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Representation categories of Mackey Lie algebras as universal monoidal categories

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To Yuri Ivanovich Manin on the occasion of his 80th birthday

Abstract: Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. We study a monoidal category $\mathbb{T}_\alpha$ which is universal among all symmetric $\mathbb{K}$-linear monoidal categories generated by two objects A and B such that A is equipped with a possibly transfinite filtration together with a pairing $A \otimes B \to \mathbb{1}$. We construct $\mathbb{T}_\alpha$ as a category of representations of the Lie algebra $\mathfrak{gl}^M(V_*, V)$ consisting of endomorphisms of a fixed diagonalizable pairing $V_* \otimes V \to \mathbb{K}$ of vector spaces V_* and V of dimension α. Here α is an arbitrary cardinal number. We describe explicitly the simple and the injective objects of $\mathbb{T}_\alpha$ and prove that the category $\mathbb{T}_\alpha$ is Koszul. We pay special attention to the case where the filtration on A is finite. In this case $\alpha = \aleph_t$ for $t \in \mathbb{Z}_{\geq 0}$.

Keywords: Mackey Lie algebra, tensor module, monoidal category, Koszul algebra, semi-artinian, Grothendieck category.

Introduction

In the last decade, monoidal categories of representations of infinite matrix algebras have been studied from various points of view. In particular, in the paper [5] the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ was introduced and investigated in detail. To recall the definition of this category, fix an algebraically closed field $\mathbb{K}$ of characteristic 0 together with a nondegenerate pairing (bilinear map) $\mathbf{p} : V_* \times V \to \mathbb{K}$ for some countable-dimensional vector spaces V_* and V over $\mathbb{K}$. Then $V_* \otimes V$ has an obvious structure of an associative algebra $\left(\left((v_*)_1 \otimes v_1\right)\left((v_*)_2 \otimes v_2\right) = \mathbf{p}\left((v_*)_2, v_1\right)(v_*)_1 \otimes v_2\right)$, and hence also of a Lie algebra. The Lie subalgebra $\ker \mathbf{p} \subset V_* \otimes V$ is isomorphic to the Lie algebra $\mathfrak{sl}(\infty)$ (in fact, the Lie algebra $\mathfrak{sl}(\infty)$ can simply be defined as $\ker \mathbf{p}$). A quick way to

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define the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is to declare it the monoidal category of all $\ker \mathbf{p}$-subquotients of finite direct sums of tensor products of the form $(V_*)^{\otimes n} \otimes V^{\otimes m}$. In [5] three other equivalent definitions of this category are given: they are all intrinsic to the Lie algebra $\mathfrak{sl}(\infty)$. From the point of view of a representation-theorist, $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is interesting as it is the "limit as $q \to \infty$" of the categories of finite-dimensional $\mathfrak{sl}(q)$-modules. Unlike the category of finite-dimensional $\mathfrak{sl}(q)$-modules, $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is not a semisimple category. The simple objects of $\mathbb{T}_{\mathfrak{sl}(\infty)}$ are parametrized by ordered pairs (μ, ν) of Young diagrams, and, based on earlier work [16] by K. Styrkas and the second author, in [5] the injective objects of $\mathbb{T}_{\mathfrak{sl}(\infty)}$ have been described and Koszul self-duality of $\mathbb{T}_{\mathfrak{sl}(\infty)}$ has been established.

In a parallel development, the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ arose in the work [18] of A. Sam and A. Snowden who took a somewhat different point of view. In particular, they showed that $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is universal among $\mathbb{K}$-linear tensor categories (see Convention 5.1 below) generated by two objects A and B together with a morphism $A \otimes B \to \mathbb{1}$ into the monoidal unit $\mathbb{1}$ (the field $\mathbb{K}$). It is important to note that $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is not a rigid tensor category, in particular there is no nonzero morphism $\mathbb{1} \to V_* \otimes V$, and $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is universal as a general (nonrigid) $\mathbb{K}$-linear tensor category. This universality property of the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ has been used in an essential way in the recent study [6] of abelianizations of Deligne categories. In addition, the boson-fermion correspondence has been categorified via $\mathbb{T}_{\mathfrak{sl}(\infty)}$ in [7].

Motivated by representation theory, in [15] a larger Lie algebra, called Mackey Lie algebra, was introduced. Let now $\mathbf{p} : V_* \times V \to \mathbb{K}$ be a nondegenerate pairing for some vector spaces, not necessarily countable dimensional or having the same dimension. A pioneering study of such pairings was undertaken by G. Mackey in his dissertation [13]. The endomorphisms of the pairing $\mathbf{p}$ form a Lie algebra which we denote by $\mathfrak{gl}^M(V_*, V)$ and call the *Mackey Lie algebra* of $\mathbf{p}$: this Lie algebra is defined by formula (3) below, and has $V_* \otimes V$ as its ideal.

As a next step, in the work [3] we showed that, if $\dim V_* = \dim V = \aleph_0$ (i.e., if V_* and V are countable dimensional as in [5]), a natural category of representations of $\mathfrak{gl}^M(V_*, V)$ has also a universality property. More precisely, in [3] we consider the category $\mathbb{T}^3_{\mathfrak{gl}^M(V_*,V)}$ of $\mathfrak{gl}^M(V_*, V)$-modules isomorphic to subquotients of finite sums of tensor products of the form $(V^*)^{\otimes n} \otimes V^{\otimes m}$ for $m, n \geq 0$ (V^* being the algebraic dual space $\mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$). Our result states that $\mathbb{T}^3_{\mathfrak{gl}^M(V_*,V)}$ is universal among (nonrigid) $\mathbb{K}$-linear tensor categories generated by two objects A and B such that A has a subobject $A_0 \hookrightarrow A$ (in the case of $\mathbb{T}^3_{\mathfrak{gl}^M(V_*,V)}$, we have $A = V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$, $B = V$, and $A_0 = V_*$, the inclusion $V_* \subset V^*$ being induced by $\mathbf{p}$). The structure of $\mathbb{T}^3_{\mathfrak{gl}^M(V_*,V)}$, both as an abelian and a tensor category, is rather elaborate. Its simple objects are parametrized by triples (λ, μ, ν) of Young diagrams, and in [3] we compute the Ext-groups between simple objects and prove Koszul self-duality for $\mathbb{T}^3_{\mathfrak{gl}^M(V_*,V)}$.

All of the above motivates the topic of our current study. Our main objective is to construct and study a $\mathbb{K}$-linear tensor category which is universal among abstract $\mathbb{K}$-linear tensor categories generated by two objects A and B such that A has an arbitrary fixed filtration. In fact, we allow the filtration on A to be transfinite. As it turns out, we can construct such a universal category as a category of tensor representations for a Mackey Lie algebra $\mathfrak{gl}^M(V_*, V)$, where the dimension of both V_* and V equals $\alpha = \aleph_a$, a being the ordinal of the maximal proper subobject of A in the fixed transfinite filtration of A.

More precisely, we consider a nondegenerate pairing $\mathbf{p} : V_* \times V \to \mathbb{K}$ where V_*, V are α-dimensional for an arbitrary cardinal number α, and suppose that the pairing is splitting in the sense that there are respective bases $\{v^*_\kappa\}$, $\{v_{\kappa'}\}$ of V_* and V such that $\mathbf{p}(v^*_\kappa, v_{\kappa'}) = \delta_{\kappa\kappa'}$ ($\delta_{\kappa\kappa'}$ being the Kronecker delta). The category $\mathbb{T}_\alpha$ is then the minimal full monoidal subcategory of the category of $\mathfrak{g}$-modules which contains V and V^* and is closed with respect to subquotients. For $\alpha = \aleph_0$, the category $\mathbb{T}_{\aleph_0}$ coincides with the category $\mathbb{T}^3_{\mathfrak{gl}^M(V_*,V)}$ studied in [3]. We show that the Grothendieck envelope $\bar{\mathbb{T}}_\alpha$ of $\mathbb{T}_\alpha$ is an ordered Grothendieck category according to a slightly more general definition than the one given in [3], and use this in a crucial way to deduce that objects of the form

$$\bigotimes_{s=t}^{0} \left(V^*/V^*_{\beta_s^+}\right)_{\lambda_s} \otimes (V^*)_\mu \otimes V_\nu \tag{1}$$

are injective in $\mathbb{T}_\alpha$. Here $\{\beta_t, \dots, \beta_1, \beta_0\}$ is a finite (possibly empty) set of infinite cardinal numbers such that $\beta_0 < \beta_1 < \dots < \beta_t \leq \alpha$, β_s^+ stands for the successor cardinal to β_s, and $\lambda_t, \dots, \lambda_1, \lambda_0, \mu, \nu$ are Young diagrams; $\bullet_\lambda$ denotes the Schur functor associated with a Young diagram λ. We show that the objects (1) have simple socles, and that the so obtained simple modules exhaust (up to isomorphism) the simple objects of $\mathbb{T}_\alpha$.

In the case when $\alpha = \aleph_t$ for some nonnegative integer t, we present an explicit combinatorial formula for the multiplicity of a simple module in an injective hull of another simple module. This generalizes the corresponding multiplicity formulas from [16] and [3]. Our next result is that, for a general α, the category $\mathbb{T}_\alpha$ is Koszul in the sense that its Ext-algebra

$$\bigoplus_{T',T \text{ simple}, p\geq 0} \mathrm{Ext}^p(T',T)$$

is generated in degree one. In the last section we use the Koszulity of $\mathbb{T}_\alpha$ to prove that $\mathbb{T}_\alpha$ possesses the universality property stated above.

Finally, we should mention that in Section 3 we present another application of the more general notion of ordered tensor category introduced in this paper: we point out that the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ introduced and studied in [14] falls under the new definition, and we prove that its injective objects are nothing but arbitrary direct sums of the indecomposable injectives described in [14].

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1. Background

Let $\mathbb{K}$ be a field. Except in Section 2, $\mathbb{K}$ is assumed to be algebraically closed and of characteristic 0. All vector spaces (including Lie algebras) are assumed to be defined over $\mathbb{K}$. If U is a vector space, we set $U^* = \mathrm{Hom}_{\mathbb{K}}(U, \mathbb{K})$ and $\mathrm{End}(U) = \mathrm{End}_{\mathbb{K}}(U)$. We also abbreviate $\otimes_{\mathbb{K}}$ to $\otimes$. All additive categories considered are understood to be linear over $\mathbb{K}$, and all additive functors are assumed to preserve this structure.

One way to define the finitary Lie algebra $\mathfrak{sl}(\infty)$ is as the inductive limit of a chain of embeddings

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \dots \tag{2}$$

where $\mathfrak{g}_q = \mathfrak{sl}(q+1)$. Similarly, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ can be defined as the inductive limits of respective chains (2) where $\mathfrak{g}_q = \mathfrak{o}(q)$ or $\mathfrak{g}_q = \mathfrak{sp}(2q)$.

A natural representation of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$ is a direct limit of natural representations V_q of $\mathfrak{g}_q$. For $\mathfrak{g} = \mathfrak{o}(\infty)$ (respectively, for $\mathfrak{sp}(\infty)$), up to isomorphism, there is a unique natural representation $V = \varinjlim V_q$ (respectively, $V = \varinjlim V_{2q}$), while for $\mathfrak{g} = \mathfrak{sl}(\infty)$ there are two nonisomorphic natural representations $V = \varinjlim V_{q+1}$ and $V_* = \varinjlim V^*_{q+1}$; here, V_{q+1} denotes the space of column vectors of length $q+1$, considered as an $\mathfrak{sl}(q+1)$-module.

Let now V_* and V be abstract vector spaces and $\mathbf{p} : V_* \times V \to \mathbb{K}$ be a nondegenerate bilinear map, or simply pairing, of the vector spaces V_* and

V. Note that $\mathbf{p}$ induces injective linear operators $V_* \hookrightarrow V^*$ and $V \hookrightarrow (V_*)^*$. The Mackey Lie algebra $\mathfrak{gl}^M(V_*, V)$ is by definition the Lie algebra of endomorphisms of the pairing $\mathbf{p}$, i.e.,

(3)
$$\mathfrak{gl}^M(V_*, V) = \{x \in \mathrm{End}\,(V_*) \mid x^*(V) \subset V\} = \{y \in \mathrm{End}(V) \,|\, y^*(V_*) \subset V_*\}\,,$$

where here * indicates dual linear operator. Note that $V_* \otimes V$ is an ideal in $\mathfrak{gl}^M\,(V_*, V)$.

If V_* and V are both countable dimensional, then it is a result of G. Mackey [13] that a nondegenerate pairing $\mathbf{p}$ is unique up to isomorphism. In this case, the Mackey Lie algebra $\mathfrak{gl}^M(V_*, V)$ is isomorphic to the Lie algebra of infinite matrices with finite rows and columns. See [15] for results concerning $\mathfrak{gl}^M(V_*, V)$ and its representations.

The above result of Mackey provides also an alternative definition of the Lie algebra $\mathfrak{sl}(\infty)$. This definition was already mentioned in the Introduction. If $\dim V_* = \dim V = \aleph_0$, the pairing $\mathbf{p}$ is unique up to isomorphism of pairings, and we can set $\mathfrak{sl}(\infty) = \ker \mathbf{p}$. Then V and V_* are the two nonisomorphic natural representations of $\mathfrak{sl}(\infty)$ which were introduced above as direct limits. In the rest of the paper we will have both of these interpretations of V and V_* in mind. Note also that V and V_* admit a pair of dual bases, i.e. bases $\{v_q\} \subset V$, $\left\{v_q^*\right\} \subset V_*$ for $q \in \mathbb{Z}_{>0}$, such that $\mathbf{p}\left(v_{q'}^*, v_q\right) = \delta_{q'q}$. In what follows we will think of V_* as a subspace of V^* (the embedding $V_* \subset V^*$ being induced by the pairing $\mathbf{p}$). If V is equipped with a nondegenerate symmetric (or antisymmetric bilinear form, then $\mathfrak{o}(\infty)$ is defined as the Lie algebra $\Lambda^2(V)$ and $\mathfrak{sp}(\infty)$ is defined as the Lie algebra $S^2(V)$. The spaces $\Lambda^2(V)$ and $S^2(V)$ are endowed with respective Lie algebra structures via the corresponding forms.

Let $\mathfrak{g}$ be any Lie algebra, M be any $\mathfrak{g}$-module, and $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra. Recall that $\mathfrak{k}$ *acts densely on* M if for any finite set of vectors $m_1, \ldots, m_q \in M$ and any $g \in \mathfrak{g}$, there is $k \in \mathfrak{k}$ such that $g \cdot m_s = k \cdot m_s$ for $s = 1, \ldots, q$. Below we use the fact that if $\mathfrak{k}$ acts densely on M then $\mathfrak{k}$ acts densely on any $\mathfrak{g}$-subquotient of the tensor algebra $\mathrm{T}(M)$ [15, Lemma 7.3].

We recall also that there is a well-defined Schur functor $\bullet_\lambda$ for any Young diagram, or partition, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q)$,

$$\bullet_\lambda : \mathrm{Vect} \to \mathrm{Vect}\,,$$

Vect being the category of vector spaces over $\mathbb{K}$. By definition, V_λ is a direct summand of the tensor power $V^{\otimes|\lambda|}$, where $|\lambda| = \lambda_1 + \ldots + \lambda_q$. For the precise definition see [8]. If V is a $\mathfrak{g}$-module for some Lie algebra $\mathfrak{g}$, then V_λ has a natural structure of a $\mathfrak{g}$-module (as a $\mathfrak{g}$-submodule of $V^{\otimes|\lambda|}$).

