

The stringy Euler number of Calabi-Yau hypersurfaces in toric varieties and the Mavlyutov duality

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Dedicated to Yuri Ivanovich Manin on the occasion of his 80th birthday

Abstract: We show that minimal models of nondegenerated hypersurfaces defined by Laurent polynomials with a d -dimensional Newton polytope Δ are Calabi-Yau varieties X if and only if the Fine interior of the polytope Δ consists of a single lattice point. We give a combinatorial formula for computing the stringy Euler number of such Calabi-Yau variety X via the lattice polytope Δ . This formula allows us to test mirror symmetry in cases when Δ is not a reflexive polytope. In particular, we apply this formula to pairs of lattice polytopes (Δ, Δ^\vee) that appear in the Mavlyutov's generalization of the polar duality for reflexive polytopes. Some examples of Mavlyutov's dual pairs (Δ, Δ^\vee) show that the stringy Euler numbers of the corresponding Calabi-Yau varieties X and X^\vee may not satisfy the expected topological mirror symmetry test: $e_{\text{st}}(X) = (-1)^{d-1}e_{\text{st}}(X^\vee)$. This shows the necessity of an additional condition on Mavlyutov's pairs (Δ, Δ^\vee) .

1. Introduction

Many examples of pairs of mirror symmetric Calabi-Yau manifolds X and X^* can be obtained using Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties corresponding to pairs (Δ, Δ^*) of d -dimensional reflexive lattice polytopes Δ and Δ^* that are polar dual to each other [Bat94].

A d -dimensional convex polytope $\Delta \subset \mathbb{R}^d$ is called a *lattice polytope* if $\Delta = \text{Conv}(\Delta \cap \mathbb{Z}^d)$, i. e., all vertices of Δ belong to the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. If a d -dimensional polytope Δ contains the zero $0 \in \mathbb{Z}^d$ in its interior, one defines the polar polytope $\Delta^* \subset \mathbb{R}^d$ as

$$\Delta^* := \{y \in \mathbb{R}^d : \langle x, y \rangle \geq -1 \ \forall x \in \Delta\},$$

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where $\langle *, * \rangle : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the standard scalar product on \mathbb{R}^d . A d -dimensional lattice polytope $\Delta \subset \mathbb{R}^d$ containing 0 in its interior is called *reflexive* if the polar polytope Δ^* is also a lattice polytope. If Δ is reflexive then Δ^* is also reflexive and one has $(\Delta^*)^* = \Delta$.

For a d -dimensional reflexive polytope Δ one considers the family of Laurent polynomials

$$f(x) = \sum_{m \in \mathbb{Z}^d \cap \Delta} a_m x^m \in \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$$

with sufficiently general coefficients $a_m \in \mathbb{C}$. Using the theory of toric varieties (see e.g. [Ful93, CLS11]), one can prove that the affine hypersurface $Z \subset \mathbb{T}_d := (\mathbb{C}^*)^d$ defined by $f = 0$ is birational to a $(d-1)$ -dimensional Calabi-Yau variety X . In the same way one obtains another $(d-1)$ -dimensional Calabi-Yau variety X^* corresponding to the polar reflexive polytope Δ^* .

The polar duality $\Delta \leftrightarrow \Delta^*$ between the reflexive polytopes Δ and Δ^* defines a duality between their proper faces $\Theta \leftrightarrow \Theta^*$ ($\Theta \prec \Delta$, $\Theta^* \prec \Delta^*$) satisfying the condition $\dim \Theta + \dim \Theta^* = d-1$, where the dual face $\Theta^* \prec \Delta^*$ is defined as

$$\Theta^* := \{y \in \Delta^* : \langle x, y \rangle = -1 \ \forall x \in \Theta\}.$$

There is a simple combinatorial formula for computing the stringy Euler number of the Calabi-Yau manifold X [BD96, Corollary 7.10]:

$$(1.1) \quad e_{\text{str}}(X) = \sum_{k=1}^{d-2} (-1)^{k-1} \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta = k}} v(\Theta) \cdot v(\Theta^*),$$

where $v(P) := (\dim P)! \text{Vol}(P) \in \mathbb{Z}$ denotes the integral volume of a lattice polytope P . An alternative proof of the formula (1.1) together with its generalizations for Calabi-Yau complete intersections is contained in [BS17]. The formula (1.1) and the duality $\Theta \leftrightarrow \Theta^*$ between faces of reflexive polytopes Δ and Δ^* immediately imply the equality

$$e_{\text{str}}(X) = (-1)^{d-1} e_{\text{str}}(X^*)$$

which is a consequence of a stronger topological mirror symmetry test for the stringy Hodge numbers [BB96]:

$$h_{\text{str}}^{p,q}(X) = h_{\text{str}}^{d-1-p,q}(X^*), \quad 0 \leq p, q \leq d-1.$$

It is important to mention another combinatorial mirror construction suggested by Berglund and Hübsch [BHü93] and generalized by Krawitz [Kra09].

This mirror construction considers $(d - 1)$ -dimensional Calabi-Yau varieties X which are birational to affine hypersurfaces $Z \subset \mathbb{T}_d$ defined by Laurent polynomials

$$f(x) = \sum_{m \in \mathbb{Z}^d \cap \Delta} a_m x^m \in \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}],$$

whose Newton polytope $\Delta \subset \mathbb{R}^d$ is a lattice simplex, but it is important to stress that this simplex may be not a reflexive simplex. The mirror duality for the stringy (or orbifold) Hodge numbers of Calabi-Yau varieties obtained by Berglund-Hübsch-Krawitz mirror construction was proved by Chiodo and Ruan [CR11] and Borisov [Bor13].

The Batyrev mirror construction [Bat94] and the Berglund-Hübsch-Krawitz mirror construction [BHü93, Kra09] can be applied to different classes of hypersurfaces in toric varieties, but they coincide for Calabi-Yau hypersurfaces of Fermat-type. So it is natural to expect that there must be a generalization of two mirror constructions that includes both as special cases (see [AP15, Bor13, ACG16, Pum11, BHü16]). Moreover, it is natural to expect the existence of a generalization of combinatorial formula (1.1) for the stringy Euler number $e_{\text{str}}(X)$ of Calabi-Yau varieties X which holds true for projective varieties coming from a wider class of nondegenerate affine hypersurfaces $Z \subset \mathbb{T}_d$ defined by Laurent polynomials.

Recall that the stringy Euler number $e_{\text{str}}(X)$ can be defined for an arbitrary n -dimensional normal projective \mathbb{Q} -Gorenstein variety X with at worst log-terminal singularities using a desingularization $\rho : Y \rightarrow X$ whose exceptional locus is a union of smooth irreducible divisors D_1, \dots, D_s with only normal crossings [Bat98]. For this purpose, one sets $I := \{1, \dots, s\}$, $D_\emptyset := Y$ and for any nonempty subset $J \subseteq I$ one defines $D_J := \bigcap_{j \in J} D_j$. Using the rational coefficients a_1, \dots, a_s from the formula

$$K_Y = \rho^* K_X + \sum_{i=1}^s a_i D_i,$$

one defines the stringy Euler number

$$e_{\text{str}}(X) := \sum_{\emptyset \subseteq J \subseteq I} e(D_J) \prod_{j \in J} \left(\frac{-a_j}{a_j + 1} \right).$$

Using methods of a nonarchimedean integration (see e.g. [Bat98, Bat99]), one can show that $e_{\text{str}}(X)$ is independent of the choice of the desingularization $\rho : Y \rightarrow X$ and one has $e_{\text{str}}(X) = e_{\text{str}}(X')$ if two projective Calabi-Yau varieties

X and X' with at worst canonical singularities are birational. More generally, the stringy Euler number $e_{\text{str}}(X)$ of any minimal projective algebraic variety X does not depend on the choice of this model and coincides with the stringy Euler number of its canonical model, because all these birational models are K -equivalent to each other. There exist some versions of the stringy Euler number that are conjectured to have minimum exactly on minimal models in a given birational class [BG18].

We remark that in general the stringy Euler number may not be an integer, and so far no example of mirror symmetry is known if the stringy Euler number $e_{\text{str}}(X)$ of a Calabi-Yau variety X is not an integer.

In Section 2, we give a review of results of Ishii [Ish99] on minimal models of nondegenerate hypersurfaces and give a combinatorial criterion that describes all d -dimensional lattice polytopes Δ such that minimal models of Δ -nondegenerate hypersurfaces $Z \subset \mathbb{T}_d$ are Calabi-Yau varieties. We show that a Δ -nondegenerate hypersurface $Z \subset \mathbb{T}_d$ is birational to a Calabi-Yau variety X with at worst \mathbb{Q} -factorial terminal singularities if and only if the Fine interior Δ^{FI} of its Newton polytope Δ consists of a single lattice point (Theorem 2.26). We remark that there exist many d -dimensional lattice polytopes Δ such that $\Delta^{FI} = 0 \in \mathbb{Z}^d$ which are not reflexive if $d \geq 3$.

In Section 3, we discuss the generalized combinatorial duality suggested by Mavlyutov in [Mav11]. Lattice polytopes Δ that appear in the Mavlyutov duality satisfy not only the condition $\Delta^{FI} = 0$, but also the additional condition $[[\Delta^*]^*] = \Delta$, where P^* denotes the polar polytope of P and $[P]$ denotes the convex hull of all lattice points in P .

The lattice polytopes Δ with $\Delta^{FI} = 0$ satisfying the condition $[[\Delta^*]^*] = \Delta$ we call *pseudoreflexive*. A lattice polytope Δ with $\Delta^{FI} = 0$ may not be a pseudoreflexive, but its *Mavlyutov dual polytope* $\Delta^\vee := [\Delta^*]$ and the lattice polytope $[[\Delta^*]^*]$ are always pseudoreflexive. Moreover, if $\Delta^{FI} = 0$ then $[[\Delta^*]^*]$ is the smallest pseudoreflexive polytope containing Δ . For this reason we call d -dimensional lattice polytopes Δ with the only condition $\Delta^{FI} = 0$ *almost pseudoreflexive*.

If the lattice polytope Δ is pseudoreflexive, then one has $((\Delta^\vee)^\vee) = \Delta$. Any reflexive polytope Δ is pseudoreflexive, because in this case one has $\Delta^\vee = [\Delta^*] = \Delta^*$. Therefore, the Mavlyutov duality $\Delta \leftrightarrow \Delta^\vee$ is a generalization of the polar duality $\Delta \leftrightarrow \Delta^*$ for reflexive polytopes.

