{Q}$, where $\mathrm{Crys}(X)$ is the category of crystals of coherent sheaves on X. For $M \in \mathrm{Isoc}(X)$ we set $R\Gamma_{\mathrm{crys}}(X, M) := R\Gamma_{\mathrm{crys}}(X, M_0) \otimes \mathbb{Q}$, where M_0 is any object of $\mathrm{Crys}(X)$ equipped with an isomorphism $M_0 \otimes \mathbb{Q} \xrightarrow{\sim} M$. If $M \in F\text{-Isoc}(X)$ then the complex of $\mathbb{Q}_p$-vector spaces $R\Gamma_{\mathrm{crys}}(X, M)$ is equipped with an action of the Frobenius endomorphism F.

Lemma 3.1.2. $\mathrm{Ext}^1(M_1, M_2)$ *is canonically isomorphic to the first cohomology of the complex*

$$(3.1) \qquad \mathrm{Cocone}(R\Gamma_{\mathrm{crys}}(X, N) \xrightarrow{F-1} R\Gamma_{\mathrm{crys}}(X, N)),$$

where $N := \underline{\mathrm{Hom}}(M_1, M_2) = M_1^* \otimes M_2$ and $\mathrm{Cocone} := \mathrm{Cone}[-1]$. □

The lemma is well known. We give a proof for completeness. Note that for our purposes it is enough to know that $\mathrm{Ext}^1(M_1, M_2)$ is isomorphic to the first cohomology of (3.1) in the case that X is *affine*, and this weaker statement can be easily checked by choosing a lift of X to a smooth formal scheme over $W(k)$.

Proof. Let $\mathcal{O}$ denote the unit object of the tensor category $F\text{-Isoc}(X)$. (Later we will use the same symbol for the unit objects of some other tensor categories.) The composition

$$\mathrm{Ext}^1(\mathcal{O}, N) \to \mathrm{Ext}^1(M_1, M_1 \otimes N) \to \mathrm{Ext}^1(M_1, M_2)$$

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[2]E.g., see [Ke6, Thm. 4.1 and Cor. 4.2] as well as [Ke6, Thm. 5.1 and Rem. 5.2]. In the case that $\dim X = 1$ the slope filtration goes back to N. Katz [Ka1, Corollary 2.6.3].

[3]See [Ber, §2.4]. The main point is that F-equivariant isocrystals are automatically convergent; this is proved in [Ber] using an argument which goes back to Dwork.

is an isomorphism: its inverse is the composition

$$
\begin{aligned}
\operatorname{Ext}^1(M_1, M_2) \quad &\to \quad \operatorname{Ext}^1(M_1 \otimes M_1^*, M_2 \otimes M_1^*) = \operatorname{Ext}^1(M_1 \otimes M_1^*, N) \\
&\to \quad \operatorname{Ext}^1(\mathcal{O}, N).
\end{aligned}
$$

So it remains to compute $\operatorname{Ext}^1(\mathcal{O}, N)$.

Let $\operatorname{Isoc}(X)$ denote the category of isocrystals on X. Let $\mathscr{E}xt_{F\text{-Isoc}}(\mathcal{O}, N)$ (resp. $\mathscr{E}xt_{\operatorname{Isoc}}(\mathcal{O}, N)$) denote the Picard groupoid of extensions of $\mathcal{O}$ by N in the category $F\text{-Isoc}(X)$ (resp. $\operatorname{Isoc}(X)$).

Let us first compute $\mathscr{E}xt_{\operatorname{Isoc}}(\mathcal{O}, N)$. By definition, $\operatorname{Isoc}(X) = \operatorname{Crys}(X) \otimes \mathbb{Q} = \operatorname{Crys}_{\mathbb{Z}_p\text{-flat}}(X) \otimes \mathbb{Q}$, where $\operatorname{Crys}(X)$ is the category of crystals of coherent sheaves on X and $\operatorname{Crys}_{\mathbb{Z}_p\text{-flat}}(X)$ is the full subcategory of those objects of $\operatorname{Crys}(X)$ that have no non-zero subcrystals killed by p. The fiber over N of the functor $\operatorname{Crys}_{\mathbb{Z}_p\text{-flat}}(X) \to \operatorname{Isoc}(X)$ is the poset of *lattices* in N, denoted by $\operatorname{Lat}(N)$. One has

$$
\mathscr{E}xt_{\operatorname{Isoc}}(\mathcal{O}, N) = \varinjlim_{L \in \operatorname{Lat}(N)} \mathscr{E}xt_{\operatorname{Crys}}(\mathcal{O}, L)
$$

(in the right-hand side $\mathcal{O}$ denotes the unit object of $\operatorname{Crys}(X)$). An object of $\mathscr{E}xt_{\operatorname{Crys}}(\mathcal{O}, L)$ is the same as an L-torsor on the crystalline site of X, so the Picard groupoid $\mathscr{E}xt_{\operatorname{Crys}}(\mathcal{O}, L)$ corresponds (in the sense of [SGA4], Exposé XVIII, §1.4]) to the complex $\tau^{\leq 0} R\Gamma_{\operatorname{crys}}(X, L[1])$. Therefore $\mathscr{E}xt_{\operatorname{Isoc}}(\mathcal{O}, N)$ corresponds to $\tau^{\leq 0} R\Gamma_{\operatorname{crys}}(X, N[1])$.

The Picard groupoid $\mathscr{E}xt_{F\text{-Isoc}}(\mathcal{O}, N)$ is the groupoid of Frobenius-equivariant objects of $\mathscr{E}xt_{\operatorname{Isoc}}(\mathcal{O}, N)$, so it corresponds to the complex

$$
\tau^{\leq 0} \operatorname{Cocone}(R\Gamma_{\operatorname{crys}}(X, N[1]) \xrightarrow{F-1} R\Gamma_{\operatorname{crys}}(X, N[1])).
$$

Therefore $\operatorname{Ext}^1(\mathcal{O}, N)$ is the 0-th cohomology of this complex or equivalently, the first cohomology of the complex (3.1). $\qquad \square$

Lemma 3.1.2 shows that Proposition 3.1.1 is a particular case of the following statement in the spirit of [Ke3, §5.4].

Proposition 3.1.3. *Suppose that $N \in F\text{-Isoc}(X)$ is isoclinic of slope s at each point of X. Then the i-th cohomology of the complex (3.1) vanishes for all $i < -s$.*

Proposition 3.1.3 is equivalent to part (a) of the following one.

Proposition 3.1.4. *Let $K \supset \mathbb{Q}_p$ be a finite extension and $\gamma \in K^\times$. Suppose that $N \in F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ is unit-root. Then*

  (a) the i-th cohomology of the complex

$$(3.2) \qquad \mathrm{Cocone}(R\Gamma_{\mathrm{crys}}(X, N) \xrightarrow{F - \gamma} R\Gamma_{\mathrm{crys}}(X, N))$$

vanishes for all $i < v(\gamma)$;

  (b) if X is affine then the i-th cohomology of (3.2) also vanishes for all $i > v(\gamma) + 1$;

  (c) if $v(\gamma) < 0$ then all cohomology groups of (3.2) vanish;

  (d) if $v(\gamma) > \dim X$ then all cohomology groups of (3.2) vanish.

(As usual, $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ denotes the category of objects of $F\text{-Isoc}(X)$ equipped with K-action.)

If $v(\gamma)$ is a non-negative integer then Proposition 3.1.4 just says that the complex (3.2) is concentrated in degrees $v(\gamma)$ and $v(\gamma) + 1$; for a more precise statement, see Proposition 3.3.1 below.

Let us note that if $v(\gamma) > \dim X + 1$ then Proposition 3.1.4(d) immediately follows from Proposition 3.1.4(a).

3.2. Proof of Proposition 3.1.4

It suffices to prove Proposition 3.1.4 if X is affine. From now on we assume this.

3.2.1. A concrete realization of $R\Gamma_{\mathrm{crys}}(X, N)$. We fix a pair $(\mathscr{X}, \phi)$, where $\mathscr{X}$ is a smooth formal scheme over the Witt ring $W(k)$ with special fiber X and $\phi : \mathscr{X} \to \mathscr{X}$ is a lift of the absolute Frobenius of X. Let $\mathscr{X}_n$ denote the reduction of $\mathscr{X}$ modulo p^n.

By [Cr, Thm. 2.1], a unit-root object $N \in F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ is "the same as" a lisse K-sheaf $\mathcal{N}$ on $\mathscr{X}_{\mathrm{et}}$ (i.e., a $\mathbb{Q}_p$-sheaf equipped with an action of K). Let $\mathfrak{o}_K \subset K$ be the ring of integers. We have $\mathcal{N} = \mathcal{N}_0 \otimes_{\mathfrak{o}_K} K$ for some torsion-free lisse $\mathfrak{o}_K$-sheaf $\mathcal{N}_0$ on X_{et}.

Tensoring $\mathcal{N}_0/p^n\mathcal{N}_0$ by the structure sheaf $\mathcal{O}_{\mathscr{X}_n}$ (viewed as a sheaf on X_{et}), one gets a vector bundle L_n on $\mathscr{X}_n$. The vector bundles L_n on $\mathscr{X}_n$ define a vector bundle L on $\mathscr{X}$ equipped with an integrable connection ∇, an action of $\mathfrak{o}_K$, and an action of ϕ (i.e., a ϕ-linear endomorphism of $H^0(X, N)$).

Let $C^\bullet$ denote the de Rham complex of (L, ∇). This is a complex of topologically free $\mathfrak{o}_K$-modules equipped with an endomorphism F (the latter comes from the action of ϕ). The complex $C^\bullet \otimes \mathbb{Q}$ is a concrete realization of $R\Gamma_{\mathrm{crys}}(X, N)$.

3.2.2. Lemmas in the spirit of Berthelot-Ogus. The terms of the complex $C^\bullet$ are denoted by C^j. Let $\tilde{C}^\bullet \subset C^\bullet \otimes \mathbb{Q}$ denote the subcomplex whose j-th term equals $\tilde{C}^j := p^{-j} \cdot C^j \subset C^j \otimes \mathbb{Q}$. It is clear that the morphism $F : C^\bullet \otimes \mathbb{Q} \to C^\bullet \otimes \mathbb{Q}$ maps $\tilde{C}^\bullet$ to $C^\bullet$.

The following lemmas and their proofs date back to [BO] (see [BO, Lemma 1.4] and [BO, Props 1.5-1.7]).

Lemma 3.2.3. *(i) The complex $\tilde{C}^\bullet/p\tilde{C}^\bullet$ has zero differential.*
(ii) The morphism $F : \tilde{C}^\bullet \to C^\bullet$ is a quasi-isomorphism.

Proof. Statement (i) is clear. The terms of the complexes $\tilde{C}^\bullet$ and $C^\bullet$ are topologically free. So to prove (ii), it suffices to check that the morphism $\tilde{C}^\bullet/p\tilde{C}^\bullet \to C^\bullet/pC^\bullet$ induced by F is a quasi-isomorphism. This is a well-known interpretation of the inverse of the Cartier isomorphism due to Mazur [Maz]. $\qquad\square$

Lemma 3.2.4. *Let $\gamma \in K$, $i \in \mathbb{Z}$, $i < v(\gamma)$. Let $C^\bullet_{\leq i}$ (resp. $\tilde{C}^\bullet_{\leq i}$) denote the complex obtained from $C^\bullet$ (resp. $\tilde{C}^\bullet$) by replacing the terms of degree $> i$ with zeros. Then*
(i) the morphism $F - \gamma : \tilde{C}^\bullet_{\leq i} \otimes \mathbb{Q} \to C^\bullet_{\leq i} \otimes \mathbb{Q}$ maps $\tilde{C}^\bullet_{\leq i}$ to $C^\bullet_{\leq i}$;

(ii) the cohomology of the complex $\mathrm{Cocone}(\tilde{C}^\bullet_{\leq i} \xrightarrow{F-\gamma} C^\bullet_{\leq i})$ is concentrated in degree $i + 1$; this cohomology is zero if $i \geq \dim X$.

Proof. Since $v(\gamma) > i$ we have

$$(3.3) \qquad\qquad \gamma(\tilde{C}^\bullet_{\leq i}) \subset m_K \cdot C^\bullet_{\leq i},$$

where m_K is the maximal ideal of $\mathfrak{o}_K$. Statement (i) follows from (3.3) and the inclusion $F(\tilde{C}^\bullet_{\leq i}) \subset C^\bullet_{\leq i}$.

$\mathrm{Cocone}(\tilde{C}^\bullet_{\leq i} \xrightarrow{F-\gamma} C^\bullet_{\leq i})$ is a complex of topologically free $\mathfrak{o}_K$-modules. By Lemma 3.2.3 and formula (3.3), the cohomology of its reduction modulo m_K is concentrated in degree $i + 1$, and if $i \geq \dim X$ this cohomology is zero. Statement (ii) follows. $\qquad\square$

3.2.5. End of the proof. Let $\gamma \in K$, $i \in \mathbb{Z}$, $i < v(\gamma)$. By Lemma 3.2.4(ii), the cohomology of the complex $\mathrm{Cocone}(\tilde{C}^\bullet_{\leq i} \otimes \mathbb{Q} \xrightarrow{F-\gamma} C^\bullet_{\leq i} \otimes \mathbb{Q})$ is concentrated in degree $i+1$. So the cohomology of the complex $\mathrm{Cocone}(\tilde{C}^\bullet \otimes \mathbb{Q} \xrightarrow{F-\gamma} C^\bullet \otimes \mathbb{Q})$ is concentrated in degrees $> i$. This is equivalent to Proposition 3.1.4(a).

Now suppose that $i > v(\gamma) + 1$ and $j \geq i - 1$. Then

$$\gamma^{-1} F(C^j) \subset m_K \cdot C^j.$$

So the operator $1 - \gamma^{-1}F : C^j \to C^j$ is invertible: its inverse equals

$$1 + \gamma^{-1}F + (\gamma^{-1}F)^2 + \dots .$$

So $F - \gamma : C^j \otimes \mathbb{Q} \to C^j \otimes \mathbb{Q}$ is invertible for all $j \geq i - 1$. Therefore $\mathrm{Cocone}(C^\bullet \otimes \mathbb{Q} \xrightarrow{F-\gamma} C^\bullet \otimes \mathbb{Q})$ is quasi-isomorphic to the complex

$$\mathrm{Cocone}(C^\bullet_{<i-1} \otimes \mathbb{Q} \xrightarrow{F-\gamma} C^\bullet_{<i-1} \otimes \mathbb{Q}).$$

The latter complex is concentrated in degrees $< i$, and if $i = 1$ the complex is zero. This implies Proposition 3.1.4(b-c).

Proposition 3.1.4(d) follows from the second part of Lemma 3.2.4(ii) applied for $i = \dim X$. $\square$

The next subsection is not used in the rest of the article.

3.3. A refinement of Proposition 3.1.4

If $v(\gamma)$ is a non-negative integer then Proposition 3.1.4 says that the complex (3.2) is concentrated in degrees $v(\gamma)$ and $v(\gamma) + 1$. Here is a more precise statement, whose proof given below was explained to us by L. Illusie and K. Kato.

Proposition 3.3.1. *Let $r \in \mathbb{Z}$, $r \geq 0$. Suppose that in the situation of Proposition 3.1.4 one has $v(\gamma) = r$. Then there exists a projective system*

$$\dots \to \mathcal{G}_3 \to \mathcal{G}_2 \to \mathcal{G}_1$$

of sheaves of abelian groups on X_{et} such that

(i) each $\mathcal{G}_n$ is a flat sheaf of $(\mathbb{Z}/p^n\mathbb{Z})$-modules, and the morphism $\mathcal{G}_{n+1} \to \mathcal{G}_n$ identifies $\mathcal{G}_n$ with $\mathcal{G}_{n+1}/p^n\mathcal{G}_{n+1}$;

(ii) if X' is any scheme etale over X and $N' \in F\text{-Isoc}(X') \otimes_{\mathbb{Q}_p} K$ is the pullback of N then one has a canonical isomorphism

$$\mathrm{Cocone}(R\Gamma_{\mathrm{crys}}(X', N') \xrightarrow{F-\gamma} R\Gamma_{\mathrm{crys}}(X', N'))$$

(3.4)
$$\simeq \varprojlim_n R\Gamma(X'_{\mathrm{et}}, \mathcal{G}_n)[-r] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p .$$

We will see that the sheaves $\mathcal{G}_n$ are not constructible if $r > 0$.

Remark 3.3.2. It suffices to prove Proposition 3.3.1 if $\gamma = p^r$ (otherwise twist N by a suitable unit-root object of $F\text{-Isoc}(\mathrm{Spec}\,k) \otimes_{\mathbb{Q}_p} K$).

3.3.3. Constructing the sheaves $\mathcal{G}_n$. We will assume that $\gamma = p^r$ (see Remark 3.3.2). Under this assumption, the sheaves $\mathcal{G}_n$ from Proposition 3.3.1 are constructed as follows.

On X_{et} we have the projective system of de Rham-Witt complexes $W_n\Omega_X^\bullet$. For each n and r we have the "logarithmic" subsheaf $W_n\Omega_{X,\log}^r \subset W_n\Omega_X^r$ defined in §5.7 of Ch. I of Illusie's article [Il] (p. 596-597). This is a sheaf of $(\mathbb{Z}/p^n\mathbb{Z})$-modules. For fixed r and variable n the sheaves $W_n\Omega_{X,\log}^r$ form a projective system. It is known that the sheaf $W_n\Omega_{X,\log}^r$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and the morphism $W_{n+1}\Omega_{X,\log}^r \to W_n\Omega_{X,\log}^r$ identifies $W_n\Omega_{X,\log}^r$ with $W_{n+1}\Omega_{X,\log}^r/p^nW_{n+1}\Omega_{X,\log}^r$ (see Lemma 3 on p. 779 of [CSS]).

Let us note that $W_n\Omega_{X,\log}^r$ has a K-theoretic description: by Theorem 5.1 of [Mor] (which is due to many authors), $W_n\Omega_{X,\log}^r$ identifies with the sheaf K_r/p^nK_r on X_{et} and also with $K_r^M/p^nK_r^M$, where K_r^M is the sheaf of Milnor K-groups[4]. For instance, $W_n\Omega_{X,\log}^0$ is the constant sheaf with fiber $\mathbb{Z}/p^n\mathbb{Z}$, and $W_n\Omega_{X,\log}^1 = \mathcal{O}_{X_{\mathrm{et}}}^\times/(\mathcal{O}_{X_{\mathrm{et}}}^\times)^{p^n}$.

Now set

$$(3.5) \qquad \mathcal{G}_n := \mathcal{N}_0 \otimes_{\mathbb{Z}_p} W_n\Omega_{X,\log}^r \,,$$

where $\mathcal{N}_0$ is as in §3.2.1.

3.3.4. Constructing the isomorphism (3.4). As before, we assume that $\gamma = p^r$ and $\mathcal{G}_n$ is defined by (3.5). We will also assume that the scheme X' from Proposition 3.3.1(ii) equals X.