If $\mathcal{C}$ is an abelian category, a *chain* of objects of $\mathcal{C}$ is a set of objects $\{A_\sigma\}$ such that, for any pair of objects A_{σ_1} and A_{σ_2}, precisely one noninvertible monomorphism $A_{\sigma_1} \to A_{\sigma_2}$ or $A_{\sigma_2} \to A_{\sigma_1}$ is fixed. This endows the set of indices $\{\sigma\}$ with a linear order: $\sigma_1 < \sigma_2$ if a noninvertible monomorphism $A_{\sigma_1} \to A_{\sigma_2}$ is fixed. Given an object A of $\mathcal{C}$, we say that A is endowed with a *transfinite filtration* if a chain of subobjects $\{A_\sigma\}$ of A is given such that the linear order on the set of indices $\{\sigma\}$ is a well-order.

For background on Grothendieck categories we refer the reader to [9, §1.1] or [17, §2.8]. In any such category every object X has a *transfinite socle filtration*: the socle of X, $\operatorname{soc} X$, is the maximal semisimple subobject of X (the sum of all semisimple subobjects of X), and then the socle filtration is built by transfinite induction each time taking the pullback in X of the socle of the relevant quotient. We have

$$\begin{aligned} 0 \subset \operatorname{soc} X = \operatorname{soc}^1 X \subset \operatorname{soc}^2 X = \pi_1^{-1}(X/(\operatorname{soc} X)) \subset \ldots \subset \operatorname{soc}^{\aleph_0} X \\ = \pi_{\aleph_0}^{-1}\left(\varinjlim (\operatorname{soc}^q X)\right) \subset \ldots \end{aligned}$$

where $\pi_1 : X \to X/(\operatorname{soc} X)$ and $\pi_{\aleph_0} : X \to X/\left(\varinjlim_{q<\aleph_0} (\operatorname{soc}^q X)\right)$ are the canonical projections. For $q \in \mathbb{Z}_{\geq 1}$, we denote by $\underline{\operatorname{soc}}^q X$ the q-th layer $\operatorname{soc}^q X/\operatorname{soc}^{q-1} X$ of the transfinite socle filtration of X.

2. Ordered Grothendieck categories

Here we extend the notion of ordered Grothendieck category introduced in [3] to the infinite-length setting.

First, recall that a family of objects $\{Z_\kappa\}$ of an abelian category $\mathcal{C}$ is a *family of generators* if for any two distinct morphisms $A \xrightarrow{\varphi} B$ and $A \xrightarrow{\psi} B$ in the category $\mathcal{C}$, there is an object Z_κ together with a morphism $\gamma : Z_\kappa \to A$ so that the compositions $\varphi \circ \gamma$ and $\psi \circ \gamma$ are distinct, see [11, §V.7] or [17, discussion preceding Proposition 1.2.2]. Next, recall the following notion (see [17, §5.6] or [1, §41] for a discussion in the context of module categories).

Definition 2.1 An object X in a Grothendieck category is *semi-artinian* if every nonzero quotient of X has nonzero socle. A Grothendieck category is *semi-artinian* if it has a set of semi-artinian generators. ♦

Let $\mathcal{C}$ be Grothendieck category and $(I, \preceq)$ a poset. Let also $X_i \in \mathcal{C}$, $i \in I$ be a collection of semi-artinian objects, and $\mathcal{S}_i$ be the set of isomorphism classes of simple subobjects of X_i.

Throughout, we assume that the opposite poset $(I, \preceq)^{\mathrm{op}}$ (i.e. the set I equipped with the partial order opposite to $\preceq$) is *well ordered*. This means that every nonempty totally ordered subset of I has a largest element with respect to $\preceq$.

Remark 2.2 Note that our usage of the term 'well-ordered' is somewhat nonstandard, as the posets we consider are not required to be totally ordered; see e.g. [10, Definition 3.2.3]. ♦

The following definition generalizes Definition 2.1 in [3].

Definition 2.3 The above structure makes $\mathcal{C}$ an *ordered Grothendieck category* provided the following conditions hold:

(i) every object in $\mathcal{C}$ is isomorphic to a subquotient of a direct sum $\bigoplus_{i \in I' \subset I} X_i^{\oplus \gamma_i}$ for some subset $I' \subset I$ and some cardinal numbers γ_i;
(ii) the sets $\mathcal{S}_i$ are disjoint, and they exhaust the isomorphism classes of simple objects of $\mathcal{C}$;
(iii) simple subquotients of X_i which are not subobjects of $\operatorname{soc} X_i$ belong to $\mathcal{S}_j$ for $j \prec i$;
(iv) each X_i decomposes as a direct sum of subobjects with simple socle;
(v) for all $i \succ j$ the maximal subobject $Y_{i \succ j} \subseteq X_i$ whose simple constituents are in various $\mathcal{S}_k$ for $i \succeq k \succ j$, is the joint kernel of a family of morphisms $X_i \to X_j$. ♦

If U is a semisimple subobject in $\operatorname{soc} X_i$, then by $\widetilde{U}$ we denote the direct summand of X_i such that $\operatorname{soc} \widetilde{U} = U$; the existence of $\widetilde{U}$ is guaranteed by condition (iv).

Remark 2.4 An ordered Grothendieck category is semi-artinian. This follows from the observation that condition (i) of Definition 2.3 together with our assumption that X_i are semi-artinian ensure that $\mathcal{C}$ is semi-artinian. The proof can be found in [1, 41.10 (4)] (or in the original source [20, 27.5, 32.5] cited therein) for module categories; the general result is analogous. ♦

Definition 2.5 If the X_i in Definition 2.3 are of finite length (or, equivalently, if X_i satisfy Definition 2.1 in [3]), we say that $\mathcal{C}$ is a *finite ordered Grothendieck category.*

The next proposition shows that checking the properties (i)–(v) on a set of objects $\{X_i\}$ suffices to describe, up to isomorphism, all injective hulls of simple objects in $\mathcal{C}$ as direct summands of X_i (cf. [3, Proposition 2.3]).

Proposition 2.6 *In the setup of Definition 2.3, for any $i \in I$ and any simple object $U \in \mathcal{S}_i$ the object $\widetilde{U}$ is an injective hull of U.*

Proof Let $U \subset J$ be an essential extension such that J is a subquotient of a direct sum $X = \bigoplus_{j \in I' \subset I} X_j^{\oplus \gamma_j}$ for some cardinal numbers γ_j. It suffices to show that J admits a monomorphism into $\widetilde{U}$.

Writing J as a subobject of a suitable quotient Z of X, we can factor out a direct complement of U in $\operatorname{soc} Z$ thus reducing to the case where Z itself has socle U. Therefore, upon substituting Z for J, we can assume that J is a quotient of X.

Now consider the largest subobject K of X whose simple constituents lie in various $\mathcal{S}_k$ for $k \succ i$. First, since no subquotient of K is in $\mathcal{S}_i$, K is automatically a subobject of the kernel of the epimorphism $X \to J$. Additionally, condition (v) in Definition 2.3 ensures the existence of a morphism from X into a product X_i^{γ} with kernel equal to K.

Condition (iii) of Definition 2.3 and the fact that the socle of J is $U \in \mathcal{S}_i$ now imply that a subobject of a single factor X_i of X_i^{γ} admits an epimorphism to J. Finally, this implies via the decomposition in condition (iv) of Definition 2.3 that a subobject of $\widetilde{U} \subset X_i$ admits an isomorphism with J. ■

Corollary 2.7 *Under the conditions of Proposition 2.6, the indecomposable injective objects of $\mathcal{C}$ are, up to isomorphism, precisely the indecomposable summands of the various objects X_i.*

Proof The indecomposable injectives are injective hulls of the simple objects, and these are precisely the objects in $\mathcal{S}_i$, $i \in I$. The conclusion now follows from Proposition 2.6. ■

One characteristic of ordered Grothendieck categories that will be important for us below is a certain "upper triangular" character of Ext-groups between simple objects, which limits the possibilities for such Ext-groups to be nontrivial.

In order to state the result, we need the following definition (cf. e.g. [3, §2.2]).

Definition 2.8 Let $i \preceq j$ be two elements in a poset $(I, \preceq)$. The *defect* $d(i,j)$ is the supremum in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ of the set of nonnegative integers q for which there is a chain

$$i = i_0 \prec i_1 \prec \cdots \prec i_q = j$$

in I. ♦

Remark 2.9 Note that for $i \preceq j \preceq k$ we have $d(i,j) \geq d(i,k) + d(k,j)$. ♦

We can now state the result alluded to above.

Proposition 2.10 *Let $T \in \mathcal{S}_j$ and $T' \in \mathcal{S}_i$ be two simple objects and suppose* $\mathrm{Ext}^p(T', T) \neq 0$ *for some $p \geq 0$. Then $i \preceq j$ and $d(i, j) \geq p$.*

Before delving into the proof, it will be convenient to introduce a notation and a definition.

Notation 2.11 For an arbitrary object $U \in \mathcal{C}$ we denote by $\widetilde{U}$ an injective hull of U in $\mathcal{C}$. By Proposition 2.6, this is consistent with the previous usage of the notation $\widetilde{U}$. ♦

Definition 2.12 For an object $U \in \mathcal{C}$ its *corona* $\mathrm{cor}\, U$ in $\mathcal{C}$ is $\widetilde{U}/U$. ♦

Proof of Proposition 2.10 The case $p = 0$ is not interesting, so suppose $p \geq 1$. If $p = 1$, then the desired conclusion (namely, $i \prec j$) follows from condition (iii) of Definition 2.3.

Now suppose $p > 1$. Then the long exact sequence of Ext-groups corresponding to the exact sequence

$$0 \to T \to \tilde{T} \to \mathrm{cor}\, T \to 0$$

yields an identification

$$\mathrm{Ext}^p(T', T) \cong \mathrm{Ext}^{p-1}(T', \mathrm{cor}\, T). \tag{4}$$

By condition (iii) of Definition 2.3 again, all simple subquotients of $\mathrm{cor}\, T$ belong to various $\mathcal{S}_\ell$ for $\ell \prec j$.

If $p - 1 = 1$, then some of these indices ℓ will be strictly larger than i (because of the nonvanishing of the right-hand side Ext^1 of (4)). Therefore

$$i \prec \ell \prec j\,,$$

in particular, $i \prec j$ and $d(i, j) \geq 2$. Otherwise, we can continue the process, passing to the double corona $\mathrm{cor}(\mathrm{cor}\, T)$, until p has been whittled down to 1. ■

The reason why we referred to the result of Proposition 2.10 as 'upper triangularity' is the following fragment of the statement, which we isolate for emphasis; it says that nonvanishing Ext-functors are unidirectional with respect to the poset $(I, \preceq)$.

Corollary 2.13 *If $i \succ j \in I$ and $T \in \mathcal{S}_j$, $T' \in \mathcal{S}_i$, then*

$$\mathrm{Ext}^p(T', T) = 0 \textit{ for } p > 0\,. \qquad ■$$

Note that, although all indecomposable injective objects of $\mathcal{C}$ are described by Corollary 2.7, arbitrary injective objects need in general not be sums of indecomposable injectives. We now identify some sufficient conditions that ensure that $\mathcal{C}$ is better behaved in this sense; we then apply these results to specific ordered Grothendieck categories.

Proposition 2.14 *If each X_i is a union of its finite-length subobjects then, up to isomorphism, the injective objects in $\mathcal{C}$ are precisely arbitrary direct sums of indecomposable direct summands of the X_i.*

Before going into the proof, recall the following notions (see e.g. [17, §5.7 and §5.8]).

Definition 2.15 Let X be an object of a Grothendieck category $\mathcal{C}$.

(1) X is *noetherian* if it satisfies the ascending chain condition on subobjects;
(2) $\mathcal{C}$ is *locally noetherian* if it has a set of noetherian generators. ♦

The reason why Definition 2.15 is relevant to Proposition 2.14 is that it precisely captures the conditions that give us the kind of control over arbitrary injective objects alluded to above, as the following result shows (this is an abbreviated version of [17, Theorems 5.8.7, 5.8.11]).

Proposition 2.16 *For a Grothendieck category $\mathcal{C}$ the following conditions are equivalent:*

(1) $\mathcal{C}$ is locally noetherian;
(2) the injective objects in $\mathcal{C}$ are precisely the arbitrary direct sums of indecomposable injective objects. ■

Proof of Proposition 2.14 The hypothesis ensures that the finite-length subquotients of the X_i form a set of generators, and hence $\mathcal{C}$ is locally noetherian. Our claim follows now from Proposition 2.16. ■

We end this section with a discussion of how the present material relates to the notion of a *highest weight category* in the sense of [4, Definition 3.1]. First, note that Definition 2.3 specializes to [3, Definition 2.1] when the poset I has the property that down-sets

$$I_{\preceq i} := \{j \in I \mid j \preceq i\}$$

are finite. Moreover, [3, Proposition 2.16] shows that in that case an ordered Grothendieck category is a highest weight category. That result extends virtually verbatim to the present setting provided I is *interval-finite*, i.e. all intervals

$$[i, k] = \{j \in I \mid i \preceq j \preceq k\}, \ i, k \in I$$

are finite. We record the resulting statement here.

Proposition 2.17 *An ordered Grothendieck category based on an interval-finite poset $(I, \preceq)$ is a highest weight category.* ■

3. A first application

Before moving on to our main object of study, we would like to point out that the material in Section 2 applies to an interesting category studied in [14]. Specifically, recall that for

$$\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$$

the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is defined as the full subcategory of $\mathfrak{g}$-mod consisting of integrable modules M of finite Loewy length such that the algebraic dual M^* is also an integrable $\mathfrak{g}$-module of finite Loewy length. It can be shown that $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is closed under dualization, taking subobjects, quotient objects, and extensions (and hence also finite direct sums).

Definition 3.1 For $\mathfrak{g}$ as above, we denote by $\mathcal{C}_{\mathfrak{g}}$ the smallest full, exact Grothendieck subcategory of $\mathfrak{g}$-mod containing $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. ♦

The category $\mathcal{C}_{\mathfrak{g}}$ is simply the full subcategory of $\mathfrak{g}$-mod whose objects are sums of objects in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$.