Unfortunately, Mavlyutov dual pseudoreflexive polytopes Δ and Δ^\vee are not necessarily combinatorially dual to each other. For this reason, we can not expect a natural duality between k -dimensional faces of pseudoreflexive polytope Δ and $(d - 1 - k)$ -dimensional faces of its dual pseudoreflexive polytope Δ^\vee . Mavlyutov observed that a natural duality can be obtained

if one restricts attention to some part of faces of Δ [Mav13]. A proper k -dimensional face $\Theta \prec \Delta$ of a pseudoreflexive polytope Δ will be called *regular* if $\dim[\Theta^*] = d - k - 1$, where Θ^* is the dual face of the polar polytope Δ^* . If $\Theta \prec \Delta$ is a regular face of a pseudoreflexive polytope Δ then $\Theta^\vee := [\Theta^*]$ is a regular face of the Mavlyutov dual pseudoreflexive polytope Δ^\vee and one has $(\Theta^\vee)^\vee = \Theta$, so that one obtains a natural duality between k -dimensional regular faces of Δ and $(d - k - 1)$ -dimensional regular faces of Δ^\vee . Mavlyutov hoped that this duality could help to find a mirror symmetric generalization of the formula (1.1) for arbitrary pairs (Δ, Δ^\vee) of pseudoreflexive polytopes [Mav13].

In Section 4, we are interested in a combinatorial formula for the stringy E -function $E_{\text{str}}(X; u, v)$ of a canonical Calabi-Yau model X of a Δ -nondegenerated hypersurface for an arbitrary d -dimensional almost pseudoreflexive polytope Δ . Using the results of Danilov and Khovanskii [DKh86], we obtain such a combinatorial formula for the stringy function $E_{\text{str}}(X; u, 1)$ (Theorem 4.10) and for the stringy Euler number $e_{\text{str}}(X) := E_{\text{str}}(X; 1, 1)$ (Theorem 4.11):

$$(1.2) \quad e_{\text{str}}(X) = \sum_{k=0}^{d-1} \sum_{\substack{\Theta \subseteq \Delta \\ \dim \Theta = d-k}} (-1)^{d-1-k} v(\Theta) \cdot v(\sigma^\Theta \cap \Delta^*).$$

In this formula, the polar polytope Δ^* is in general a rational polytope, the integer $v(\Theta)$ denotes the integral volume of a $(d - k)$ -dimensional face $\Theta \preceq \Delta$ and the rational number $v(\sigma^\Theta \cap \Delta^*)$ denotes the integral volume of the k -dimensional rational polytope $\sigma^\Theta \cap \Delta^*$ contained in the k -dimensional normal cone σ^Θ corresponding to the face $\Theta \preceq \Delta$ in the normal fan of the polytope Δ . One can easily see that the formula (1.1) can be considered as a particular case of the formula (1.2) if Δ is a reflexive polytope.

In Section 5, we consider examples of Mavlyutov pairs (Δ, Δ^\vee) of pseudoreflexive polytopes obtained from Newton polytopes of polynomials defining Calabi-Yau hypersurfaces X of degree $a + d$ in the d -dimensional weighted projective spaces $\mathbb{P}(a, 1, \dots, 1)$ of dimension $d \geq 5$ such that the weight a does not divide the degree $a + d$ and $a < d/2$. These pseudoreflexive polytopes Δ and Δ^\vee are not reflexive. If $d = ab + 1$ for an integer $b \geq 2$ then X is quasi-smooth and one can apply the Berglund-Hübsch-Krawitz mirror construction. We compute the stringy Euler numbers of Calabi-Yau hypersurfaces X and their mirrors X^\vee . In particular, we show that the equality $e_{\text{str}}(X) = (-1)^{d-1} e_{\text{str}}(X^\vee)$ holds if $d = ab + 1$ and in this case one obtains quasi-smooth Calabi-Yau hypersurfaces. However, if $d = ab + l$ ($2 \leq l \leq a - 1$), then the Calabi-Yau hypersurfaces $X \subset \mathbb{P}(a, 1, \dots, 1)$ are not quasi-smooth.

Using our formulas for the stringy Euler numbers $e_{\text{str}}(X)$ and $e_{\text{str}}(X^\vee)$ we show that the equality $e_{\text{str}}(X) = (-1)^{d-1}e_{\text{str}}(X^\vee)$ can not be satisfied if e.g. $d = ab + 2$, where a, b are two distinct odd prime numbers (Theorem 5.5).

In Section 6, we investigate the Mavlyutov duality $\Delta \leftrightarrow \Delta^\vee$ together with an additional condition on singular facets of the pseudoreflexive polytopes Δ and Δ^\vee . This condition can be considered as a version of a quasi-smoothness condition on Mavlyutov's pairs (Δ, Δ^\vee) suggested in some form by Borisov [Bor13]. For Calabi-Yau varieties X and X^\vee corresponding to Mavlyutov's pairs (Δ, Δ^\vee) satisfying this additional condition we prove another generalization of the formula (1.1) such that the equality $e_{\text{str}}(X) = (-1)^{d-1}e_{\text{str}}(X^\vee)$ holds (Theorem 6.3).

2. Minimal models of nondegenerate hypersurfaces

Let $M \cong \mathbb{Z}^d$ be a free abelian group of rank d and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Denote by $N_{\mathbb{R}}$ the dual space $\text{Hom}(M, \mathbb{R})$ with the natural pairing $\langle *, * \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$.

Definition 2.1. Let $P = \text{Conv}(x_1, \dots, x_k) \subset M_{\mathbb{R}}$ be a convex polytope obtained as the convex hull of a finite subset $\{x_1, \dots, x_k\} \subset M_{\mathbb{R}}$. Define the piecewise linear function

$$\text{ord}_P : N_{\mathbb{R}} \rightarrow \mathbb{R}$$

as

$$\text{ord}_P(y) := \min_{x \in P} \langle x, y \rangle = \min_{i=1}^k \langle x_i, y \rangle.$$

We associate with P its *normal fan* Σ^P which is finite collection of *normal cones* σ^Q in the dual space $N_{\mathbb{R}}$ parametrized by faces $Q \preceq P$. Then the cone σ^Q is defined as

$$\sigma^Q := \{y \in N_{\mathbb{R}} : \text{ord}_P(y) = \langle x, y \rangle, \forall x \in Q\}.$$

The zero $0 \in N_{\mathbb{R}}$ is considered as the normal cone to P . One has

$$N_{\mathbb{R}} = \bigcup_{Q \preceq P} \sigma^Q.$$

If $P \subset M_{\mathbb{R}}$ is a d -dimensional polytope containing $0 \in M$ in its interior, then we call

$$P^* := \{y \in N_{\mathbb{R}} : \text{ord}_P(y) \geq -1\}$$

the *polar polytope* of P . The polar polytope P^* is the union over all proper faces $Q \prec P$ of the subsets

$$P^* \cap \sigma^Q = \{y \in \sigma^Q : \langle x, y \rangle \geq -1, \forall x \in Q\},$$

i.e.,

$$P^* = \bigcup_{Q \prec P} (P^* \cap \sigma^Q).$$

Definition 2.2. Let $n \in N$ be a primitive lattice vector and let $l \in \mathbb{Z}$. We consider an affine hyperplane $H_n(l) \subset M_{\mathbb{R}}$ defined by the equation $\langle x, n \rangle = l$. If $m \in M$ then the nonnegative integer

$$|\langle m, n \rangle - l|$$

is called the *integral distance* between m and the hyperplane $H_n(l)$.

Definition 2.3. Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polytope, i.e., all vertices of Δ belong to M . Then ord_{Δ} has integral values on N .

For any nonzero lattice point $n \in N$ one defines the following two half-spaces in $M_{\mathbb{R}}$:

$$\begin{aligned} \Gamma_0^{\Delta}(n) &:= \{x \in M_{\mathbb{R}} : \langle x, n \rangle \geq \text{ord}_{\Delta}(n)\}, \\ \Gamma_1^{\Delta}(n) &:= \{x \in M_{\mathbb{R}} : \langle x, n \rangle \geq \text{ord}_{\Delta}(n) + 1\}. \end{aligned}$$

For all $n \in N$ we have obvious the inclusion $\Gamma_1^{\Delta}(n) \subset \Gamma_0^{\Delta}(n)$ and the lattice polytope Δ can be written as intersection

$$\Delta = \bigcap_{0 \neq n \in N} \Gamma_0^{\Delta}(n).$$

Definition 2.4. Let Δ be a d -dimensional lattice polytope. The *Fine interior* of Δ is defined as

$$\Delta^{FI} := \bigcap_{0 \neq n \in N} \Gamma_1^{\Delta}(n).$$

Remark 2.5. It is clear that Δ^{FI} is a convex subset in the interior of Δ . We remark that the interior of a d -dimensional polytope Δ is always nonempty, but the Fine interior of Δ may be sometimes empty. I was told that the subset $\Delta^{FI} \subset \Delta$ first has appeared in the PhD thesis of Jonathan Fine [Fine83].

Remark 2.6. Since Δ^{FI} is defined as an intersection of countably many half-spaces $\Gamma_1^{\Delta}(n)$ it is not immediately clear that the polyhedral set Δ^{FI} has only finitely many faces. The latter follows from the fact that for any proper face $\Theta \prec \Delta$ the semigroup $S_{\Theta} := N \cap \sigma^{\Theta}$ of all lattice points in the cone σ^{Θ} is finitely generated (Gordan's lemma). One can show that Δ^{FI} can be obtained

as a finite intersection of those half-spaces $\Gamma_1^\Delta(n)$ such that the lattice vector n appears as a minimal generator of the semigroup S_Θ for some face $\Theta \prec \Delta$. Indeed, if $n', n'' \in \sigma^\Theta$, i.e., if two lattice vectors n', n'' are in the same cone σ^Θ , and if $x \in \Delta$ is a point in $\Gamma_1^\Delta(n') \cap \Gamma_1^\Delta(n'')$, then we have

$$\langle x, n'+n'' \rangle = \langle x, n' \rangle + \langle x, n'' \rangle \geq \text{ord}_\Delta(n') + 1 + \text{ord}_\Delta(n'') + 1 > \text{ord}_\Delta(n'+n'') + 1,$$

i.e., $\Gamma_1^\Delta(n') \cap \Gamma_1^\Delta(n'')$ is contained in $\Gamma_1^\Delta(n'+n'')$

Remark 2.7. Let $\Delta \subset M_\mathbb{R}$ be a d -dimensional lattice polytope. If $m \in \Delta$ is an interior lattice point, then $m \in \Delta^{FI}$. Indeed, if $m \in \Delta$ is an interior lattice point, then for any lattice point $n \in N$ one has $\langle m, n \rangle > \text{ord}_\Delta(n)$. Since both numbers $\langle m, n \rangle$ and $\text{ord}_\Delta(n)$ are integers, we obtain

$$\langle m, n \rangle \geq \text{ord}_\Delta(n) + 1, \quad \forall n \in N,$$

i.e., m belongs to Δ^{FI} . This implies the inclusion

$$\text{Conv}(\text{Int}(\Delta) \cap M) \subseteq \Delta^{FI},$$

i.e., the Fine interior of Δ contains the convex hull of the set interior lattice points in Δ .