According to [Il], the de Rham complex $C^\bullet$ introduced in §3.2.1 is canonically quasi-isomorphic to $\mathbb{Q} \otimes \varprojlim_n R\Gamma(X_{\mathrm{et}}, \mathcal{N}_0 \otimes_{\mathbb{Z}_p} W_n\Omega_X^\bullet)$. So the problem is to compute the complex

$$(3.6) \qquad \mathbb{Q} \otimes \varprojlim_n R\Gamma(X_{\mathrm{et}}, \mathcal{N}_0 \otimes_{\mathbb{Z}_p} \mathrm{Cocone}(W_n\Omega_X^\bullet \xrightarrow{F-p^r} W_n\Omega_X^\bullet)).$$

In addition to F, we have the "de Rham-Witt" operators V and F satisfying $\mathsf{VF} = \mathsf{FV} = p$ (see [Il]). Unlike F, they do *not* commute with the differential d. Note that while V is an endomorphism of $W_n\Omega_X^i$, the operator F is a morphism from $W_{n+1}\Omega_X^i$ to its quotient $W_n\Omega_X^i$. However, the

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[4]Theorem 5.1 of [Mor] involves the "improved" Milnor K-groups rather than the usual ones. However, for local rings with *infinite* residue fields the "improved" Milnor K-groups are equal to the usual ones, and the residue field of each stalk of $\mathcal{O}_{X_{\mathrm{et}}}$ is infinite (because it is separably closed).

operator $p\mathsf{F} : W_n\Omega^i_X \to W_n\Omega^i_X$ is well-defined and nilpotent. Moreover, if $i = 0$ then $\mathsf{F} : W_n\Omega^i_X \to W_n\Omega^i_X$ is well-defined. Recall that the morphism $F : W_n\Omega^i_X \to W_n\Omega^i_X$ equals $p^i\mathsf{F}$.

Lemma 3.3.5. *If $i \neq r$ then the kernel and cokernel of $F - p^r : W_n\Omega^i_X \to W_n\Omega^i_X$ are killed by a power of p independent of n.*

Proof. If $i > r$ write $F - p^r = p^r(p^{i-r} \cdot \mathsf{F} - 1)$ and note that $p^{i-r} \cdot \mathsf{F} - 1$ is invertible because $p^{i-r} \cdot \mathsf{F}$ is nilpotent.

If $i < r$ write $F - p^r = F(1 - p^{r-i-1}\mathsf{V})$. The operator $1 - p^{r-i-1}\mathsf{V}$ is invertible because V is nilpotent. Finally, the kernel and cokernel of $F : W_n\Omega^i_X \to W_n\Omega^i_X$ are killed by p^{i+1} because $F\mathsf{V} = \mathsf{V}F = p^{i+1}$. $\qquad\square$

The lemma implies that the complex (3.6) is canonically isomorphic to

$$\mathbb{Q} \otimes \varprojlim_n R\Gamma(X_{\mathrm{et}}, \mathcal{N}_0 \otimes_{\mathbb{Z}_p} \mathrm{Cocone}(W_n\Omega^r_X \xrightarrow{F-p^r} W_n\Omega^r_X))[-r]$$

and therefore to

$$\mathbb{Q} \otimes \varprojlim_n R\Gamma(X_{\mathrm{et}}, \mathcal{N}_0 \otimes_{\mathbb{Z}_p} \mathrm{Cocone}(W_n\Omega^r_X \xrightarrow{\mathsf{F}-1} W_n\Omega^r_X/\mathsf{F}(A^r_n))[-r],$$

where $A^r_n := \mathrm{Ker}(W_{n+1}\Omega^r_X \twoheadrightarrow W_n\Omega^r_X)$. Finally, Lemma 2 on p. 779 of [CSS] tells us[5] that

$$\mathrm{Cocone}(W_n\Omega^r_X \xrightarrow{\mathsf{F}-1} W_n\Omega^r_X/\mathsf{F}(A^r_n)) = W_n\Omega^r_{X,\log}.$$

Thus we get the desired isomorphism (3.4). This finishes the proof of Proposition 3.3.1.

Example 3.3.6. If $X = (\mathbb{G}_m)^n$, N is the unit object of the tensor category $F\text{-Isoc}(X)$, and $0 \leq r < n$ then a direct computation shows that both H^r and H^{r+1} of the complex

$$\mathrm{Cocone}(R\Gamma_{\mathrm{crys}}(X, N) \xrightarrow{F-p^r} R\Gamma_{\mathrm{crys}}(X, N))$$

are nonzero, and H^{r+1} has infinite dimension over $\mathbb{Q}_p$.

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[5]To see this, note that the sheaf $d\mathsf{V}^{n-1}W_n\Omega^{r-1}_X$ from Lemma 2 on p. 779 of [CSS] is equal to $\mathsf{F}(A^r_n)$. This follows from the formula $A^r_n = \mathsf{V}^n W_{n+1}\Omega^r_X + d\mathsf{V}^n W_{n+1}\Omega^r_X$ (see Proposition 3.2 on p.568 of [Il]).

4. Some reformulations of Theorem 1.1.5

4.1. The canonical decomposition corresponding to a big gap between the slopes

Theorem 1.1.5 is clearly equivalent to the following

Proposition 4.1.1. *Let $M \in F\text{-Isoc}(X)$. Suppose that $a_i^\eta(M) - a_{i+1}^\eta(M) > 1$ for some i. Then M admits a decomposition*

$$(4.1) \qquad\qquad M = M_1 \oplus M_2$$

such that the slopes of M_1 (resp. M_2) at η are the numbers $a_j^\eta(M)$ for $j \leq i$ (resp. $j > i$). □

Remark 4.1.2. It is clear that in the situation of Proposition 4.1.1 one has

$$(4.2) \qquad\qquad \text{Hom}(M_1, M_2) = \text{Hom}(M_2, M_1) = 0.$$

This implies that the decomposition (4.1) is unique.

Proposition 4.1.3. *Let $M \in F\text{-Isoc}(X)$. Suppose that $\text{End}\, M$ has no non-trivial central idempotents. Then $a_i^\eta(M) - a_{i+1}^\eta(M) \leq 1$ for all $i < n$, where n is the rank of M.*

Proof. Suppose that $a_i^\eta(M) - a_{i+1}^\eta(M) > 1$ for some i. Consider the corresponding decomposition (4.1). Let $\pi \in \text{End}\, M$ be the projection to M_1. Then π is a non-trivial idempotent. It is central by (4.2). □

4.2. The categories $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ and $F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$

4.2.1. The definitions. For K a finite extension of $\mathbb{Q}_p$, we define $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ to be the category of objects of $F\text{-Isoc}(X)$ equipped with an action of K; define $F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} K$ similarly. Set

$$F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p := \varinjlim_K F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K,$$

$$F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p := \varinjlim_K F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} K,$$

where K runs through the set of all subfields of $\overline{\mathbb{Q}}_p$ finite over $\mathbb{Q}_p$. For an object M in $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ or in $F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$, the slopes $a_i^x(M)$, $x \in X$, are defined similarly to §1.1.4.

Remark 4.2.2. For K a finite extension of $\mathbb{Q}_p$, one has the forgetful functor

$$\mathrm{Forg}_{K/\mathbb{Q}_p} : F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K \to F\text{-Isoc}(X).$$

If $M \in F\text{-Isoc}(X)$ has rank n then $\mathrm{Forg}_{K/\mathbb{Q}_p}(M)$ has rank dn, where $d :=$ $[K : \mathbb{Q}_p]$. The set of slopes of $\mathrm{Forg}_{K/\mathbb{Q}_p}(M)$ is equal to that of M, but the multiplicity of each slope is multiplied by d.

Proposition 4.2.3. *Theorem 1.1.5 and Propositions 4.1.1, 4.1.3 remain valid for $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$, where K is a finite extension of $\mathbb{Q}_p$.*

Proof. By Remark 4.2.2, Proposition 4.1.3 for $M \in F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ follows from Proposition 4.1.3 for $\mathrm{Forg}_{K/\mathbb{Q}_p}(M) \in F\text{-Isoc}(X)$. Theorem 1.1.5 and Proposition 4.1.1 for $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$ follow from Proposition 4.1.3 for $M \in F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} K$. $\qquad\square$

Corollary 4.2.4. *Theorem 1.1.5 and Propositions 4.1.1, 4.1.3 remain valid for $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.* $\qquad\square$

5. Proof of Theorem 1.3.3

Recall that $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$. We fix a valuation $v : \overline{\mathbb{Q}}^\times \to \mathbb{Q}$ such that $v(p) = 1$; slopes of algebraic $\overline{\mathbb{Q}}_\ell$-sheaves are defined using v.

5.1. Proof of Theorem 1.3.3 for irreducible sheaves on curves

For any subfield $E \subset \overline{\mathbb{Q}}$, let E_v denote the completion of E with respect to v. The union of the fields E_v corresponding to all subfields $E \subset \overline{\mathbb{Q}}$ finite over $\mathbb{Q}$ is an algebraic closure of $\mathbb{Q}_p$; we denote it by $\overline{\mathbb{Q}}_p$.

Assume that $\dim X = 1$ and $\mathcal{E}$ is irreducible. Then Theorem 4.4.1 of Abe's work [Ab] provides an irreducible object $M \in F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ of rank n such that for every $x \in |X|$, the characteristic polynomials of the geometric Frobenius acting on $M_{\bar{x}}$ and $\mathcal{E}_{\bar{x}}$ are equal to each other. Then the multiset of slopes of $\mathcal{E}$ at any $x \in |X|$ is equal to that of M. Define the numbers $a_i^\eta(\mathcal{E})$ to be the slopes of M at the generic point $\eta \in X$. Applying semicontinuity of the Newton polygon to M, we see that the numbers $a_i^\eta(\mathcal{E})$ satisfy properties (i)–(ii) from Theorem 1.3.3.

To check (iii), recall that the functor $F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}(X)$ is fully faithful, so by Lemma 1.1.3, M is indecomposable as an object of $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. It remains to apply Theorem 1.1.5 to M, which is possible by Corollary 4.2.4. $\qquad\square$

5.2. Proof of Theorem 1.3.3 for curves

In §5.1 we proved Theorem 1.3.3 assuming that $\mathcal{E}$ is irreducible. This immediately implies Theorem 1.3.3(i-ii) for any $\mathcal{E}$. Theorem 1.3.3(iii) for any indecomposable $\mathcal{E}$ will be proved in §5.2.6.

5.2.1. A result of Deligne. As before, slopes are defined using a fixed valuation $v : \overline{\mathbb{Q}}^{\times} \to \mathbb{Q}$.

Proposition 5.2.2. *Let X be a scheme of finite type over $\mathbb{F}_p$; let π denote the morphism $X \to \operatorname{Spec} \mathbb{F}_p$. Let $\mathcal{E}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X.*

(i) Suppose that $\mathcal{E}$ is algebraic in the sense of §1.3.1 and the slopes of $\mathcal{E}$ (with respect to some p-adic place of $\overline{\mathbb{Q}}$) are in the interval $[r, s]$. Then for each i the sheaf $R^i \pi_! \mathcal{E}$ is algebraic, and its slopes are in the interval

$$(5.1) \qquad [r + \max\{0, i - n\}, s + \min\{i, n\}], \quad \text{where } n := \dim X.$$

(ii) If X is smooth then the same statements hold for $R^i \pi_* \mathcal{E}$.

Proof. Statement (i) is a reformulation of Theorems 5.2.2 and 5.4 of [SGA7, exposé XXI], which are due to Deligne. Statement (ii) follows by Verdier duality. □

Example 5.2.3. Let X be the n-th power of a non-supersingular elliptic curve. Let $\mathcal{E} = (\overline{\mathbb{Q}}_\ell)_X$ and $r = s = 0$. Then all integers from the interval (5.1) appear as slopes of $R^i \pi_* \mathcal{E}$.

Remark 5.2.4. The slope estimates from Proposition 5.2.2 remain valid for overconvergent F-isocrystals and rigid cohomology instead of lisse $\overline{\mathbb{Q}}_\ell$-sheaves and ℓ-adic cohomology, see [Ke3, Thm. 5.4.1].

Corollary 5.2.5. *Let X be an irreducible smooth variety over $\mathbb{F}_p$. Let $\mathcal{E}_1, \mathcal{E}_2$ be algebraic lisse $\overline{\mathbb{Q}}_\ell$-sheaves on X. Suppose that one of the following assumptions holds:*

(i) for some non-empty open $U \subset X$ all the slopes of $\mathcal{E}_1^* \otimes \mathcal{E}_2$ at all points of $|U|$ are < -1;

(ii) for some non-empty open $U \subset X$ all the slopes of $\mathcal{E}_1^* \otimes \mathcal{E}_2$ at all points of $|U|$ are > 0.
Then $\operatorname{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) = 0$.

Proof. Let $j : U \hookrightarrow X$ be the embedding. Let π be the morphism $X \to \operatorname{Spec} \mathbb{F}_p$.

Since X is normal, the map $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) \to \mathrm{Ext}^1(j^*\mathcal{E}_1, j^*E_2)$ is injective. So we can assume that $U = X$. Then all the slopes of the sheaves

$$R^i\pi_*(\mathcal{E}_1^* \otimes \mathcal{E}_2), \quad i \in \{0, 1\}$$

are non-zero by Proposition 5.2.2(ii). So $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) = H^1(\mathrm{Spec}\,\mathbb{F}_p, R\pi_*(\mathcal{E}_1^* \otimes \mathcal{E}_2)) = 0$. $\qquad\square$

5.2.6. Proof of Theorem 1.3.3(iii) if dim $X = 1$. The case where $\mathcal{E}$ is irreducible was treated in §5.1. The case where $\mathcal{E}$ is indecomposable but not necessarily irreducible follows by Corollary 5.2.5. $\qquad\square$

5.3. The case dim $X > 1$

Lemma 5.3.1. *There exists $N \in \mathbb{N}$ such that $a_i^x(\mathcal{E}) \in N^{-1}\mathbb{Z}$ for all $x \in |X|$ and $i \in \{1, \ldots, n\}$.*

Proof. By [De3, Thm. 3.1], there exists a subfield $E \subset \overline{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}$ such that for every $x \in |X|$ all the coefficients of the characteristic polynomial of the geometric Frobenius acting on $\mathcal{E}_{\bar{x}}$ belong to E. Our p-adic valuation v maps $E^\times$ to $d^{-1}\mathbb{Z}$ for some $d \in \mathbb{N}$. Now one can set $N := n! \cdot d$. $\qquad\square$

Lemma 5.3.2. *For every $x \in |X|$ there exists a smooth connected curve over $\mathbb{F}_p$ and a morphism $f : C \to X$ such that $x \in f(C)$ and $f^*\mathcal{E}$ is indecomposable.*

Proof. This is a consequence of [Dr, Prop. 2.17]. $\qquad\square$

Lemma 5.3.3. *For each $r \in \{1, \ldots, n\}$ the function*

$$(5.2) \qquad\qquad x \mapsto \sum_{i=1}^{r} a_i^x(\mathcal{E}), \quad x \in |X|$$

is bounded above.

Proof. Without loss of generality, one can assume that $\mathcal{E}$ is indecomposable. By [De1, Prop. 1.3.4(i)], after tensoring $\mathcal{E}$ by a rank 1 local system on $\mathrm{Spec}\,\mathbb{F}_p$ one can also assume that $(\det \mathcal{E})^{\otimes m}$ is trivial for some m, so $\sum_{i=1}^{n} a_i^x(\mathcal{E}) = 0$. In this situation $\sum_{i=1}^{r} a_i^x(\mathcal{E}) \leq r(n-r)/2$ by Lemma 5.3.2 and Remark 1.3.6 (the latter is applicable because we already proved Theorem 1.3.3 for curves). $\qquad\square$

Lemma 5.3.4. *Let $\mathcal{T}$ denote the following topology on $|X|$: a subset $F \subset |X|$ is $\mathcal{T}$-closed if and only if $F \cap |C|$ is closed for all curves $C \subset X$. Then $|X|$ is irreducible with respect to $\mathcal{T}$.*

Proof. Suppose that $F_1, F_2 \subset |X|$ are $\mathcal{T}$-closed and different from $|X|$. Choose $x_1, x_2 \in |X|$ so that $x_i \notin F_i$. By Hilbert irreducibility (e.g., by [Dr, Thm. 2.15(i)]), there exists an irreducible curve $C \subset X$ containing x_1 and x_2. Then the sets $|C| \cap F_i$ are finite, so $|C| \not\subset F_1 \cup F_2$. Therefore $F_1 \cup F_2 \neq |X|$. $\quad\square$

Proof of Theorem 1.3.3. By Lemmas 5.3.1 and 5.3.3, the function (5.2) has a maximal value s_r; let U_r denote the set of all $x \in |X|$ for which this value is attained, and let $U := \bigcap_r U_r$. Define the numbers $a_i^\eta(\mathcal{E})$ as follows: $a_1^\eta(\mathcal{E}) := s_1$, $a_i^\eta(\mathcal{E}) := s_i - s_{i-1}$ for $i > 1$.

Let us prove that the numbers $a_i^\eta(\mathcal{E})$ have the properties stated in Theorem 1.3.3. Each U_r is clearly non-empty, and by §5.2.6 it is $\mathcal{T}$-open. So U is non-empty and $\mathcal{T}$-open by Lemma 5.3.4. For every $i < n$ one has $a_i^\eta(\mathcal{E}) \geq a_{i+1}^\eta(\mathcal{E})$ because $a_j^\eta(\mathcal{E}) = a_j^x(\mathcal{E})$ for all $x \in U$ and all j. By construction, $s_r^x(\mathcal{E}) \leq s_r^\eta(\mathcal{E})$ for all r. Finally, if $\mathcal{E}$ is indecomposable then for $x \in U$ one has $a_i^x(\mathcal{E}) - a_{i+1}^x(\mathcal{E}) \leq 1$: to see this, apply §5.2.6 to the curve C from Lemma 5.3.2.

The uniqueness part of Theorem 1.3.3(i) follows from Lemma 5.3.4. $\quad\square$

6. Slopes for automorphic representations of $GL(n)$

6.1. Definition of slopes

6.1.1. Some notation. Suppose that X is a smooth irreducible curve over $\mathbb{F}_p$ (it is *not* assumed to be projective). The order of the residue field of $x \in |X|$ will be denoted by q_x. Let F denote the field of rational functions on X and $\mathbb{A}_F$ its adele ring. Let F_x denote the completion of F at $x \in |X|$ and $O_x \subset F_x$ its ring of integers.

We fix an algebraic closure $\overline{\mathbb{Q}}_p \supset \mathbb{Q}_p$. Let $\overline{\mathbb{Z}}_p$ denote the ring of integers of $\overline{\mathbb{Q}}_p$. Let $v : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ denote the p-adic valuation normalized so that $v(p) = 1$. For each $x \in |X|$, let $v_x : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ denote the p-adic valuation normalized so that $v_x(q_x) = 1$ (so $v_x = v/\deg x$, where $\deg x := \log_p q_x$).

6.1.2. $p^{1/2}$ and the Satake parameter. We fix a square root of p in $\overline{\mathbb{Q}}_p$ and denote it by $p^{1/2}$. For each $x \in |X|$ we set $q_x^{1/2} := (p^{1/2})^{\deg x}$; we use this square root of q_x in the definition of the Satake parameter of an unramified irreducible representation of $GL(n, F_v)$ over $\overline{\mathbb{Q}}_p$.

6.1.3. Slopes. Let π be an irreducible admissible representation of $GL(n, \mathbb{A}_F)$ over $\overline{\mathbb{Q}}_p$. Let S_π denote the set of all $x \in |X|$ such that π_x is ramified.

For every $x \in |X| \setminus S_\pi$ the Satake parameter of π_x is an n-tuple $(\gamma_1, \ldots, \gamma_n) \in (\overline{\mathbb{Q}}_p^\times)^n$ defined up to permutations. Set $a_i^x(\pi) := v_x(\gamma_i)$. Note that $a_i^x(\pi)$ does not depend on the choice of $p^{1/2}$ in §6.1.2. We order the numbers $a_i^x(\pi) \in \mathbb{Q}$ so that $a_i^x(\pi) \geq a_{i+1}^x(\pi)$. These numbers will be called the *slopes* of π at x.