Remark 3.2 For an algebra A, the smallest Grothendieck category of modules containing a given A-module M (constructed essentially as we have just described) is sometimes denoted by $\sigma(M)$ in the literature, e.g. [1, §41]. We can think of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as $\sigma(M)$ where A is the enveloping algebra of $\mathfrak{g}$ and M is the direct sum of a set of generators for $\mathcal{C}_{\mathfrak{g}}$. ♦

We now explain how $\mathcal{C}_{\mathfrak{g}}$ fits into the framework of Section 2. Recall [5] that $\mathbb{T}_{\mathfrak{g}}$ is the full subcategory of $\mathfrak{g}$-mod which consists of $\mathfrak{g}$-modules isomorphic to finite-length subquotients of finite direct sums of the form $\mathrm{T}\,(V \oplus V_*)^{\oplus q}$ for $q \in \mathbb{Z}_{>0}$ (for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, one can replace $\mathrm{T}\,(V \oplus V_*)^{\oplus q}$ simply by $\mathrm{T}(V)^{\oplus q}$). The category $\mathbb{T}_{\mathfrak{g}}$ is equipped with an auto-equivalence $M \mapsto M_*$ (which is the identity in the case of $\mathfrak{g} = \mathfrak{o}$ or $\mathfrak{sp}$) induced by the automorphism of $\mathfrak{g}$ arising from switching V and V_*. Let F denote the composition of functors

$$M \mapsto M_* \mapsto (M_*)^* .$$

The poset $(I, \preceq)$, relevant for the category $\mathcal{C}_{\mathfrak{g}}$, consists of all pairs (n, m) of nonnegative integers, where

$$(n, m) \preceq (n', m') \iff n \leq n' \text{ and } m \leq m'.$$

For $(n,m) \in I$ we define

$$X_{n,m} := F\big((V_*)^{\otimes n} \otimes V^{\otimes m}\big).$$

Then the contents of [14, §6] amount to the fact that $\mathcal{C}_{\mathfrak{g}}$ satisfies the conditions of Definition 2.3; we leave the easy verification to the reader. Condition (v), for instance, which is perhaps the least obvious, follows from [14, Lemma 6.6] by taking the family of morphisms required by condition (v) to be the family of all morphisms $X_i \to X_j$ where $i = (n', m') \succ j = (n, m)$.

The poset $(I, \preceq)$ is interval-finite, hence $\mathcal{C}_{\mathfrak{g}}$ is a highest-weight category according to Proposition 2.17.

Next, note that the finite-length subobjects of finite direct sums of objects $X_{n,m}$ form a set of generators for $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, and hence the same is true of $\mathcal{C}_{\mathfrak{g}}$ by the definition of the latter. This shows that $\mathcal{C}_{\mathfrak{g}}$ is locally noetherian in the sense of Definition 2.15, and therefore according to Proposition 2.16 we have

Proposition 3.3 *The injective objects in $\mathcal{C}_{\mathfrak{g}}$ are precisely arbitrary direct sums of indecomposable injectives.* ■

In [14, Corollary 6.7], the indecomposable injectives in the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ have been described explicitly as being isomorphic to direct summands of $X_{n,m}$, and hence Proposition 3.3 classifies the injective objects in $\mathcal{C}_{\mathfrak{g}}$. Moreover, Proposition 3.3 is an essential improvement of Theorem 6.15 in [14] which claims the existence of a certain finite filtration on any injective object in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$.

4. The categories $\mathbb{T}_\alpha$ and $\bar{\mathbb{T}}_\alpha$

We now introduce a series of ordered Grothendieck categories which we study throughout the rest of the paper. The general setting is as follows: α is an arbitrary infinite cardinal number, V and V_* are α-dimensional complex vector spaces, and

$$\mathbf{p} : V_* \otimes V \to \mathbb{K}$$

is a nondegenerate pairing diagonalizable in the sense that there are bases $\{v_\kappa^*\}$ of V_* and $\{v_{\kappa'}\}$ of V such that $\mathbf{p}\,(v_\kappa^*, v_{\kappa'}) = \delta_{\kappa\kappa'}$.

Fixing the bases $\{v_\kappa^*\}$ and $\{v_\kappa\}$, and an arbitrary total order on the set Σ of indices κ, allows us to think of the elements of V as size-α column vectors with finitely many nonzero entries, and of the elements of V^* as arbitrary size-α row vectors; of those, the elements of V_* are precisely the row vectors with finitely many nonzero entries.

By α^+ we denote the successor cardinal to α. For each infinite cardinal $\beta \leq \alpha^+$ we denote by $V^*_\beta \subset V^*$ the subspace consisting of row vectors with strictly fewer than β nonzero entries. In this way we have a transfinite filtration

$$0 \subset V^*_{\aleph_0} \subset \ldots \subset V^*_\alpha \subset V^* . \tag{5}$$

Note that $V^*_{\alpha^+} = V^*$ and $V^*_{\aleph_0} = V_*$ by definition.

Let $\mathfrak{gl}^M$ be the Mackey Lie algebra of the pairing $\mathbf{p}$. Using the bases $\{v^*_\kappa\}$ and $\{v_\kappa\}$, and the total order on Σ, we can identify $\mathfrak{gl}^M(V_*, V)$ with $\alpha \times \alpha$-matrices with finite rows and columns. Every infinite cardinal $\beta \leq \alpha^+$ yields an ideal $\mathfrak{gl}^M_\beta$ in $\mathfrak{gl}^M$: it consists of matrices in $\mathfrak{gl}^M$ with strictly fewer than β nonzero rows and columns, or equivalently with strictly fewer than β nonzero entries. The action of $\mathfrak{gl}^M$ on V is nothing but multiplication of matrices. On V^* the action of $\mathfrak{gl}^M$ is given by the formula

$$g \cdot v = -vg \text{ for } g \in \mathfrak{gl}^M, v \in V^* .$$

Clearly $\mathfrak{gl}^M \cdot V^*_\beta \subseteq V^*_\beta$, i.e., the filtration (5) of V^* is $\mathfrak{gl}^M$-stable.

Recall that, for a Young diagram μ and an object $M \in \mathbb{T}_\alpha$, we denote by M_μ the image of M through the Schur functor associated to μ. Moreover, given Young diagrams μ and ν, we denote by $V_{\mu,\nu}$ the space of *traceless tensors* in $(V_*)_\mu \otimes V_\nu$, i.e. those annihilated by all compositions

$$(V_*)_\mu \otimes V_\nu \subseteq (V_*)^{\otimes|\mu|} \otimes V^{\otimes|\nu|} \to (V_*)^{\otimes(|\mu|-1)} \otimes V^{\otimes(|\nu|-1)}$$

where the right-hand arrow ranges over the $|\mu| \cdot |\nu|$ possible applications of $\mathbf{p}$.

Definition 4.1 $\mathbb{T}_\alpha$ is the smallest full monoidal subcategory (with respect to $\otimes$) of $\mathfrak{g}$-mod which contains V and V^*, and is closed under taking subquotients.

We will also work with the Grothendieck envelope $\overline{\mathbb{T}}_\alpha$ obtained as the full subcategory of $\mathfrak{gl}^M$-mod with objects arbitrary sums of objects in $\mathbb{T}_\alpha$ (see Remark 3.2). We embark below on a study of $\mathbb{T}_\alpha$ and $\overline{\mathbb{T}}_\alpha$, our first goal being to show that the latter category fits into the framework of Section 2.

4.1. Simple objects

Our aim here is to prove the following classification of the simple objects in the category $\mathbb{T}_\alpha$.

Proposition 4.2 *Let t be a nonnegative integer such that there exist infinite cardinal numbers $\beta_t, \dots, \beta_0$ with $\beta_0 < \cdots < \beta_t \le \alpha$. Then, given Young diagrams*

$$\lambda_t, \cdots, \lambda_0, \mu, \nu,$$

the object[1]

$$V_{(\beta_t,\lambda_t),\cdots,(\beta_0,\lambda_0),\mu,\nu} := \bigotimes_{s=t}^{0} (V^*_{\beta_s^+}/V^*_{\beta_s})_{\lambda_s} \otimes V_{\mu,\nu} \tag{6}$$

is simple over $\mathfrak{gl}^M$, and its endomorphism algebra in $\mathbb{T}_\alpha$ is $\mathbb{K}$. Moreover, the objects obtained for distinct choices of cardinals or Young diagrams are mutually nonisomorphic.

We work in stages towards a proof. First, we have

Lemma 4.3 *The object $V_{\mu,\nu}$ has no nonzero proper subobjects and its endomorphism algebra over $\mathfrak{gl}^M$ is $\mathbb{K}$.*

Proof As a consequence of [16, Theorem 2.2] and [15, Theorem 5.5], $V_{\mu,\nu}$ is simple over the ideal $\mathfrak{gl}^M_{\aleph_0} = V_* \otimes V \subset \mathfrak{gl}^M$: the former result handles the case of countable-dimensional V and V_*, whereas the latter result transports this to the general case via a categorical equivalence. In conclusion, $V_{\mu,\nu}$ is also simple over $\mathfrak{gl}^M$.

As for the statement regarding the endomorphism algebra, we can again assume that we are in the countable-dimensional setup of [16], as the general case follows then by [15, Theorem 5.5]. Then the Lie algebra $V_* \otimes V$ is the union of a chain of upper-left-hand-corner inclusions

$$\mathfrak{gl}(2) \subset \mathfrak{gl}(3) \subset \cdots \subset \mathfrak{gl}(q) \subset \cdots,$$

and $V_{\mu,\nu}$ is a direct limit of irreducible $\mathfrak{gl}(q)$-modules $(V_{\mu,\nu})_q$.

Now let ψ be an $\mathfrak{gl}^M$-endomorphism of $V_{\mu,\nu}$. Consider a vector $0 \ne v \in V_{\mu,\nu}$. Then, $v \in (V_{\mu,\nu})_q$ for some q. The vector $\psi(v)$ lies in $(V_{\mu,\nu})_{q'}$ for some $q' > q$, hence $\psi(v)$ generates $(V_{\mu,\nu})_{q'}$ over $U(\mathfrak{gl}(q'))$. Consequently, $\psi|_{(V_{\mu,\nu})_{q'}}$ is a well-defined automorphism of $(V_{\mu,\nu})_{q'}$, and equals a constant by Schur's

[1]Since in the expression $V_{(\beta_t,\lambda_t),\cdots,(\beta_0,\lambda_0),\mu,\nu} = \bigotimes_{s=t}^{0} (V^*_{\beta_s^+}/V^*_{\beta_s})_{\lambda_s} \otimes V_{\mu,\nu}$ the indices of the cardinal numbers β_s decrease from left to right, in what follows we will often see tensor product or summation formulas with indices ranging from $t > 0$ to 0.

Lemma. The fact that the constants obtained in this way for all $q'' > q'$ coincide is obvious. The statement follows. ∎

The following result is a direct corollary of [2, Lemma 3] and [3, Lemma 3.1].

Lemma 4.4 *Let $\mathfrak{G}$ be a Lie algebra and $\mathfrak{I} \subseteq \mathfrak{G}$ be an ideal. If W is a $\mathfrak{G}$-module on which $\mathfrak{I}$ acts densely and irreducibly with $\mathrm{End}_{\mathfrak{I}}(W) = \mathbb{K}$, then the functor*

$$\bullet \otimes W : \mathfrak{G}/\mathfrak{I}\text{-mod} \to \mathfrak{G}\text{-mod}$$

is fully faithful and preserves simplicity. ∎

We will also need the following auxiliary Schur-Weyl-type result.

Proposition 4.5 *Let W be a vector space and $\mathfrak{G} \subseteq \mathrm{End}(W)$ be a Lie subalgebra which acts densely on W. Then, for any partition λ, the $\mathfrak{G}$-module W_λ is simple and $\mathrm{End}_{\mathfrak{G}}(W_\lambda) \cong \mathbb{K}$.*

Proof As we noted in Section 1, $\mathfrak{G}$ acts densely of W_λ. Moreover, it suffices to prove the statement for the case $\mathfrak{G} = \mathrm{End}(W)$, as both simplicity and the endomorphism ring are preserved by passing to a Lie subalgebra acting densely (see [15, Lemma 7.3, Theorem 7.4]). The simplicity of W_λ as $\mathrm{End}(W)$-module is obvious as W_λ is the direct limit of all subspaces F_λ for finite-dimensional subspaces $F \subset W$, and the latter are simple $\mathrm{End}(F)$-modules.

Let now $\psi : W_\lambda \to W_\lambda$ be an automorphism. Choose a decomposition $W = F \oplus \bar{F}$ where $\dim F < \infty$. Note that, for large enough $\dim F$, the $\mathrm{End}(F)$-module F_λ is a submodule of $W_\lambda\big|_{\mathrm{End}(F)}$ of multiplicity 1. Hence, $\psi\big|_{\mathrm{End}(F)}$ is well defined and $\psi\big|_{F_\lambda} = c$ for some $c \in \mathbb{K}$. The fact that c does not depend on the choice of the decomposition $F \oplus \bar{F}$ follows from the fact that, for any two decompositions $W = F' \oplus \bar{F}' = F'' \oplus \bar{F}''$, there is a decomposition $W = F''' \oplus \bar{F}'''$ with $F', F'' \subset F'''$. ∎

The next two results highlight the relevance of Proposition 4.5 to our setup.

Lemma 4.6 *The Lie algebra $\mathfrak{gl}^M/\mathfrak{gl}^M_\alpha$ acts densely on the quotient V^*/V^*_α.*

Proof Since

$$\mathfrak{gl}^M_\alpha \cdot V^* \subseteq V^*_\alpha,$$

the Lie algebra $\mathfrak{gl}^M/\mathfrak{gl}^M_\alpha$ does indeed act on the quotient V^*/V^*_α. We will henceforth focus on showing that $\mathfrak{gl}^M$ acts densely. For this purpose, let v_s

for $0 \leq s \leq q$ be linearly independent vectors in V^*/V^*_α, and $w_s \in V^*/V^*_\alpha$ be q other vectors. We have to show that there exists $g \in \mathfrak{gl}^M$ such that

$$g \cdot \tilde{v}_s = \tilde{w}_s \text{ for any } s, \tag{7}$$

where tilde indicates a preimage in V^*.

We think of $\tilde{v}_s$ and $\tilde{w}_s$ as row vectors. The coordinates of row vectors in V^* are indexed by a totally ordered set Σ of cardinality α. For the duration of the proof, we identify Σ with the well-ordered set of all ordinals b such that $b < \alpha$. The matrices in $\mathfrak{gl}^M$ act on row vectors in V^* as $-vg$, where vg is the product of matrices. The linear independence of the vectors $v_s \in V^*/V^*_\alpha$ ensures that their representatives $\tilde{v}_s$ in V^* have at least α nonzero entries. This implies that we can partition the set Σ of indices of cardinality α into finite sets Σ_b parametrized by all ordinals $b < \alpha$, and such that each finite set $\{\tilde{v}_s|_{\Sigma_b}\}$ is a linearly independent set of finite vectors; here $\tilde{v}_s|_{\Sigma_b}$ denotes the finite vector formed by making all entries of $\tilde{v}_s$ outside of Σ_b equal to zero.

The latter claim is proved by transfinite induction. We start by finding Σ_0 corresponding to the ordinal 0: a set Σ_0 exists such that $\{\tilde{v}_s|_{\Sigma_0}\}$ is a linearly independent set, otherwise the images v_s of $\tilde{v}_s$ in V^*/V^*_α will be linearly dependent. The transfinite induction step is carried out in the same way: Let $\Sigma' := \Sigma \setminus \bigsqcup_{b'<b} \Sigma_{b'}$ for some ordinal $b < \alpha$. If the vectors $\left\{\tilde{v}_s|_{\Sigma'_b}\right\}$ are linearly dependent for all choices of a finite set $\Sigma'_b \subseteq \Sigma'$, then their images in V^*/V^*_α are linearly dependent, a contradiction.