We see below that for 2-dimensional lattice polytopes this inclusion is equality. In order to find the Fine interior of an arbitrary 2-dimensional lattice polytope, we will use the following well-known fact:

Proposition 2.8. *Let $\Delta \subset \mathbb{R}^2$ be a lattice triangle such that $\Delta \cap \mathbb{Z}^2$ consists of vertices of Δ . Then Δ is isomorphic to the standard triangle with vertices $(0, 0), (1, 0), (0, 1)$. In particular, the integral distance between a vertex of Δ and its opposite side of Δ is always 1.*

Proposition 2.9. *If Δ is a 2-dimensional lattice polytope, then Δ^{FI} is exactly the convex hull of interior lattice points in Δ .*

Proof. Let Δ be a 2-dimensional lattice polytope. If Δ has no interior lattice points, then Δ is isomorphic to either a lattice polytope in \mathbb{R}^2 contained in the strip $0 \leq x_1 \leq 1$, or to the lattice triangle with vertices $(0, 0), (2, 0), (0, 2)$ (see e. g. [Kho97]). In both cases, one can easily check that $\Delta^{FI} = \emptyset$.

If Δ has exactly one interior lattice point then Δ is isomorphic to one of 16 reflexive polygons and one can check that this interior lattice point is exactly the Fine interior of Δ , because, by 2.8, this interior lattice point has integral distance 1 to its sides.

If Δ is a 2-dimensional lattice polytope with at least two interior lattice points then we denote $\Delta' := \text{Conv}(\text{Int}(\Delta) \cap M)$. One has $\dim \Delta' \in \{1, 2\}$.

If $\dim \Delta' = 1$ then by 2.8 the integral distance from the affine line L containing Δ' and any lattice vertex of Δ outside of this line must be 1. This implies that the Fine interior Δ^{FI} is contained in L . By 2.7, if A and B are two vertices of the segment Δ' then $A, B \in \Delta^{FI}$. By 2.8, there exist a side of Δ with the integral distance 1 from A having nonempty intersection with the line L . Therefore, A is vertex of Δ^{FI} . Analogously, B is also a vertex of Δ^{FI} .

Assume now that $\dim \Delta' = 2$, i.e., Δ' is k -gon. Then Δ' is an intersection of k half-planes $\Gamma_1, \dots, \Gamma_k$ whose boundaries are k lines L_1, \dots, L_k through the k sides of Δ' . By 2.8, for any $1 \leq i \leq k$ all vertices of Δ outside the half-plane Γ_i must have integral distance 1 from L_i . Therefore Γ_i contains the Fine interior Δ^{FI} and $\Delta^{FI} \subseteq \bigcap_{i=1}^k \Gamma_i = \Delta'$. The opposite inclusion $\Delta' \subseteq \Delta^{FI}$ follows from 2.7. \square

Remark 2.10. The convex hull of all interior lattice points in a lattice polytope Δ of dimension $d \geq 3$ must not coincide with the Fine interior Δ^{FI} in general. For example, there exist 3-dimensional lattice polytopes Δ without interior lattice points such that Δ^{FI} is not empty [Tr08]. The simplest well-known example of such a situation is the 3-dimensional lattice simplex corresponding to Newton polytope of the Godeaux surface obtained as a free cyclic group of order 5 quotient of the Fermat surface $w^5 + x^5 + y^5 + z^5 = 0$ by the mapping $(w : x : y : z) \rightarrow (w : \rho x : \rho^2 y : \rho^3 z)$, where ρ is a fifth root of 1.

Definition 2.11. Assume that the lattice polytope Δ has a nonempty Fine interior Δ^{FI} . We define the *support* of Δ^{FI} as

$$\text{Supp}(\Delta^{FI}) := \{n \in N : \langle x, n \rangle = \text{ord}_\Delta(n) + 1 \text{ for some } x \in \Delta^{FI}\} \subset N.$$

The convex rational polytope

$$\Delta^{can} := \bigcap_{n \in \text{Supp}(\Delta^{FI})} \Gamma_0^\Delta(n)$$

containing Δ we call the *canonical hull* of Δ .

Remark 2.12. The support of Δ^{FI} is a finite subset in the lattice N , because it is contained in the union over all faces $\Theta \prec \Delta$ of all minimal generating subsets for the semigroups $N \cap \sigma^\Theta$. In particular, $\text{Supp}(\Delta^{FI})$ always consists of finitely many primitive nonzero lattice vectors $v_1, \dots, v_l \in N$ such that $\sum_{i=1}^l \mathbb{R}_{\geq 0} v_i = N_{\mathbb{R}}$.

Let us now identify the lattice $M \cong \mathbb{Z}^d$ with the set of monomials $x^m = x_1^{m_1} \cdots x_d^{m_d}$ in the Laurent polynomial ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}] \cong \mathbb{C}[M]$. For simplicity, we will consider only algebraic varieties X over the algebraically closed field \mathbb{C} .

Definition 2.13. The *Newton polytope* $\Delta(f)$ of a Laurent polynomial $f(x) = \sum_m a_m x^m \in \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ is the convex hull of all lattice points $m \in M$ such that $a_m \neq 0$. For any face $\Theta \subseteq \Delta(f)$ one defines the Θ -part of the Laurent polynomial f as

$$f_\Theta(x) := \sum_{m \in M \cap \Theta} a_m x^m.$$

A Laurent polynomial $f \in \mathbb{C}[M]$ with a Newton polytope Δ is called Δ -*nondegenerate* or simply *nondegenerate* if for every face $\Theta \subseteq \Delta$ the zero locus $Z_\Theta := \{x \in \mathbb{T}_d : f_\Theta(x) = 0\}$ of the Θ -part of f is empty or a smooth affine hypersurface in the d -dimensional algebraic torus \mathbb{T}_d .

The theory of toric varieties allows to construct a smooth projective algebraic variety \widehat{Z}_Δ that contains the affine Δ -nondegenerate hypersurface $Z_\Delta \subset T$ as a Zariski open subset. For this purpose, one first considers the closure \overline{Z}_Δ of Z_Δ in the projective toric variety \mathbb{P}_Δ associated with the normal fan Σ^Δ . Then one chooses a regular simplicial subdivision $\widehat{\Sigma}$ of the fan Σ^Δ and obtains a projective morphism $\rho : \mathbb{P}_{\widehat{\Sigma}} \rightarrow \mathbb{P}_\Delta$ from a smooth toric variety $\mathbb{P}_{\widehat{\Sigma}}$ to \mathbb{P}_Δ such that, by Bertini theorem, its restriction to the Zariski closure \widehat{Z}_Δ of Z_Δ in $\mathbb{P}_{\widehat{\Sigma}}$ is a smooth and projective desingularization of \overline{Z}_Δ .

Now one can apply the Minimal Model Program of Mori to the smooth projective hypersurface \widehat{Z}_Δ (see e.g. [Mat02]). One can show that for Δ -nondegenerate hypersurfaces a minimal model of \widehat{Z}_Δ can be obtained via the toric Mori theory due to Miles Reid [Reid83, Wi02, Fu03, FS04] applied to pairs (V, D) consisting of a projective toric variety V and the Zariski closure D of the nondegenerate hypersurface Z_Δ in V [Ish99]. Therefore, minimal models of nondegenerate hypersurfaces Z_Δ can be constructed by combinatorial methods.

Recall the following standard definitions from the Minimal Model Program [Ko13].

Definition 2.14. Let X be a normal projective algebraic variety over \mathbb{C} , and let K_X be its canonical class. A birational morphism $\rho : Y \rightarrow X$ is called a *log-desingularization* of X if Y is smooth and the exceptional locus of ρ consists of smooth irreducible divisors D_1, \dots, D_k with simple normal crossings. Assume that X is \mathbb{Q} -Gorenstein, i.e., some integral multiple of K_X is a Cartier divisor

on X . We set $I := \{1, \dots, k\}$, $D_\emptyset := Y$, and for any nonempty subset $J \subseteq I$ we denote by D_J the intersection of divisors $\bigcap_{j \in J} D_j$, which is either empty or a smooth projective subvariety in Y of codimension $|J|$. Then the canonical classes of X and Y are related by the formula

$$K_Y = \rho^* K_X + \sum_{i=1}^k a_i D_i,$$

where a_1, \dots, a_k are rational numbers which are called *discrepancies*. The singularities of X are said to be

$$\begin{array}{lll} \textit{log-terminal} & \text{if} & a_i > -1 \quad \forall i \in I; \\ \textit{canonical} & \text{if} & a_i \geq 0 \quad \forall i \in I; \\ \textit{terminal} & \text{if} & a_i > 0 \quad \forall i \in I. \end{array}$$

It is known that \mathbb{Q} -Gorenstein toric varieties always have at worst log-terminal singularities [Reid83].

Definition 2.15. A projective normal \mathbb{Q} -Gorenstein algebraic variety Y is called *canonical model* of X if Y is birationally equivalent to X , Y has at worst canonical singularities and the linear system $|mK_Y|$ is base point free for sufficiently large integer $m \in \mathbb{N}$.

Definition 2.16. A projective algebraic variety Y is called *minimal model* of X if Y is birationally equivalent to X , Y has at worst terminal \mathbb{Q} -factorial singularities, the canonical class K_Y is numerically effective, and the linear system $|mK_Y|$ is base point free for sufficiently large integer $m \in \mathbb{N}$.