6.1.4. Slopes and the central character. Let $\eta : \mathbb{A}_F^\times \to \overline{\mathbb{Q}}_p^\times$ denote the central character of π. Then for every $x \in |X| \setminus S_\pi$ and every $u \in F_x^\times$ one has

$$(6.1) \qquad v(\eta(u)) = -v(|u|) \cdot \sum_{i=1}^n a_i^x(\pi),$$

where $|u| \in \mathbb{Q}^\times$ is the normalized absolute value. If η is trivial on $F^\times \subset \mathbb{A}_F^\times$ then there exists $c \in \mathbb{Q}^\times$ such that $v(\eta(u)) = c \cdot v(|u|)$ for all $u \in \mathbb{A}_F^\times$. By (6.1), this implies that the number $\sum_{i=1}^n a_i^x(\pi)$ does not depend on $x \in |X| \setminus S_\pi$.

6.2. The result

As before, let π be an irreducible representation of $GL(n, \mathbb{A}_F)$ over $\overline{\mathbb{Q}}_p$.

Theorem 6.2.1. *Suppose that π is cuspidal automorphic. Then*

(i) there exist rational numbers $a_i^\eta(\pi)$, $1 \leq i \leq n$, such that $a_i^x(\pi) = a_i^\eta(\pi)$ for all but finitely many $x \in |X|$;

(ii) for all $x \in |X| \setminus S_\pi$ and $r \in \{1, \ldots, n-1\}$ one has

$$\sum_{i=1}^r a_i^x(\pi) \leq \sum_{i=1}^r a_i^\eta(\pi);$$

(iii) $a_i^\eta(\pi) - a_{i+1}^\eta(\pi) \leq 1$ for all $i \in \{1, \ldots, n-1\}$.

Proof. By Theorem 4.2.2 of Abe's work [Ab], π corresponds (in the sense of Langlands) to some irreducible object of $F\text{-Isoc}^\dagger(X \setminus S_\pi) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. So statements (i)–(ii) hold by semicontinuity of the Newton polygon, and (iii) holds by Theorem 1.1.5 and Corollary 4.2.4. $\qquad \square$

Remark 6.2.2. Let π be as in Theorem 6.2.1. Set $A(\pi) := \frac{1}{n} \cdot \sum_{i=1}^{n} a_i^x(\pi)$; by §6.1.4, this number does not depend on x. Similarly to Corollary 1.1.7, Theorem 6.2.1 implies that

$$\sum_{i=1}^{r} a_i^x(\pi) - rA(\pi) \le r(n-r)/2$$

(6.2)
$$\text{for all } x \in x \in |X| \setminus S_\pi \text{ and } r \in \{1, \ldots, n-1\}.$$

On the other hand, V. Lafforgue explained in [Laf2] that the inequality (6.2) has an easy direct proof. We will recall it in §6.3.3.

Question 6.2.3. *Can Theorem 6.2.1 be proved directly (i.e., without passing to F-isocrystals)?*

As already mentioned, the inequality (6.2) has a direct proof. If $n = 2$ then formula (6.2) means that $a_1^x(\pi) - a_2^x(\pi) \le 1$ for all $x \in |X| \setminus S_\pi$. So for $n = 2$ the only question is whether statements (i)–(ii) of Theorem 6.2.1 have a direct proof.

6.3. Reformulation in terms of Hecke eigenvalues

6.3.1. Hecke eigenvalues in terms of the Satake parameter. Let π be a smooth representation of $GL(n, \mathbb{A}_F)$ over $\overline{\mathbb{Q}}_p$. Then for each $x \in |X| \setminus S_\pi$ one has the usual Hecke operators T_i^x, $1 \le i \le n$, acting on the subspace of $GL(n, O_x)$-invariants of π. If π is irreducible and admissible then T_i^x acts as multiplication by some number $t_i^x(\pi) \in \overline{\mathbb{Q}}_p$. The numbers $t_i^x(\pi)$ (where $x \in |X| \setminus S_\pi$ and $1 \le i \le n$) are called *Hecke eigenvalues.* Formula (3.14) of [Gr] (which goes back to T. Tamagawa) tells us that

(6.3)
$$1 + \sum_{i=1}^{n} (q_x^{1/2})^{-i(n-i)} t_i^x(\pi) z^i = \prod_{i=1}^{n} (1 + \gamma_i z),$$

where $(\gamma_1, \ldots, \gamma_n)$ is the Satake parameter of π_x, and z is a variable.

Lemma 6.3.2. *For all $r \in \{0, \ldots, n\}$ and $x \in |X| \setminus S_\pi$ one has*

(6.4)
$$\sum_{i=1}^{n-r} a_i^x(\pi) = v_x(t_n^x(\pi)) - \text{Newt}_\pi^x(r),$$

where $\mathrm{Newt}_\pi^x : \{0, 1, \ldots, n\} \to \mathbb{Q}$ *is the biggest convex function such that*

(6.5)
$$\mathrm{Newt}_\pi^x(0) = 0,$$
$$\mathrm{Newt}_\pi^x(r) \le v_x(t_r^x(\pi)) + \frac{r(r-n)}{2} \ \text{for } r \in \{1, \ldots, n\}.$$

Proof. This follows from (6.3) and the usual relation (via Newton polygons) between the absolute values of the roots and coefficients of a polynomial over $\overline{\mathbb{Q}}_p$ (e.g., see [Neu, Ch. II, Prop. 6.3]). □

6.3.3. V. Lafforgue's proof of (6.2). We think of π as a subspace of the space of automorphic forms. Since π is cuspidal, the automorphic forms from π are compactly supported modulo the center of $GL(n, \mathbb{A})$.

After twisting π, we can assume that

(6.6)
$$\eta(\mathbb{A}_F^\times) \subset \overline{\mathbb{Z}}_p^\times,$$

where $\eta : \mathbb{A}_F^\times \to \overline{\mathbb{Q}}_p^\times$ is the central character of π. Let $N \subset \pi$ be the $\overline{\mathbb{Z}}_p$-submodule of those automorphic forms from π whose values belong to $\overline{\mathbb{Z}}_p$. Then $N \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p = \pi$; this follows from (6.6) because automorphic forms from π are compactly supported modulo the center. For each $x \in |X|$ the submodule $N^{GL(n, O_x)}$ is stable under the Hecke operators at x. For any open subgroup $U \subset GL(n, \mathbb{A})$, the $\overline{\mathbb{Z}}_p$-module N^U has finite type. So $t_i^x(\pi) \in \overline{\mathbb{Z}}_p$ for all $i \in \{1, \ldots, n\}$ and $x \in |X| \backslash S_\pi$. Moreover, $t_n^x(\pi) \in \overline{\mathbb{Z}}_p^\times$ by (6.6). So the function Newt_π^x from Lemma 6.3.2 satisfies the inequality $\mathrm{Newt}_\pi^x(r) \ge r(r-n)/2$, and by (6.4) we have $\sum_{i=1}^{n-r} a_i^x(\pi) = - \mathrm{Newt}_\pi^x(r) \le r(n-r)/2$. This proves (6.2). □

6.3.4. Reformulation of Theorem 6.2.1 in terms of Hecke eigenvalues. Let π be a cuspidal automorphic representation of $GL(n, \mathbb{A}_F)$. For each $x \in |X| \setminus S_\pi$ let $\mathrm{Newt}_\pi^x : \{0, 1, \ldots, n\} \to \mathbb{Q}$ be the biggest convex function satisfying (6.5). By Lemma 6.3.2, one can reformulate Theorem 6.2.1 as follows:

(i) there exists a function $\mathrm{Newt}_\pi^\eta : \{0, 1, \ldots, n\} \to \mathbb{Q}$ such that $\mathrm{Newt}_\pi^x = \mathrm{Newt}_\pi^\eta$ for almost all $x \in |X|$;

(ii) $\mathrm{Newt}_\pi^x \ge \mathrm{Newt}_\pi^\eta$ for all $x \in |X| \setminus S_\pi$;

(iii) $\mathrm{Newt}_\pi^\eta(r+1) - 2 \, \mathrm{Newt}_\pi^\eta(r) + \mathrm{Newt}_\pi^\eta(r-1) \le 1$ for all $r \in \{1, \ldots, n-1\}$.

Example 6.3.5. Let π be a cuspidal automorphic representation of $PGL(3, \mathbb{A}_F)$. For $x \in |X| \setminus S_\pi$ and $i \in \{1, 2, 3\}$ set $c_i^x := v_x(t_i^x(\pi))$. As explained in §6.3.3, it is clear that for all $x \in |X| \setminus S_\pi$ one has $c_1^x, c_2^x \ge 0$ and

$c_3^x = 0$. According to Theorem 6.2.1(i,iii), for *almost* all $x \in |X| \setminus S_\pi$ the point (c_1^x, c_2^x) belongs to $A \cup B$, where

$$A := \{(y_1, y_2) \in \mathbb{Q}^2 \mid y_1 \geq \frac{1}{3}, \, y_2 \geq \frac{1}{3}\},$$
$$B := \{(y_1, y_2) \in \mathbb{Q}^2 \mid 0 \leq y_1/2 \leq y_2 \leq 2y_1\}.$$

7. Lemmas on algebraic groups

We fix an algebraically closed field E of characteristic 0. All group schemes will be over E. All vector spaces and representations are assumed finite-dimensional and over E.

7.1. The group scheme $\widetilde{\mathbb{G}}_m$

Set $\widetilde{\mathbb{G}}_m := \mathrm{Hom}(\mathbb{Q}, \mathbb{G}_m) = \varprojlim_n \mathrm{Hom}(n^{-1}\mathbb{Z}, \mathbb{G}_m)$. For each $n \in \mathbb{N}$ one has

$$\mathrm{Hom}(n^{-1}\mathbb{Z}, \mathbb{G}_m) \simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m.$$

So for any algebraic group H the set $\mathrm{Hom}(\widetilde{\mathbb{G}}_m, H)$ canonically identifies with the quotient of the product $\mathrm{Hom}(\mathbb{G}_m, H) \times \mathbb{N}$ by the following equivalence relation: a pair $(f_1, n_1) \in \mathrm{Hom}(\mathbb{G}_m, H) \times \mathbb{N}$ is equivalent to (f_2, n_2) if $f_1^{n_2} = f_2^{n_1}$. Note that $\mathrm{Hom}(\widetilde{\mathbb{G}}_m, \mathbb{G}_m) = \mathbb{Q}$, so the weights of $\widetilde{\mathbb{G}}_m$ are rational numbers.

7.2. The small gaps condition

7.2.1. We say that a $\widetilde{\mathbb{G}}_m$-module V has *small gaps* if the gap between any consecutive weights of $\widetilde{\mathbb{G}}_m$ in V is ≤ 1.

7.2.2. Let G be a connected reductive group and $\check{\Lambda}_G^{+,\mathbb{Q}}$ the set of its dominant rational coweights. We say that $\check{\lambda} \in \check{\Lambda}_G^{+,\mathbb{Q}}$ has *small gaps* if $(\check{\lambda}, \alpha_i) \leq 1$ for every simple root α_i of G. This is equivalent to the condition $\check{\rho} - \check{\lambda} \in \check{\Lambda}_G^{+,\mathbb{Q}}$, where $\check{\rho} \in \check{\Lambda}_G^{+,\mathbb{Q}}$ is one half of the sum of the positive coroots of G (to see this, recall that $(\check{\lambda}, \alpha_i) = 1$ for all i).

Now assume that G is reductive but not necessarily connected. Let G° be the neutral connected component of G. One has a canonical bijection between $\check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ and the set of G°-conjugacy classes of elements $\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G) = \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G^\circ)$; the class of μ in $\check{\Lambda}_G^{+,\mathbb{Q}}$ will be denoted by $[\mu]$. We say that $\mu : \widetilde{\mathbb{G}}_m \to G$ has *small gaps* if $[\mu]$ has small gaps. Note that if $G = GL(V)$ this is equivalent to §7.2.1.

7.3. The parabolics $\mathfrak{p}_\mu^\pm$ and $P_\mu^\pm$

Somewhat informally, the parabolics defined below are related to the "big gaps" of $\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G)$.

7.3.1. The subalgebras $\mathfrak{p}_\mu^\pm \subset \mathfrak{g}$. Let G be a reductive group and G° its neutral connected component. Let $\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G)$.

Set $\mathfrak{g} := \mathrm{Lie}(G)$. Then $\widetilde{\mathbb{G}}_m$ acts on $\mathfrak{g}$ via μ. Consider the weight decomposition

$$\mathfrak{g} = \bigoplus_{r \in \mathbb{Q}} \mathfrak{g}_r$$

corresponding to this $\widetilde{\mathbb{G}}_m$-action. Let $\mathfrak{p}_\mu^+$ (resp. $\mathfrak{p}_\mu^-$) denote the Lie subalgebra of $\mathfrak{g}$ generated by the subspaces $\mathfrak{g}_r$ for $r \geq -1$ (resp. $r \leq 1$).

Lemma 7.3.2. *(i) $\mathfrak{p}_\mu^+$ and $\mathfrak{p}_\mu^-$ are parabolic subalgebras of $\mathfrak{g}$ opposite to each other.*

(ii) The Lie subalgebra $\mathfrak{p}_\mu^+ \cap \mathfrak{p}_\mu^- \subset \mathfrak{g}$ is generated by the subspaces $\mathfrak{g}_r$ for $-1 \leq r \leq 1$.

(iii) Let $T \subset G^\circ$ be a maximal torus containing $\mu(\widetilde{\mathbb{G}}_m)$. Choose a basis α_i, $i \in I$, in the root system of (G°, T) so that $(\alpha_i, \mu) \geq 0$. Set $I_{\leq 1} := \{i \in I \mid (\alpha_i, \mu) \leq 1\}$. Then the Lie algebra $\mathfrak{p}_\mu^+$ is generated by $\mathrm{Lie}(T)$, the root spaces $\mathfrak{g}_{\alpha_i}$ for $i \in I$, and the spaces $\mathfrak{g}_{-\alpha_i}$ for $i \in I_{\leq 1}$; the Lie algebra $\mathfrak{p}_\mu^-$ is generated by $\mathrm{Lie}(T)$, the root spaces $\mathfrak{g}_{-\alpha_i}$ for $i \in I$, and the spaces $\mathfrak{g}_{\alpha_i}$ for $i \in I_{\leq 1}$; finally, the Lie algebra $\mathfrak{p}_\mu^+ \cap \mathfrak{p}_\mu^-$ is generated by $\mathrm{Lie}(T)$ and the spaces $\mathfrak{g}_{\pm\alpha_i}$ for $i \in I_{\leq 1}$.

Proof. Statement (iii) is clear. Statements (i)-(ii) follow. $\qquad\square$

7.3.3. The subgroups $P_\mu^\pm$ and M_μ. Let G and μ be as in §7.3.1. The group $\pi_0(G) = G/G^\circ$ acts on $\check\Lambda_{G^\circ}^{+;\mathbb{Q}}$. From now on *we will assume that the class $[\mu] \in \check\Lambda_{G^\circ}^{+;\mathbb{Q}}$ is (G/G°)-invariant.* This means that the G°-conjugacy class of μ is equal to its G-conjugacy class or equivalently, that

(7.1) $$G = G^\circ \cdot Z_G(\mu),$$

where $Z_G(\mu)$ is the centralizer of μ in G.

Set

$$P_\mu^+ := \{g \in G \mid \mathrm{Ad}_g(\mathfrak{p}_\mu^+) = \mathfrak{p}_\mu^+\}, \quad P_\mu^- := \{g \in G \mid \mathrm{Ad}_g(\mathfrak{p}_\mu^-) = \mathfrak{p}_\mu^-\},$$
$$M_\mu := P_\mu^+ \cap P_\mu^-.$$

Then $\mathrm{Lie}(P_\mu^\pm) = \mathfrak{p}_\mu^\pm$. The groups $P_\mu^\pm \cap G^\circ$ and $M_\mu \cap G^\circ$ are connected. It is easy to check that

$$(7.2) \qquad\qquad\qquad Z_G(\mu) \subset M_\mu.$$

Combining this with (7.1), we see that the map $G \to \pi_0(G)$ induces isomorphisms

$$\pi_0(P_\mu^\pm) \xrightarrow{\sim} \pi_0(G), \quad \pi_0(M_\mu) \xrightarrow{\sim} \pi_0(G).$$

By Lemma 7.3.2(iii), $M_\mu = G$ if and only if μ has small gaps in the sense of §7.2.2.

Example 7.3.4. Let $G = GL(V)$, where V is a vector space. Let $\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G)$, then $\widetilde{\mathbb{G}}_m$ acts on V. It is clear that the $\widetilde{\mathbb{G}}_m$-module V has a unique decomposition into a direct sum of $\widetilde{\mathbb{G}}_m$-submodules $V_1, \ldots, V_m$ with the following properties:

(a) each of the the $\widetilde{\mathbb{G}}_m$-modules V_i has small gaps;

(b) if $r \in \mathbb{Q}$ is a weight of $\widetilde{\mathbb{G}}_m$ on V_i and $r' \in \mathbb{Q}$ is a weight of $\widetilde{\mathbb{G}}_m$ on V_{i+1} then $r - r' > 1$.

In this situation the parabolic P_μ^+ defined in §7.3.3 is the stabilizer of the flag formed by the subspaces $V_{\leq j} := \bigoplus_{i \leq j} V_i$, and P_μ^- is the stabilizer of the flag formed by the subspaces $V_{\geq j}$.

Remark 7.3.5. Let G and μ be as in §7.3.3. Let G' be a reductive group and $\rho : G \to G'$ a homomorphism. We claim that

$$\rho(P_\mu^\pm) \subset P_{\rho \circ \mu}^\pm.$$

Indeed, since $P_\mu^\pm = (P_\mu^\pm \cap G^\circ) \cdot Z_G(\mu)$ it suffices to check that $\rho(P_\mu^\pm \cap G^\circ) \subset P_{\rho \circ \mu}^\pm$. This follows from the inclusion $\rho(\mathfrak{p}_\mu^\pm) \subset \mathfrak{p}_{\rho \circ \mu}^\pm$, which holds by the definition of $\mathfrak{p}_\mu^\pm$.

7.3.6. Tannakian approach to $P_\mu^\pm$. Let G and μ be as in §7.3.3. Then we have the parabolics $P_\mu^\pm$ defined in §7.3.3. On the other hand, for every representation $\rho : G \to GL(V)$ we have the parabolics $P_{\rho \circ \mu}^\pm \subset GL(V)$ corresponding to $\rho \circ \mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, GL(V))$ (see Example 7.3.4 for their explicit description).

Lemma 7.3.7. $P_\mu^\pm = \bigcap_{\rho \in \hat{G}} \rho^{-1}(P_{\rho \circ \mu}^\pm)$, *where $\hat{G}$ is the set of isomorphism classes of irreducible representations of G.*

Proof. For any $\rho \in \hat{G}$ one has $P_\mu^\pm \subset \rho^{-1}(P_{\rho\circ\mu}^\pm)$ by Remark 7.3.5.

Let us construct $\rho_+, \rho_- \in \hat{G}$ such that

$$(7.3) \qquad P_\mu^+ \supset \rho_+^{-1}(P_{\rho_+\circ\mu}^+), \quad P_\mu^- \supset \rho_-^{-1}(P_{\rho_+\circ\mu}^-).$$

Let T, I and $I_{\leq 1}$ be as in Lemma 7.3.2(iii). The subset $I_{\leq 1} \subset I$ is stable under the action of the group $G/G^\circ = Z_G(\mu)/Z_{G^\circ}(\mu)$. So there exists a (G/G°)-invariant dominant weight ω of G° such that

$$\{i \in I \mid (\omega, \check{\alpha}_i) = 0\} = I_{\leq 1}.$$

Let $\rho^\circ \in \widehat{G^\circ}$ have highest weight ω. Then there exists an irreducible G-module V whose restriction to G° is a multiple of ρ°. We have a homomorphism $\rho_+ : G \to GL(V)$ and the dual homomorphism $\rho_- : G \to GL(V^*)$. We claim that the inclusions (7.3) hold.