The linear independence of the vectors $\tilde{v}_s|_{\Sigma_b}$ means that we can select column vectors g_b with finitely many nonzero entries indexed by Σ_b, and such that for the product of matrices $-(\tilde{v}_s|_{\Sigma_b})g_b$ we have

$$-(\tilde{v}_s|_{\Sigma_b})g_b = \text{ the } b\text{-indexed entry of } \tilde{w}_s \text{ for } 0 \leq s \leq q.$$

Now simply take g to be the matrix having the g_b as its columns. It has finite rows and columns by construction, and satisfies the desired condition (7). ■

We can generalize Lemma 4.6 as follows.

Lemma 4.7 *For every infinite cardinal number $\beta \leq \alpha$ the quotient $\mathfrak{gl}^M_{\beta+}/\mathfrak{gl}^M_\beta$ acts densely on $V^*_{\beta+}/V^*_\beta$.*

Proof The case $\beta = \alpha$ is treated in Lemma 4.6. Assume $\beta < \alpha$. We have to show that for any choice of finitely many linearly independent vectors $v_s \in V^*_{\beta+}/V^*_\beta$ and any choice of w_s in the same vector space, there exists $g \in \mathfrak{gl}^M$ such that

$$g \cdot v_s = w_s \text{ for all } s.$$

We can lift v_s and w_s to row vectors in V^* with β nonzero entries. Having done so, denote by Σ' the union of the sets of indices of nonzero entries of all these lifted vectors. We can now restrict our attention to only those vectors in V and V^* and matrices in $\mathfrak{gl}^M$ whose nonzero coordinates have indices in Σ'.

This is equivalent to working with the pairing between the β-dimensional subspace $V_{\Sigma'} \subseteq V$ spanned by Σ'-entry vectors and the subspace $(V_{\Sigma'})_* \subseteq V_*$, and with the corresponding Mackey Lie algebra. To complete the proof, we simply apply Lemma 4.6 to this pairing of lower-dimensional vector spaces.■

Lemma 4.8 *For any cardinal number $\beta \leq \alpha$ and any partition λ, the module $(V^*_{\beta+}/V^*_\beta)_\lambda$ is irreducible over $\mathfrak{gl}^M_{\beta+}/\mathfrak{gl}^M_\beta$, and its endomorphism ring is $\mathbb{K}$.*

Proof This is an immediate application of Proposition 4.5 and Lemma 4.7.■

Proof of Proposition 4.2 We split the proof into two portions.

Part 1: Simplicity and endomorphism algebra of the object (6). We prove this by induction on t, the case $t = 0$ being a consequence of [15, Theorem 4.1].

Now assume that the statement holds for $t-1$ and set $\beta = \beta_t$ and $\lambda = \lambda_t$. Lemma 4.7 and Proposition 4.5 ensure that the tensorand $(V^*_{\beta+}/V^*_\beta)_\lambda$ of (6) is simple over $\mathfrak{gl}^M_{\beta+}/\mathfrak{gl}^M_\beta$ with scalar endomorphism algebra. Setting

$$W = V_{(\beta_{t-1},\lambda_{t-1}),\cdots,(\beta_0,\lambda_0),\mu,\nu},$$

we can then apply Lemma 4.4 to the ideal $\mathfrak{gl}^M_\beta \subseteq \mathfrak{gl}^M_{\beta+}$ (in the role of $\mathfrak{I} \subseteq \mathfrak{G}$) to finish the proof.

Part 2: The simple objects are mutually nonisomorphic. Suppose that the modules $V_{(\beta_t,\lambda_t),\cdots,(\beta_0,\lambda_0),\mu,\nu}$ and $V_{(\beta'_q,\lambda'_q),\cdots,(\beta'_0,\lambda'_0),\mu',\nu'}$ are isomorphic. Restricting first to $\mathfrak{gl}^M_{\aleph_0} = V_* \otimes V$, over which the modules are direct sums of copies of $V_{\mu,\nu}$ and $V_{\mu',\nu'}$ respectively, we get $\mu' = \mu$ and $\nu' = \nu$. We can now proceed recursively in the following fashion.

Assume that for some $u \leq \min(q, t)$ we have shown that

$$\beta_s = \beta'_s \text{ and } \lambda_s = \lambda'_s \text{ for } 0 \leq s \leq u.$$

Then, setting

$$W = V_{(\beta_u,\lambda_u),\cdots,(\beta_0,\lambda_0),\mu,\nu}$$

and

$$\mathfrak{I} \subseteq \mathfrak{G} \text{ to be } \mathfrak{gl}^M_{\beta_{u+1}} \subseteq \mathfrak{gl}^M,$$

we conclude from Lemma 4.4 that

$$\bigotimes_{s=t}^{u+1}(V^*_{\beta_s^+}/V^*_{\beta_s})_{\lambda_s} \cong \bigotimes_{s=q}^{u+1}(V^*_{\beta_s'^+}/V^*_{\beta_s'})_{\lambda_s'}.$$

Restricting this isomorphism to $\mathfrak{gl}^M_{\beta^+}/\mathfrak{gl}^M_{\beta}$ for $\beta = \min(\beta_{u+1}, \beta'_{u+1})$ we conclude that $\beta_{u+1} = \beta'_{u+1}$ and $\lambda_{u+1} = \lambda'_{u+1}$. We can now repeat the procedure with u in place of $u+1$, until the process terminates. This can only happen if $q = t$ and the corresponding β_s and β'_s are equal, and similarly for λ_s and λ'_s.■

4.2. Ordering $\bar{\mathbb{T}}_{\alpha}$

Here we explain how the category $\bar{\mathbb{T}}_\alpha$ fits into the setting of Section 2.

Our objects X_i will be finite tensor products of the form

$$\left(\bigotimes_{\beta} \left(V^*/V^*_\beta\right)^{\otimes n_\beta}\right) \otimes (V^*)^{\otimes n} \otimes V^{\otimes m} \tag{8}$$

for infinite cardinal numbers $\beta \leq \alpha$. In this way, the underlying set I of the poset indexing the objects X_i consists of all finite tuples

$$(n_\beta, n, m)_{\beta\leq\alpha}$$

of nonnegative integers where almost all n_β vanish. We define a partial order on I by setting

$$(n_\beta, n, m)\preceq(n'_\beta, n', m')$$

if and only the following conditions hold:

(a) if β is the largest cardinal with $n_\beta \neq n'_\beta$ then $n_\beta > n'_\beta$;
(b) $n \leq n'$ and $m \leq m'$;
(c) $\sum_\beta n_\beta + n - m = \sum_\beta n'_\beta + n' - m'$.

It is easy to check that the opposite poset $(I, \preceq)^{\mathrm{op}}$ is well-ordered. In order to show that the above choice of objects X_i for $i \in I$ makes $\bar{\mathbb{T}}_\alpha$ an ordered Grothendieck category, we start with

Lemma 4.9 *Let $\beta_0 < \cdots < \beta_t \leq \alpha$ be infinite cardinal numbers, and $\lambda_t, \ldots, \lambda_0, \mu, \nu$ be arbitrary Young diagrams.*

Then, the object of $\mathbb{T}_\alpha$

$$\bigotimes_{s=t}^{0}(V^*/V^*_{\beta_s})_{\lambda_s}\otimes(V^*)_\mu\otimes V_\nu \tag{9}$$

is an essential extension of

$$V_{(\beta_t,\lambda_t),\cdots,(\beta_0,\lambda_0),\mu,\nu}=\bigotimes_{s=t}^{0}(V^*_{\beta_s^+}/V^*_{\beta_s})_{\lambda_s}\otimes V_{\mu,\nu}. \tag{10}$$

Proof Step 1. In first instance we argue that (10) is essential in

$$\bigotimes_{s=t}^{0}(V^*_{\beta_s^+}/V^*_{\beta_s})_{\lambda_s}\otimes(V^*)_\mu\otimes V_\nu. \tag{11}$$

To this end, note first that it suffices to show that (10) is essential in (11) when regarded as a module over the ideal $V_*\otimes V$ of $\mathfrak{gl}^M$. Since $V_*\otimes V$ annihilates $\bigotimes_{s=t}^{0}(V^*_{\beta_s^+}/V^*_{\beta_s})_{\lambda_s}$, this in turn reduces to showing that $V_{\mu,\nu}$ is essential in $(V^*)_\mu\otimes V_\nu$ as a module over $V_*\otimes V$.

The inclusions

$$(V_*)^{\otimes n}\otimes V^{\otimes m}\subset(V^*)^{\otimes n}\otimes V^{\otimes m}$$

are essential: for any $v\neq 0$ belonging to the right-hand side, the $V_*\otimes V$-module generated by v is nonzero and contained in the left-hand side.

Now simply apply this remark to a traceless v in the image of the Young symmetrization operator sending $(V^*)^{\otimes n}\otimes V^{\otimes m}$ to $(V^*)_\mu\otimes V_\nu$.

This concludes the proof that (10) is essential in (11).

Step 2. We next argue that (11) is essential in

$$\bigotimes_{s=t}^{1}(V^*_{\beta_s^+}/V^*_{\beta_s})_{\lambda_s}\otimes(V^*/V^*_{\beta_0})_{\lambda_0}\otimes(V^*)_\mu\otimes V_\nu. \tag{12}$$

This is very similar in spirit to the proof of Step 1: it is enough to prove that the extension in question is essential over the Lie subalgebra $\mathfrak{gl}^M_{\beta_0^+}\subseteq\mathfrak{gl}^M$ which annihilates the s-indexed tensorands in (12) and maps the middle tensorand $(V^*/V^*_{\beta_0})_{\lambda_0}$ into $(V^*_{\beta_0^+}/V^*_{\beta_0})_{\lambda_0}$.

As before, it suffices to observe that $\mathfrak{gl}^M_{\beta_0^+}$ does not annihilate any nonzero elements of $(V^*/V^*_{\beta_0})_{\lambda_0}\otimes(V^*)_\mu\otimes V_\nu$.

Step 3: conclusion. We now repeat the argument in Step 2 inductively, each time replacing one $V^*_{\beta_s^+}$ by V^* and working over the ever-larger Lie algebra $\mathfrak{gl}^M_{\beta_s^+}$. After exhausting all tensorands, we obtain a tower of essential extensions with the simple object (9) at the bottom and the object (8) at the top. The desired conclusion follows. ■

Corollary 4.10 *The transfinite filtration (5) is the transfinite socle filtration of the object V^* of $\mathbb{T}_\alpha$.*

Proof This follows immediately from the simplicity of the objects $V^*_{\beta+}/V^*_\beta$ for cardinals $\beta \le \alpha$ and from the fact that V^*/V^*_β is an essential extension of $V^*_{\beta+}/V^*_\beta$. ■

Proposition 4.11 *For the above choice of the poset I and the objects X_i for $i \in I$, the Grothendieck envelope $\bar{\mathbb{T}}_\alpha$ is an ordered Grothendieck category.*

Proof Part (i) of Definition 2.3 is implicit in Definition 4.1. For part (ii), note that X_i is a direct sum of objects of the form (9). Since (9) is an essential extension of (10) by Lemma 4.9, and the object (10) is simple by Proposition 4.2, $\operatorname{soc} X_i$ is a direct sum of objects (10) with $|\lambda_\beta| = n_\beta$, $|\mu| = n$, $|\nu| = m$. This implies (ii). Now, part (iv) is also clear as the object (9) is indecomposable with simple socle (10).

For part (iii), one has to check that any simple subquotient of an object (9) satisfies conditions a), b), and c) from the definition of the partial order $\preceq$ on I. This is straightforward if we note that the injective object (9) has a filtration obtained by tensoring the filtration of $(V_*)_\mu \otimes V_\nu$ from [16, Theorem 2.2] and those of the various $V^*/V^*_{\beta_s}$ with simple subquotients of the form $V^*_{\beta_r^+}/V^*_{\beta_r}$. We leave the details to the reader. See also Example 4.32 below for a particular case where all simple subquotients of (9) are displayed explicitly.

Finally, for part (v), it is a rather routine verification that having fixed i and j in I as in that portion of Definition 2.3, the morphisms $X_i \to X_j$ obtained by composing and tensoring surjections $V^*/V^*_\beta \to V^*/V^*_\gamma$ for $\gamma \ge \beta$ and contractions $V^* \otimes V \to \mathbb{C}$ will satisfy the condition. ■

Our next result classifies the indecomposable injective objects of $\overline{\mathbb{T}}_\alpha$; these happen to already be contained in $\mathbb{T}_\alpha$.

Theorem 4.12 *The indecomposable injectives in the category $\overline{\mathbb{T}}_\alpha$ are, up to isomorphism, the objects (9) with respective socles (10), where the choices range over tuples of infinite cardinal numbers $\beta_0 < \cdots < \beta_t \le \alpha$ and Young diagrams $\lambda_t, \ldots, \lambda_0, \mu, \nu$.*

Proof This is a consequence of Proposition 4.11 and Corollary 2.7 together with the classification of simple objects from Proposition 4.2 and the fact that (10) is essential in (9) via Lemma 4.9. ■

4.3. Blocks of $\bar{\mathbb{T}}_\alpha$

We will show that the integers appearing on the two sides of the equality (c) in the definition of $(I, \preceq)$ in Section 4.2 in fact parametrize the blocks of the category $\overline{\mathbb{T}}_\alpha$. First, let us recall

Definition 4.13 Suppose the class $\mathrm{Indec}(\mathcal{C})$ of isomorphism classes of indecomposable objects of an abelian category $\mathcal{C}$ is a set. The *blocks* of $\mathcal{C}$ are the classes of the finest equivalence relation on $\mathrm{Indec}(\mathcal{C})$ requiring that objects Z, Y with $\mathrm{Hom}_{\mathcal{C}}(Z, Y) \neq 0$ belong to the same class. ♦

Theorem 4.14 *The blocks of* $\overline{\mathbb{T}}_\alpha$ *are parametrized by* $\mathbb{Z}$*, and the simple object* $V_{(\beta_t,\lambda_t),\cdots,(\beta_0,\lambda_0),\mu,\nu}$ *belongs to the block indexed by* $\sum_{s=t}^{0} |\lambda_s| + |\mu| - |\nu|$.

Proof For a simple object $U = V_{(\beta_t,\lambda_t),\cdots,(\beta_0,\lambda_0),\mu,\nu}$ as in the statement, let us refer to the integer

$$\sum_{s=t}^{0} |\lambda_s| + |\mu| - |\nu| \in \mathbb{Z}$$

as the *content* of U. We split the proof into two halves.

(1) Different content ⇒ different blocks. Since the content of a simple object U is nothing but the expression in part c) of the definition of the partial order on I (where $n_\beta = |\lambda_\beta|$, $n = |\mu|$, $m = |\nu|$), the fact that $\bar{\mathbb{T}}_\alpha$ is an ordered Grothendieck category (Proposition 4.11) implies that all simple subquotients of the injective hull (9) of a simple object U as in (10) have the same content as U.

(2) Same content ⇒ same block. Consider a simple object U of the form (10). Its injective hull (9) surjects onto

$$\bigotimes_{s=t}^{0} (V^*/V^*_{\beta_t})_{\lambda_s} \otimes (V^*/V^*_{\beta_t})_\mu \otimes V_\nu,$$

so U is in the same block as an injective object of the form

$$(V^*/V^*_{\beta_t})_\lambda \otimes V_\nu, \tag{13}$$

for some Young diagram λ. In turn, the object (13) is a quotient of the indecomposable injective $(V^*)_\lambda \otimes V_\nu$. Finally, the classification of blocks in $\mathbb{T}_{\mathfrak{sl}(\infty)}$ from [5, Corollary 6.6], together with the equivalence of categories

$$\mathbb{T}_{\mathrm{End}(V)} \simeq \mathbb{T}_{\mathfrak{sl}(\infty)}$$

established in [15], shows that the block of $(V^*)_\lambda \otimes V_\nu$ depends only on the difference $|\lambda| - |\nu|$. The result follows. ■

4.4. Vanishing Ext-functors and Koszulity

Our main aim in the present subsection is to prove an analogue of [3, Theorem 3.11], improving on Proposition 2.10 and describing necessary conditions for nonvanishing Ext-functors between the simple objects of $\mathbb{T}_\alpha$ described in Proposition 4.2.