The main result of the Minimal Model Program for nondegenerate hypersurfaces in toric varieties in [Ish99] can be reformulated using combinatorial interpretations of Zariski decompositions of effective divisors on toric varieties [OP91] (see also [HKP06, Appendix A]) and their applications to log minimal models of polarized pairs [BiH14]

Definition 2.17. Let \mathbb{P}_Σ be a d -dimensional projective toric variety defined by a fan Σ whose 1-dimensional cones $\sigma_i = \mathbb{R}_{\geq 0} v_i \in \Sigma(1)$ are generated by primitive lattice vectors $v_1, \dots, v_s \in N$. Denote by V_i ($1 \leq i \leq s$) torus invariant divisors on \mathbb{P} corresponding to v_i . Let $D = \sum_{i=1}^s a_i V_i$ be an arbitrary torus invariant \mathbb{Q} -divisor on \mathbb{P} such that the rational polytope

$$\Delta_D := \{x \in M_{\mathbb{R}} : \langle x, v_i \rangle \geq -a_i, 1 \leq i \leq s\}$$

is not empty. Then the rational numbers

$$\text{ord}_{\Delta_D}(v_i) := \min_{x \in \Delta_D} \langle x, v_i \rangle, \quad 1 \leq i \leq r,$$

satisfy the inequalities

$$\text{ord}_{\Delta_D}(v_i) + a_i \geq 0, \quad 1 \leq i \leq r.$$

Without loss of generality we can assume that the equality $\text{ord}_{\Delta_D}(v_i) + a_i = 0$ holds if and only if $1 \leq i \leq r$ ($r \leq s$). We define the *support of the polytope* Δ_D as

$$\text{Supp}(\Delta_D) := \{v_1, \dots, v_r\}$$

and we write the divisor $D = \sum_{i=1}^l a_i V_i$ as the sum

$$D = P + N, \quad P = \sum_{i=1}^s (-\text{ord}_{\Delta_D}(v_i)) V_i, \quad N := \sum_{i=r+1}^s (\text{ord}_{\Delta_D}(v_i) + a_i) V_i,$$

where $\text{ord}_{\Delta_D}(v_i) + a_i > 0$ for all $r + 1 \leq i \leq s$. Then there exists a d -dimensional \mathbb{Q} -factorial projective toric variety \mathbb{P}' defined by a simplicial fan Σ' whose 1-dimensional cones are generated by the lattice vectors $v_i \in \text{Supp}(\Delta_D)$ together with a birational toric morphism $\varphi : \mathbb{P} \rightarrow \mathbb{P}'$ that contracts the divisors V_{r+1}, \dots, V_s such the nef divisor P on \mathbb{P} is the pull back of the nef divisor

$$P' := \sum_{v_i \in \text{Supp}(\Delta_D)} (-\text{ord}_{\Delta_D}(v_i)) V'_i$$

on \mathbb{P}' . The decomposition $D = P + N$ together with the nef divisor P' on the \mathbb{Q} -factorial projective toric variety \mathbb{P}' we call *toric Zariski decomposition* of D .

Theorem 2.18. *A Δ -nondegenerate hypersurface $Z_\Delta \subset \mathbb{T}_d$ has a minimal model if and only if the Fine interior Δ^{FI} is not empty. In the latter case, a canonical model of the nondegenerate hypersurface Z_Δ is its closure X in the toric variety $\mathbb{P}_{\Delta^{can}}$ associated to the canonical hull Δ^{can} of the lattice polytope Δ . The birational isomorphism between \overline{Z}_Δ and X is induced by the birational isomorphism of toric varieties $\alpha : \mathbb{P}_\Delta \dashrightarrow \mathbb{P}_{\Delta^{can}}$, it can be included in a diagram*

$$\begin{array}{ccc} & \mathbb{P}_{\Sigma} & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ \mathbb{P}_{\Delta} & \overset{\alpha}{\dashrightarrow} & \mathbb{P}_{\Delta^{can}} \end{array}$$

where $\widehat{\Sigma}$ denotes the common regular simplicial subdivision of the normal fans Σ^Δ and $\Sigma^{\Delta^{can}}$. In particular, one obtains two birational morphisms $\rho_1 : \widehat{Z}_\Delta \rightarrow \overline{Z}_\Delta$, $\rho_2 : \widehat{Z}_\Delta \rightarrow X$ in the diagram

$$\begin{array}{ccc} & \widehat{Z}_\Delta & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ \overline{Z}_\Delta & \text{---} \alpha \text{---} & X \end{array}$$

where \widehat{Z}_Δ denotes the Zariski closure of Z_Δ in $\mathbb{P}_{\widehat{\Sigma}}$.

Proof. Let L be the ample Cartier divisor on the d -dimensional toric variety \mathbb{P}_Δ corresponding to a d -dimensional lattice polytope Δ . We apply the toric Zariski decomposition to the adjoint divisor $D := \rho^*L + K_{\mathbb{P}_{\widehat{\Sigma}}}$ for some toric desingularization $\rho : \mathbb{P}_{\widehat{\Sigma}} \rightarrow \mathbb{P}_\Delta$ defined by a fan $\widehat{\Sigma}$ which is a regular simplicial subdivision of the normal fan Σ^Δ . Let $\{v_1, \dots, v_s\}$ be the set of primitive lattice vectors in N generating 1-dimensional cones in $\widehat{\Sigma}$.

Since one has $K_{\mathbb{P}_{\widehat{\Sigma}}} = -\sum_{i=1}^s V_i$ and $\rho^*L = \sum_{i=1}^s (-\text{ord}_\Delta(v_i))V_i$, we obtain that the rational polytope Δ_D corresponding to the adjoint divisor on $\mathbb{P}_{\widehat{\Sigma}}$

$$D = \rho^*L + K_{\mathbb{P}_{\widehat{\Sigma}}} = \sum_{i=1}^s (-\text{ord}_\Delta(v_i) - 1)V_i$$

is exactly the Fine interior Δ^{FI} of Δ .

We can assume that $\text{Supp}(\Delta^{FI}) = \{v_1, \dots, v_r\}$ ($r \leq s$) and the first l lattice vectors v_1, \dots, v_l ($l \leq r$) form the set of generators of 1-dimensional cones $\mathbb{R}v_i$ ($1 \leq i \leq l$) in the normal fan $\Sigma^{\Delta^{can}}$ so that one has

$$\Delta^{can} = \bigcap_{i=1}^r \Gamma_0^\Delta(v_i) = \bigcap_{i=1}^l \Gamma_0^\Delta(v_i)$$

and

$$\text{ord}_\Delta(v_i) + 1 = \text{ord}_{\Delta^{FI}}(v_i), \quad \forall i = 1, \dots, r.$$

The toric Zariski decomposition of $D = \rho^*L + K_{\mathbb{P}_{\widehat{\Sigma}}} = \sum_{i=1}^s (-\text{ord}_\Delta(v_i) - 1)V_i$ is the sum $P + N$ where

$$\begin{aligned} P &= \sum_{i=1}^s (-\text{ord}_{\Delta^{FI}}(v_i))V_i, \\ N &= \sum_{i=r+1}^s (\text{ord}_{\Delta^{FI}}(v_i) - \text{ord}_\Delta(v_i) - 1)V_i \end{aligned}$$

where $(\text{ord}_{\Delta^{FI}}(v_i) - \text{ord}_{\Delta}(v_i) - 1) > 0$ for all $i > r$. Moreover, there exists a projective \mathbb{Q} -factorial toric variety \mathbb{P}' such that v_1, \dots, v_r is the set of primitive lattice generators of 1-dimensional cones in the fan Σ' defining the toric variety \mathbb{P}' and

$$\sum_{i=1}^r (-\text{ord}_{\Delta^{FI}}(v_i)) V'_i$$

is a nef \mathbb{Q} -Cartier divisor on \mathbb{P}' .

Therefore, the canonical divisor $K_{\mathbb{P}_{\Delta^{can}}}$ equals $-\sum_{i=1}^l V_i$ where V_1, \dots, V_l the set of torus invariant divisors on $\mathbb{P}_{\Delta^{can}}$. Let X be the Zariski closure of the affine Δ -nondegenerated hypersurface Z_{Δ} in $\mathbb{P}_{\Delta^{can}}$. Then X is linearly equivalent to a linear combination $\sum_{i=1}^l b_i V_i$, where $b_i = -\text{ord}_{\Delta}(v_i)$ ($1 \leq i \leq l$). So we obtain

$$K_{\mathbb{P}_{\Delta^{can}}} + X \sim \sum_{i=1}^l (-\text{ord}_{\Delta}(v_i) - 1) V_i = \sum_{i=1}^l (-\text{ord}_{\Delta^{FI}}(v_i)) V_i.$$

On the other hand, we have

$$\Delta^{FI} = \bigcap_{n \in \text{Supp}(\Delta^{FI})} \Gamma_1^{\Delta}(n) = \bigcap_{i=1}^l \Gamma_1^{\Delta}(v_i).$$

So $K_{\mathbb{P}_{\Delta^{can}}} + X$ is a semiample \mathbb{Q} -Cartier divisor on the projective toric variety $\mathbb{P}_{\Delta^{can}}$ corresponding to the rational convex polytope Δ^{FI} .

For nondegenerate hypersurfaces one can apply the adjunction and obtain that the canonical class K_X is the restriction to X of the semiample \mathbb{Q} -Cartier divisor $K_{\mathbb{P}_{\Delta^{can}}} + X$. The log-discrepancies of the toric pair $(\mathbb{P}_{\Delta^{can}}, X)$ are equal to the discrepancies of X , because of inversion of the adjunction for non-degenerate hypersurfaces [Amb03]. \square

Corollary 2.19. *For the above birational morphism $\rho_2 : \mathbb{P}_{\widehat{\Sigma}} \rightarrow \mathbb{P}_{\Delta^{can}}$ one has*

$$K_{\widehat{\mathbb{P}}_{\Sigma}} + \widehat{Z}_{\Delta} = \rho_2^*(K_{\mathbb{P}_{\Delta^{can}}} + X) + \sum_{i=l+1}^s a_i V_i$$

and

$$K_{\widehat{Z}_{\Delta}} = \rho_2^* K_X + \sum_{i=l+1}^s a_i D_i, \quad D_i := V_i \cap \widehat{Z}_{\Delta},$$

where V_i denotes the torus invariant divisor on $\mathbb{P}_{\widehat{\Sigma}}$ corresponding to the lattice point $v_i \in N$ and

$$a_i = -\text{ord}_{\Delta}(v_i) + \text{ord}_{\Delta^{FI}}(v_i) - 1 \geq 0.$$

By 2.7, one immediately obtains

Corollary 2.20. *It a d -dimensional lattice polytope Δ contains an interior lattice point, then a Δ -nondegenerated affine hypersurface $Z_\Delta \subset \mathbb{T}_d$ has a minimal model.*

Example 2.21. As we have already mentioned in 2.10 there exist 3-dimensional lattice polytopes Δ without interior lattice points such that Δ^{FI} is not empty. One of such examples is the 3-dimensional lattice simplex Δ such that Δ -nondegenerated hypersurface is birational to the Godeaux surface obtained as a quotient of the Fermat quintic $z_0^5 + z_1^5 + z_2^5 + z_3^5 = 0$ in \mathbb{P}^3 by the action of a cyclic group of order 5

$$(z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : \rho z_1 : \rho^2 z_2 : \rho^3 z_3)$$

where ρ is a 5-th root of unity.

We apply the above general results to Δ -nondegenerate hypersurfaces whose minimal models are Calabi-Yau varieties. It is known that the number of interior lattice points in Δ equals the geometric genus of the Δ -nondegenerate hypersurface [Kho78]. Therefore, if a nondegenerate hypersurface Z_Δ is birational to a Calabi-Yau variety, then Δ must contain exactly one interior lattice point. However, this condition for Δ is not sufficient.

Example 2.22. In [CG11] Corti and Golyshev gave 9 examples of 3-dimensional lattice simplices Δ with a single interior lattice point 0 such that the corresponding nondegenerate hypersurfaces Z_Δ are not birational to a $K3$ -surface. For example, they consider hypersurfaces of degree 20 in the weighted projective space $\mathbb{P}(1, 5, 6, 8)$. The corresponding 3-dimensional lattice simplex Δ is the convex hull of the lattice points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-5, -6, -8)$. The Fine interior Δ^{FI} of Δ is a 1-dimensional polytope on the ray generated by the lattice vector $(-1, -1, -2)$.