Let us check this for ρ_+. Let $V_\omega \subset V$ be the highest weight subspace (i.e., the maximal subspace on which T acts by ω), and let $\mathrm{Stab}(V_\omega) \subset G$ be the stabilizer of V_ω in G. Then $\rho_+^{-1}(P_{\rho_+\circ\mu}^+) \subset \mathrm{Stab}(V_\omega)$ because $(\omega - \omega', \mu) > 1$ for any weight ω' of T in V_ω such that $\omega' \neq \omega$. On the other hand, $\mathrm{Stab}(V_\omega) \cap G^\circ = P_\mu^+ \cap G^\circ$, so $\mathrm{Stab}(V_\omega)$ is contained in the normalizer of $P_\mu^+ \cap G^\circ$ in G, which equals P_μ^+. $\square$

Corollary 7.3.8. *Let G and μ be as in §7.3.3. Suppose that every irreducible G-module has small gaps as a $\widetilde{\mathbb{G}}_m$-module (with the $\widetilde{\mathbb{G}}_m$-action being defined via $\mu : \widetilde{\mathbb{G}}_m \to G$). Then μ has small gaps.* $\square$

Let us note that the converse of Corollary 7.3.8 is also true (and easy to prove).

Corollary 7.3.9. *Let $\rho : G \to G'$ be a homomorphism of reductive groups. In addition, assume that G is semisimple. If $\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G)$ has small gaps then so does $\rho \circ \mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G')$.*

Proof. We can assume that G and G' are connected. Applying Corollary 7.3.8 to G', we reduce the proof to the case that $G' = GL(V)$.

Since G is semisimple the biggest weight of $\widetilde{\mathbb{G}}_m$ in any G-module is non-negative and the smallest one is non-positive (because the sum of all weights is zero). This allows us to reduce to the case that $\rho : G \to GL(V)$ is irreducible. This case is clear. $\square$

7.3.10. Elements of $\mathrm{Hom}(\widetilde{\mathbb{G}}_{\boldsymbol{m}}, \boldsymbol{G})$ centralizing each other. Let G, μ, $\mathfrak{g}$, and $\mathfrak{g}_r$ be as in §7.3.1. Let G_0 denote the centralizer of μ in G°; then $\mathrm{Lie}(G_0)$ equals the weight space $\mathfrak{g}_0$. The group G_0 is connected and reductive (in fact, it is a Levi of G°).

Now let $\nu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G_0)$. Then the product $\mu \cdot \nu$ is a homomorphism $\widetilde{\mathbb{G}}_m \to G$. Set $\mu' := \mu \cdot \nu$.

Lemma 7.3.11. *Suppose that $\nu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G_0)$ has small gaps (see §7.2.2) and $\nu(\widetilde{\mathbb{G}}_m) \subset [G_0, G_0]$. Then*
(i) $\mathfrak{p}_{\mu'}^{\pm} \supset \mathfrak{p}_{\mu}^{\pm}$;
(ii) if μ has small gaps then so does μ'.

Proof. It suffices to prove (i). We will show that $\mathfrak{p}_{\mu'}^+ \supset \mathfrak{p}_{\mu}^+$ (the inclusion $\mathfrak{p}_{\mu'}^- \supset \mathfrak{p}_{\mu}^-$ is proved similarly). This is equivalent to proving that $\mathfrak{p}_{\mu'}^+ \supset \mathfrak{g}_r$ for all $r \geq -1$.

It is clear that for every $r \geq -1$ one has $\mathfrak{p}_{\mu'}^+ \supset \mathfrak{g}_r \cap \mathfrak{g}^{\geq 0}$, where $\mathfrak{g}^{\geq 0} \subset \mathfrak{g}$ is the sum of the non-negative weight spaces of the action $\widetilde{\mathbb{G}}_m \xrightarrow{\nu} G_0 \to \mathrm{Aut}\,\mathfrak{g}$. Since ν has small gaps one has $\mathfrak{p}_{\mu'}^+ \supset \mathfrak{g}_0$. So for each $r \geq -1$ one has $\mathfrak{p}_{\mu'}^+ \supset V_r$, where $V_r \subset \mathfrak{g}_r$ is the G_0-submodule generated by $\mathfrak{g}_r \cap \mathfrak{g}^{\geq 0}$. It remains to show that $V_r = \mathfrak{g}_r$. Assume the contrary; then the composition

$$\widetilde{\mathbb{G}}_m \xrightarrow{\nu} G_0 \to \mathrm{Aut}(\mathfrak{g}/V_r) \xrightarrow{\det} \mathbb{G}_m$$

is strictly negative, which contradicts the assumption $\nu(\widetilde{\mathbb{G}}_m) \subset [G_0, G_0]$. ☐

7.4. The key lemma

The goal of this subsection is to prove Lemma 7.4.2, which will be used in the proof of Theorem 8.6.3.

Lemma 7.4.1. *Let G and μ be as in §7.3.3. Let $H \subset G$ be a subgroup with the following properties:*
(i) $H \supset \mu(\widetilde{\mathbb{G}}_m)$;
(ii) every indecomposable H-module has small gaps as a $\widetilde{\mathbb{G}}_m$-module (with the $\widetilde{\mathbb{G}}_m$-action being defined via $\mu : \widetilde{\mathbb{G}}_m \to H$);
Then $H \subset M_\mu$, $Z_G(H) \subset M_\mu$, and

$$(7.4) \qquad\qquad Z(M_\mu) \subset Z(Z_G(H)).$$

Here $Z(M_\mu)$ is the center of M_μ, and $Z(Z_G(H))$ is the center of the centralizer of H.

Proof. By property (ii), for every representation $\rho : G \to GL(V)$ one has $\rho(H) \subset M_{\rho \circ \mu}$. By Lemma 7.3.7, this implies that $H \subset M_\mu$.

By property (i), $Z_G(H) \subset Z_G(\mu)$. By (7.2), $Z_G(\mu) \subset M_\mu$. So $Z_G(H) \subset M_\mu$. This implies that $Z(M_\mu)$ centralizes $Z_G(H)$. On the other hand, $Z(M_\mu) \subset Z_G(H)$ because $H \subset M_\mu$. So $Z(M_\mu) \subset Z(Z_G(H))$. $\square$

Lemma 7.4.2. *Let G, μ, H be as in Lemma 7.4.1. In addition, assume that the inclusion*

$$\mathrm{Hom}(\mathbb{G}_m, Z(G)) \subset \mathrm{Hom}(\mathbb{G}_m, Z(Z_G(H)))$$

is an equality. Then $\mu : \widetilde{\mathbb{G}}_m \to G$ has small gaps.

Proof. By (7.4), the inclusion $\mathrm{Hom}(\mathbb{G}_m, Z(G)) \subset \mathrm{Hom}(\mathbb{G}_m, Z(M_\mu))$ is an equality. Therefore

$$(7.5) \qquad \mathrm{Hom}(\widetilde{\mathbb{G}}_m, Z(G)) = \mathrm{Hom}(\widetilde{\mathbb{G}}_m, Z(M_\mu)).$$

We have to prove that $M_\mu = G$. Assume the contrary. It is clear that $\mu(\widetilde{\mathbb{G}}_m) \subset M_\mu^\circ$, where $M_\mu^\circ := M_\mu \cap G^\circ$. Let $\bar\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, Z(M_\mu^\circ))$ denote the image of $\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, M_\mu^\circ)$ under the composition

$$\mathrm{Hom}(\widetilde{\mathbb{G}}_m, M_\mu^\circ) \to \mathrm{Hom}(\widetilde{\mathbb{G}}_m, M_\mu^\circ/[M_\mu^\circ, M_\mu^\circ]) \xrightarrow{\sim} \mathrm{Hom}(\widetilde{\mathbb{G}}_m, Z(M_\mu^\circ)).$$

To get a contradiction, we will show that $\bar\mu$ belongs to the r.h.s. of (7.5) but not to the l.h.s.

It is clear that $\bar\mu$ is invariant under all automorphisms of G preserving μ, so $Z_G(\bar\mu) \supset Z_G(\mu)$. Since $\bar\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, Z(M_\mu^\circ))$ we also have $Z_G(\bar\mu) \supset M_\mu^\circ$. So by (7.1), we get $Z_G(\bar\mu) \supset M_\mu$. Therefore $\bar\mu \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, Z(M_\mu))$.

The coweight μ is strictly dominant with respect to P_μ^+ (i.e., $(\alpha, \bar\mu) > 0$ for all roots α such that $\mathfrak{g}_\alpha$ is contained in the Lie algebra of the unipotent radical of P_μ^+). So $\bar\mu$ is strictly dominant with respect to P_μ^+ (indeed, one can get $\bar\mu$ from μ by averaging with respect to the action of the Weyl group of M_μ). Since we assumed that $M_\mu \neq G$, we get $\bar\mu(\widetilde{\mathbb{G}}_m) \not\subset Z(G)$. $\square$

8. On Newton coweights for homomorphisms $\pi_1(X) \to G(\overline{\mathbb{Q}}_\ell)$, where G is reductive

The goal of this section is to generalize Theorem 1.3.3 by replacing $GL(n)$ with an arbitrary reductive group.

We fix a universal cover $\tilde X \to X$ (this is a scheme which usually has infinite type over $\mathbb{F}_p$). Set $\Pi := \mathrm{Aut}(\tilde X/X)$.

If one also chooses a geometric point ξ of X and a lift of ξ to $\tilde X$ then Π identifies with $\pi_1(X, \xi)$.

8.1. Frobenius elements

Let $|\tilde{X}|$ denote the set of closed points of $\tilde{X}$. For each $\tilde{x} \in |\tilde{X}|$ we have the geometric Frobenius $\mathrm{Fr}_{\tilde{x}} \in \Pi$: this is the unique automorphism of $\tilde{X}$ over X whose restriction to $\{\tilde{x}\}$ equals the composition $\{\tilde{x}\} \xrightarrow{\varphi} \{\tilde{x}\} \hookrightarrow \tilde{X}$, where $\varphi : \{\tilde{x}\} \to \{\tilde{x}\}$ is the Frobenius morphism with respect to x (this means that for any regular function f on $\{\tilde{x}\}$ one has $\varphi * (f) = f^{q_x}$, where q_x is the order of the residue field of x).

The map $|\tilde{X}| \to \Pi$ defined by $\tilde{x} \mapsto \mathrm{Fr}_{\tilde{x}}$ is Π-equivariant (we assume that Π acts on itself by conjugation). So for $x \in X$ one has the geometric Frobenius $\mathrm{Fr}_x \in \Pi$, which is defined only up to conjugacy.

8.2. A class of homomorphisms $\Pi \to G(\overline{\mathbb{Q}}_{\boldsymbol{\ell}})$

Let G be an algebraic group over $\overline{\mathbb{Q}}_\ell$.

Remark 8.2.1. Let $g \in G(\overline{\mathbb{Q}}_\ell)$. Let $g_{\mathrm{ss}} \in G(\overline{\mathbb{Q}}_\ell)$ be the semisimple part of the Jordan decomposition of g, and let $\langle g_{\mathrm{ss}} \rangle$ be the smallest algebraic subgroup of G containing g_{ss}. Then the following conditions are equivalent:

(i) the eigenvalues of the image of g in any representation of G are in $\overline{\mathbb{Q}}$;
(ii) $\chi(g_{\mathrm{ss}}) \in \overline{\mathbb{Q}}$ for all $\chi \in \mathrm{Hom}(\langle g_{\mathrm{ss}} \rangle, \mathbb{G}_m)$.

8.2.2. "Algebraic" homomorphisms $\Pi \to G(\overline{\mathbb{Q}}_{\boldsymbol{\ell}})$.   Let $\sigma : \Pi \to G(\overline{\mathbb{Q}}_\ell)$ be a continuous homomorphism.[6] Similarly to §1.3.1, we say that σ is *"algebraic"* if for any $x \in |X|$ the element $\sigma(\mathrm{Fr}_x) \in G(\overline{\mathbb{Q}}_\ell)$ (which is well defined up to conjugacy) satisfies the equivalent conditions of Remark 8.2.1.

8.3. The slope homomorphisms

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$. Fix a valuation $v : \overline{\mathbb{Q}}^\times \to \mathbb{Q}$ such that $v(p) = 1$. Let $\widetilde{\mathbb{G}}_m$ be the pro-torus over $\overline{\mathbb{Q}}_\ell$ with group of characters $\mathbb{Q}$ (see §7.1).

8.3.1. The homomorphism $\boldsymbol{\mu_{g,w}}$.   If $g \in G(\overline{\mathbb{Q}}_\ell)$ satisfies the equivalent conditions of Remark 8.2.1 then each homomorphism $w : \overline{\mathbb{Q}}^\times \to \mathbb{Q}$ defines a homomorphism

$$\mathrm{Hom}(\langle g_{\mathrm{ss}} \rangle, \mathbb{G}_m) \to \mathbb{Q}, \quad \chi \mapsto w(\chi(g_{\mathrm{ss}})),$$

which is the same as a homomorphism $\mu_{g,w} : \widetilde{\mathbb{G}}_m \to \langle g_{\mathrm{ss}} \rangle \subset G$.

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[6]It is well known that the image of any continuous homomorphism $\Pi \to G(\overline{\mathbb{Q}}_\ell)$ is defined over some *finite* extension of $\mathbb{Q}_\ell$ (e.g., see [Dr2, Prop. 3.2.2]).

8.3.2. The slope homomorphisms. Let $\sigma : \Pi \to G(\overline{\mathbb{Q}}_\ell)$ be a continuous homomorphism, which is "algebraic" in the sense of §8.2.2. Let $\tilde{x} \in |\tilde{X}|$, and let $x \in |X|$ be its image. Applying §8.3.1 to $g = \sigma(\mathrm{Fr}_{\tilde{x}})$ and $w = v/(\deg x)$, one gets a homomorphism $\mu_{\tilde{x}} : \widetilde{\mathbb{G}}_m \to G$. We call it the *slope homomorphism* corresponding to $\tilde{x}$. (This agrees with the terminology of [RR, Thm. 1.8].)

8.4. The Newton coweights

From now on we assume that G is reductive. Let G° be the neutral connected component of G. Let $\sigma : \Pi \to G(\overline{\mathbb{Q}}_\ell)$ be a continuous homomorphism, which is "algebraic" in the sense of §8.2.2.

For each $\tilde{x} \in |\tilde{X}|$ one has the element $\mu_{\tilde{x}} \in \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G) = \mathrm{Hom}(\widetilde{\mathbb{G}}_m, G^\circ)$ defined in §8.3.2. Recall that the set of conjugacy classes of homomorphisms $\widetilde{\mathbb{G}}_m \to G^\circ$ identifies with the set of dominant rational coweights of G°, denoted by $\check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$. The class of $\mu_{\tilde{x}}$ in $\check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ is denoted by $a^{\tilde{x}}(\sigma)$ and called the *Newton coweight* of σ at $\tilde{x}$.

The map $|\tilde{X}| \to \check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ defined by $\tilde{x} \mapsto a^{\tilde{x}}(\sigma)$ is Π-equivariant, where Π acts on $\check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ via the composition $\Pi \to G \to G/G^\circ \to \mathrm{Aut}(\check{\Lambda}_{G^\circ}^{+,\mathbb{Q}})$. In particular, if $G = G^\circ$ then $a^{\tilde{x}}(\sigma)$ depends only on the image of $\tilde{x}$ in $|X|$, so we can write $a^x(\sigma)$ for $x \in |X|$. In the case $G = GL(n)$ a homomorphism $\sigma : \Pi \to GL(n, \overline{\mathbb{Q}}_\ell)$ defines a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{E}$, and $a^x(\sigma)$ is just the collection of the slopes $a_i^x(\mathcal{E})$ from §1.3.2.

8.5. A generalization of Theorem 1.3.3(i-ii)

For any reductive group G, we equip $\check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ with the following partial order: $\check{\lambda}_1 \leq \check{\lambda}_2$ if $\check{\lambda}_2 - \check{\lambda}_1$ is a linear combination of simple coroots with non-negative rational coefficients. Equivalently, this means that $(\omega, \check{\lambda}_1) \leq (\omega, \check{\lambda}_2)$ for all dominant weights of G°.

Proposition 8.5.1. *Let G and $\sigma : \Pi \to G(\overline{\mathbb{Q}}_\ell)$ be as in §8.4. In addition, assume that the composition*

$$\Pi \xrightarrow{\ \sigma\ } G \to G/G^\circ$$

is surjective.

(i) There exists a unique element $a^\eta(\sigma) \in (\check{\Lambda}_{G^\circ}^{+,\mathbb{Q}})^{G/G^\circ}$ with the following property: let U denote the set of all $x \in |X|$ such that $a^{\tilde{x}}(\sigma) = a^\eta(\sigma)$ for all $\tilde{x} \in |\tilde{X}|$ mapping to x, then U is non-empty, and for any curve $C \subset X$ the subset $U \cap |C|$ is open in $|C|$.

(ii) For all $\tilde{x} \in |\tilde{X}|$ one has $a^{\tilde{x}}(\sigma) \leq a^\eta(\sigma)$.

Proof. Let $\Pi_1 := \sigma^{-1}(G^\circ) \subset \Pi$, and let $\sigma_1 : \Pi_1 \to G^\circ$ be the restriction of σ. Let $X_1 := \tilde{X}/\Pi_1$ and $\eta_1 \in X_1$ the generic point. We claim that it suffices to prove the proposition for X_1, Π_1, σ_1 instead of X, Π, σ. To show this, one only has to check that the rational coweight $a^{\eta_1}(\sigma_1)$ provided by statement (i) is (G/G°)-invariant. Indeed, uniqueness in statement (i) implies (Π/Π_1)-invariance of $a^{\eta_1}(\sigma_1)$, and $\Pi/\Pi_1 = G/G^\circ$ by the surjectivity assumption.

So it suffices to prove the proposition in the case $G = G^\circ$.

Uniqueness in (i) follows from Lemma 5.3.4. To construct $a^\eta(\sigma)$, we need the following observation.

For each dominant weight ω of G, let ρ_ω be an irreducible representation of G with highest weight ω, and let $\mathcal{E}_\omega$ be the lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X corresponding to $\rho_\omega \circ \sigma$. Then

$$(8.1) \qquad (\omega, a^x(\sigma)) = a_1^x(\mathcal{E}_\omega),$$

where $a_1^x(\mathcal{E}_\omega)$ is the maximal slope of $\mathcal{E}_\omega$ at x.

Choose dominant weights $\omega_1, \ldots, \omega_n$ of G so that each dominant weight of G can be written as a linear combination of ω_i's with non-negative rational coefficients. Let U_i denote the set of all $x \in |X|$ such that $a_1^x(\mathcal{E}_{\omega_i}) = a_1^\eta(\mathcal{E}_{\omega_i})$, where $a_1^\eta(\mathcal{E}_{\omega_i})$ is as in Theorem 1.3.3. By Lemma 5.3.4, $\bigcap_i U_i \neq \emptyset$. By (8.1), the numbers

$$(\omega_i, a^x(\sigma)), \quad x \in \bigcap_i U_i,$$

do not depend on x. So for $x \in \bigcap_i U_i$ the coweight $a^x(\sigma)$ does not depend on x. Take $a^\eta(\sigma)$ to be this coweight. Property (i) is clear (note that $U = \bigcap_i U_i$). By (8.1), we have $(\omega_i, a^x(\sigma)) \leq (\omega_i, a^\eta(\sigma))$ for all $x \in |X|$ and all i. This is equivalent to (ii). □

8.6. A generalization of Theorem 1.3.3(iii)

8.6.1. The group $Z_G(\sigma)$.

Let G and $\sigma : \Pi \to G(\overline{\mathbb{Q}}_\ell)$ be as in Proposition 8.5.1. Let $Z_G(\sigma) \subset G$ denote the centralizer of σ in G.