We start with a formula for the defect $d(i,j)$. Consider two elements i and j of I: $i = (n_\beta, n, m)$ and $j = \left(n'_\beta, n', m'\right)$. Let $\left\{\beta'_0 < \beta'_1 < \ldots < \beta'_{q'}\right\}$ be the union of all infinite cardinals for which $n_\beta \neq 0$ or $n'_{\beta'} \neq 0$. Extend this set to a minimal set of cardinals which is *interval-closed* in the sense that whenever $\beta_{j'}$ is a finite iterated successor $\beta_j^{++\cdots+}$, all intermediate successors β_j^+, β_j^{++}, etc. belong to the set. Denote by $\{\beta_0 < \ldots < \beta_q\}$ the resulting finite set.

Proposition 4.15 *Given $i = (n_\beta, n, m) \preceq j = (n'_\beta, n', m')$ with finite defect $d(i,j)$, we have*

$$d(i,j) = n' - n + \sum_{s=0}^{q} s\left(n_{\beta_s} - n'_{\beta_s}\right). \tag{14}$$

Proof We argue by induction on $d(i,j)$. If $d(i,j) = 1$, then there are two possibilities:

- $n_\beta = n'_\beta$ for all cardinals β, and $n' - n = m' - m = 1$

or

- $m' = m$, $n' = n$, $n'_\beta \neq n_\beta$ for precisely two cardinals β of the form β_s, $\beta_{s+1} = \beta_s^+$, and $n'_{\beta_s} - n_{\beta_s} = n_{\beta_{s+1}} - n'_{\beta_{s+1}} = 1$.

In both cases formula (14) holds. For the induction step, consider a maximal chain

$$i = i_0 \prec i_1 \cdots \prec i_u = j$$

for $u = d(i,j)$. Then $d\left(i_{u-1}, i_u\right) = 1$, and using the above observation and the induction assumption, one immediately checks formula (14) for the pair (i,j). ■

Corollary 4.16 *(a) If $d(i,j) < \infty$, the length of any finite chain in I linking i and j equals $d(i,j)$.*

(b) Under the assumption of (a), we have

$$d(i,k) = d(i,j) + d(j,k).$$

Proof Part (a) emerged over the course of the proof of Proposition 4.15, while part (b) is a consequence of (a): if all maximal chains have the same length, then that common length must be equal to the length of a chain $i \to k$ obtained by concatenating chains $i \to j$ and $j \to k$. ■

Remark 4.17 In the language of [19, Section 3.1], intervals $[i,j]$ for which $d(i,j) < \infty$ are *graded* posets (some authors refer to such posets as *ranked*).♦

Notation 4.18 For simple objects $U \in \mathcal{S}_i$ and $W \in \mathcal{S}_j$ with $i \preceq j$ we sometimes write $d(U,W)$ for $d(i,j)$. ♦

We will use the following observations in the proof of Theorem 4.20 below.

Corollary 4.19 *Suppose $U \in \mathcal{S}_i$ and $W \in \mathcal{S}_j$ are simple objects of $\mathbb{T}_\alpha$ such that $d(i,j) = p$. Then for every infinite cardinal γ and any simple objects*

$$\widetilde{U} \subset \mathrm{soc}((V^*/V^*_\gamma) \otimes U), \ \widetilde{W} \subset \mathrm{soc}((V^*/V^*_\gamma) \otimes W)$$

we have $d(\widetilde{U}, \widetilde{W}) = p$. Similarly, for a simple object

$$\widetilde{\widetilde{U}} \subset \mathrm{soc}((V^*/V^*_{\gamma^+}) \otimes U)$$

we have $d\left(\widetilde{\widetilde{U}}, \widetilde{W}\right) = p + 1$.

Proof Let $i = (n_\beta, n, m)$, $j = \left(n'_\beta, n', m'\right)$. Then $\widetilde{U} \in \mathcal{S}_{\tilde{i}}$, for $\tilde{i} = (\tilde{n}_\beta, n, m)$, where $\tilde{n}_\beta = n_\beta$ for $\beta \neq \gamma$, and $\tilde{n}_\gamma = n_\gamma + 1$. Similarly, $\widetilde{W} \in \mathcal{S}_{\tilde{j}}$ for $\tilde{j} = \left(\tilde{n}'_\beta, n', m'\right)$, where $\tilde{n}'_\beta = n'_\beta$ for $\beta \neq \gamma$, and $\tilde{n}'_\gamma = n'_\gamma + 1$. Finally, $\widetilde{\widetilde{U}} \in \mathcal{S}_{\tilde{\tilde{i}}}$, for $\tilde{\tilde{i}} = (\tilde{\tilde{n}}_\beta, n, m)$, where $\tilde{\tilde{n}}_\beta = n_\beta$ for $\beta \neq \gamma^+$, and $\tilde{\tilde{n}}_{\gamma^+} = n_{\gamma^+} + 1$. Therefore both claims follow from immediate application of Proposition 4.15. ■

The first main result of the present subsection is

Theorem 4.20 *Let $T' \in \mathcal{S}_i$ and $T \in \mathcal{S}_j$ be two simple objects in $\mathbb{T}_\alpha$, and suppose $\mathrm{Ext}^p(T', T) \neq 0$ for some $p \geq 0$. Then $d(i,j) = p$.*

Before beginning the proof, we introduce

Notation 4.21 For $i = (n_\beta, n, m) \in I$ as in Section 4.2 we denote

$$m_i = m, n_i = n, \quad n_{\beta,i} = n_\beta.$$

when we wish to extract the components n_β, n, and m from i. ♦

Proof of Theorem 4.20 We already know from Proposition 2.10 that $\mathrm{Ext}^p(T', T) \neq 0$ implies $i \preceq j$ and $d(i, j) \geq p$; the inequality $i \preceq j$ will be implicit throughout the proof.

We do simultaneous induction on p and the nonnegative integer

$$N_j := \sum_\beta n_{\beta,j} \tag{15}$$

(see Notation 4.21). For $p = 0$ the statement is trivial as there are no nonzero morphisms between nonisomorphic irreducible objects. Similarly, the case $N_j = 0$ is also immediate, since T will then be both simple and injective by Theorem 4.12.

We now address the induction step, assuming that both p and N_j are strictly positive. In that case, we can find some cardinal β, which is infinite or equals zero, such that T is a direct summand of the socle of $(V^*/V^*_\beta) \otimes U$ for some simple object U. By Lemma 4.9 the socle of $(V^*/V^*_\beta) \otimes U$ is $(V^*_{\beta+}/V^*_\beta) \otimes U$, and hence T is a direct summand of the latter. Consider the short exact sequence

$$0 \to (V^*_{\beta+}/V^*_\beta) \otimes U \to (V^*/V^*_\beta) \otimes U \to (V^*/V^*_{\beta+}) \otimes U \to 0, \tag{16}$$

in which we set $V^*_\beta = 0$, $V^*_{\beta+} = V_*$ for $\beta = 0$.

The assumption $\mathrm{Ext}^p(T', T) \neq 0$ implies

$$\mathrm{Ext}^p(T', (V^*_{\beta+}/V^*_\beta) \otimes U) \neq 0,$$

and via the long exact sequence for Ext-groups, this entails

$$\mathrm{Ext}^p(T', (V^*/V^*_\beta) \otimes U) \neq 0 \tag{17}$$

or

$$\mathrm{Ext}^{p-1}(T', (V^*/V^*_{\beta+}) \otimes U) \neq 0. \tag{18}$$

Consider first the case (17). Theorem 4.12 makes it clear that tensor products of injective objects with finite-length socle are again injective, and hence tensoring an injective resolution of U

$$0 \to U \to J^0 \to J^1 \to \cdots$$

with V^*/V^*_β produces an injective resolution $(V^*/V^*_\beta) \otimes J^\bullet$ of $(V^*/V^*_\beta) \otimes U$. Since $U \in \mathcal{S}_{j'}$ with $N_{j'} = N_j - 1$, the induction hypothesis ensures that the socle of J^p consists of simple objects W in various $\mathcal{S}_t$ with $d(W,U) = p$. Therefore, by Corollary 4.19, $d(W',T) = p$ for all simple objects W' in $\mathrm{soc}(V^*/V_\beta^* \otimes J^p)$, and the claim follows.

In the case of (18), the proof is very similar an uses the second statement of Corollary 4.19. ■

We next use Theorem 4.20 to show that the Grothendieck category $\overline{\mathbb{T}}_\alpha$ is Koszul according to the following definition.

Definition 4.22 Let $\mathcal{C}$ be a semi-artinian Grothendieck category such that the class of isomorphism classes of simple objects is a set. We denote by $E\mathcal{C}$ the *Ext-algebra* of $\mathcal{C}$, by definition equal to

$$\bigoplus \mathrm{Ext}^p(T',T),$$

made into an algebra via the Yoneda product; the summation ranges over the isomorphism classes of simple objects T, T', and integers $p \geq 0$. Note that $E\mathcal{C}$ is graded by placing Ext^p-groups in degree p.

The category $\mathcal{C}$ is said to be *Koszul* if $E\mathcal{C}$ is generated in degree one. ♦

We can now state

Theorem 4.23 *The category $\overline{\mathbb{T}}_\alpha$ is Koszul for any infinite cardinal α.*

Proof We have to prove that for each nonnegative integer p the degree-p component of $E\overline{\mathbb{T}}_\alpha$ is contained in the span of Yoneda products of elements in the degree-1 and degree-$(p-1)$ components.

Let T' and T be simple objects in $\mathbb{T}_\alpha$, with $d(T',T) = p$ (see Notation 4.18), and let $\widetilde{T}$ be an injective hull of T. Consider a nonzero element $x \in \mathrm{Ext}^p(T',T)$. It is obtained, via the long exact sequence of Exts corresponding to the exact sequence

$$0 \to T \to \widetilde{T} \to \mathrm{cor}\, T \to 0,$$

determined by a nonzero element of $\mathrm{Ext}^{p-1}(T', \mathrm{cor}\, T)$. In turn, Theorem 4.20 implies that this is actually a nonzero element of

$$\mathrm{Ext}^{p-1}\left(T', \mathrm{soc}(\mathrm{cor}\, T)\right) \cong \bigoplus \mathrm{Ext}^{p-1}(T',U)\,, \tag{19}$$

where the summation ranges over the isomorphism classes of the simple submodules U of $\mathrm{cor}\, T$. Finally, this means that x is in the span of the Yoneda product of the space (19) and $\bigoplus \mathrm{Ext}^1(U,T)$. These spaces are subspaces of $(E\overline{\mathbb{T}}_\alpha)_{p-1}$ and $(E\overline{\mathbb{T}}_\alpha)_1$ respectively. ■

4.5. The case $\alpha = \aleph_t$ for $t \in \mathbb{Z}_{\geq 0}$

In this section, $\alpha = \aleph_t$ for $t \in \mathbb{Z}_{\geq 0}$.

Proposition 4.24 *Under the above assumption, the objects X_i have finite length, and thus $\bar{\mathbb{T}}_\alpha$ is a finite ordered Grothendieck category.*

Proof Since V^* has finite length, the object $(V^*)^{\otimes n}$ has a finite filtration whose successive quotients are direct sums of modules of the form

$$(V^*/V^*_{\aleph_t})^{\otimes q_t} \otimes \cdots \otimes (V^*_{\aleph_1}/V^*_{\aleph_0})^{\otimes q_0} \otimes (V_*)^{\otimes q} \otimes V^{\otimes q'} .$$

By [15] and [16], $(V_*)^{\otimes q} \otimes V^{\otimes q'}$ has finite length with irreducible subquotients of the form $V_{\mu,\nu}$. Furthermore, modules of the form $\left(V^*/V^*_{\aleph_t}\right)^{\otimes q_t} \otimes \cdots \otimes \left(V^*_{\aleph_1}/V^*_{\aleph_0}\right)^{\otimes q_0}$ have finite filtrations with irreducible subquotients of the form $V_{\lambda_t,\ldots,\lambda_0,\emptyset,\emptyset}$ for Young diagrams $\lambda_t, \ldots, \lambda_0$ (some of which may be empty). Finally, Proposition 4.2 implies that tensor products of the form $V_{\lambda_t,\ldots,\lambda_0,\emptyset,\emptyset} \otimes V_{\mu,\nu}$ are irreducible. The statement follows. ■

Proposition 4.24 implies in particular that the discussion in [3, Section 2] applies verbatim.

Corollary 4.25 *(a) $\bar{\mathbb{T}}_{\aleph_t}$ is a highest weight category.*

(b) The indecomposable injectives in the category $\bar{\mathbb{T}}_{\aleph_t}$ are (up to isomorphism)

$$\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu} = \bigotimes_{s=t}^{0} (V^*/V^*_{\aleph_s})_{\lambda_s} \otimes (V^*)_\mu \otimes V_\nu, \tag{20}$$

with respective socles

$$V_{\lambda_t,\cdots,\lambda_0,\mu,\nu} = \bigotimes_{s=t}^{0} (V^*_{\aleph_{s+1}}/V^*_{\aleph_s})_{\lambda_s} \otimes V_{\mu,\nu} \tag{21}$$

for arbitrary Young diagrams $\lambda_t, \ldots, \lambda_0, \mu, \nu$;

(c) The injective objects of $\bar{\mathbb{T}}_{\aleph_t}$ are precisely the direct sums of indecomposable injectives of the form (20).

(d) The category $\overline{\mathbb{T}}_{\aleph_t}$ is equivalent to the category of comodules over a Koszul semiperfect coalgebra C. Moreover, any such equivalence identifies $\mathbb{T}_{\aleph_t}$ with the category of finite-dimensional C-comodules.

Proof Part (a) follows from Proposition Proposition 2.17. Part (b) is a particular instance of Theorem 4.12. For part (c), we note that Proposition 2.14 applies to the category $\bar{\mathbb{T}}_{\aleph_t}$. Part (d) is a consequence of Theorem 4.23 and [3, Theorem 2.13] (cf. [3, Corollary 3.17]). ■

Note that the module (21) is nothing but the module (10). When $\alpha = \aleph_t$, the infinite cardinal β_s can be read off from the position of λ_s in the subscript of the left-hand side of (21). Therefore we can simply write $V_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ instead of $V_{(\beta_t,\lambda_t),\dots,(\beta_0,\lambda_0),\mu,\nu}$.