Theorem 2.23. *A canonical model of Δ -nondegenerate affine hypersurface $Z_\Delta \subset \mathbb{T}_d$ is birational to a Calabi-Yau variety X with at worst Gorenstein canonical singularities if and only if the Fine interior Δ^{FI} of the lattice polytope Δ consists of a single lattice point. If $\Delta^{FI} = 0$ then X can be obtained as a Zariski closure of Z_Δ in the toric \mathbb{Q} -Fano variety $\mathbb{P}_{\Delta^{can}}$ so that X is an anticanonical divisor on $\mathbb{P}_{\Delta^{can}}$. There exists an embedded desingularization $\rho_2 : \widehat{Z}_\Delta \rightarrow X$ and*

$$K_{\widehat{Z}_\Delta} = \rho_2^* K_X + \sum_{i=l+1}^s a_i D_i,$$

where the discrepancy a_i of the exceptional divisor $D_i := V_i \cap \widehat{Z}_\Delta$ on \widehat{Z}_Δ can be computed by the formula

$$a_i = -\text{ord}_\Delta(v_i) - 1 \geq 0.$$

Proof. First of all we remark that this statement has been partially proved in [ACG16, Prop. 2.2.], but the application of Mori theory for nondegenerate hypersurfaces (see 2.18 and 2.19) imply stronger statements. The above formula for the discrepancies a_i is not new and it has appeared already in [CG11] for resolutions of canonical singularities of Calabi-Yau hypersurfaces X in weighted projective spaces. In general case, one can make a direct computation of a_i using the global nowhere vanishing differential $(d-1)$ -form ω obtained as the Poincaré residue $\omega = \text{Res} \Omega$ of the rational differential form [Bat93]:

$$\Omega := \frac{1}{f} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_d}{x_d}.$$

Since Δ is the Newton polytope of the Laurent polynomial f , the order of zero of ω along the exceptional divisor E_i corresponding to the lattice point $v_i \in N$ equals $-\text{ord}_\Delta(v_i) - 1$. \square

Remark 2.24. We note that a d -dimensional lattice polytope Δ with $\Delta^{FI} = 0$ is reflexive if and only if $\Delta = \Delta^{\text{can}}$.

If Δ is reflexive then the canonical singularities of the projective Calabi-Yau hypersurface $\overline{Z}_\Delta \subset \mathbb{P}_\Delta$ have a MPCP (maximal projective crepant partial) resolution obtained from a simplicial fan $\widehat{\Sigma}$ whose generators of 1-dimensional cones are lattice points on the boundary of the polar reflexive polytope Δ^* [Bat94]. This fact can be generalized to an arbitrary d -dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ such that $\Delta^{FI} = 0$. For this we need the following statement:

Proposition 2.25. *Let Δ be a d -dimensional polytope with $\Delta^{FI} = 0$. Then one has*

$$\text{Supp}(\Delta^{FI}) = \{\Delta^* \cap N\} \setminus \{0\},$$

where Δ^* is the polar polytope.

Proof. By Definition 2.11, a lattice point $n \in N$ belongs to the support of the Fine interior $\Delta^{FI} = 0$ if and only if $\text{ord}_\Delta(n) = -1$. The polar polytope $\Delta^* \subset N_{\mathbb{R}}$ is defined by the condition $\text{ord}_\Delta(x) \geq -1$. Therefore, we obtain $\text{Supp}(\Delta^{FI}) \subset \Delta^*$. Since 0 is an interior lattice point of Δ one has $0 > \text{ord}_\Delta(n) \in \mathbb{Z}$ for any nonzero lattice vector $n \in N$. In particular, one has $\text{ord}_\Delta(n) = -1$ for any nonzero lattice point $n \in \Delta^*$, i.e., $\{\Delta^* \cap N\} \setminus \{0\} \subset \text{Supp}(\Delta^{FI})$. \square

Theorem 2.26. *A minimal model of a Δ -nondegenerate affine hypersurface $Z_\Delta \subset \mathbb{T}_d$ is birational to a Calabi-Yau variety X' with at worst \mathbb{Q} -factorial Gorenstein terminal singularities if and only if the Fine interior of Δ is 0.*

Proof. By Theorem 2.23, it remains to explain how to construct a maximal projective crepant partial resolution $\rho' : X' \rightarrow X$. We consider the finite set $\text{Supp}(\Delta^{FI}) = \{v_1, \dots, v_r\} \subset N$ consisting of all nonzero lattice points in $\Delta^* \subset N_{\mathbb{R}}$. (see 2.25). Denote by Σ the fan of cones over all faces of the lattice polytope $[\Delta^*]$ obtained as convex hull of all lattice points in $\text{Supp}(\Delta^{FI})$. The fan Σ admits a maximal simplicial projective subdivision Σ' which consists of simplicial cones whose generators are nonzero lattice vectors in Δ^* . Thus we obtain a projective crepant toric morphism $\rho' : \mathbb{P}_{\Sigma'} \rightarrow \mathbb{P}_{\Sigma}$. Since Σ is the normal fan to the polytope $\Delta^{can} = [\Delta^*]^*$, the morphism ρ' induces a projective crepant morphism of Calabi-Yau varieties $\rho' : X' \rightarrow X$, where X' is the Zariski closure of Z_Δ in $\mathbb{P}_{\Sigma'}$. Since the toric singularities of $\mathbb{P}_{\Sigma'}$ are \mathbb{Q} -factorial and terminal, the same is true for the singularities of X' . \square

3. The Mavlyutov duality

In [Mav11] Mavlyutov has proposed a generalization the Batyrev-Borisov duality [BB97]. In particular, his generalization includes the polar duality for reflexive polytopes [Bat94]. We reformulate the ideas of Mavlyutov about Calabi-Yau hypersurfaces in toric varieties in some equivalent more convenient form.

For simplicity we denote by $[P]$ the convex hull $\text{Conv}(P \cap \mathbb{Z}^d)$ for any subset $P \subset \mathbb{R}^d$. As above, we denote by P^* the polar set of P if 0 is an interior lattice point of P .

Let $\Delta \subset M_{\mathbb{R}}$ be d -dimensional lattice polytope such that the Fine interior of Δ is zero, i.e., $\Delta^{FI} = 0 \in M$. By 2.25, the support of the Fine interior $\text{Supp}(\Delta^{FI})$ is equal to the set of nonzero lattice points in the polar polytope $\Delta^* \subset N_{\mathbb{R}}$ and the zero lattice point $0 \in N$ is the single interior lattice point of $[\Delta^*]$. Therefore, the inclusion $[\Delta^*] \subseteq \Delta^*$ implies the inclusions

$$\Delta = (\Delta^*)^* \subseteq [(\Delta^*)^*] = \Delta^{can}$$

and

$$\Delta \subseteq [[\Delta^*]^*] = [\Delta^{can}],$$

because Δ is a lattice polytope.

Definition 3.1. We call a d -dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ with $\Delta^{FI} = 0$ *pseudoreflexive* if one has the equality

$$\Delta = [[\Delta^*]^*].$$

Remark 3.2. The above definition of pseudoreflexive polytopes has been discovered by Mavlyutov in 2004 (see [Mav05, Remark 4.7]). In the paper [Mav11] Mavlyutov called these polytopes \mathbb{Z} -*reflexive* (or *integrally reflexive*). Independently, this definition has been discovered by Kreuzer [Kr08, Definition 3.11] who called such polytopes *IPC-closed*.

Remark 3.3. Every reflexive polytope Δ is pseudoreflexive, because for reflexive polytopes Δ we have $\Delta = [\Delta]$ and $\Delta^* = [\Delta^*]$. The converse is not true if $\dim \Delta \geq 5$. For instance the convex hull $\text{Conv}(e_0, e_1, \dots, e_5)$ of the standard basis e_1, \dots, e_5 in \mathbb{Z}^5 and the lattice vector $e_0 = -e_1 - e_2 - e_3 - e_4 - 2e_5$ is a 5-dimensional pseudoreflexive simplex which is not reflexive.

There exist a close connection between lattice polytopes Δ with $\Delta^{FI} = 0$ and pseudoreflexive polytopes:

Proposition 3.4. *Let $\Delta \subset M_{\mathbb{R}}$ a d -dimensional lattice polytope. Then the following conditions are equivalent:*

- (i) $\Delta^{FI} = 0$;
- (ii) *the polytopes Δ and $[\Delta^*]$ contain 0 in their interior;*
- (iii) *Δ contains 0 in its interior and Δ is contained in a pseudoreflexive polytope.*

Proof. (i) \Rightarrow (ii). Assume that $\Delta^{FI} = 0$. Then 0 is an interior lattice point of Δ and the support of the Fine interior $\text{Supp}(\Delta^{FI})$ is exactly the set of nonzero lattice points in the polar polytope Δ^* . Moreover, one has

$$0 = \{x \in M_{\mathbb{R}} : \langle x, v \rangle \geq 0 \quad \forall v \in \text{Supp}(\Delta^{FI})\}.$$

Hence, $[\Delta^*]$ also contains 0 in its interior.

(ii) \Rightarrow (i). If Δ contains 0 in its interior, then $0 \in \Delta^{FI}$. For any nonzero lattice point $v \in \Delta^*$ the minimum of $\langle *, v \rangle$ on Δ equals -1 . If $[\Delta^*]$ contains 0 in its interior, then there exists lattice points $v_1, \dots, v_l \in [\Delta^*]$ generating $M_{\mathbb{R}}$ such that for some positive numbers λ_i ($1 \leq i \leq l$) one has

$$\lambda_1 v_1 + \dots + \lambda_l v_l = 0.$$

On the other hand, Δ^{FI} is contained in the intersection of the half-spaces $\langle x, v_i \rangle \geq 0$ $1 \leq i \leq l$. Therefore, one has $\Delta^{FI} = 0$.

(iii) \Rightarrow (ii). Assume that Δ is contained in a pseudoreflexive lattice polytope $\tilde{\Delta}$. Then we obtain the inclusions $\tilde{\Delta}^* \subseteq \Delta^*$ and $[\tilde{\Delta}^*] \subseteq [\Delta^*]$. Since $\tilde{\Delta}$ is pseudoreflexive, its Fine interior is zero and it follows from (i) \Rightarrow (ii) that $[\tilde{\Delta}^*]$ contains 0 in its interior. Therefore, the lattice polytope $[\Delta^*]$ also contains 0 in its interior.