One can think of $Z_G(\sigma)$ as follows. The homomorphism σ defines a G-local system $\mathcal{E}$ on X (i.e., a G-torsor in the Tannakian category of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on X, see §9.1.1). Then $Z_G(\sigma) = \operatorname{Aut} \mathcal{E}$.

8.6.2. Formulation of the theorem. The center of any group H will be denoted by $Z(H)$. It is clear that $Z(G) \subset Z(Z_G(\sigma))$, so

$$(8.2) \qquad \mathrm{Hom}(\mathbb{G}_m, Z(G)) \subset \mathrm{Hom}(\mathbb{G}_m, Z(Z_G(\sigma))).$$

Theorem 8.6.3. *Suppose that in the situation of Proposition 8.5.1 the inclusion* (8.2) *is an equality. Then $a^\eta(\sigma)$ has small gaps in the sense of §7.2.2.*

The proof will be given in §8.6.7. Similarly to Corollary 1.1.7 and Remark 1.3.6, one has the following

Corollary 8.6.4. *In the situation of Theorem 8.6.3 suppose that G is semisimple. Then $a^x(\sigma) \leq \check{\rho}$ for all $x \in |X|$. (As usual, $\check{\rho} \in \check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ is one half of the sum of the positive coroots of G.)*

Proof. By Theorem 8.6.3, $\check{\rho} - a^\eta(\sigma)$ is dominant. Since G is semisimple, this implies that $\check{\rho} - a^\eta(\sigma) \geq 0$. It remains to use Proposition 8.5.1(ii). $\qquad\square$

Remark 8.6.5. If G has type A_n, Corollary 8.6.4 is due to V. Lafforgue (see Remark 1.3.6). If G has type B_n or C_n, Corollary 8.6.4 easily follows from V.Lafforgue's result. This does not seem to be the case for other groups G.

8.6.6. The case $G = GL(n)$. In this case Theorem 8.6.3 is equivalent to Theorem 1.3.3(iii). To deduce it from Theorem 1.3.3(iii), assume that $a_i^\eta(\sigma) - a_{i+1}^\eta(\sigma) > 1$ for some i. Then Theorem 1.3.3(iii) implies that the representation σ admits a decomposition $\sigma = \sigma_1 \oplus \sigma_2$ such that the slopes of σ_1 (resp. σ_2) at η are the numbers $a_j^\eta(\sigma)$ for $j \leq i$ (resp. $j > i$). This decomposition is unique, so it is preserved by automorphisms of σ. So it yields a homomorphism $\mathbb{G}_m \to Z(\mathrm{Aut}\,\sigma) = Z(Z_{GL(n)}(\sigma))$ whose image is not contained in $Z(GL(n))$.

8.6.7. Proof of Theorem 8.6.3. Let $H \subset G$ be the Zariski closure of $\sigma(\Pi)$. Let U be as in Proposition 8.5.1. Fix $\tilde{x} \in |\tilde{X}| \times_{|X|} U$ and set $\mu := \mu_{\tilde{x}}$, where $\mu_{\tilde{x}} : \widetilde{\mathbb{G}}_m \to G$ is the slope homomorphism (see §8.3.2). Let us check that μ and H satisfy the conditions of Lemma 7.4.2.

Let $[\mu] \in \check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ be the class of μ. Since $\tilde{x} \in U$ we have

$$(8.3) \qquad\qquad [\mu] = a^{\tilde{x}}(\sigma) = a^\eta(\sigma).$$

By Proposition 8.5.1, this implies that $[\mu]$ is (G/G°)-invariant, as required in §7.3.3.

Clearly $H \supset \mu(\widetilde{\mathbb{G}}_m)$. We have $Z_G(\sigma) = Z_G(H)$, so (8.2) means that

$$\mathrm{Hom}(\mathbb{G}_m, Z(G)) = \mathrm{Hom}(\mathbb{G}_m, Z(Z_G(H))).$$

Finally, by Theorem 1.3.3(iii), every indecomposable H-module has small gaps as a $\widetilde{\mathbb{G}}_m$-module (if the $\widetilde{\mathbb{G}}_m$-action is defined via μ).

Applying Lemma 7.4.2, we see that μ has small gaps. By (8.3), this means that $a^\eta(\sigma)$ has small gaps. □

8.7. A corollary related to elliptic Arthur parameters

8.7.1. A class of homomorphisms $\Pi \times SL(2, \overline{\mathbb{Q}}_\ell) \to G(\overline{\mathbb{Q}}_\ell)$. We will be considering homomorphisms $\psi : \Pi \times SL(2, \overline{\mathbb{Q}}_\ell) \to G(\overline{\mathbb{Q}}_\ell)$ satisfying the following conditions:

(i) the restriction of ψ to Π is continuous and "algebraic" in the sense of §8.2.2;

(ii) the restriction of ψ to $SL(2, \overline{\mathbb{Q}}_\ell)$ is a homomorphism of algebraic groups;

(iii) the map $\Pi \to G/G^\circ$ induced by ψ is surjective;

(iv) the centralizer of ψ in G is finite modulo the center of G.

8.7.2. Relation to Arthur parameters. Suppose that $\dim X = 1$ and that G is equipped with a splitting $G/G^\circ \to G$. According to [Laf3, §12.2.2], a homomorphism ψ satisfying (i)-(iii) is called an *Arthur parameter* if the restriction of ψ to Π becomes pure of weight 0 when composed with any representation of G. An Arthur parameter satisfying (iv) is said to be *elliptic* (or *discrete*, see the Appendix of [BC]).

8.7.3. The homomorphism $\phi_\psi : \Pi \to G(\overline{\mathbb{Q}}_\ell)$. Let $\tau : \Pi \to \mathbb{Q}_\ell^\times$ be the cyclotomic character. Fix a square root $\tau^{1/2} : \Pi \to \overline{\mathbb{Q}}_\ell^\times$ whose restriction to the geometric part of Π is trivial. Given ψ as in §8.7.1, define a homomorphism $\phi_\psi : \Pi \to G(\overline{\mathbb{Q}}_\ell)$ by

$$\phi_\psi(\gamma) = \psi(\gamma, \xi(\tau^{1/2}(\gamma))),$$

where $\xi : \mathbb{G}_m \to SL(2)$ is the homomorphism

(8.4) $$\xi(t) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Let us note that if ψ is an Arthur parameter (see §8.7.2) then ϕ_ψ is called the *Langlands parameter associated to ψ*.

Corollary 8.7.4. *Let $\psi : \Pi \times SL(2, \overline{\mathbb{Q}}_\ell) \to G(\overline{\mathbb{Q}}_\ell)$ be a homomorphism satisfying conditions (i)-(iv) of §8.7.1. Then $a^\eta(\phi_\psi)$ has small gaps in the sense of §7.2.2.*

Proof. Let $\sigma : \Pi \to G(\overline{\mathbb{Q}}_\ell)$ and $f : SL(2) \to G$ be the restrictions of ψ. Then $\operatorname{Im} f \subset Z_G(\sigma)$, so

$$Z(Z_G(\sigma)) = Z_G(\sigma) \cap Z_G(Z_G(\sigma)) \subset Z_G(\sigma) \cap Z_G(f) = Z_G(\psi).$$

Therefore condition (iv) of §8.7.1 implies that $\operatorname{Hom}(\mathbb{G}_m, Z(Z_G(\sigma))) = \operatorname{Hom}(\mathbb{G}_m, Z(G))$. So by Theorem 8.6.3, $a^\eta(\sigma)$ has small gaps.

To deduce from this that $a^\eta(\phi_\psi)$ has small gaps, we will apply Lemma 7.3.11. Let $\mu : \widetilde{\mathbb{G}}_m \to G$ be as in §8.6.7; we already know that μ has small gaps. Let G_0 be the centralizer of μ in G°. The homomorphism ψ maps $SL(2)$ to G_0. Let ξ be as in (8.4), and let $\nu : \widetilde{\mathbb{G}}_m \to G_0$ be the composition of $\xi^{1/2} : \widetilde{\mathbb{G}}_m \to SL(2)$ and $\psi|_{SL(2)} : SL(2) \to G_0$. By the theory of $\mathfrak{sl}(2)$-triples, ν has small gaps (see [Bo, §VIII.11, Prop. 5]); this also follows from Corollary 7.3.9. Since $SL(2)$ is semisimple, we have $\nu(\widetilde{\mathbb{G}}_m) \subset [G_0, G_0]$. It remains to apply Lemma 7.3.11(ii). $\square$

9. An analog of Theorem 1.1.5 for arbitrary reductive groups

9.1. Generalities on Tannakian categories

References: [DM, De2, De4].

9.1.1. G-torsors in Tannakian categories. If $\mathcal{T}$ is a Tannakian category over a field E and G is an algebraic group over E, then a *G-torsor in $\mathcal{T}$* is an exact tensor functor $\operatorname{Rep}(G) \to \mathcal{T}$, where $\operatorname{Rep}(G)$ is the tensor category of finite-dimensional representations of G. If $\mathcal{T}$ is the category of vector spaces this is the usual notion of G-torsor. A $GL(n)$-torsor in any Tannakian category $\mathcal{T}$ is the same as an n-dimensional object of $\mathcal{T}$.

For any G, all G-torsors in $\mathcal{T}$ form a groupoid enriched over the category of E-schemes. So for any group scheme H over E there is a notion of H-action on a G-torsor.

9.1.2. Example. A G-torsor in $\operatorname{Rep}(H)$ is the same as a usual G-torsor equipped with H-action. So if E is algebraically closed then G-torsors in $\operatorname{Rep}(H)$ are classified by conjugacy classes of homomorphisms $H \to G$.

Let us apply this for $H = \widetilde{\mathbb{G}}_m$, where $\widetilde{\mathbb{G}}_m$ is as in §7.1. If G is a connected reductive group and E is algebraically closed then the set of conjugacy classes

of homomorphisms $\widetilde{\mathbb{G}}_m \to G$ identifies with the set of dominant rational coweights of G, denoted by $\check{\Lambda}_G^{+,\mathbb{Q}}$, so G-torsors in $\mathrm{Rep}(\widetilde{\mathbb{G}}_m)$ are classified by $\check{\Lambda}_G^{+,\mathbb{Q}}$.

More generally, let $\mathcal{T}$ be a Tannakian category over any field E and $\mathcal{E}$ be a G-torsor in $\mathcal{T}$ equipped with a $\widetilde{\mathbb{G}}_m$-action. Then one defines the class of $\mathcal{E}$ in $\check{\Lambda}_G^{+,\mathbb{Q}}$ to be the class of $F(\mathcal{E})$, where F is any fiber functor on $\mathcal{T}$ over an algebraically closed extension of E.

9.2. G-torsors in F-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$

Clearly F-Isoc(X) and F-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ are Tannakian categories over $\mathbb{Q}_p$ and $\overline{\mathbb{Q}}_p$, respectively.

9.2.1. Grading by slopes. For any $x \in X$, let x_{perf} denote the spectrum of the perfection of the residue field of x. The Tannakian category F-Isoc(x_{perf}) has a canonical $\mathbb{Q}$-grading by slopes. The pullback of $M \in F$-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ to x_{perf} will be denoted by M_x.

9.2.2. Definition of F-Isoc$_G(X)$. Let G be an algebraic group over $\overline{\mathbb{Q}}_p$. The groupoid of G-torsors in the Tannakian category F-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ (see §9.1.1) will be denoted by F-Isoc$_G(X)$. In [RR] objects of F-Isoc$_G(X)$ are called *F-isocrystals with G-structure.*

9.2.3. Examples. (i) A $GL(n)$-torsor in F-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ is the same as an n-dimensional object of F-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.

(ii) If G is finite then by Proposition B.4.1, a G-torsor in F-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ is the same as a G-torsor on X_{et}.

9.2.4. Newton coweights if G is connected. Let G be a connected reductive group over $\overline{\mathbb{Q}}_p$ and $\mathcal{E} \in F$-Isoc$_G(X)$. Let $x \in X$. Since F-Isoc(x_{perf}) is $\mathbb{Q}$-graded, the G-torsor $\mathcal{E}_x$ is equipped with an action of $\widetilde{\mathbb{G}}_m := \mathrm{Hom}(\mathbb{Q}, \mathbb{G}_m)$. So by §9.1.2, it has a class $a^x(\mathcal{E}) \in \check{\Lambda}_G^{+,\mathbb{Q}}$. Following [RR, Ko2], we call it the *Newton coweight* of $\mathcal{E}_x$ (or the Newton coweight of $\mathcal{E}$ at x). If $G = GL(n)$ then $\mathcal{E}$ is just an n-dimensional object of F-Isoc$(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ and $a^x(\mathcal{E})$ is the collection of its slopes $a_i^x(\mathcal{E})$, $1 \leq i \leq n$.

9.2.5. Newton coweights without assuming connectedness of G. We fix a universal cover $\tilde{X} \to X$ and set $\Pi := \mathrm{Aut}(\tilde{X}/X)$.

Now let G be a not necessarily connected reductive group over $\overline{\mathbb{Q}}_p$; let G° be its neutral connected component. An object $\mathcal{E} \in F$-Isoc$_G(X)$ defines an

object $\bar{\mathcal{E}} \in F\text{-Isoc}_{\pi_0(G)}(X)$, which is just a $\pi_0(G)$-torsor on X_{et}, see §9.2.3(ii). Fix a trivialization of the pullback of $\bar{\mathcal{E}}$ to $\tilde{X}$. Then the torsor $\bar{\mathcal{E}}$ is described by a homomorphism

$$(9.1) \qquad\qquad\qquad \Pi \to \pi_0(G).$$

Let U be its kernel. The pullback of $\bar{\mathcal{E}}$ to $\tilde{X}/U$ is trivialized, so the pullback of $\mathcal{E}$ to $\tilde{X}/U$ comes from a G°-torsor. Applying §9.2.4 to this G°-torsor on $\tilde{X}/U$, we get a map $\tilde{X}/U \to \check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$. The corresponding map $\tilde{X} \to \check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ is denoted by

$$(9.2) \qquad\qquad\qquad \tilde{x} \mapsto a^{\tilde{x}}(\mathcal{E}), \quad \tilde{x} \in \tilde{X}.$$

The composition of (9.1) and the homomorphism $\pi_0(G) \to \operatorname{Aut}\check{\Lambda}_{G^\circ}$ defines an action of Π on $\Lambda_{G^\circ}^{+,\mathbb{Q}}$. The map (9.2) is Π-equivariant.

Let $\eta, \tilde{\eta}$ denote the generic points of $X, \tilde{X}$. We write $a^\eta(\mathcal{E})$ instead of $a^{\tilde{\eta}}(\mathcal{E})$. By Π-equivariance of (9.2), we have

$$(9.3) \qquad\qquad\qquad a^\eta(\mathcal{E}) \in (\Lambda_{G^\circ}^{+,\mathbb{Q}})^\Pi.$$

9.2.6. Remark. Here is a reformulation of the definition of $a^{\tilde{x}}(\mathcal{E})$ in terms of §B.5 of Appendix B. An object $\mathcal{E} \in F\text{-Isoc}_G(X)$ equipped with a trivialization of the pullback of $\bar{\mathcal{E}}$ to $\tilde{X}$ is the same as a homomorphism $\sigma : \pi_1^{F\text{-Isoc}}(X) \to G$ up to G°-conjugation. We also have a homomorphism $\nu_{\tilde{x}} : \widetilde{\mathbb{G}}_m \to \pi_1^{F\text{-Isoc}}(X)$ defined up to conjugation by $(\pi_1^{F\text{-Isoc}}(X))^\circ$, see the end of §B.5.3. Set $\mu_{\tilde{x}} := \sigma \circ \nu_{\tilde{x}} \in \operatorname{Hom}(\widetilde{\mathbb{G}}_m, G^\circ)$. Then $a^{\tilde{x}}(\mathcal{E}) \in \check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$ is the conjugacy class of $\mu_{\tilde{x}}$.

9.2.7. Semicontinuity. The set $\check{\Lambda}_G^{+,\mathbb{Q}}$ is equipped with the following partial order: $\check{\lambda}_1 \leq \check{\lambda}_2$ if $\check{\lambda}_2 - \check{\lambda}_1$ is a linear combination of simple coroots with non-negative rational coefficients. The map (9.2) is *lower-semicontinuous*. It suffices to check this if G is connected; in this case this is Theorem 3.6 of [RR] (as explained in [RR], it immediately follows from semicontinuity of usual Newton polygons, see [Ka1, Cor. 2.3.2]). In particular, lower semicontinuity implies that

$$(9.4) \qquad\qquad a^{\tilde{x}}(\mathcal{E}) \leq a^\eta(\mathcal{E}) \quad \text{for all } \tilde{x} \in \tilde{X}.$$

Let us note that the map (9.2) takes finitely many values. It suffices to check this for $G = GL(n)$; in this case it follows from (9.4) and boundedness of the denominators of the slopes.

Theorem 9.2.8. *Let G be a reductive group and $\mathcal{E} \in F\text{-Isoc}_G(X)$. Suppose that*

(i) the map (9.1) *is surjective;*
(ii) the inclusion

$$(9.5) \qquad \operatorname{Hom}(\mathbb{G}_m, Z(G)) \subset \operatorname{Hom}(\mathbb{G}_m, Z(\operatorname{Aut}\mathcal{E}))$$

is an equality (here $Z(G)$ and $Z(\operatorname{Aut}\mathcal{E})$ are the centers of G and $\operatorname{Aut}\mathcal{E}$, respectively).

Then $a^\eta(\mathcal{E})$ has small gaps in the sense of §7.2.2.

Proof. We will mimic the proof of Theorem 8.6.3 given in §8.6.7.

Our $\mathcal{E}$ corresponds to a homomorphism $\sigma : \pi_1^{F\text{-Isoc}}(X) \to G$, where $\pi_1^{F\text{-Isoc}}(X)$ is the group defined in §B.5 of Appendix B. Let $H := \operatorname{Im}(\sigma) \subset G$.

In §9.2.6 we defined $\mu_{\tilde{x}} \in \operatorname{Hom}(\widetilde{\mathbb{G}}_m, G^\circ)$ for each $\tilde{x} \in \tilde{X}$. Let $\mu := \mu_{\tilde{\eta}}$, where $\tilde{\eta} \in \tilde{X}$ is the generic point. Then $a^\eta(\mathcal{E})$ is equal to the conjugacy class $[\mu] \in \check{\Lambda}_{G^\circ}^{+,\mathbb{Q}}$.

It remains to check that μ and H satisfy the conditions of Lemma 7.4.2.

Combining (9.3) and assumption (i) of the theorem, we see that $[\mu]$ is $\pi_0(G)$-invariant, as required in §7.3.3. Clearly $H \supset \mu(\widetilde{\mathbb{G}}_m)$. Assumption (ii) of the theorem means that

$$\operatorname{Hom}(\mathbb{G}_m, Z(G)) = \operatorname{Hom}(\mathbb{G}_m, Z(Z_G(H))).$$

Finally, by Theorem 1.1.5 combined with Corollary 4.2.4, every indecomposable H-module has small gaps as a $\widetilde{\mathbb{G}}_m$-module (if the $\widetilde{\mathbb{G}}_m$-action is defined via μ). $\qquad\square$

The interested reader can formulate and prove an analog of Corollary 8.7.4 for $F\text{-Isoc}_G(X)$.

10. Newton weights of automorphic representations of reductive groups

10.1. Notation

We keep the notation of §6.1.1-6.1.2. In particular, $v : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ is the p-adic valuation normalized so that $v(p) = 1$, and for each $x \in |X|$ we define $v_x : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ by $v_x := v/\deg x$.