Remark 4.26 The coalgebra C in Corollary 4.25(d) is not unique, being determined only up to Morita equivalence. Nevertheless, for a specific choice, see Remark 5.6 below. ♦

Our next goal is to describe the socle filtration of the injectives (20). We start with

Lemma 4.27 *Fix $\lambda_t, \dots, \lambda_0, \mu, \nu$ as in Corollary 4.25. Then $V_{\kappa_t,\dots,\kappa_0,\gamma,\delta}$ is a direct summand of $\underline{\mathrm{soc}}^s\, \tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ if and only if $V_{\kappa_t,\dots,\kappa_0,\gamma,\delta}$ is a constituent of $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ and $d(i,j) = s-1$, where $i = (|\kappa_t|,\dots,|\kappa_0|,|\gamma|,|\delta|)$ and $j = (|\lambda_t|,\dots,|\lambda_0|,|\mu|,|\nu|)$.*

Proof Theorem 4.20 implies that $V_{\kappa_t,\dots,\kappa_0,\gamma,\delta}$ is a submodule of $\underline{\mathrm{soc}}^2\, \tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ if and only if $V_{\kappa_1,\dots,\kappa_0,\gamma,\delta}$ is a constituent of $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ and $d(i,j) = 1$. The general case follows by a straightforward induction argument. ■

We now describe the socle filtration of $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ explicitly. Our approach is to break down the problem into manageable pieces.

As in [3, §3.2], we use Sweedler notation $\Delta : \lambda \mapsto \lambda_{(1)} \otimes \lambda_{(2)}$ for the comultiplication in the Hopf algebra Sym of symmetric functions ([12]) with the usual basis consisting of Schur functions indexed by partitions λ. This notation propagates to multiple iterations of the comultiplication, as in

$$(\Delta \otimes \mathrm{id}) \circ \Delta : \lambda \mapsto \lambda_{(1)} \otimes \lambda_{(2)} \otimes \lambda_{(3)},$$

etc. We also reprise the notation for truncating such sums according to the number of boxes in the partitions indexing the basis elements. For instance,

$$\lambda^q_{(1)} \otimes \lambda^{|\lambda|-q}_{(2)}$$

denotes the sum of all of those summands in $\Delta(\lambda) = \lambda_{(1)} \otimes \lambda_{(2)}$ whose left-hand tensorand corresponds to a partition with k boxes.

Furthermore, in order to shorten the notation, we will sometimes denote the simple object $V^*_{\aleph_{u+1}}/V^*_{\aleph_u}$ by U_u.

Lemma 4.28 *Let* $0 \le u \le t$ *and let* λ *be a partition. Then* $\underline{\mathrm{soc}}^q \left((V^*/V^*_{\aleph_u})_\lambda\right)$ *is isomorphic to*

$$\bigoplus (U_t)_{\lambda_{(t-u+1)}^{\ell_{t-u+1}}} \otimes \cdots \otimes (U_u)_{\lambda_{(1)}^{\ell_1}}, \tag{22}$$

where the summation is over all choices of nonnegative integers ℓ_x *such that*

$$\sum_{x=t-u+1}^{1} \ell_x = |\lambda| \tag{23}$$

and

$$\sum_{x=t-u+1}^{1} (x-1)\ell_x = q-1. \tag{24}$$

Proof The fact that the simple modules in the sum (22) satisfying (23) are constituents of $\left(V^*/V^*_{\aleph_u}\right)_\lambda$ follows from the general remark that given an exact sequence

$$0 \to U \to W \to Y \to 0$$

of vector spaces, W_λ admits a filtration

$$0 \subseteq U_\lambda \subseteq U_{\lambda_{(1)}^{|\lambda|-1}} \otimes Y_{\lambda_{(2)}^1} \subseteq \cdots \subseteq U_{\lambda_{(1)}^1} \otimes Y_{\lambda_{(2)}^{|\lambda|-1}} \subseteq Y_\lambda.$$

According to Lemma 4.27, the equality (24) singles out the simple constituents of $\underline{\mathrm{soc}}^q \left((V^*/V^*_{\aleph_0})_\lambda\right)$, see Proposition 4.15. ■

We have an analogous result regarding the tensorand $(V^*)_\mu \otimes V_\nu$ in (20). Before stating it, we introduce the notation $\langle \bullet, \bullet \rangle$ for the bilinear form on Sym making the basis $\{\lambda\}$ orthonormal.

Lemma 4.29 *The subquotient* $\underline{\mathrm{soc}}^q \left((V^*)_\mu \otimes V_\nu\right)$ *is isomorphic to*

$$\bigoplus \left((U_t)_{\mu_{(t+1)}^{\ell_{t+1}}} \otimes \cdots \otimes (U_0)_{\mu_{(1)}^{\ell_1}} \otimes V_{\mu_{(t+2)}^{\ell_{t+2}}, \nu_{(1)}^{|\nu|-\tau}}\right)^{\oplus \langle \mu_{(t+3)}, \nu_{(2)} \rangle},$$

with summation over those choices of τ *and* ℓ_x *such that*

$$\tau + \sum_{x=t+2}^{1} \ell_x = |\mu|$$

and

$$\tau + \sum_{x=t+1}^{1} x\ell_x = q - 1. \tag{25}$$

Proof First, filtering $(V^*)_\mu$ alone using the same technique as in the proof of Lemma 4.28, we obtain a coarser filtration of $(V^*)_\mu \otimes V_\nu$ by

$$0 = F^0 \subseteq F^1 \subseteq F^2 \subseteq \cdots$$

with the subquotient F^q/F^{q-1} isomorphic to

$$\bigoplus (U_t)_{\mu_{(t+1)}^{\ell_{t+1}}} \otimes \cdots \otimes (U_0)_{\mu_{(1)}^{\ell_1}} \otimes (V_*)_{\mu_{(t+2)}^{\ell_{t+2}}} \otimes V_\nu$$

for

$$\sum_{x=t+2}^{1} \ell_x = |\mu| \text{ and } \sum_{x=t+1}^{1} x\ell_x = q - 1.$$

Then we filter each subobject $(V_*)_\lambda \otimes V_\nu$ of one of the subquotients F^q/F^{q-1} by its socle filtration. In the comultiplication-based notation and language, we use [16, Theorem 2.3] which says that $\underline{\mathrm{soc}}^p((V_*)_\lambda \otimes V_\nu)$ is isomorphic to

$$\left(V_{\lambda_{(1)}^{|\mu|-p+1}, \nu_{(1)}^{|\nu|-p+1}}\right)^{\oplus \langle \lambda_{(2)}, \nu_{(2)} \rangle}.$$

Splicing together these two filtration processes produces the claimed result. Here (25) is responsible for identifying the submodule of $\underline{\mathrm{soc}}^q\left((V*)_\mu \otimes V_\nu\right)$.■

The following are more explicit versions of Lemmas 4.28 and 4.29, involving the Littlewood-Richardson coefficients $N_{\mu,\nu}^{\eta}$.

Lemma 4.28 bis *Let $u \leq t$, and let λ and μ_x, $0 \leq x \leq t-u$, be partitions with*

$$\sum_{x=t-u}^{0} |\mu_x| = |\lambda|.$$

Set

$$q := 1 + \sum_{x=t-u}^{0} x|\mu_x|.$$

Then the multiplicity of the simple object

$$(U_t)_{\mu_{t-u}} \otimes \cdots \otimes (U_u)_{\mu_0}$$

in the subquotient $\underline{\mathrm{soc}}^q((V^*/V^*_{\aleph_u})_\lambda)$ *equals*

$$\sum N^{\alpha_1}_{\mu_0,\mu_1} N^{\alpha_2}_{\alpha_1,\mu_2} \cdots N^{\lambda}_{\alpha_{t-u-1},\mu_{t-u}}$$

with summation over repeated indices.

No other simples appear as constituents of $(V^*/V^*_{\aleph_u})_\lambda$.

Proof This is a reformulation of Lemma 4.28, identifying ℓ_x from that result to $|\mu_{x-1}|$ in the present one.

Indeed, the multiplicity in question is the coefficient of

$$\mu_{t-u} \otimes \cdots \otimes \mu_0$$

in

$$\Delta^{t-u}(\lambda) = \lambda_{(t-u+1)} \otimes \cdots \otimes \lambda_{(1)}.$$

The very definition of the comultiplication in the ring of symmetric functions implies that this number is the multiplicity of U_λ in the tensor product

$$U_{\mu_{t-u}} \otimes \cdots \otimes U_{\mu_0}.$$

In turn, this multiplicity is expressible in terms of Littlewood-Richardson coefficients as in the statement. ■

Lemma 4.29 bis *Let* μ, ν *and* η_x, $0 \le x \le t$, ξ, ζ *be partitions, and set*

$$q := 1 + (|\nu| - |\zeta|) + \sum_{x=0}^{t} (x+1)|\eta_j|.$$

Then the multiplicity of the simple object $V_{\eta_t,\cdots,\eta_0,\xi,\zeta}$ *in the subquotient* $\underline{\mathrm{soc}}^q((V^*)_\mu \otimes V_\nu)$ *equals*

$$\sum N^{\pi_1}_{\eta_0,\eta_1} N^{\pi_2}_{\pi_1,\eta_2} \cdots N^{\mu}_{\pi_{t-1},\eta_t} N^{\pi_{t-1}}_{\xi,\delta} N^{\nu}_{\zeta,\delta},$$

with summation over repeated indices.

No other simples appear as constituents of $(V^*)_\mu \otimes V_\nu$.

Proof The deduction of this statement from that of Lemma 4.29 is analogous to the previous proof: once more using the definition of the comultiplication, the multiplicity we are after is the sum

$$\sum_{\delta}(\text{multiplicity of } \mu \text{ in } \eta_0\cdots\eta_t\xi\delta)\cdot(\text{multiplicity of } \nu \text{ in } \zeta\delta) \tag{26}$$

where as before the Young diagram symbols stand for the corresponding Schur functions in Sym and juxtaposition means multiplication therein.

In turn, (26) is expressible as a sum of products of Littlewood-Richardson coefficients as claimed. ■

Finally, the general result on the socle filtrations of the indecomposable simple objects (20) is obtained by tensoring together instances of Lemmas 4.28 and 4.29.

Proposition 4.30 *The subquotient* $\underline{\mathrm{soc}}^q(\widetilde{V}_{\lambda_t,\cdots,\lambda_0,\mu,\nu})$ *of (20) is isomorphic to*

$$\sum_{u_x,y}\bigotimes_{x=t}^{0}\underline{\mathrm{soc}}^{u_x}((V^*/V^*_{\aleph_x})_{\lambda_x})\otimes\underline{\mathrm{soc}}^{y}((V^*)_{\mu}\otimes V_{\nu}),$$

with summation over all choices of u_x *and* y *such that* $\sum_{x=t}^{0}(u_x-1)+(y-1)=q-1$.

Proof The ingredients are contained in Lemmas 4.28 and 4.29 and their proofs. ■

Remark 4.31 In [3, §3.7] we show that when $\alpha=\aleph_0$, the category $\overline{\mathbb{T}}_\alpha$ is not only Koszul, but is in fact *self-dual*: the quadratic algebra $E\overline{\mathbb{T}}_{\aleph_0}$ is anti-isomorphic to its quadratic dual.

One consequence of the self-duality is that dimensions of Ext-groups can be read off from socle filtrations of indecomposable injectives: when $\alpha=\aleph_0$ the simples (10) are indexed by three Young diagrams, and [3, Corollary 3.34] then says that

$$\dim\mathrm{Ext}^q(V_{\lambda_0,\mu,\nu},V_{\lambda_0',\mu',\nu'})$$

equals the multiplicity of $V_{\lambda_0,\mu^\perp,\nu}$ in the $\underline{\mathrm{soc}}^{q+1}$ subquotient of the injective hull of $V_{\lambda_0',(\mu')^\perp,\nu'}$; here $^\perp$ indicates passage to the transposed Young diagram. In other words, in order to compute the Ext-group $\mathrm{Ext}^q(V_{\lambda_0,\mu,\nu},V_{\lambda_0',\mu',\nu'})$, one simply passes to the socle filtration of $\widetilde{V}_{\lambda_0',(\mu')^\perp,\nu'}$.

For $\alpha=\aleph_t$ for $t\geq 1$, no such "transposing pattern" computes $\mathrm{Ext}^q(V_{\lambda_t,\ldots,\lambda_0,\mu,\nu},V_{\lambda_t',\ldots,\lambda_0',\mu',\nu'})$. Indeed, the subquotient $\underline{\mathrm{soc}}^3\,V^*$ of the injective hull V^* of V_* is nonzero, while the quotient V^*/V_* is injective and hence all $\mathrm{Ext}^2(\bullet, V_*)$ vanish. The problem of computing the Ext-groups of pairs of simple objects in $\mathbb{T}_{\aleph_t}$ is thus open for $t \geq 1$. ♦

We conclude this subsection by

Example 4.32 Consider the objects $\widetilde{V}_{\emptyset,(1,1),\emptyset,\emptyset}$ and $\widetilde{V}_{\emptyset,\emptyset,(1,1),(1)}$ in $\mathbb{T}_{\aleph_1}$. They have respective socle filtrations

$$\begin{array}{c} V_{(1,1),\emptyset,\emptyset,\emptyset} \\ \hline V_{(1),(1),\emptyset,\emptyset} \\ \hline V_{\emptyset,(1,1),\emptyset,\emptyset} \end{array} \qquad\qquad \begin{array}{c} V_{(1,1),\emptyset,\emptyset,(1)} \\ \hline V_{(1),(1),\emptyset,(1)} \oplus V_{(1),\emptyset,\emptyset,\emptyset} \\ \hline V_{\emptyset,(1,1),\emptyset,(1)} \oplus V_{(1),\emptyset,(1),(1)} \oplus V_{\emptyset,(1),\emptyset,\emptyset} \\ \hline V_{\emptyset,(1),(1),(1)} \oplus V_{\emptyset,\emptyset,(1),\emptyset} \\ \hline V_{\emptyset,\emptyset,(1,1),(1)} \end{array}$$

where $\underline{\mathrm{soc}}^s$ is displayed at level s from the bottom (the bottom being the socle). The object

$$\tilde{V}_{\emptyset,(1,1),(1,1),(1)} = \tilde{V}_{\emptyset,(1,1),\emptyset,\emptyset} \otimes \tilde{V}_{\emptyset,\emptyset,(1,1),(1)}$$

has socle filtration

$$\begin{array}{c} V_{(2,2),\emptyset,\emptyset,(1)} \oplus V_{(2,1,1),\emptyset,\emptyset,(1)} \oplus V_{(1,1,1,1),\emptyset,\emptyset,(1)} \\ \hline 2\,V_{(2,1),(1),\emptyset,(1)} \oplus 2\,V_{(1,1,1),(1),\emptyset,(1)} \oplus V_{(2,1),\emptyset,\emptyset,\emptyset} \oplus V_{(1,1,1),\emptyset,\emptyset,\emptyset} \\ \hline 3\,V_{(1,1),(1,1),\emptyset,(1)} \oplus V_{(2),(1,1),\emptyset,(1)} \oplus V_{(1,1),(2),\emptyset,(1)} \oplus V_{(2),(2),\emptyset,(1)} \\ \oplus V_{(2),(1),\emptyset,\emptyset} \oplus 2\,V_{(1,1),(1),\emptyset,\emptyset} \oplus V_{(2,1),\emptyset,(1),(1)} \oplus V_{(1,1,1),\emptyset,(1),(1)} \\ \hline V_{(2),(1),(1),(1)} \oplus 2\,V_{(1,1),(1),(1),(1)} \oplus V_{(1),(2),\emptyset,\emptyset} \oplus 2\,V_{(1),(1,1),\emptyset,\emptyset} \\ \oplus V_{(1,1),\emptyset,(1),\emptyset} \oplus 2\,V_{(1),(2,1),\emptyset,(1)} \oplus 2\,V_{(1),(1,1,1),\emptyset,(1)} \\ \hline V_{\emptyset,(2,1),\emptyset,\emptyset} \oplus V_{\emptyset,(1,1,1),\emptyset,\emptyset} \oplus V_{(1),(2),(1),(1)} \oplus 2\,V_{(1),(1,1),(1),(1)} \oplus V_{(1),(1),(1),\emptyset} \\ \oplus V_{\emptyset,(2,2),\emptyset,(1)} \oplus V_{\emptyset,(2,1,1),\emptyset,(1)} \oplus V_{\emptyset,(1,1,1,1),\emptyset,(1)} \oplus V_{(1,1),\emptyset,(1,1),(1)} \\ \hline V_{\emptyset,(2,1),(1),(1)} \oplus V_{\emptyset,(1,1,1),(1),(1)} \oplus V_{\emptyset,(1,1),(1),\emptyset} \oplus V_{(1),(1),(1,1),(1)} \\ \hline V_{\emptyset,(1,1),(1,1),(1)} \end{array}$$

♦

5. Universality

Let $(\mathcal{D}, \otimes, \mathbb{1}_{\mathcal{D}})$ be a $\mathbb{K}$-linear, abelian, symmetric monoidal category with monoidal unit $\mathbb{1}_{\mathcal{D}}$, such that all tensor functors $x \otimes \bullet$ are exact.