(i) \Rightarrow (iii). Assume that $\Delta^{FI} = 0$. Then we obtain the inclusion $\Delta \subseteq [[\Delta^*]^*]$. It is sufficient to show that $[[\Delta^*]^*]$ is pseudoreflexive. The latter follows from the equality $[[[\Delta^*]^*]^*] = [\Delta^*]$. Indeed, the inclusion $\Delta \subseteq [[\Delta^*]^*]$ implies the inclusions $[[\Delta^*]^*]^* \subseteq \Delta^*$ and $[[[\Delta^*]^*]^*] \subseteq [\Delta^*]$. On the other hand, the Fine interior of $[\Delta^*]$ is also zero, because Δ contains 0 in its interior. This implies the opposite inclusion $[\Delta^*] \subseteq [[[\Delta^*]^*]^*]$. \square

Corollary 3.5. *Let $\Delta \subset M_{\mathbb{R}}$ be a d -dimensional lattice polytope with $\Delta^{FI} = 0$. Then the following statements hold.*

- (i) *The lattice polytopes $[\Delta^*]$ and $[[\Delta^*]^*]$ are pseudoreflexive;*
- (ii) *$[[\Delta^*]^*]$ is the smallest pseudoreflexive polytope containing Δ .*

Proof. The statement (i) follows from the equality $[[[\Delta^*]^*]^*] = [\Delta^*]$ in the proof of 3.4. If $\tilde{\Delta}$ is a pseudoreflexive polytope containing Δ , then the inclusion $\Delta \subseteq \tilde{\Delta}$ implies the sequence of inclusions $\tilde{\Delta}^* \subseteq \Delta^*$, $[\tilde{\Delta}^*] \subseteq [\Delta^*]$, $[\Delta^*]^* \subseteq [\tilde{\Delta}^*]^*$, $[[\Delta^*]^*] \subseteq [[\tilde{\Delta}^*]^*] = \tilde{\Delta}$. This implies (ii). \square

The statements in 3.4 and 3.5 motivate another names for lattice polytopes Δ with $\Delta^{FI} = 0$:

Definition 3.6. A d -dimensional lattice polytope is called *almost pseudoreflexive* if $\Delta^{FI} = 0$. If Δ is almost pseudoreflexive then we call the lattice polytope $[[\Delta^*]^*]$ the *pseudoreflexive closure* of Δ and the lattice polytope $[\Delta^*]$ the *pseudoreflexive dual* of Δ . The polytope Δ is called *almost reflexive* if its pseudoreflexive closure $[[\Delta^*]^*]$ (or, equivalently, its pseudoreflexive dual $[\Delta^*]$) is reflexive. In this case, we will call the lattice polytope $[[\Delta^*]^*] = [\Delta^*]^* = \Delta^{can}$ also the *canonical reflexive closure* of Δ .

Example 3.7. The 3-dimensional lattice polytope Δ obtained as the convex hull of $(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -2) \in \mathbb{Z}^3$ is a 3-dimensional almost reflexive simplex which is not reflexive. The canonical reflexive closure $[[\Delta^*]^*] = [\Delta^*]^*$ of Δ is a reflexive lattice polytope obtained from Δ by adding one more vertex $(0, 0, -1)$.

Remark 3.8. If Δ is pseudoreflexive, then $\Delta^{\vee} := [\Delta^*]$ is also pseudoreflexive. In particular, one obtains a natural duality $\Delta \leftrightarrow \Delta^{\vee}$ for pseudoreflexive polytopes that generalizes the polar duality for reflexive polytopes. This duality was suggested by Mavlyutov in [Mav11] for unifying different combinatorial mirror constructions.

Remark 3.9. Unfortunately, almost pseudoreflexive polytopes Δ do not have a natural duality, although they appear in the Berglund-Hübsch-Krawitz mirror construction. Nevertheless, the pseudoreflexive duals $[\Delta^*]$ of almost pseudoreflexive polytopes Δ allow to connect the Mavlyutov duality with the Berglund-Hübsch-Krawitz mirror construction. For instance, it may happen that two different almost pseudoreflexive polytopes $\Delta_1 \neq \Delta_2$ have the same pseudoreflexive duals, i.e., $[\Delta_1^*] = [\Delta_2^*]$. This equality is the key observation for the birationality of BHK-mirrors investigated in [Ke13, Cla14, Sh14].

Definition 3.10. Let Θ be a k -dimensional face of a d -dimensional pseudoreflexive polytope $\Delta \subset M_{\mathbb{R}}$. We call Θ *ordinary* if the following equality holds:

$$\left(\bigcap_{l \in \mathbb{Z}_{\geq 0}} l\Theta \right) \cap M = \mathbb{R}_{\geq 0}\Theta \cap M,$$

in other words, if all lattice points in the $(k+1)$ -dimensional cone $\sigma_{\Theta} = \mathbb{R}_{\geq 0}\Theta$ over the face $\Theta \prec \Delta$ are contained in the multiples $l\Theta$ ($l \in \mathbb{Z}_{\geq 0}$).

Proposition 3.11. *Let $\Theta \prec \Delta$ be a k -dimensional face of a lattice polytope Δ with $\Delta^{FI} = 0$. Assume that $[\Theta^*]$ is nonempty. Then Θ is ordinary and $[\Theta^*]$ is a face of dimension $\leq d - 1 - k$ of the pseudoreflexive polytope $[\Delta^*]$.*

Proof. Let $x \in \Theta$ be a point in the relative interior of Θ . The minimum of the linear function $\langle x, * \rangle$ on Δ^* equals -1 and it is attained exactly on the polar face $\Theta^* \prec \Delta^*$ of the rational polar polytope Δ^* .

The minimum μ_x of $\langle x, * \rangle$ on the lattice polytope $[\Delta^*]$ is attained on some lattice face $F \prec [\Delta^*]$ of the lattice polytope $[\Delta^*]$ such that $F := \{y \in [\Delta^*] : \langle x, y \rangle = \mu_x\}$. The minimum μ_x must be at least -1 , because the polytope $[\Delta^*]$ is contained in Δ^* . If $[\Theta^*]$ is not empty then the linear function $\langle x, * \rangle$ has the constant value -1 on $[\Theta^*]$. This implies that the minimum μ_x must be -1 . Therefore F must be contained in Θ^* and $F = [F] \subseteq [\Theta^*]$. Since $[\Theta^*] \subseteq \{y \in [\Delta^*] : \langle x, y \rangle = \mu_x = -1\} = F$, we conclude $F = [\Theta^*]$. \square

The next statement is a slight generalization of the results of Skarke in [Sk96].

Theorem 3.12. [Mav13] *Let Θ is a face of dimension $k \leq 3$ of a d -dimensional pseudoreflexive polytope Δ . Then Θ is ordinary. In particular, any pseudoreflexive lattice polytope Δ of dimension ≤ 4 is reflexive.*

Proof. Consider the $(k+1)$ -dimensional subspace $L := \mathbb{R}\Theta$ generated by Θ . Then Θ is contained in the k -dimensional affine hyperplane H_{Θ} in L . It is enough to show that the integral distance between H_{Θ} and 0 equals 1 . Assume

that this distance is larger than 1. Consider the pyramid $\Pi_\Theta := \text{Conv}(\Theta, 0)$. By lemma of Skarke [Sk96, Lemma 1], there exists an interior lattice point $u_0 \in M$ in $2\Pi_\Theta$ which is not contained in Π_Θ . Therefore, the lattice point u_0 is an interior lattice point in the polytope $2\Delta \subseteq 2[\Delta^*]^*$. If $\{v_1, \dots, v_l\} \subset N$ is the set of vertices of $[\Delta^*]$ then the polytope $[\Delta^*]^*$ is determined by the inequalities $\langle x, v_i \rangle \geq -1$ ($1 \leq i \leq l$). The interior lattice point $u_0 \in 2[\Delta^*]^*$ must satisfy the inequalities $\langle u_0, v_i \rangle > -2$ ($1 \leq i \leq l$). Since $\langle u_0, v_i \rangle \in \mathbb{Z}$ ($1 \leq i \leq l$), we obtain $\langle u_0, v_i \rangle \geq -1$ ($1 \leq i \leq l$), i.e., u_0 is a nonzero lattice point in $[\Delta^*]^*$. Since $\Delta = [[\Delta^*]^*]$, u_0 is a nonzero lattice point contained in Δ and in the k -dimensional cone $\mathbb{R}_{\geq 0}\Theta$ over Θ . On the other hand, $\Delta \cap \mathbb{R}_{\geq 0}\Theta = \Theta \subset \Pi_\Theta$. Contradiction. \square

Proposition 3.13. [Mav13] *Let $\Theta \prec \Delta$ be an ordinary k -dimensional face of a pseudoreflexive polytope Δ such $\dim[\Theta^*] = \dim \Theta^* = d - 1 - k \geq 0$. Then one has $[[\Theta^*]^*] = \Theta$.*

Proof. If $\dim \Theta^* = \dim[\Theta^*] = d - k - 1$ then there exists a point $y \in \Delta^*$ which is contained in the relative interior of $[\Theta^*]$ and in the relative interior of Θ^* . In particular, $y \in \Theta^*$ is contained in the relative interior of the $(d - k)$ -dimensional normal cone σ^Θ and therefore the minimum of the linear function $\langle *, y \rangle$ on Δ equals -1 and it is attained on $\Theta = \{x \in \Delta : \langle x, y \rangle = -1\}$. By definition of the polar polytope $[\Delta^*]^*$, the minimum of $\langle *, y \rangle$ on $[\Delta^*]^*$ also equals -1 , and it is attained on the k -dimensional dual face $[\Theta^*]^* \prec [\Delta^*]^*$. Hence, $[\Theta^*]^*$ contains the lattice face Θ and $[[\Theta^*]^*]$ also contains Θ . By 3.11, the lattice polytope $[[\Theta^*]^*]$ is face of $[[\Delta^*]^*] = \Delta$ of dimension $\leq k$. Since $[[\Theta^*]^*]$ contains the k -dimensional face $\Theta \prec \Delta$, the face $[[\Theta^*]^*] \prec \Delta$ must be Θ . \square

Definition 3.14. We call a k -dimensional face Θ of a pseudoreflexive polytope $\Delta \subset M_{\mathbb{R}}$ *regular*, if

$$\dim[\Theta^*] = d - k - 1.$$

A k -dimensional face Θ is called *singular* if it is not regular.

By 3.13, we immediately obtain:

Corollary 3.15. *Let $\Delta \subset M_{\mathbb{R}}$ be a d -dimensional pseudoreflexive polytope. Then there exists a natural bijection $\Theta \leftrightarrow \Theta^\vee := [\Theta^*]$ between the set of k -dimensional regular faces of Δ and $(d - k - 1)$ -dimensional regular faces of Δ^* .*

Remark 3.16. Pseudoreflexive lattice polytopes Δ satisfy a combinatorial duality $\Delta \leftrightarrow \Delta^\vee$ that extends the polar duality for reflexive lattice polytopes. However, in contrast to polar duality for reflexive polytopes there is no natural bijection between arbitrary k -dimensional faces of a pseudoreflexive polytope Δ and $(d - k - 1)$ -dimensional faces of its dual Δ^\vee . Such a natural bijection exists only for regular k -dimensional faces $\Theta \subset \Delta$.