Just as in §8, we fix a universal cover $\tilde{X} \to X$ and set $\Pi := \operatorname{Aut}(\tilde{X}/X)$. For each $\tilde{x} \in |\tilde{X}|$, one has the geometric Frobenius $\operatorname{Fr}_{\tilde{x}} \in \Pi$.

10.2. The groups G, $\check{G}$, and ^{L}G

10.2.1. The group scheme G. Let G be a smooth group scheme over X with connected reductive fibers. If A is a ring equipped with a morphism $\operatorname{Spec} A \to X$ we write $G(A) := \operatorname{Mor}_X(\operatorname{Spec} A, G)$ (this can be applied for rings F, $\mathbb{A}_F$, or O_x, $x \in |X|$).

10.2.2. Weights and coweights. Let Λ_G (resp. W) be the group of weights (resp. Weyl group) of $G \times_X \tilde{X}$. The group Π acts on Λ_G and W. As usual, let $\check{\Lambda}_G$ be the group dual to Λ_G.

Let $\Lambda_G^{\mathbb{Q}} := \Lambda_G \otimes \mathbb{Q}$, and let $\Lambda_G^{+,\mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$ be the dominant cone; one has the usual bijection $\Lambda_G^{+,\mathbb{Q}} \xrightarrow{\sim} (\Lambda_G \otimes \mathbb{Q})/W$. The set $\Lambda_G^{+,\mathbb{Q}}$ is equipped with the following partial order: $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1$ is a linear combination of simple roots with non-negative rational coefficients.

We say that $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ has *small gaps* if $(\lambda, \check{\alpha}_i) \leq 1$ for every simple coroot $\check{\alpha}_i$.

10.2.3. The groups $\check{G}$ and ^{L}G. Let $\check{G}$ be the split reductive group over $\mathbb{Q}$ canonically associated to the root datum dual to that of $G \times_X \tilde{X}$ (in particular, $\Lambda_{\check{G}} = \check{\Lambda}_G$).

Let $\Pi' := \operatorname{Ker}(\Pi \to \operatorname{Aut} \Lambda_G)$, then Π/Π' acts on $\check{G}$. Set $^{L}G := (\Pi/\Pi') \ltimes \check{G}$; this is the *Langlands dual* of G.

10.2.4. The variety $[^{L}G]$. Define $[^{L}G]$ to be the GIT quotient of ^{L}G by the conjugation action of $\check{G}$ (i.e., $[^{L}G]$ is the spectrum of the algebra of those regular functions on ^{L}G that are invariant under $\check{G}$-conjugation). This is a variety equipped with an action of Π/Π' and a (Π/Π')-equivariant map $[^{L}G] \to \Pi/\Pi'$. Moreover, the action of each element $\gamma \in \Pi/\Pi'$ on its preimage in $[^{L}G]$ is trivial.

10.3. Satake parameters

Let $x \in |X|$, and let E be an algebraically closed field of characteristic 0. Unramified irreducible representations of $G(F_x)$ over E are classified by elements of a certain set $\operatorname{Sat}_x(E)$, whose elements are called *Satake parameters* at x.

$\operatorname{Sat}_x(E)$ is defined as follows. Let $[\tilde{X}]_x$ denote the preimage of x in $[\tilde{X}]$. We have a canonical map

$$[\tilde{X}]_x \to \Pi/\Pi', \quad \tilde{x} \mapsto \overline{\operatorname{Fr}}_{\tilde{x}},$$

where $\overline{\mathrm{Fr}}_{\tilde{x}}$ is the image of $\mathrm{Fr}_{\tilde{x}}$ in Π/Π'. An element of $\mathrm{Sat}_x(E)$ is a lift of this map to a Π-equivariant map $[\tilde{X}]_x \to [{}^L G](E)$. (To specify such a map $[\tilde{X}]_x \to [{}^L G](E)$, it suffices to specify the image of a single point $\tilde{x} \in [\tilde{X}]_x$, and this image can be any element of $[{}^L G](E)$ over $\overline{\mathrm{Fr}}_{\tilde{x}}$.)

It is well known [Lan, Mo, Sp] that for every $\tilde{x} \in |\tilde{X}|_x$ one has a canonical bijection

$$\mathrm{Sat}_x(E) \xrightarrow{\sim} \mathrm{Hom}(\check{\Lambda}_G^{\mathrm{Fr}_{\tilde{x}}}, E^\times)/W^{\mathrm{Fr}_{\tilde{x}}},$$

where $\check{\Lambda}_G^{\mathrm{Fr}_{\tilde{x}}}$ and $W^{\mathrm{Fr}_{\tilde{x}}}$ are the invariants of $\mathrm{Fr}_{\tilde{x}}$ acting on $\check{\Lambda}_G$ and W.

10.4. Newton weights

Let π be an irreducible admissible representation of $G(\mathbb{A}_F)$ over $\overline{\mathbb{Q}}_p$, which is unramified at each $x \in |X|$ (i.e., the subspace of $G(O_x)$-invariant vectors is not zero). We will associate to π a Π-equivariant map

$$(10.1) \qquad\qquad |\tilde{X}| \to \Lambda_G^{+,\mathbb{Q}}, \quad \tilde{x} \mapsto a^{\tilde{x}}(\pi).$$

The element $a^{\tilde{x}}(\pi) \in \Lambda_G^{+,\mathbb{Q}}$ will be called the *Newton weight* of π at $\tilde{x}$.

To define the map (10.1), it suffices to define its restriction to $|\tilde{X}|_x$ for each $x \in |X|$. We define it to be the composition

$$[\tilde{X}]_x \to [{}^L G](\overline{\mathbb{Q}}_p) \to \Lambda_G^{+,\mathbb{Q}}$$

where the first map is the Satake parameter of π_x (see §10.3), and the second map is defined as follows.[7] An element of $[{}^L G](\overline{\mathbb{Q}}_p)$ is a $\check{G}(\overline{\mathbb{Q}}_p)$-conjugacy class of semisimple elements $g \in {}^L G(\overline{\mathbb{Q}}_p)$. For such g, let $\langle g \rangle$ be the smallest algebraic subgroup of ${}^L G \otimes \overline{\mathbb{Q}}_p$ containing g; then the homomorphism

$$\mathrm{Hom}(\langle g \rangle, \mathbb{G}_m) \to \mathbb{Q}, \quad \chi \mapsto v_x(\chi(g))$$

defines a homomorphism $\widetilde{\mathbb{G}}_m \to \langle g \rangle \subset {}^L G \otimes \overline{\mathbb{Q}}_p$, where $\widetilde{\mathbb{G}}_m$ is the pro-torus over $\overline{\mathbb{Q}}_p$ with group of characters $\mathbb{Q}$. Thus an element of $[{}^L G](\overline{\mathbb{Q}}_p)$ defines a $\check{G}(\overline{\mathbb{Q}}_p)$-conjugacy class of homomorphisms $\widetilde{\mathbb{G}}_m \to {}^L G \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$, which is the same as an element of $\check{\Lambda}_{\check{G}}^{+,\mathbb{Q}} = \Lambda_G^{+,\mathbb{Q}}$.

Remark 10.4.1. The above definition of $a^{\tilde{x}}(\pi)$ is equivalent to the following one. The Satake parameter of π_x is an element of $\mathrm{Hom}(\check{\Lambda}_G^{\mathrm{Fr}_{\tilde{x}}}, \overline{\mathbb{Q}}_p^\times)/W^{\mathrm{Fr}_{\tilde{x}}}$. Now consider the composition

$$(10.2) \qquad \mathrm{Hom}(\check{\Lambda}_G^{\mathrm{Fr}_{\tilde{x}}}, \overline{\mathbb{Q}}_p^\times) \to \mathrm{Hom}(\check{\Lambda}_G^{\mathrm{Fr}_{\tilde{x}}}, \mathbb{Q}) = (\Lambda_G^{\mathbb{Q}})_{\mathrm{Fr}_{\tilde{x}}} \xrightarrow{\sim} (\Lambda_G^{\mathbb{Q}})^{\mathrm{Fr}_{\tilde{x}}} \hookrightarrow \Lambda_G^{\mathbb{Q}},$$

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[7]The definition given below is similar to §8.3.1.

where the first map is induced by $v_x : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ and the isomorphism $(\Lambda_G^{\mathbb{Q}})_{\mathrm{Fr}_{\tilde{x}}} \xrightarrow{\sim} (\Lambda_G^{\mathbb{Q}})^{\mathrm{Fr}_{\tilde{x}}}$ is the averaging operator, i.e., the inverse of the tautological map $(\Lambda_G^{\mathbb{Q}})^{\mathrm{Fr}_{\tilde{x}}} \xrightarrow{\sim} (\Lambda_G^{\mathbb{Q}})_{\mathrm{Fr}_{\tilde{x}}}$. The composition (10.2) is $W^{\mathrm{Fr}_{\tilde{x}}}$-equivariant, so it induces a map $\mathrm{Hom}(\check{\Lambda}_G^{\mathrm{Fr}_{\tilde{x}}}, \overline{\mathbb{Q}}_p^\times)/W^{\mathrm{Fr}_{\tilde{x}}} \to \Lambda_G^{\mathbb{Q}}/W = \Lambda_G^{+,\mathbb{Q}}$. Applying this map to the Satake parameter of π_x, one gets an element of $\Lambda_G^{+,\mathbb{Q}}$. This is $a^{\tilde{x}}(\pi)$.

Let us note that the definition of the Satake parameter depends on the choice of $p^{1/2}$ (see §6.1.2), but $a^{\tilde{x}}(\pi)$ does not depend on this choice.

10.5. The case of a torus

Let T be a torus over X. Let $\eta : T(\mathbb{A}_F) \to \overline{\mathbb{Q}}_p^\times$ be a homomorphism with open kernel, which is unramified at every point of $|X|$.

Lemma 10.5.1. *Let $X' := \tilde{X}/\Pi'$, where $\Pi' := \mathrm{Ker}(\Pi \to \mathrm{Aut}\, \Lambda_T)$. Let F' be the field of rational functions on X'. Let $\tilde{x} \in |\tilde{X}|$ and $x' \in X'$ the image of $\tilde{x}$. Let $\check{\lambda} \in \check{\Lambda}_T$. Then*

(10.3)
$$(a^{\tilde{x}}(\eta), \check{\lambda}) = f_{\check{\lambda}}(\varpi_{x'})/\deg x',$$

where $f_{\check{\lambda}} : \mathbb{A}_{F'}^\times \to \mathbb{Q}$ is the composition

$$\mathbb{A}_{F'}^\times \xrightarrow{\check{\lambda}} T(\mathbb{A}_{F'}) \xrightarrow{\mathrm{Norm}} T(\mathbb{A}_F) \xrightarrow{\eta} \overline{\mathbb{Q}}_p^\times \xrightarrow{v} \mathbb{Q}$$

and $\varpi_{x'}$ is a uniformizer in $(F'_{x'})^\times \subset \mathbb{A}_{F'}^\times$.

Proof. Follows from Remark 10.4.1. $\square$

Corollary 10.5.2. *If η is trivial on $T(F)$ then $a^{\tilde{x}}(\eta)$ does not depend on $\tilde{x} \in |\tilde{X}|$.*

Proof. Let us show that the r.h.s. of (10.3) does not depend on $x' \in |X'|$. Indeed, the homomorphism $f_{\check{\lambda}} : \mathbb{A}_{F'}^\times \to \mathbb{Q}$ is trivial on $(F')^\times$, so it is proportional to $\deg : \mathbb{A}_{F'}^\times \to \mathbb{Z}$. $\square$

In the situation of Corollary 10.5.2 we will write $a(\eta)$ instead of $a^{\tilde{x}}(\eta)$.

10.6. V. Lafforgue's estimate

Let G be a split reductive group. Let π be a cuspidal automorphic irreducible representation of $G(\mathbb{A}_F)$ over $\overline{\mathbb{Q}}_p$, which is unramified at each point of $|X|$.

Let Z° be the neutral connected component of the center of G, and let $\eta : Z^\circ(\mathbb{A}_F) \to \overline{\mathbb{Q}}_p^\times$ be the central character of π. According to [Laf2, Prop. 2.1],

$$(10.4) \qquad a^{\tilde{x}}(\pi) \leq a(\eta) + \rho \quad \text{for all } \tilde{x} \in |\tilde{X}|,$$

where ρ is one half of the sum of the positive roots of G and $a(\eta) \in \Lambda_{Z^\circ}^{\mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$ is defined at the end of §10.5.

Probably the method of [Laf2] should work even without assuming G to be split (this is discussed in §5 of [Laf2]).

10.7. The result

As before, let π be an irreducible representation of $G(\mathbb{A}_F)$ over $\overline{\mathbb{Q}}_p$, which is unramified at all points $x \in |X|$.

Theorem 10.7.1. *Suppose that π is cuspidal automorphic. Then*

(i) there exists an element $a^\eta(\pi) \in (\Lambda_G^{+,\mathbb{Q}})^\Pi$ such that the set $\{\tilde{x} \in |\tilde{X}| \mid a^{\tilde{x}}(\pi) \neq a^\eta(\pi)\}$ has finite image in $|X|$;

(ii) for all $\tilde{x} \in |X|$ one has $a^{\tilde{x}}(\pi) \leq a^\eta(\pi)$;

(iii) Conjecture 12.7 of [Laf3] would imply that $a^\eta(\pi)$ has small gaps in the sense of §10.2.2.

Conjecture 12.7 of [Laf3] is a variant of Arthur's conjecture [Ar].

Proof. Let $G^{ab} := [G, G]$. After twisting π by a quasi-character of $G^{ab}(\mathbb{A})/ G^{ab}(F)$, we can assume that π is defined over the subfield $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. Choose a prime $\ell \neq p$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ equipped with a homomorphism $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. Note that $\overline{\mathbb{Q}}$ is equipped with a distinguished p-adic valuation (namely, the restriction of $v : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$).

By Theorem 12.3 of [Laf3], the canonical epimorphism $\Pi \to \Pi/\Pi' = {}^L G/\check{G}$ can be lifted to a continuous homomorphism $\sigma : \Pi \to {}^L G(\overline{\mathbb{Q}}_\ell)$ which is compatible with π in the sense of Langlands; this means that for each $x \in |X|$ the element of $\mathrm{Sat}_x(\overline{\mathbb{Q}}_\ell)$ corresponding to the map

$$\tilde{x} \mapsto \sigma(\mathrm{Fr}_{\tilde{x}}), \quad \tilde{x} \in |\tilde{X}|_x$$

is equal to the Satake parameter of π_x. Applying Proposition 8.5.1 to σ, one gets statements (i)-(ii).

Conjecture 12.7 of [Laf3] would imply that σ can be chosen to be the Langlands parameter corresponding to an elliptic Arthur parameter (see §8.7.2-8.7.3). Applying Corollary 8.7.4 to such σ, one gets statement (iii). $\square$

Appendix A. Slopes for some hypergeometric local systems

The main goal of this Appendix is to prove Proposition A.2.1, which provides the counterexamples promised in §1.2 and Remark 1.3.7. In §A.5 we explain how to produce more examples using a computer.

We will work over $\mathbb{F}_p$; in particular, $\mathbb{G}_m$ denotes the multiplicative group over $\mathbb{F}_p$.

A.1. A class of hypergeometric local systems

A.1.1. The fields E and E_λ. Let $E \subset \mathbb{Q}_p$ be the subfield generated by the roots of unity of order $p-1$. We fix a non-Archimedean place λ of E not dividing p. Let E_λ denote the completion.

Let $\tau : \mathbb{F}_p^\times \to E^\times$ denote the Teichmüller character. We have a canonical isomorphism

$$\mathbb{Z}/(p-1)\mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{F}_p^\times, E^\times), \quad j \mapsto \tau^j.$$

A.1.2. The local systems $\mathcal{E}_\mathbf{c}$. Let $\rho_1, \dots, \rho_n$ and $\chi_1, \dots, \chi_n$ be two unordered lists of characters $\mathbb{F}_p^\times \to E^\times$ such that $\rho_i \neq \chi_j$ for all i and j. Given these data, N.Katz defines in [Ka2, Ch. 8] a certain lisse E_λ-sheaf on $\mathbb{G}_m \setminus \{1\}$ of rank n, which he calls hypergeometric; he gives a brief summary in [Ka3, §4].

We will need only the case that $\chi_1 = \dots = \chi_n = 1$. In this case the input data is an unordered list $\mathbf{c}$ of elements $c_1, \dots, c_n \in \{1, \dots, p-2\}$. We set $\rho_i := \tau^{c_i}$, where τ is the Teichmüller character; note that $\rho_i \neq 1$ for all i. Katz defines the corresponding local system $\mathcal{E}_\mathbf{c}$ on $\mathbb{G}_m \setminus \{1\}$ as follows. Let $\mathcal{L}_i$ denote the lisse E_λ-sheaf on $\mathbb{G}_m$ corresponding to ρ_i in the usual sense (the "trace of geometric Frobenius" function on $\mathbb{F}_{p^m}^\times$ corresponding to $\mathcal{L}_i$ equals the composition of ρ_i with the norm map $\mathbb{F}_{p^m}^\times \to \mathbb{F}_p^\times$). Let $\mathcal{F}_i$ denote the sheaf on $\mathbb{G}_m$ defined by $\mathcal{F}_i := j_! f^* \mathcal{L}_i$, where $f : \mathbb{G}_m \setminus \{1\} \hookrightarrow \mathbb{G}_m$ is the map $x \mapsto 1-x$ and $j : \mathbb{G}_m \setminus \{1\} \hookrightarrow \mathbb{G}_m$ is the natural embedding. Finally, consider the object of the derived category of sheaves on $\mathbb{G}_m \setminus \{1\}$ defined by

$$(\mathrm{A.1}) \qquad \mathcal{E}_\mathbf{c} := j^*(\mathcal{F}_1 \star_! \dots \star_! \mathcal{F}_n)[n-1],$$

where $\star_!$ denotes multiplicative convolution with compact support. Katz proves the following statements.

Theorem A.1.3. *(i) $\mathcal{E}_\mathbf{c}$ is a lisse E_λ-sheaf on $\mathbb{G}_m \setminus \{1\}$ of rank n. It is irreducible; moreover, $\mathcal{E}_\mathbf{c} \otimes_{E_\lambda} \bar{E}_\lambda$ is geometrically irreducible.*

(ii) The canonical morphism $\mathcal{F}_1 \star_! \ldots \star_! \mathcal{F}_n \to \mathcal{F}_1 \star \ldots \star \mathcal{F}_n$ is an isomorphism (here $\star$ denotes multiplicative convolution without compact support).

(iii) $\mathcal{E}_{\mathbf{c}}$ is tamely ramified at $0, 1, \infty \in \mathbb{P}^1$.

(iv) $\mathcal{E}_{\mathbf{c}}$ is pure of weight $n - 1$.

(v) Set $N = n(n-1)/2$; then the rank 1 local system $(\det \mathcal{E}_{\mathbf{c}})(N)$ has finite order.

The theorem is proved in [Ka2, Ch. 8]: statements (i)-(iv) correspond to parts (1), (4), (5), and (8) of [Ka2, Thm. 8.4.2], and statement (v) corresponds to parts (1a)-(1b) of [Ka2, Thm. 8.12.2]. Some technical issues are explained in §A.1.6 below.

Remark A.1.4. Katz also describes the local monodromies of $\mathcal{E}_{\mathbf{c}}$ at $0, 1, \infty \in \mathbb{P}^1$. In particular, he proves that the local monodromy at $0 \in \mathbb{P}^1$ is maximally unipotent and the eigenvalues of the local monodromy at ∞ correspond to the characters $\rho_1, \ldots, \rho_n$.