Convention 5.1 A category $\mathcal{D}$ as above is a *tensor category*, and a $\mathbb{K}$-linear symmetric monoidal functor is a *tensor functor*. Similarly, a *tensor natural* *transformation* is one between tensor functors that respects all of the structure in the guessable fashion.

A tensor category *has coproducts* if it has arbitrary direct sums preserved by all functors of the form $a \otimes \bullet$. ♦

With all of this in hand, the main result of the section reads as follows.

Theorem 5.2 *Let* $\mathbf{q} : A \otimes B \to \mathbb{1}_{\mathcal{D}}$ *be a morphism in* $\mathcal{D}$ *and*

$$0 \subset A_0 \subseteq A_1 \subseteq \cdots \subseteq A_a \subset A \tag{27}$$

a transfinite filtration of A *indexed by ordinals from* 0 *to* a*. Set* $\alpha = \aleph_a$*. Then*

- *(a) up to tensor natural isomorphism, there exists a unique left exact tensor functor* $\mathcal{F} : \mathbb{T}_\alpha \rightsquigarrow \mathcal{D}$ *turning the pairing* $V^* \otimes V \to \mathbb{K}$ *into* $\mathbf{q}$ *and the transfinite socle filtration of* V^*

$$0 \subset V^*_{\aleph_0} \subset \ldots \subset V^*_\alpha \subset V^*$$

  into (27);
- *(b) if furthermore* $\mathcal{D}$ *has coproducts in the sense of Convention 5.1, then the functor* $\mathcal{F}$ *extends uniquely to a coproduct-preserving tensor functor* $\overline{\mathbb{T}}_{\aleph_t} \rightsquigarrow \mathcal{D}$*.*

We first prove Theorem 5.2 in the case $\alpha = t$ for $t \in \mathbb{Z}_{\geq 0}$. The result is as follows.

Theorem 5.3 *Let* t *be a nonnegative integer. Let* $\mathbf{q} : A \otimes B \to \mathbb{1}_{\mathcal{D}}$ *be a morphism in* $\mathcal{D}$ *and*

$$0 \subset A_0 \subseteq \cdots \subseteq A_t \subset A \tag{28}$$

be a filtration in $\mathcal{D}$ *by subobjects of* A*. Then*

- *(a) up to tensor natural isomorphism, there exists a unique left exact tensor functor* $\mathcal{F} : \mathbb{T}_{\aleph_t} \rightsquigarrow \mathcal{D}$ *turning the pairing* $V^* \otimes V \to \mathbb{K}$ *into* $\mathbf{q}$ *and the socle filtration of* V^*

$$0 \subset V^*_{\aleph_0} \subset \cdots \subset V^*_{\aleph_t} \subset V^*$$

  into (28);
- *(b) if furthermore* $\mathcal{D}$ *has coproducts in the sense of Convention 5.1, then the functor* $\mathcal{F}$ *extends uniquely to a coproduct-preserving tensor functor* $\overline{\mathbb{T}}_{\aleph_t} \rightsquigarrow \mathcal{D}$*.*

Before we embark on the proof of Theorem 5.3, it will be convenient to slightly restate Corollary 4.25(d) in the spirit of [3, §3.4]. For this purpose, we introduce the tensor algebra

$$\mathrm{T} := \mathrm{T}\left(\bigoplus_{s=t}^{0}(V^*/V^*_{\aleph_s}) \oplus V^* \oplus V\right)$$

of the direct sum of the displayed "degree-1" indecomposable injectives. For $r \in \mathbb{Z}_{\geq 0}$, we denote the degree-$r$ truncation of T by $\mathrm{T}^{\leq r}$.

Next, define

$$\mathcal{A}^r = \mathrm{End}_{\mathbb{T}_{\aleph_t}}(\mathrm{T}^{\leq r}),\ \mathcal{A} = \bigcup_{r\in\mathbb{Z}_{\geq 0}} \mathcal{A}^r,$$

where the inclusions $\mathcal{A}^r \subset \mathcal{A}^{r+1}$ are the obvious ones (extension of an endomorphism by 0). We have the following analogues of [3, Definition 3.18] and [3, Theorem 3.19] (providing an alternate version of Corollary 4.25(d)).

Definition 5.4 An $\mathcal{A}$-module is *locally unitary* if it is unitary over some $\mathcal{A}^r \subset \mathcal{A}$. ♦

Theorem 5.5 *The functor* $\mathrm{Hom}_{\mathbb{T}_{\aleph_t}}(\bullet, \mathrm{T})$ *implements a contravariant equivalence between* $\mathbb{T}_{\aleph_t}$ *and the category of finite-dimensional modules over* $\mathcal{A}$ *which are locally unitary.* ■

Remark 5.6 The inclusions $\mathcal{A}^r \subset \mathcal{A}$ split naturally, and hence give rise to inclusions $(\mathcal{A}^r)^* \subset (\mathcal{A}^{r+1})^*$ of dual finite-dimensional coalgebras. The union $\bigcup_r(\mathcal{A}^r)^*$ can be chosen for our coalgebra C from Corollary 4.25(d).

Moreover, the Koszulity of the coalgebra C translates to the fact that all algebras $\mathcal{A}^r$ are Koszul and hence quadratic. ♦

We grade $\mathcal{A}^r$ as follows:

Definition 5.7 For $d \geq 0$, the degree-d homogeneous component $\mathcal{A}^r_d$ is the direct sum of all spaces of morphisms

$$\mathrm{T}^{\leq r} \to Y \to Z \to \mathrm{T}^{\leq r},$$

where

- Y and Z are indecomposable direct summands of $\mathrm{T}^{\leq r}$ such that $\mathrm{soc}\, Z \in \mathcal{S}_i$ and $\mathrm{soc}\, Y \in \mathcal{S}_j$;
- $d(i,j) = d$ (the defect from Definition 2.8);
- the outer arrows are the surjection and inclusion realizing respectively Y and Z as direct summands.

The gradings of the various algebras $\mathcal{A}^r$ are then compatible with the inclusions $\mathcal{A}^r \subset \mathcal{A}^{r+1}$, so $\mathcal{A}$ itself acquires an $\mathbb{Z}_{\geq 0}$-grading.

We write $\deg(x)$ for the degree of an element $x \in \mathcal{A}$. ♦

The fact that Definition 5.7 does indeed define a grading follows from the triangle property of the defect, Corollary 4.16(b). These gradings make the algebras $\mathcal{A}^r$ Koszul, and the resulting grading on $\mathcal{A}$ corresponds by duality to the grading on C alluded to in Corollary 4.25(d), see Remark 5.6.

As part of our proof of Theorem 5.3, we will describe the degree-one and degree-two components of $\mathcal{A}$, as well as its relations. The latter are all quadratic by the Koszulity of the coalgebra C from Corollary 4.25(d).

Notation 5.8 For $i = (n_s, n, m)$ we set

$$i_\ell = \mathbf{n}_i = (n_t, \cdots, n_0, n), \;\; i_r = m_i = m.$$

We also set

$$\begin{aligned} i_- &= (n_t, \cdots, n_0, n-1, m-1), \\ i_\pm^s &= (n_t, \cdots, n_s+1, n_{s-1}-1, \cdots, n_0, n, m) \end{aligned}$$

and

$$i_+^s = (n_t, \cdots, n_s+1, n_{s-1}, \cdots, n_0, n, m)$$

for $0 \leq s \leq t$, where for the purpose of defining $i_\pm^0$ we regard the component n as n_{-1}. ♦

Notation 5.9 Let S_p be the symmetric group on p symbols. We denote multiple products $S_p \times S_u \times S_q \times \cdots$ by $S_{p,u,q,\ldots}$. For $i = (n_t, \ldots, n_0, n, m) \in I$, we set $S_i = S_{n_t,\ldots,n_0,n,m}$. In addition, in the rest of the paper, $\mathrm{Hom}(\bullet, \bullet) = \mathrm{Hom}_{\mathbb{T}_{\aleph_t}}(\bullet, \bullet)$ and $\mathrm{End}(\bullet) = \mathrm{End}_{\mathbb{T}_{\aleph_t}}(\bullet)$. ♦

As an immediate consequence of Proposition 4.2 and the classification of simple objects and their injective envelopes given in Theorem 4.12, we have the following analogue of [3, Lemma 3.22], describing the degree-zero component of $\mathcal{A}$.

Lemma 5.10 *Let $i = (n_t, \ldots, n_0, n, m) \in I$. The endomorphism algebra of the injective object*

$$X_i = \bigoplus_{s=t}^{0} (V^*/V_{\aleph_s}^*)^{\otimes n_s} \otimes (V^*)^{\otimes n} \otimes V^{\otimes m}$$

is isomorphic to the group algebra $\mathbb{K}S_i$. ■

According to the description of defect-one pairs $j \prec i$ given in the proof of Proposition 4.15, the morphisms that make up the degree-one component $\mathcal{A}_1$ are qualitatively of two types (matching the two bullet points in the proof of Proposition 4.15):

- morphisms $X_i \to X_{i_-}$;
- morphisms $X_i \to X_{i^s_\pm}$ for some $1 \le s \le t$.

Corresponding examples of such morphisms are:

- the evaluation morphism $\phi_{p,q} : X_i \to X_{i_-}$ of the p^{th} tensorand V^* against the q^{th} tensorand V for some choice of $1 \le p \le n$ and $1 \le q \le m$;
- the surjection morphism $\pi^s_{p,q} : X_i \to X_{i^s_\pm}$ of the p^{th} tensorand $V^*/V^*_{\aleph_{s-1}}$ onto the q^{th} tensorand $V^*/V^*_{\aleph_s}$ for a choice of

$$0 \le s \le t, \quad 1 \le p \le n_{s-1}, \quad 0 \le q \le n_s.$$

More precisely, $\pi^s_{p,q}$ first implements a surjection $V^*/V^*_{\aleph_{s-1}} \to V^*/V^*_{\aleph_s}$ defined on the p^{th} tensorand $V^*/V^*_{\aleph_{s-1}}$ in its domain, and then inserts the result of that surjection as the q^{th} tensorand of type $V^*/V^*_{\aleph_s}$ in the codomain, without altering the order of the other tensorands.

Remark 5.11 It follows from [3, Lemma 2.19] that the algebra $\mathcal{A}$ is generated by all evaluation morphisms $\phi_{p,q}$, all surjection morphisms $\pi^s_{p,q}$, and all permutations of tensorands of the objects X_i. ♦

In the following discussion we will often choose i, s, p, q as above. In such a setting, the group S_{i_-} will be understood to be embedded into S_i as the group of permutations fixing

$$p \in \{1, \cdots, n\} \quad \text{and} \quad q \in \{1, \cdots, m\}.$$

Similarly, we embed S_i and $S_{i^s_\pm}$ into $S_{i^s_+}$ in the obvious fashion.

Lemma 5.12 *Let $i = (n_t, \ldots, n_0, n, m) \in I$ and s, p, q be as above. Then, we have the following description for Hom-spaces of degree-one elements in $\mathcal{A}$.*

(a) The identification $\phi_{p,q} \mapsto 1$ extends to a bimodule isomorphism $\mathrm{Hom}(X_i, X_{i_-}) \cong \mathbb{K}S_i$, where both sides are equipped with standard bimodule structures over

$$\mathrm{End}(X_{i_-}) \cong \mathbb{K}S_{i_-} \quad \textit{and} \quad \mathrm{End}(X_i) \cong \mathbb{K}S_i.$$

(b) *The identification* $\pi^s_{p,q} \mapsto 1$ *extends to a bimodule isomorphism*

$$\mathrm{Hom}(X_i, X_{i^s_\pm}) \cong \mathbb{K}S_{i^s_+} \cong \mathrm{Ind}_{S_{n_s}}^{S_{n_s+1}} \mathbb{K}S_i,$$

where both sides are equipped with standard bimodule structures over

$$\mathrm{End}(X_{i^s_\pm}) \cong \mathbb{K}S_{i^s_\pm} \quad \textit{and} \quad \mathrm{End}(X_i) \cong \mathbb{K}S_i.$$

Proof The identifications in question give rise to morphisms of bimodules

$$\mathbb{K}S_i \to \mathrm{Hom}(X_i, X_{i_-}) \text{ and } \mathbb{K}S_{i^s_{++}} \to \mathrm{Hom}(X_i, X_{i^s_\pm})$$

which are surjective by Remark 5.11. The proof that the maps are also injective proceeds virtually identically to the corresponding argument in the proof of [3, Lemma 3.24]. ■

The following result provides a piecewise description of the degree-two component of the tensor algebra of $\mathcal{A}_1$ over $\mathcal{A}_0$. It parallels and generalizes [3, Lemma 3.25], and it follows routinely from the identifications made in Lemma 5.12.

Since we have to apply the operations $i \mapsto i_-$, $i \mapsto i^s_\pm$ and i^s_+ repeatedly, we will simply concatenate superscripts and $\pm$ subscripts. Note that with this notation, we have $i^s_{-\pm} = i^s_{\pm-}$.