Remark 3.17. By 3.11, every regular face $\Theta \prec \Delta$ is ordinary. It is easy to see that for a $(d - 1)$ -dimensional face $\Theta \prec \Delta$ the following conditions are equivalent:

- (i) Θ is regular;
- (ii) Θ is ordinary;
- (iii) the integral distance from $0 \in M$ to Θ is 1.

Remark 3.18. Reflexive polytopes of dimension 3 and 4 have been classified by Kreuzer and Skarke [KS98, KS00]. It is natural task to extend these classifications to lattice polytopes with Fine interior 0. By 3.4, a lattice polytope Δ of dimension 3 or 4 has Fine interior 0 if and only if Δ contains 0 in its interior and Δ is contained in some reflexive polytope Δ' .

All 3-dimensional lattice polytopes with the single interior lattice point 0 have been classified by Kasprzyk [Kas10]. There exists exactly 674,688 3-dimensional lattice polytopes Δ with only a single interior lattice point. However, not all these polytopes Δ have Fine interior 0. I was informed by Kasprzyk that among these 674,688 lattice polytopes there exist exactly 9,089 lattice polytopes whose Fine interior has dimension ≥ 1 . These polytopes correspond to elliptic surfaces, Todorov surfaces and some other interesting algebraic surfaces.

According to Kreuzer und Skarke [KS98, KS00], there exist exactly 4,319 3-dimensional reflexive polytopes. We remark that canonical models of $K3$ -surfaces coming from 3-dimensional reflexive polytopes have at worst toroidal quotient singularities of type A_n . However, the canonical models of $K3$ -surfaces coming from 3-dimensional lattice polytopes Δ with the weaker condition $\Delta^{FI} = 0$ may have more general Gorenstein canonical singularities of types D_n and E_n .

Analogously, we remark that canonical singularities of 3-dimensional Calabi-Yau varieties obtained as hypersurfaces in 4-dimensional Gorenstein toric Fano varieties defined by 4-dimensional reflexive polytopes are toroidal. They admit smooth crepant resolutions, because any 3-dimensional \mathbb{Q} -factorial terminal Gorenstein toric variety is smooth. Singularities of 3-dimensional Calabi-Yau varieties X obtained as minimal models of Δ -nondegenerate hypersurfaces with $\Delta^{FI} = 0$ generally can not be resolved crepantly, because

\mathbb{Q} -factorial Gorenstein terminal singularities in dimension 3 are cDV -points that may cause that the stringy Euler number of X will be a rational number [DR01]. So the classification of 4-dimensional lattice polytopes Δ with Fine interior 0 would give many new examples of 3-dimensional Calabi-Yau varieties with isolated terminal cDV -points that need additionally to be smoothed by a deformation [Na94] in order to get a smooth Calabi-Yau 3-fold.

It would be very interesting to know what rational numbers can appear as stringy Euler numbers of minimal 3-dimensional Calabi-Yau varieties coming from 4-dimensional lattice polytopes Δ with $\Delta^{FI} = 0$.

4. The stringy Euler number

Definition 4.1. If V is a smooth projective algebraic variety over \mathbb{C} , then its *E-polynomial* (or *Hodge polynomial*) is defined as

$$E(V; u, v) := \sum_{0 \leq p, q \leq \dim V} (-1)^{p+q} h^{p,q}(V) u^p v^q,$$

where $h^{p,q}(V)$ are Hodge numbers of V .

For any quasi-projective variety W one can use the mixed Hodge structure in k -th cohomology group $H_c^k(W)$ with compact supports and define $E(W; u, v)$ by the formula

$$E(W; u, v) := \sum_{p,q} e^{p,q}(W) u^p v^q,$$

where the coefficients

$$e^{p,q}(W) = \sum_k (-1)^k h^{p,q}(H_c^k(W))$$

are called *Hodge-Deligne numbers* of W [DKh86].

Definition 4.2. Let X be a normal projective variety over \mathbb{C} with at worst \mathbb{Q} -Gorenstein log-terminal singularities. Denote by r the minimal positive integer such that rK_X is a Cartier divisor. Let $\rho : Y \rightarrow X$ be a log-desingularization together with smooth irreducible divisors D_1, \dots, D_k with simple normal crossings whose support covers the exceptional locus of ρ . We can uniquely write

$$K_Y = \rho^* K_X + \sum_{i=1}^k a_i D_i.$$

for some rational numbers $a_i \in \frac{1}{r}\mathbb{Z}$ satisfying the additional condition $a_i = 0$ if D_i is not in the exceptional locus of ρ . We set $I := \{1, \dots, k\}$ and, for any $\emptyset \subseteq J \subseteq I$, we define

$$D_J := \begin{cases} Y & \text{if } J = \emptyset, \\ \bigcap_{j \in J} D_j & \text{if } J \neq \emptyset, \end{cases} \quad D_J^\circ := D_J \setminus \bigcup_{j \in I \setminus J} D_j.$$

$$E_{\text{str}}(X; u, v) := \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1} \right) \cdot E(D_J^\circ; u, v)$$

$$= \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1} - 1 \right) \cdot E(D_J; u, v),$$

where $E(D_J; u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q}(D_J) u^p v^q$ is the E -polynomial of the smooth projective variety D_J . The rational function $E_{\text{str}}(X; u, v)$ is called *stringy E -function* of the algebraic variety X .

Let $x \in X$ be a point on X . We define the *local stringy E -function of X at $x \in X$* by the formula

$$E_{\text{str}}(X, x; u, v) := \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1} - 1 \right) \cdot E(\rho^{-1}(x) \cap D_J; u, v).$$

In particular, we define the *local stringy Euler number of X at point $x \in X$* as

$$e_{\text{str}}(X, x) := \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{-a_j}{a_j + 1} - 1 \right) \cdot e(\rho^{-1}(x) \cap D_J).$$

Our goal is to derive a combinatorial formula for the stringy E -function $E_{\text{str}}(X; u, v)$ of a minimal Calabi-Yau model X of an affine Δ -nondegenerate hypersurface $Z \subset \mathbb{T}_d$ corresponding to a d -dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ such that $\Delta^{FI} = 0$. For this purpose, we need a rational function $R(C, m, t)$ associated with an arbitrary d -dimensional rational polyhedral cone $C \subset N_{\mathbb{R}}$ and a primitive lattice point $m \in M$ in the interior of the dual cone $C^* \subset M_{\mathbb{R}}$.

Definition 4.3. Let $C \subset N_{\mathbb{R}}$ be an arbitrary d -dimensional rational polyhedral cone with vertex $0 = C \cap (-C)$ and let $m \in M$ be a primitive lattice point such that $C(1) := \{y \in C : \langle m, y \rangle \leq 1\}$ is a d -dimensional compact polytope with rational vertices. Let C° be the interior of the cone C . We define two power series

$$R(C, m, t) := \sum_{n \in C \cap N} t^{(m, n)}$$

and

$$R(C^\circ, m, t) := \sum_{n \in C^\circ \cap N} t^{\langle m, n \rangle}.$$

Example 4.4. If $M = N = \mathbb{Z}^d$, $C = \mathbb{R}_{\geq 0}^d \subset \mathbb{R}^d$ and $m = (1, \dots, 1)$. Then $C(1)$ is a d -dimensional simplex in \mathbb{R}^d defined by the conditions $x_i \geq 0$ ($1 \leq i \leq d$), $\sum_{i=1}^d x_i \leq 1$. We have

$$R(C, m, t) = \left(\sum_{k=0}^{\infty} t^k \right)^d = \frac{1}{(1-t)^d}$$

and

$$R(C^\circ, m, t) = \left(\sum_{k=1}^{\infty} t^k \right)^d = \frac{t^d}{(1-t)^d}$$

Proposition 4.5. *The power series $R(C, m, t)$ and $R(C^\circ, m, t)$ are rational functions satisfying the equation*

$$R(C, m, t) = (-1)^d R(C^\circ, -m, t).$$

Moreover, two limits

$$\lim_{t \rightarrow 1} (1-t)^d R(C, m, t), \quad \lim_{t \rightarrow 1} (t-1)^d R(C^\circ, -m, t)$$

equal the integral volume $v(C(1)) = d! \text{Vol}(C(1))$, where $\text{Vol}(C(1))$ denotes the usual volume of the d -dimensional compact set $C(1)$.

Proof. First we remark that $R(C, m, t)$ is a rational function, because $R(C, m, t)$ can be considered as the Poincaré series of the graded finitely graded commutative semigroup algebra $\mathbb{C}[C \cap N]$ such that the degree of an element $n \in C \cap N$ equals $\langle m, n \rangle$. The function $R(C^\circ, m, t)$ is also rational, because it is the Poincaré series of a graded homogeneous ideal in $\mathbb{C}[C \cap N]$.

In order to explicitly compute the rational functions $R(C, m, t)$ and $R(C^\circ, m, t)$ we use a regular simplicial subdivision of the cone C defined by a finite fan $\Sigma = \{\sigma\}$ consisting of cones σ generated by parts of \mathbb{Z} -bases of N . Denote by σ° the relative interior of a cone $\sigma \in \Sigma$. Then we obtain

$$(4.1) \quad R(C, m, t) = \sum_{\sigma \in \Sigma} R(\sigma^\circ, m, t)$$

and

$$(4.2) \quad R(C^\circ, m, t) = \sum_{\substack{\sigma \in \Sigma \\ \sigma^\circ \subseteq C^\circ}} R(\sigma^\circ, m, t).$$

If $\sigma \in \Sigma$ is a k -dimensional cone, the semigroup $\sigma \cap N$ is freely generated by some elements $v_1, \dots, v_k \in N$ such that $\langle m, v_i \rangle = c_i \in \mathbb{Z}_{>0}$ ($1 \leq i \leq k$). Therefore, we obtain

$$R(\sigma, m, t) = \prod_{i=1}^k \frac{1}{1 - t^{c_i}}$$

and

$$\begin{aligned} R(\sigma^\circ, -m, t) &= \prod_{i=1}^k \frac{t^{-c_i}}{1 - t^{-c_i}} = \prod_{i=1}^k \frac{1}{t^{c_i} - 1} \\ &= (-1)^k \prod_{i=1}^k \frac{1}{1 - t^{c_i}} = (-1)^k R(\sigma, m, t). \end{aligned}$$

In order to prove the equation $R(C, m, t) = (-1)^d R(C^\circ, -m, t)$ for the whole d -dimensional cone C we note that for any $\sigma \in \Sigma$ one has

$$R(\sigma^\circ, m, t) = \sum_{\tau \preceq \sigma} (-1)^{\dim \sigma - \dim \tau} R(\tau, m, t) = \sum_{\tau \preceq \sigma} (-1)^{\dim \sigma} R(\tau^\circ, -m, t).$$

Using the equalities (4.1) and (4.2), we get

$$\begin{aligned} R(C^\circ, m, t) &= \sum_{\substack{\sigma \in \Sigma \\ \sigma^\circ \subseteq C^\circ}} R(\sigma^\circ, m, t) = \sum_{\substack{\sigma \in \Sigma \\ \sigma^\circ \subseteq C^\circ}} \sum_{\tau \preceq \sigma} (-1)^{\dim \sigma} R(\tau^\circ, -m, t) = \\ &= \sum_{\tau \in \Sigma} R(\tau^\circ, -m, t) \sum_{\tau \preceq \sigma \in \Sigma} (-1)^{\dim \sigma} = \\ &= (-1)^d \sum_{\tau \subseteq C} R(\tau^\circ, -m, t) = (-1)^d R(C, -m, t), \end{aligned}$$

because for any cone $\tau \in \Sigma$ one has $\sum_{\tau \preceq \sigma} (-1)^{\dim \sigma} = (-1)^d$.