Remark A.1.5. Theorem A.1.3(ii) implies that the local system dual to $\mathcal{E}_{\mathbf{c}}$ is isomorphic to $\mathcal{E}_{\mathbf{c}'}(n-1)$, where $\mathbf{c}'$ is the n-uple formed by the numbers $c_i' := p - 1 - c_i$.

The next subsection should be skipped by the reader unless he wants to check that Theorem A.1.3 is indeed proved in [Ka2, Ch. 8].

A.1.6. Some comments on [Ka2] and [Ka3]. In [Ka2, §8.2.2] Katz introduces a complex of sheaves on $\mathbb{G}_m$ denoted by $\mathrm{Hyp}(!, \psi; \chi\text{'s}; \rho\text{'s })$ (here ψ is a non-trivial additive character of $\mathbb{F}_p$). Its multiplicative translate by $\lambda \in \mathbb{G}_m$ is denoted in [Ka2, §8.2.13] by $\mathrm{Hyp}_\lambda(!, \psi; \chi\text{'s}; \rho\text{'s})$. In [Ka2, Thm. 8.4.2] he sets $\mathcal{H}_\lambda(!, \psi; \chi\text{'s}; \rho\text{'s}) := \mathrm{Hyp}_\lambda(!, \psi; \chi\text{'s}; \rho\text{'s})[-1]$ and shows that $\mathcal{H}_\lambda$ is a sheaf.

In [Ka3, §4] Katz considers only $\lambda = 1$ and writes $\mathcal{H}$ instead of $\mathcal{H}_1$. On p.98 of [Ka3] he introduces a sheaf $\mathcal{H}^{can}(\chi\text{'s}; \rho\text{'s})$, which is a tensor product of $\mathcal{H}$ and a rank 1 local system on $\mathrm{Spec}\,\mathbb{F}_p$; unlike $\mathcal{H}$, the sheaf $\mathcal{H}^{can}$ does not depend on ψ. On p.99 of [Ka3] he writes a formula[8] for $\mathcal{H}^{can}$ which does not involve ψ at all. This formula shows that if the χ_i's are trivial then the restriction of $\mathcal{H}^{can}(\chi\text{'s}; \rho\text{'s})$ to $\mathbb{G}_m \setminus \{1\}$ is the sheaf that we denote by $\mathcal{E}_{\mathbf{c}}$.

To deduce Theorem A.1.3(iv-v) from the results of [Ka2], one has to take into account that our $\mathcal{E}_{\mathbf{c}}$ corresponds to $\mathcal{H}^{can}$, while Theorems 8.4.2 and 8.12.2 of [Ka2] are formulated in terms of $\mathcal{H}$; one also has to keep in mind that in our situation the χ_i's are trivial.

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[8] By [Ka2, §8.2.3], it suffices to prove this formula for $n = 1$. This can be done by comparing the "trace of Frobenius" functions.

A.1.7. The crystalline companions of $\mathcal{E}_{\mathbf{c}}$. According to K. Miyatani [Mi], for any $\mathbf{c}$ as in §A.1.2 there exists an irreducible object $M_{\mathbf{c}} \in F\text{-Isoc}^\dagger(\mathbb{G}_m \setminus \{1\})$ such that for every closed point $x \in \mathbb{G}_m \setminus \{1\}$ the Frobenius characteristic polynomial of $(M_{\mathbf{c}})_x$ is equal to that of $(\mathcal{E}_{\mathbf{c}})_x$ (the word "equal" makes sense because the coefficients of the Frobenius characteristic polynomial of $(\mathcal{E}_{\mathbf{c}})_x$ are in the number field E, which is a subfield of $\mathbb{Q}_p$). The construction of $M_{\mathbf{c}}$ given in [Mi, §3.2] is parallel to the one from §A.1.2.

A.2. The counterexamples

For any $x \in \bar{\mathbb{F}}_p^\times \setminus \{1\}$ the characteristic polynomial of the geometric Frobenius (with respect to the field $\mathbb{F}_p(x) \subset \bar{\mathbb{F}}_p$) acting on the stalk $(\mathcal{E}_{\mathbf{c}})_x$ has coefficients in E. Since $E \subset \mathbb{Q}_p$ we can talk about the slopes of $\mathcal{E}_{\mathbf{c}}$ at x. We denote them by $a_i^x(\mathcal{E}_{\mathbf{c}})$, where $a_1^x(\mathcal{E}_{\mathbf{c}}) \geq \ldots \geq a_n^x(\mathcal{E}_{\mathbf{c}})$.

Proposition A.2.1. *Assume that $p \geq 5$. Let $n = 3$ and $\mathbf{c} = (c_1, c_2, c_3)$, where $c_1 = 1$, $c_2 = p - 2$, and c_3 is any element of $\{1, \ldots, p-2\}$ different from $(p-1)/2$. Let $\mathcal{E}_{\mathbf{c}}$ be the lisse E_λ-sheaf of rank 3 on $\mathbb{G}_m \setminus \{1\}$ defined in §A.1.2. Then there is a unique $x \in \mathbb{F}_p^\times \setminus \{1\}$ such that $a_1^x(\mathcal{E}_{\mathbf{c}}) - a_2^x(\mathcal{E}_{\mathbf{c}}) > 1$; namely, x is the residue class of $-(2c_3)^{-1}$.*

The proof will be given in §A.4.

The local systems $\mathcal{E}_{\mathbf{c}}$ from Proposition A.2.1 are the counterexamples promised in Remark 1.3.7, and their crystalline companions (see §A.1.7) are the counterexamples promised in §1.2.

A.3. The key computation

Just as in §A.1.2, let n be any positive integer and $\mathbf{c} := (c_1, \ldots, c_n)$, where $c_i \in \{1, \ldots, p-2\}$. Given $m \in \mathbb{N}$ and $x \in \mathbb{F}_{p^m}^\times \setminus \{1\}$ set

$$P_x^{(m)}(t) := \det(1 - F_x t, (\mathcal{E}_{\mathbf{c}})_x),$$

where $(\mathcal{E}_{\mathbf{c}})_x$ is the stalk of $\mathcal{E}_{\mathbf{c}}$ at $x \in \bar{\mathbb{F}}_p^\times \setminus \{1\}$ and F_x is the geometric Frobenius of the field $\mathbb{F}_{p^m}$. By (A.1), the coefficients of the polynomial $P_x^{(m)}$ belong to the ring of integers O_E, which is a subring of $\mathbb{Z}_p$. Let $\bar{P}_x^{(m)}$ be the image of $P_x^{(m)}$ in $\mathbb{Z}_p[t]/p\mathbb{Z}_p[t] = \mathbb{F}_p[t]$.

Proposition A.3.1. *(i) One has*

$$(A.2) \qquad -\frac{d}{dt} \log \bar{P}_x^{(m)}(t) = \alpha_x^{(m)}(1 - \alpha_x^{(m)}t)^{-1}, \quad \alpha_x^{(m)} := N_{\mathbb{F}_{p^m}/\mathbb{F}_p}(u_{\mathbf{c}}(x)),$$

where $N_{\mathbb{F}_{p^m}/\mathbb{F}_p} : \mathbb{F}_{p^m} \to \mathbb{F}_p$ is the norm map and $u_{\mathbf{c}} \in \mathbb{F}_p[X]$ is the following polynomial:

$$(A.3) \qquad u_{\mathbf{c}}(X) := \sum_{r \geq 0} (-1)^{nr} \binom{c_1}{r} \cdot \ldots \cdot \binom{c_n}{r} X^r.$$

(ii) If $p > n$ then $\bar{P}_x^{(m)}(t) = 1 - \alpha_x^{(m)} t$.

Later we will show that the assumption $p > n$ in statement (ii) is unnecessary, see Corollary A.3.2.

The proof given below is somewhat similar to that of the Chevalley-Warning theorem.

Proof. $\bar{P}_x^{(m)}$ is a polynomial in t of degree $\leq n$, so it can be reconstructed from its logarithmic derivative if $p > n$. Therefore it suffices to prove (i).

For $m \in \mathbb{N}$ and $x \in \mathbb{F}_p^\times$ set

$$V_x^{(m)} := \{(x_1, \ldots, x_n) \in (\mathbb{F}_{p^m}^\times)^n \mid x_1 \cdot \ldots \cdot x_n = x\},$$
$$S^{(m)}(x) := \sum_{V_x^{(m)}} (1 - x_1)^{\tilde{c}_1} \cdot \ldots \cdot (1 - x_n)^{\tilde{c}_n},$$

where $\tilde{c}_i := c_i(1 + p + \ldots + p^{m-1})$. By formula (A.1), statement (i) is equivalent to the following one: for any $m \in \mathbb{N}$ and $x \in \mathbb{F}_{p^m}^\times$ one has $S^{(m)}(x) = (-1)^{n-1} \alpha_x^{(m)}$.

One has $\tilde{c}_i < p^m - 1$ for all i. Note that if $r_1, \ldots r_n \in \{0, 1, \ldots, p^m - 2\}$ then the sum

$$\sum_{V_x^{(m)}} \prod_{i=1}^{n} x_i^{r_i}$$

is non-zero only if $r_1 = \ldots = r_n$; in this case it equals $(-1)^{n-1} x^r$, where $r := r_1 = \ldots = r_n$. So $S^{(m)}(x) = u_{\tilde{\mathbf{c}}}(x)$, where $u_{\tilde{\mathbf{c}}}$ is defined by formula (A.3) with $\mathbf{c}$ replaced by $\tilde{\mathbf{c}} := (\tilde{c}_1, \ldots, \tilde{c}_n)$. On the other hand,

$$\alpha_x^{(m)} := N_{\mathbb{F}_{p^m}/\mathbb{F}_p}(u_{\mathbf{c}}(x)) = \prod_{j=0}^{m-1} u_{\mathbf{c}}(x)^{p^j}.$$

So it remains to prove the following identity in $\mathbb{F}_p[X]$:

$$u_{\tilde{\mathbf{c}}}(X) = \prod_{j=0}^{m-1} u_{\mathbf{c}}(X)^{p^j}.$$

To show this, use the following well known property of binomial coefficients: if $N = \sum_{j=0}^{m-1} N_j p^j$ and $r = \sum_{j=0}^{m-1} r_j p^j$, where $N_j, r_j \in \{0, 1, \ldots, p-1\}$, then

$$\binom{N}{r} \equiv \prod_j \binom{N_j}{r_j} \mod p.$$

This property follows from the identity $(1+X)^N = \prod_j (1+X^{p^j})^{N_j}$ in $\mathbb{F}_p[X]$. $\quad\square$

Corollary A.3.2. *Let $n > 1$ and $\mathbf{c} := (c_1, \ldots, c_n)$, where $c_i \in \{1, \ldots, p-2\}$. The equality*

(A.4) $$\bar{P}_x^{(m)}(t) = 1 - \alpha_x^{(m)} t$$

from Proposition A.3.1 holds without assuming that $p > n$. The slopes $a_i^x(\mathcal{E}_{\mathbf{c}})$, $x \in \bar{\mathbb{F}}_p^\times \setminus \{1\}$, have the following properties:

(a) $a_i^x(\mathcal{E}_{\mathbf{c}}) + a_{n+1-i}^x(\mathcal{E}_{\mathbf{c}'}) = n - 1$, where $\mathbf{c}'$ is as in Remark A.1.5;

(b) $\sum_{i=1}^n a_i^x(\mathcal{E}_{\mathbf{c}}) = n(n-1)/2$;

(c) $a_n^x(\mathcal{E}_{\mathbf{c}}) \geq 0$, $a_{n-1}^x(\mathcal{E}_{\mathbf{c}}) > 0$, and

$$a_n^x(\mathcal{E}_{\mathbf{c}}) > 0 \Leftrightarrow u_{\mathbf{c}}(x) = 0,$$

where $u_{\mathbf{c}}$ is the polynomial defined by (A.3).

(d) $a_1^x(\mathcal{E}_{\mathbf{c}}) \leq n - 1$, $a_2^x(\mathcal{E}_{\mathbf{c}}) < n - 1$, and

$$a_1^x(\mathcal{E}_{\mathbf{c}}) < n - 1 \Leftrightarrow u_{\mathbf{c}'}(x) = 0,$$

where $\mathbf{c}'$ is as in Remark A.1.5.

(e) For all but finitely many $x \in \bar{\mathbb{F}}_p^\times \setminus \{1\}$ one has $a_i^x(\mathcal{E}_{\mathbf{c}}) = n - i$.

Proof. Statement (a) follows from Remark A.1.5. Statement (b) follows from Theorem A.1.3(v). By (A.1), the eigenvalues of the geometric Frobenius acting on the stalks of $\mathcal{E}_{\mathbf{c}}$ are algebraic integers, so

(A.5) $$a_n^x(\mathcal{E}_{\mathbf{c}}) \geq 0 \quad \text{for all } x.$$

Let us prove (e). The polynomial $u_{\mathbf{c}}$ is non-zero (in fact, its constant term equals 1). So formula (A.2) implies that for all but finitely many x one has $a_n^x(\mathcal{E}_{\mathbf{c}}) = 0$. By (a), this implies that $a_1^x(\mathcal{E}_{\mathbf{c}}) = n - 1$ for almost all x. On the other hand, combining Theorem 1.3.3(iii) with Theorem A.1.3(i), we see that $a_i^x(\mathcal{E}_{\mathbf{c}}) - a_{i+1}^x(\mathcal{E}_{\mathbf{c}}) \leq 1$ for almost all x. Statement (e) follows.

By (b),(e), and Theorem 1.3.3(ii), $a_{n-1}^x(\mathcal{E}_\mathbf{c}) + a_n^x(\mathcal{E}_\mathbf{c}) > 0$ for all x. Since $a_{n-1}^x(\mathcal{E}_\mathbf{c}) \geq a_n^x(\mathcal{E}_\mathbf{c})$, we see that $a_{n-1}^x(\mathcal{E}_\mathbf{c}) > 0$ for all x. This means that all polynomials $\bar{P}_x^{(m)}$ have degree ≤ 1. Combining this with (A.2), we get formula (A.4).

Statement (c) follows from (A.5) and (A.4). Statement (d) follows from (a) and (c).   □

A.4. Proof of Proposition A.2.1

We apply Corollary A.3.2. In the situation of Proposition A.2.1 the polynomial $u_\mathbf{c}$ defined by (A.3) has degree 1. Its unique root x_0 equals $-(2c_3)^{-1}$, which is not equal to 1 and not a root of $u_{\mathbf{c}'}$. So by Corollary A.3.2(c,d) we have

$$a_3^{x_0}(\mathcal{E}_\mathbf{c}) > 0, \quad a_3^x(\mathcal{E}_\mathbf{c}) = 0 \text{ for } x \neq x_0,$$
$$a_1^{x_0}(\mathcal{E}_\mathbf{c}) = 2, \quad a_1^x(\mathcal{E}_\mathbf{c}) \leq 2 \text{ for all } x.$$

By Corollary A.3.2(b), $a_1^x(\mathcal{E}_\mathbf{c}) + a_2^x(\mathcal{E}_\mathbf{c}) + a_3^x(\mathcal{E}_\mathbf{c}) = 3$. Thus $a_1^{x_0}(\mathcal{E}_\mathbf{c}) = 2$ and $a_2^{x_0}(\mathcal{E}_\mathbf{c}) < 1$; on the other hand, if $x \neq x_0$ then $a_1^x(\mathcal{E}_\mathbf{c}) + a_2^x(\mathcal{E}_\mathbf{c}) = 3$ and $a_1^x(\mathcal{E}_\mathbf{c}) \leq 2$. So $a_1^x(\mathcal{E}_\mathbf{c}) - a_2^x(\mathcal{E}_\mathbf{c}) > 1$ if and only if $x = x_0$.   □

A.5. Using a computer to produce more examples

In the situation of Proposition A.2.1 the field generated by the coefficients of the Frobenius characteristic polynomials is not equal to $\mathbb{Q}$. One can produce examples of sheaves $\mathcal{E}_\mathbf{c}$ such that this field equals $\mathbb{Q}$ and $a_i^x(\mathcal{E}_\mathbf{c}) - a_{i+1}^x(\mathcal{E}_\mathbf{c}) > 1$ for some i and x. However, this requires taking $n \geq 4$, which seems to necessitate the use of computer calculations.

For example[9], take $n = 4$ and

$$c_i = i \cdot (p-1)/5, \quad 1 \leq i \leq 4,$$

where $p \equiv 1 \mod 5$ (so $\rho_1, \ldots, \rho_4$ are the primitive characters of order 5). Note that by Remark A.1.5 one has

$$(A.6) \qquad\qquad\qquad \mathcal{E}_\mathbf{c}^* \simeq \mathcal{E}_\mathbf{c}(3).$$

---

[9]The sheaf $\mathcal{E}_\mathbf{c}$ from this example is related to the Dwork pencil of quintic threefolds, see [Ka3, Thm. 5.3].

For $x \in \mathbb{F}_p^{\times} \setminus \{1\}$ write

$$\det(1 - F_x t, (\mathcal{E}_{\mathbf{c}})_x) = \sum_{i=0}^{4} b_i^x t^i, \quad b_i^x \in \mathbb{Z}, \, b_0^x = 1.$$

Using Proposition A.3.1(ii) and a simple computer search, one easily identifies pairs (p, x) for which $p | b_1^x$; this also ensures that $p | b_2^x$ by the same Proposition A.3.1(ii). Using (A.6), one sees that for such pairs (p, x) the slopes are $\left(\frac{5}{2}, \frac{5}{2}, \frac{1}{2}, \frac{1}{2}\right)$ provided that b_2^x is not divisible by p^2. One cannot check this non-divisibility using Proposition A.3.1(ii), but a related p-adic calculation is implemented in the "hypergeometric motives" package[10] of the MAGMA computer algebra system (see [Mag]). For example, MAGMA confirms that for $p = 31$ and x equal to 4 or 17, the slopes equal $\left(\frac{5}{2}, \frac{5}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Appendix B. Recollections on the Tannakian categories F-Isoc(X) and F-Isoc$^{\dagger}(X)$

In this Appendix we recall some results of R. Crew [Cr2].

B.1. Notation

B.1.1.   We fix a universal cover $\tilde{X} \to X$ and set $\Pi := \mathrm{Aut}(\tilde{X}/X)$.

B.1.2.   The category of finite-dimensional vector spaces over a field E is denoted by Vec_E. The category of finite-dimensional representations of G is denoted by $\mathrm{Rep}_E(G)$.

B.1.3.   If a finite group Γ acts on a Tannakian category $\mathcal{T}$ then Γ-equivariant objects of $\mathcal{T}$ form a Tannakian category; we denote it by $\mathcal{T}^{\Gamma}$. For instance, $(\mathrm{Vec}_E)^{\Gamma} = \mathrm{Rep}_E(\Gamma)$.

B.2. Isotrivial F-isocrystals

B.2.1. Definition of isotriviality.   An object $M \in F$-Isoc(X) is said to be *trivial* if it is isomorphic to a direct sum of several copies of the unit object.

---

[10]To specify a rank n hypergeometric local system for MAGMA, one enters two sequences of rational numbers of length n (they correspond to the eigenvalues of the local monodromy at 0 and ∞). In our situation the sequences are $[0, 0, 0, 0]$ and $[1/5, 2/5, 3/5, 4/5]$.

An object $M \in F\text{-Isoc}(X)$ is said to be *isotrivial* if its pullback to $\tilde{X}/U$ is trivial for some open subgroup $U \subset \Pi$. Isotrivial objects form a Tannakian subcategory[11] of $F\text{-Isoc}(X)$.

B.2.2.   Let $U \subset \Pi$ be an open normal subgroup. Then Π/U acts on $\tilde{X}/U$ and therefore on $F\text{-Isoc}(\tilde{X}/U)$. So we have the Tannakian category $F\text{-Isoc}(\tilde{X}/U)^{\Pi/U}$ (see §B.1.3).