Lemma 5.13 *Let* $i \in I$. *The tensor products of the spaces described in Lemma 5.12 are as follows.*

(a) *The space*

$$\mathrm{Hom}(X_{i_-}, X_{i_{--}}) \otimes_{\mathrm{End}(X_{i_-})} \mathrm{Hom}(X_i, X_{i_-}) \tag{29}$$

is isomorphic to $\mathbb{K}S_i$ *as an* $(S_{i_{--}}, S_i)$*-bimodule.*

(b) *The spaces*

$$\mathrm{Hom}(X_{i_-}, X_{i^s_{-\pm}}) \otimes_{\mathrm{End}(X_{i_-})} \mathrm{Hom}(X_i, X_{i_-}) \tag{30}$$

and

$$\mathrm{Hom}(X_{i^s_\pm}, X_{i^s_{\pm-}}) \otimes_{\mathrm{End}(X_{i^s_\pm})} \mathrm{Hom}(X_i, X_{i^s_\pm}) \tag{31}$$

are both isomorphic to

$$\mathbb{K}S_{i^s_+} \cong \mathrm{Ind}_{S_{n_s}}^{S_{n_s+1}} \mathbb{K}S_i$$

as $(S_{i^s_{\pm-}}, S_i)$*-bimodules.*

(c) *Now let* $0 \le r, s \le t$ *and set* $j = i^{rs}_{\pm\pm}$. *Then the space*

$$\mathrm{Hom}(X_{i^r_\pm}, X_j) \otimes_{\mathrm{End}(X_{i^r_\pm})} \mathrm{Hom}(X_i, X_{i^r_\pm}) \tag{32}$$

is isomorphic to $\mathbb{K}S^{rs}_{++}$ *as an* (S^{rs}_{++}, S_i)*-bimodule.* ■

Next, we describe the subspaces of the tensor products from Lemma 5.13 that are annihilated upon composing morphisms in $\mathcal{A}$. We have three types of relations, corresponding to parts (a) to (c) of Lemma 5.13.

Convention 5.14 We drop the p and q subscripts from the morphisms $\phi_{p,q}$ and $\pi^s_{p,q}$ when the latter are understood to involve only the rightmost relevant tensorands. So for instance, $\phi : X_i \to X_{i_-}$ is the evaluation of the n^{th} tensorand V^* against the m^{th} tensorand V in X_i.

The generators of the spaces in parts (a) and (b) of Lemma 5.12 are always assumed (unless specified otherwise) to be ϕ and π respectively. ♦

Notation 5.15 For $i \in I$ and $j = i^{rr'}_{\pm\pm}$ for some $r, r' \in \{0, \cdots, t\}$, we denote by $\Theta_{i,j}$ the set of those s for which

$$d(j, i^s_\pm) = d(i^s_\pm, i) = 1.$$

♦

In other words, $\Theta_{i,j}$ is the set of all s for which $X_{i^s_\pm}$ can appear as an intermediate object for morphisms $X_i \to X_j = X_{i^{rr'}_{\pm\pm}}$ obtained by composition from tensor products as in part (c) of Lemma 5.13.

Lemma 5.16 *Let* $i = (n_t, \ldots, n_0, n, m) \in I$. *The degree-two relations of* $\mathcal{A}$ *can be described as follows.*

(a) *The map*

$$(29) \to \mathrm{Hom}(X_i, X_{i_{--}})$$

is surjective, with kernel generated by

$$\phi \otimes \phi - (\phi \otimes \phi) \circ (n, n-1)(m, m-1)$$

as an $(S_{i_{--}}, S_i)$*-bimodule, where* $(n, n-1)$ *and* $(m, m-1)$ *are transpositions in the factors* S_n *and* S_m *respectively of*

$$S_i = S_{n_t} \times \cdots \times S_{n_0} \times S_n \times S_m.$$

(b) *The map*

$$(30) \oplus (31) \to \mathrm{Hom}(X_i, X_{i^s_{-\pm}})$$

is surjective, with kernel generated by

$$\pi^s \otimes \phi - (\phi \otimes \pi^s) \circ \tau$$

as an $(S_{i^s_{\pm-}}, S_i)$*-bimodule, where*

$$\tau = (n, n-1) \in S_n \subset S_i$$

if $s = 0$ *and* $\tau = \mathrm{id}$ *otherwise.*

(c) *Let* $j = i^{ss'}_{\pm\pm}$ *for some* $0 \leq s \leq s' \leq t$*. The map*

$$\bigoplus_{r \in \Theta_{i,j}} (32) \to \mathrm{Hom}(X_i, X_j) \tag{33}$$

is a surjective morphism of (S_j, S_i)*-bimodules, and we have the following cases:*

(1) *if* $s' - s \geq 2$ *then* $\Theta_{i,j} = \{s, s'\}$*, and the kernel of the bimodule map (33) is generated by*

$$\pi^s \otimes \pi^{s'} - \pi^{s'} \otimes \pi^s;$$

(2) *if* $s' = s + 1$ *then* $\Theta_{i,j} = \{s, s+1\}$ *again, and the kernel of (33) is generated by*

$$\pi^s \otimes \pi^{s+1} - \sigma \circ (\pi^{s+1} \otimes \pi^s), \tag{34}$$

where

$$\begin{aligned}\sigma = (n_s + 1, n_s) &\in S_{n_s+1} \subset S_{i^{ss'}_{++}} \\ &= S_{n_t} \times \cdots \times S_{n_{s+1}+1} \times S_{n_s+1} \times \cdots \times S_{n_0} \times S_n \times S_m;\end{aligned}$$

(3) *if* $s = s'$ *then* $\Theta_{i,j} = \{s\}$*, and the kernel of (33) is generated by*

$$\sigma \circ (\pi^s \otimes \pi^s) - (\pi^s \otimes \pi^s) \circ \tau,$$

where

$$\tau = (n_{s-1}, n_{s-1} - 1) \in S_{n_{s-1}} \subset S_i = S_{n_t} \times \cdots \times S_{n_0} \times S_n \times S_m$$

and

$$\begin{aligned}\sigma = (n_s + 2, n_s + 1) &\in S_{n_s+2} \subset S_{i^{ss}_{++}} \\ &= S_{n_t} \times \cdots \times S_{n_s+2} \times \cdots \times S_{n_0} \times S_n \times S_m.\end{aligned}$$

Proof Surjectivity follows in all cases from the fact that $\mathcal{A}$ is generated in degree one; in turn, this is a consequence of Theorem 4.23.

As for the description of the kernels, we will prove one of the several cases listed in the statement, the rest being entirely analogous. Note that the types of compositions covered by parts (a) and (b) of the lemma are qualitatively similar to those in [3, Lemma 3.26, (a) and (b)] ((a) refers to compositions of evaluations while (b) refers to composing evaluations and surjections with source and target of the form $V^*/V^*_\bullet$). For this reason, we focus on part (c). Point (3) of the latter is analogous to [3, Lemma 3.26 (c)], so the new phenomena occur in parts (1) and (2) of (c). Finally, we will tackle part (2) as representative of the essence of the argument.

Since the kernel of (33) is easily seen to contain the (S_j, S_i)-bimodule generated by (34), it suffices to prove the opposite containment. As in [3, Lemma 3.26] we do this by a dimension count, i.e. by proving a lower bound on the dimension of the image $\mathrm{Hom}(X_i, X_j)$ of (33).

The two summands of (34) belong respectively to the two direct summands of

$$\left(\mathrm{Hom}(X_{i_\pm^{s+1}}, X_j) \otimes \mathrm{Hom}(X_i, X_{i_\pm^{s+1}})\right) \oplus \left(\mathrm{Hom}(X_{i_\pm^{s}}, X_j) \otimes \mathrm{Hom}(X_i, X_{i_\pm^{s}})\right),$$

which upon making the identification from point (c) of Lemma 5.13 is isomorphic to

$$\left(\mathbb{K}S_{i_{++}^{ss'}}\right)^{\oplus 2} = \mathbb{K}[S_{n_t} \times \cdots \times S_{n_{s+1}+1} \times S_{n_s+1} \times \cdots \times S_{n_0} \times S_n \times S_m]^{\oplus 2}.$$

Under this identification, (34) is simply $1 \oplus (-\sigma)$, with $\sigma = (n_s + 1, n_s)$ in the S_{n_s+1} factor of the second $\mathbb{K}S_{i_{++}^{ss'}}$ summand.

The (S_j, S_i)-bimodule generated by $1 \oplus (-\sigma)$ coincides with the left $S_{i_{++}^{ss'}}$-module generated by it, and hence has dimension

$$\dim\left(\mathbb{K}S_{i_{++}^{ss'}}\right) = n_t! \cdots (n_{s+1}+1)!(n_s+1)! \cdots n_0! n! m!$$

It thus suffices to show that the dimension of the image $\mathrm{Hom}(X_i, X_j)$ of (33) is at least

$$\dim\left(\mathbb{K}S_{i_{++}^{ss'}}\right)^{\oplus 2} - \dim\left(\mathbb{K}S_{i_{++}^{ss'}}\right) = \dim\left(\mathbb{K}S_{i_{++}^{ss'}}\right).$$

Since the tuples i and j only differ in their index-$(s+1)$ and index-$(s-1)$-entries, we can argue as in Lemmas 5.10 and 5.12 that we have

$$\mathrm{Hom}(X_i, X_j) \cong \mathbb{K}[S_{n_t} \cdots S_{n_{s+2}} \times S_{n_{s-2}} \cdots S_{n_0} \times S_n \times S_m] \otimes \mathrm{Hom}(X_{i'}, X_{j'}),$$

where i' and j' are the tuples obtained from i and j respectively by substituting zeros for m and n_r, $r \notin \{s, s \pm 1\}$. We can thus assume that the only possibly nonzero entries in i and j are those with indices $s-1$, s and $s+1$; in other words, we suppose that

$$i = (0, \cdots, n_{s+1}, n_s, n_{s-1}, \cdots, 0), \quad j = (0, \cdots, n_{s+1}+1, n_s, n_{s-1}-1, \cdots, 0).$$

In this setting, we then have to prove

$$\mathrm{Hom}(X_i, X_j) \geq (n_{s+1}+1)!(n_s+1)!(n_{s-1})! \tag{35}$$

Now consider the $(n_s+1)!$ elements of $\mathrm{Hom}(X_i, X_j)$ obtained as follows:

- surject the rightmost tensorand $V^*/V^*_{\aleph_{s-1}}$ onto the rightmost tensorand $V^*/V^*_{\aleph_s}$;
- apply a permutation in S_{n_s+1} to the resulting tensorand $(V^*/V^*_{\aleph_s})^{\otimes(n_s+1)}$;
- surject the rightmost tensorand $V^*/V^*_{\aleph_s}$ onto the rightmost tensorand $V^*/V^*_{\aleph_{s+1}}$.

By choosing a basis for V^* compatible with the filtration by the $V^*_{\aleph_r}$ and examining the action of $\mathrm{Hom}(X_i, X_j)$ on the resulting tensor product basis, it is easy to see that the $(n_s+1)!$ morphisms we have just described span a free $(S_{n_{s+1}+1}, S_{n_{s-1}})$-sub-bimodule of $\mathrm{Hom}(X_i, X_j)$. This proves the desired dimension count (35). ■

Proof of Theorem 5.3 We follow the same strategy as in the proof of [3, Theorem 3.22].

First, note that the conclusion would follow from an application of [3, Theorem 2.22] provided we can show that we can define a symmetric monoidal functor (also denoted by $\mathcal{F}$ by a slight notational abuse) to $\mathcal{D}$ from the tensor category with objects X_i and Hom-spaces generated by the morphisms from Lemma 5.12.

Note that such a tensor functor is constrained up to tensor natural isomorphism by the requirements of the statement:

$$\mathcal{F}(V) = B, \quad \mathcal{F}(V_s^*) = A_s, \quad \mathcal{F}(V^* \otimes V \to \mathbb{K}) = \mathbf{q},$$

as well as

$$\mathcal{F}(V^*/V^*_{\aleph_{s-1}} \to V^*/V^*_{\aleph_s}) = A/A_{s-1} \to A/A_s\,.$$

The only thing to check is that such a definition is consistent, i.e. that the relations satisfied in $\mathbb{T}_{\aleph_t}$ by the morphisms in Lemma 5.12 are satisfied by the corresponding morphisms in $\mathcal{D}$.

The Koszulity result proven in Theorem 4.23, reinterpreted via Corollary 4.25(d) and Remark 5.6, shows that it suffices to prove that the quadratic relations described by Lemma 5.16 are satisfied in $\mathcal{D}$. But this follows directly from the definition of $\mathcal{D}$ as a tensor category. ∎

The next lemma is a small amplification of Theorem 5.3 providing necessary and sufficient conditions for the functors $\mathcal{F}$ therein to be exact (rather than only left exact).

Lemma 5.17 *The functor $\mathcal{F} : \mathbb{T}_{\aleph_t} \rightsquigarrow \mathcal{D}$ from Theorem 5.3 is exact if and only if its restriction to the full subcategory on the indecomposable injective objects (20) of $\mathbb{T}_{\aleph_t}$ is exact.*

Proof One implication (exactness of $\mathcal{F}$ ⇒ exactness of its restriction) is immediate, whereas the other one follows from the description of the right derived functors $R^*\mathcal{F}$ via injective resolutions in $\mathbb{T}_{\aleph_t}$: the hypothesis ensures that the injective resolution

$$x \to J^0 \to J^1 \to \cdots$$

of any object in $\mathbb{T}_{\aleph_t}$ is turned by $\mathcal{F}$ into an exact sequence, hence the conclusion that the higher derived functors $R^p\mathcal{F}(x) = H^p\big(\mathcal{F}(J^\bullet)\big)$ vanish. ∎

Finally, we are ready for the proof of Theorem 5.2.

Proof of Theorem 5.2 For each tuple $\mathbf{s}$

$$\beta_0 < \cdots < \beta_t \le \alpha$$

of infinite cardinal numbers Theorem 5.3 provides a functor $\mathbb{T}_{\aleph_t} \rightsquigarrow \mathbb{T}_\alpha$ that easily seen to be a full exact embedding (exactness follows from Lemma 5.17).

Applying Theorem 5.3 to all such finite-length subcategories $\mathbb{T}_{\mathbf{s}} \cong \mathbb{T}_{\aleph_t} \subseteq \mathbb{T}_\alpha$ we obtain functors $\mathcal{F}_{\mathbf{s}} : \mathbb{T}_{\mathbf{s}} \rightsquigarrow \mathcal{D}$ turning V^* into A, the filtration

$$V^*_{\beta_0} \subset \cdots \subset V^*_{\beta_t}$$

into

$$A_{\beta_0} \subseteq \cdots \subseteq A_{\beta_t}$$

and the pairing $V^* \otimes V \to \mathbb{K}$ into $\mathbf{q}$.

We regard tuples $\mathbf{s}$ as forming a directed system by inclusion. The assignment $\mathbf{s} \to \mathbb{T}_{\mathbf{s}}$ is increasing in the sense that if $\mathbf{s}'$ contains $\mathbf{s}$ then $\mathbb{T}_{\mathbf{s}} \subset \mathbb{T}_{\mathbf{s}'}$, and $\mathbb{T}_\alpha$ can be expressed as the direct limit

$$\mathbb{T}_\alpha = \varinjlim_{s} \mathbb{T}_{\mathbf{s}}. \tag{36}$$

The uniqueness statements in Theorem 5.3 ensure that the functors $\mathcal{F}_{\mathbf{s}}$ are compatible, in the sense that $\mathcal{F}_{\mathbf{s}}$ and $\mathcal{F}_{\mathbf{s}'}$ are (up to tensor natural isomorphism) restrictions of $\mathcal{F}_{\mathbf{s}''}$ for any tuple $\mathbf{s}''$ that contains $\mathbf{s}$ and $\mathbf{s}'$.

The conclusion follows from (36) by taking $\mathcal{F}$ to be the unique (up to tensor natural isomorphism) tensor functor restricting to $\mathcal{F}_{\mathbf{s}}$ on each $\mathbb{T}_{\mathbf{s}}$. ■

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