Lemma B.2.3. *Pullback from X to $\tilde{X}/U$ defines a tensor equivalence*

$$(B.1) \qquad\qquad F\text{-Isoc}(X) \xrightarrow{\sim} F\text{-Isoc}(\tilde{X}/U)^{\Pi/U}.$$

Proof. Follows from etale descent for F-isocrystals. $\square$

Remark B.2.4. Note that $F\text{-Isoc}(\tilde{X}/U)^{\Pi/U} \supset (\text{Vec}_{\mathbb{Q}_p})^{\Pi/U} = \text{Rep}_{\mathbb{Q}_p}(\Pi/U)$. It is clear that the equivalence (B.1) identifies $\text{Rep}_{\mathbb{Q}_p}(\Pi/U)$ with the full subcategory of those objects of $F\text{-Isoc}(X)$ whose pullback to $\tilde{X}$ is trivial.

Remark B.2.5. By the previous remark, the Tannakian subcategory of isotrivial objects of $F\text{-Isoc}(X)$ canonically identifies with $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi)$, where $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi)$ is the Tannakian category of smooth finite-dimensional representations of Π over $\mathbb{Q}_p$, i.e.,

$$(B.2) \qquad\qquad \text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi) := \varinjlim_{U} \text{Rep}_{\mathbb{Q}_p}(\Pi/U).$$

So we get a fully faithful embedding $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi) \hookrightarrow F\text{-Isoc}(X)$ and therefore a fully faithful embedding

$$
\begin{aligned}
(B.3) \qquad & \text{Rep}_{\overline{\mathbb{Q}}_p}^{\text{smooth}}(\Pi) \hookrightarrow F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \,, \\
& \text{Rep}_{\overline{\mathbb{Q}}_p}^{\text{smooth}}(\Pi) := \text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p,
\end{aligned}
$$

whose essential image is a Tannakian subcategory of $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.

B.3. Crew's theorem on unit-root F-isocrystals

Recall that an object $M \in F\text{-Isoc}(X)$ is said to be *unit-root* (or *etale*) if all its slopes at all points of X (or equivalently, at the generic point $\eta \in X$) are

---

[11]Following [An, §2.3.5], by a *Tannakian subcategory* of a Tannakian category $\mathcal{T}$ we mean a strictly full subcategory $\mathcal{T}' \subset \mathcal{T}$ stable under tensor products, direct sums, dualization, and passing to subobjects.

zero. The Tannakian subcategory of unit-root objects of F-Isoc(X) is denoted by F-Isoc$_{\text{et}}(X)$. All isotrivial F-isocrystals are unit-root.

Let $\text{Rep}_{\mathbb{Q}_p}^{\text{cont}}(\Pi)$ denote the Tannakian category of *continuous* representations of Π in finite-dimensional vector spaces over $\mathbb{Q}_p$. It contains $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi)$ as a Tannakian subcategory. In Remark B.2.5 we defined a tensor equivalence between $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi)$ and the category of isotrivial F-isocrystals. According to [Cr, Thm. 2.1], it *extends to a canonical tensor equivalence*

$$\text{Rep}_{\mathbb{Q}_p}^{\text{cont}}(\Pi) \xrightarrow{\sim} F\text{-Isoc}_{\text{et}}(X).$$

B.4. Crew's characterization of the Tannakian subcategory $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi) \subset F\text{-Isoc}(X)$

By (B.2), $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi)$ is a union of an increasing family of Tannakian categories of the form $\text{Rep}_{\mathbb{Q}_p}(\Gamma)$, where Γ is a finite group. On the other hand, one has the following result.

Proposition B.4.1. *Let Γ be a finite group. Then any tensor functor $\text{Rep}_{\mathbb{Q}_p}(\Gamma) \to F\text{-Isoc}(X)$ factors through $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi) \subset F\text{-Isoc}(X)$. Moreover, any tensor functor $\text{Rep}_{\overline{\mathbb{Q}}_p}(\Gamma) \to F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ factors through $\text{Rep}_{\overline{\mathbb{Q}}_p}^{\text{smooth}}(\Pi) \subset F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.*

Proof. It suffices to prove the second statement. Let $\Phi : \text{Rep}_{\overline{\mathbb{Q}}_p}(\Gamma) \to F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ be a tensor functor and $\text{Im}\,\Phi$ its essential image. We will first prove that

(B.4) $$\text{Im}\,\Phi \subset F\text{-Isoc}_{\text{et}}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p.$$

Proving this amounts to showing that for any algebraically closed field L and any morphism $\alpha : \text{Spec}\,L \to X$ one has

$$\text{Im}(\alpha^* \circ \Phi) \subset F\text{-Isoc}_{\text{et}}(\text{Spec}\,L) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p.$$

Note that $F\text{-Isoc}(\text{Spec}\,L) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \simeq \text{Rep}_{\overline{\mathbb{Q}}_p}(\widetilde{\mathbb{G}}_m)$, where $\widetilde{\mathbb{G}}_m$ is the pro-torus with group of characters $\mathbb{Q}$. So $\alpha^* \circ \Phi$ is a tensor functor

$$\text{Rep}_{\overline{\mathbb{Q}}_p}(\Gamma) \to \text{Rep}_{\overline{\mathbb{Q}}_p}(\widetilde{\mathbb{G}}_m).$$

It corresponds to a homomorphism $\widetilde{\mathbb{G}}_m \to \Gamma$. It remains to note that all such homomorphisms are trivial.

Thus we have proved (B.4). So by §B.3, it remains to show that any tensor functor

$$(B.5) \qquad \operatorname{Rep}_{\overline{\mathbb{Q}}_p}(\Gamma) \to \operatorname{Rep}_{\mathbb{Q}_p}^{\operatorname{cont}}(\Pi) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$$

factors through $\operatorname{Rep}_{\overline{\mathbb{Q}}_p}^{\operatorname{smooth}}(\Pi)$. Define an affine group scheme $\hat{\Pi}$ over $\mathbb{Q}_p$ as follows: $\hat{\Pi}$ is the projective limit of all affine algebraic group schemes H over $\mathbb{Q}_p$ equipped with a continuous homomorphism $\Pi \to H(\mathbb{Q}_p)$ with Zariski-dense image. Then $\operatorname{Rep}_{\mathbb{Q}_p}^{\operatorname{cont}}(\Pi) = \operatorname{Rep}_{\mathbb{Q}_p}(\hat{\Pi})$. So a tensor functor (B.5) corresponds to a homomorphism $\hat{\Pi} \to \Gamma$ or equivalently, to a continuous homomorphism $\Pi \to \Gamma$. The latter has open kernel, which finishes the proof. $\qquad \square$

B.5. The group $\pi_1^{F\text{-Isoc}}(X)$

Warning: the group $\pi_1^{F\text{-Isoc}}(X)$ defined below is *different* from (but closely related to) the group denoted by $\pi_1^{F\text{-Isoc}}(X)$ in formula (2.5.4) on p. 446 of [Cr2].

Set

$$F\text{-Isoc}(\tilde{X}) := \varinjlim_U F\text{-Isoc}(\tilde{X}/U),$$

where U runs through the set of open subgroups of Π. Fix a fiber functor $\tilde{\xi} : F\text{-Isoc}(\tilde{X}) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \operatorname{Vec}_{\overline{\mathbb{Q}}_p}$. The existence of $\tilde{\xi}$ is guaranteed by a general theorem of Deligne [De4]; one can also construct $\tilde{\xi}$ by choosing a closed point $\tilde{x} \in \tilde{X}$ and a fiber functor on $F\text{-Isoc}(\tilde{x}) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.

Let $\xi : F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \operatorname{Vec}_{\overline{\mathbb{Q}}_p}$ be the composition of $\tilde{\xi}$ with the pullback functor from X to $\tilde{X}$. We set $\pi_1^{F\text{-Isoc}}(X) := \operatorname{Aut} \xi$; this is an affine group scheme over $\overline{\mathbb{Q}}_p$, and one has a canonical equivalence

$$F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \operatorname{Rep}_{\overline{\mathbb{Q}}_p}(\pi_1^{F\text{-Isoc}}(X)).$$

The fully faithful functor (B.3) defines a canonical epimorphism

$$(B.6) \qquad \pi_1^{F\text{-Isoc}}(X) \twoheadrightarrow \Pi.$$

For any open subgroup $U \subset \Pi$ one has a similar group $\pi_1^{F\text{-Isoc}}(\tilde{X}/U)$ equipped with an epimorphism $\pi_1^{F\text{-Isoc}}(\tilde{X}/U) \twoheadrightarrow U$. If $U \subset U'$ then pullback from $\tilde{X}/U'$ to $\tilde{X}/U$ defines a homomorphism $\pi_1^{F\text{-Isoc}}(\tilde{X}/U) \to \pi_1^{F\text{-Isoc}}(\tilde{X}/U')$ over Π. In particular, one has a canonical homomorphism

$$(B.7) \qquad \pi_1^{F\text{-Isoc}}(\tilde{X}/U) \to \pi_1^{F\text{-Isoc}}(X).$$

Proposition B.5.1. *The homomorphism* (B.7) *identifies* $\pi_1^{F\text{-Isoc}}(\tilde{X}/U)$ *with* $\pi_1^{F\text{-Isoc}}(X) \times_\Pi U$.

Proof. Choose an open normal subgroup $V \subset \Pi$ so that $V \subset U$. Lemma B.2.3 implies that the sequences

$$0 \to \pi_1^{F\text{-Isoc}}(\tilde{X}/V) \to \pi_1^{F\text{-Isoc}}(X) \to \Pi/V \to 0,$$

$$0 \to \pi_1^{F\text{-Isoc}}(\tilde{X}/V) \to \pi_1^{F\text{-Isoc}}(\tilde{X}/U) \to U/V \to 0$$

are exact. The proposition follows. $\square$

Proposition B.5.2. *The kernel of the canonical epimorphism* $\pi_1^{F\text{-Isoc}}(X) \twoheadrightarrow \Pi$ *is connected. In other words,* $\mathrm{Ker}(\pi_1^{F\text{-Isoc}}(X) \twoheadrightarrow \Pi) = ((\pi_1^{F\text{-Isoc}}(X))^\circ$.

Proof. By Proposition B.4.1, every morphism from $\pi_1^{F\text{-Isoc}}(X)$ to a finite group factors through Π. $\square$

B.5.3. Some homomorphisms $\widetilde{\mathbb{G}}_{\boldsymbol{m}} \to \boldsymbol{\pi_1^{F\text{-Isoc}}(X)}$. Let $x \in X$, and let x_{perf} denote the spectrum of the perfection of the residue field of x. The Tannakian category $F\text{-Isoc}(x_{\mathrm{perf}})$ has a canonical $\mathbb{Q}$-grading by slopes, so we have a canonical central homomorphism $\widetilde{\mathbb{G}}_m \to \pi_1^{F\text{-Isoc}}(x_{\mathrm{perf}})$. Composing it with the homomorphism

$$(B.8) \qquad \pi_1^{F\text{-Isoc}}(x_{\mathrm{perf}}) \to \pi_1^{F\text{-Isoc}}(X),$$

we get a homomorphism

$$(B.9) \qquad \widetilde{\mathbb{G}}_m \to \pi_1^{F\text{-Isoc}}(X);$$

note that the homomorphisms (B.8)-(B.9) are defined only up to $\pi_1^{F\text{-Isoc}}(X)$-conjugacy.

Let $U \subset \Pi$ be an open subgroup and suppose that we fix $x' \in \tilde{X}/U$ such that $x' \mapsto x$. Applying the previous procedure to $(\tilde{X}/U, x')$ instead of (X, x), one gets a homomorphism (B.9) defined up to $\pi_1^{F\text{-Isoc}}(\tilde{X}/U)$-conjugacy.

Passing to the limit with respect to U, we get for each $\tilde{x} \in \tilde{X}$ a homomorphism

$$(B.10) \qquad \nu_{\tilde{x}} : \widetilde{\mathbb{G}}_m \to \pi_1^{F\text{-Isoc}}(X)$$

defined up to conjugation by elements of the group $\mathrm{Ker}(\pi_1^{F\text{-Isoc}}(X) \twoheadrightarrow \Pi)$. This group is the neutral connected component $(\pi_1^{F\text{-Isoc}}(X))^\circ$, see Proposition B.5.2.

B.6. A general lemma on $\mathrm{Rep}_E(G)$

Lemma B.6.1. *Let $f : H \to G$ be a homomorphism of affine group schemes over a field E and $f^* : \mathrm{Rep}_E(G) \to \mathrm{Rep}_E(H)$ the corresponding tensor functor.*

(i) f^* is fully faithful if and only if every regular function on $G/f(H)$ is constant.

(ii) f is an epimorphism[12] if and only if f^* has the following property: for every $V \in \mathrm{Rep}_E(G)$ every H-submodule of V is a G-submodule.

Probably the lemma is well known: e.g., statement (ii) is [DM,Prop.2.21 (a)]. We give a proof for completeness.

Proof. (i) Full faithfulness means that for every $V, W \in \mathrm{Rep}(G)$ the map

(B.11) $$\mathrm{Hom}_G(V, W) \to \mathrm{Hom}_H(V, W)$$

is an isomorphism. Let L denote the space of regular functions on $G/f(H)$, then the map (B.11) is just the map

$$\mathrm{Hom}_G(V \otimes W^*, E) \to \mathrm{Hom}_G(V \otimes W^*, L)$$

induced by the inclusion $E \hookrightarrow L$. So the map (B.11) is an isomorphism if and only if $L = E$.

(ii) It suffices to prove the "if" statement. Suppose that for every $V \in \mathrm{Rep}_E(G)$ every H-submodule $W \subset V$ is a G-submodule. Then this is true even if V is an infinite-dimensional G-module. Take V to be the space of regular functions on G (on which G acts by left translations), and let $W \subset V$ be the ideal of $f(H)$. We see that W is a G-submodule, which means that $f(H) = G$. $\qquad\square$

Example B.6.2. Let G be a connected reductive group over a field E and $P \subset G$ a parabolic subgroup. By Lemma B.6.1(i), the restriction functor $\mathrm{Rep}_E(G) \to \mathrm{Rep}_E(P)$ is fully faithful, so one can think of $\mathrm{Rep}_E(G)$ as a full subcategory of $\mathrm{Rep}_E(P)$. However, if $P \neq G$ this subcategory is *not closed under passing to subobjects*: this follows from Lemma B.6.1(ii) or by direct inspection. So if $P \neq G$ then $\mathrm{Rep}_E(G)$ is *not a Tannakian subcategory* of $\mathrm{Rep}_E(P)$ in the sense of [An, §2.3.5] (although it is a full subcategory which is a Tannakian category).

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[12]By definition, "epimorphism" means that $f(H)$ equals G as a scheme. If E has characteristic 0 this just means that f is surjective.

B.7. Overconvergent F-isocrystals and the group $\pi_1^{F\text{-}\mathbf{Isoc}^\dagger}(X)$

Overconvergent F-isocrystals form a Tannakian category $F\text{-Isoc}^\dagger(X)$. It is equipped with a tensor functor $F\text{-Isoc}^\dagger(X) \to F\text{-Isoc}(X)$. As already mentioned in §1.1.1, this functor is known to be fully faithful, so we view $F\text{-Isoc}^\dagger(X)$ as a full subcategory of $F\text{-Isoc}(X)$.

B.7.1. Warning. It often happens that the full subcategory $F\text{-Isoc}^\dagger(X) \subset F\text{-Isoc}(X)$ is not closed with respect to passing to subobjects (see [Ke6, Rem. 5.12]). In this case $F\text{-Isoc}^\dagger(X)$ is *not a Tannakian subcategory* of $F\text{-Isoc}(X)$ in the sense of [An, §2.3.5] (this is similar to Example B.6.2).

B.7.2. The embedding $\mathbf{Rep}_{\mathbb{Q}_p}^{\mathbf{smooth}}(\Pi) \hookrightarrow F\text{-}\mathbf{Isoc}^\dagger(X)$. The essential image of the fully faithful functor $\text{Rep}_{\mathbb{Q}_p}^{\text{smooth}}(\Pi) \hookrightarrow F\text{-Isoc}(X)$ from Remark B.2.5 is contained in $F\text{-Isoc}^\dagger(X)$ (by etale descent for overconvergent F-isocrystals, see [Et, Thm. 1]). So the essential image of the functor (B.3) is contained in $F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.

Proposition B.4.1 remains valid if one replaces $F\text{-Isoc}$ by $F\text{-Isoc}^\dagger$ (this follows from Proposition B.4.1 itself).

B.7.3. The homomorphism $\pi_1^{F\text{-}\mathbf{Isoc}}(X) \to \pi_1^{F\text{-}\mathbf{Isoc}^\dagger}(X)$. Let

$$\tilde{\xi} : F\text{-Isoc}(\tilde{X}) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \text{Vec}_{\overline{\mathbb{Q}}_p} \quad \text{and} \quad \xi : F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \text{Vec}_{\overline{\mathbb{Q}}_p}$$

be as in §B.5. Recall that $\pi_1^{F\text{-Isoc}}(X) := \text{Aut}\,\xi$. Let $\xi^\dagger : F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \text{Vec}_{\overline{\mathbb{Q}}_p}$ be the restriction of ξ, and set

$$\pi_1^{F\text{-Isoc}^\dagger}(X) := \text{Aut}\,\xi^\dagger.$$

Both $\pi_1^{F\text{-Isoc}}(X)$ and $\pi_1^{F\text{-Isoc}^\dagger}(X)$ are affine group schemes over $\overline{\mathbb{Q}}_p$. Restriction from $F\text{-Isoc}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ to $F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ defines a canonical homomorphism

$$(\text{B.12}) \qquad\qquad \pi_1^{F\text{-Isoc}}(X) \to \pi_1^{F\text{-Isoc}^\dagger}(X).$$

One has a canonical equivalence

$$F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \text{Rep}_{\overline{\mathbb{Q}}_p}(\pi_1^{F\text{-Isoc}^\dagger}(X)).$$

By §B.7.1, the homomorphism (B.12) is not always surjective. But it has the following weaker property.

Lemma B.7.4. *Let H denote the image of the homomorphism* (B.12). *Then every regular function on $\pi_1^{F\text{-Isoc}^\dagger}(X)/H$ is constant.*

Proof. Follows from Lemma B.6.1(i). □

B.7.5. The epimorphism $\pi_1^{F\text{-Isoc}^\dagger}(X) \twoheadrightarrow \Pi$. The fully faithful functor

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_p}^{\mathrm{smooth}}(\Pi) \hookrightarrow F\text{-Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$$

induces an epimorphism $\pi_1^{F\text{-Isoc}^\dagger}(X) \twoheadrightarrow \Pi$, whose composition with (B.12) is equal to the epimorphism (B.6).

Proposition B.7.6. *(i) The kernel of the canonical epimorphism $\pi_1^{F\text{-Isoc}^\dagger}(X) \twoheadrightarrow \Pi$ is connected.*

(ii) For any open subgroup $U \subset \Pi$, the canonical homomorphism $\pi_1^{F\text{-Isoc}^\dagger}(\tilde{X}/U) \to \pi_1^{F\text{-Isoc}^\dagger}(X)$ induces an isomorphism $\pi_1^{F\text{-Isoc}^\dagger}(\tilde{X}/U) \xrightarrow{\sim} \pi_1^{F\text{-Isoc}^\dagger}(X) \times_\Pi U$.

Proof. The proposition can be proved similarly to Propositions B.5.1-B.5.2. Statement (i) also follows from Proposition B.5.2 because by Lemma B.7.4, the homomorphism $\pi_0(\pi_1^{F\text{-Isoc}}(X)) \to \pi_0(\pi_1^{F\text{-Isoc}^\dagger}(X))$ is surjective. □

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