We note that the limit

$$\lim_{t \rightarrow 1} (1-t)^d R(\sigma^\circ, m, t) = \lim_{t \rightarrow 1} (1-t)^d \prod_{i=1}^k \frac{t^{c_i}}{1 - t^{c_i}}$$

is zero if $k = \dim \sigma < d$. If $\sigma \in \Sigma(d)$ is a d -dimensional cone, then

$$\lim_{t \rightarrow 1} (1-t)^d R(\sigma^\circ, m, t) = \lim_{t \rightarrow 1} (1-t)^d \prod_{i=1}^d \frac{t^{c_i}}{1 - t^{c_i}} = \prod_{i=1}^d \frac{1}{c_i} = d! \text{Vol}(\sigma(1)),$$

because $\sigma(1)$ is a d -dimensional simplex which is the convex hull of vectors $\frac{1}{c_i}v_i$, where v_1, \dots, v_d is a \mathbb{Z} -basis of M . Using (4.1), we get

$$\lim_{t \rightarrow 1} (1-t)^d R(C, m, t) = \sum_{\sigma \in \Sigma} \lim_{t \rightarrow 1} (1-t)^d R(\sigma^\circ, m, t) = \sum_{\sigma \in \Sigma(d)} v(\sigma(1)) = v(C(1)),$$

because

$$\text{Vol}(C(1)) = \sum_{\sigma \in \Sigma(d)} \text{Vol}(\sigma(1)).$$

It follows now from the equation $R(C, m, t) = (-1)^d R(C^\circ, -m, t)$ that

$$\lim_{t \rightarrow 1} (t-1)^d R(C^\circ, -m, t) = v(C(1)). \quad \square$$

The following results of Danilov and Khovanskii allow us to compute the polynomial $E(Z; u, v)$ for any $(d-1)$ -dimensional Δ -nondegenerate affine hypersurface $Z_\Delta \subset T$ [DKh86, Remark 4.6].

Theorem 4.6. *Let $\Delta \subset M_{\mathbb{R}}$ be a d -dimensional lattice polytope. The power series*

$$P(\Delta, t) := \sum_{k=0}^{\infty} |k\Delta \cap M| t^k$$

is a rational function of the form

$$P(\Delta, t) = \frac{\psi_0(\Delta) + \psi_1(\Delta)t + \dots + \psi_d(\Delta)t^d}{(1-t)^{d+1}},$$

where $\psi_i(\Delta)$ ($0 \leq i \leq d$) are nonnegative integers satisfying the conditions $\psi_0(\Delta) = 1$, $\sum_{i=1}^d \psi_i(\Delta) = v(\Delta)$.

Let $E(Z_\Delta; u, 1)$ be the E -polynomial of a $(d-1)$ -dimensional Δ -nondegenerate hypersurface $Z_\Delta \subset \mathbb{T}_d$. Then one has

$$E(Z_\Delta; u, 1) = \frac{(u-1)^d - (-1)^d}{u} + (-1)^{d-1} \sum_{i=1}^d \psi_i(\Delta) u^{i-1}.$$

In particular, the Euler number $e(Z_\Delta) = E(Z_\Delta; 1, 1)$ equals $(-1)^{d-1}v(\Delta)$.

In particular, one obtains

Corollary 4.7. [Kho78] *The Euler number $e(Z_\Delta) = E(Z_\Delta; 1, 1)$ of a $(d-1)$ -dimensional affine Δ -nondegenerate hypersurface Z_Δ equals $(-1)^{d-1}v(\Delta)$.*

Definition 4.8. Let $\Delta \subset M_{\mathbb{R}}$ is an arbitrary d -dimensional almost pseudoreflexive polytope. For any k -dimensional face $\Theta \prec \Delta$ we define the polynomial

$$\begin{aligned} E(\Theta, u) &:= \frac{(u-1)^{\dim \Theta}}{u} + \frac{(u-1)^{\dim \Theta + 1}}{u} \sum_{l \in \mathbb{Z}_{\geq 0}} |l\Theta \cap M| u^l \\ &= \frac{(u-1)^k - (-1)^k}{u} + (-1)^{k-1} \sum_{i=1}^k \psi_i(\Theta) u^{i-1}. \end{aligned}$$

Definition 4.9. Let $\Delta \subset M_{\mathbb{R}}$ be an arbitrary d -dimensional almost pseudoreflexive polytope and let σ^{Θ} be the $(d-k)$ -dimensional cone in the normal fan Σ^{Δ} that correspond to the k -dimensional face $\Theta \prec \Delta$. We choose an arbitrary lattice point $m \in \Theta \cap M$ and set

$$R(\sigma^{\Theta}, u) := R(\sigma^{\Theta}, -m, u)$$

where the rational function $R(\sigma^{\Theta}, -m, u)$ defined in 4.3. It is easy to see that the rational function $R(\sigma^{\Theta}, -m, u)$ does not depend on the choice of the lattice point $m \in \Theta \cap M$.

We prove the following theorem:

Theorem 4.10. *Let Δ be an arbitrary d -dimensional almost pseudoreflexive polytope. Denote by X a canonical $(d-1)$ -dimensional Calabi-Yau model of a Δ -nondegenerate hypersurface $Z_{\Delta} \subset T$ in the d -dimensional algebraic torus T . Then the stringy function $E_{\text{st}}(X; u, 1)$ can be computed as follows:*

$$E_{\text{st}}(X; u, 1) = \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta \geq 1}} E(\Theta, u) \cdot R(\sigma^{\Theta}, u) \cdot (1-u)^{d-\dim \Theta}.$$

Proof. Let $\widehat{\Sigma}$ be a common regular simplicial subdivision of the normal fans Σ^{Δ} and $\Sigma^{\Delta^{can}}$. As in Theorem 2.18, we obtain birational morphisms $\rho_1 : \widehat{Z}_{\Delta} \rightarrow \overline{Z}_{\Delta}$, $\rho_2 : \widehat{Z}_{\Delta} \rightarrow X$ in the diagram

$$\begin{array}{ccc} & \widehat{Z}_{\Delta} & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ \overline{Z}_{\Delta} & \overset{\alpha}{\dashrightarrow} & X \end{array}$$

where \widehat{Z}_{Δ} is a smooth variety obtained as Zariski closure of Z_{Δ} in the smooth projective toric variety \widehat{P} defined by the fan $\widehat{\Sigma}$. For computing the stringy

E -function $E_{\text{str}}(X, u, 1)$ we use the formula (4.2)

$$E_{\text{str}}(X; u, 1) = \sum_{\emptyset \subseteq J \subseteq I} \left(\prod_{j \in J} \frac{u-1}{u^{a_j+1}-1} \right) \cdot E(D_J^\circ; u, 1)$$

where the strata D_J° are intersections of the hypersurface $\widehat{Z}_\Delta \subset \widehat{P}$ with torus orbits corresponding to cones $\sigma \in \widehat{\Sigma}$.

Let $\sigma \in \widehat{\Sigma}(k)$ be a k -dimensional simplicial cone generated by primitive lattice vectors v_{i_1}, \dots, v_{i_k} . Then the relative interior σ° of σ is contained in the relative interior σ_\circ^Θ of some cone σ^Θ of the normal fan Σ^Δ corresponding to a face $\Theta \prec \Delta$, $\dim \Theta \leq d-k$. We set $J := \{i_1, \dots, i_k\}$. By 2.23, the discrepancy coefficients a_j of smooth divisors D_j ($j \in J$) on \widehat{Z}_Δ can be computed by the formula $a_j = -\langle m, v_j \rangle - 1$ ($j \in J$) where $m \in M$ is any lattice point in the face Θ . Since the fiber $\rho_1^{-1}(p)$ of the birational toric morphism ρ_1 over every point $p \in \mathbb{P}_\Delta$ consists of torus orbits, the codimension k stratum D_J° is isomorphic to the product of a torus $(\mathbb{C}^*)^{d-k-\dim \Theta}$ and a Θ -nondegenerate hypersurface $Z_\Theta \subset (\mathbb{C}^*)^{\dim \Theta}$. By 4.6 and 4.8 we have

$$E(D_J^\circ; u, 1) = E(\Theta, u) \cdot (u-1)^{d-k-\dim \Theta}$$

and

$$\begin{aligned} & \left(\prod_{j \in J} \frac{u-1}{u^{a_j+1}-1} \right) \cdot E(D_J^\circ; u, 1) \\ &= \left(\prod_{j \in J} \frac{u-1}{u^{-\langle m, v_j \rangle} - 1} \right) \cdot E(\Theta, u) \cdot (u-1)^{d-k-\dim \Theta} \\ &= (-1)^{\dim \sigma} R(\sigma, -m, u) \cdot E(\Theta, u) \cdot (u-1)^{d-\dim \Theta}. \end{aligned}$$

Using 4.5, we obtain $(-1)^{\dim \sigma} R(\sigma, -m, u) = R(\sigma^\circ, m, u)$ and

$$\sum_{\substack{\sigma \in \widehat{\Sigma} \\ \sigma^\circ \subseteq \sigma_\circ^\Theta}} R(\sigma^\circ, m, u) = R(\sigma_\circ^\Theta, m, u) = (-1)^{\dim \sigma^\Theta} R(\sigma^\Theta, -m, u).$$

Since $\dim \sigma^\Theta = d - \dim \Theta$, we conclude

$$\begin{aligned} E_{\text{str}}(X; u, 1) &= \sum_{\Theta \preceq \Delta} E(\Theta, u) \cdot (u-1)^{d-\dim \Theta} \cdot \sum_{\substack{\sigma \in \widehat{\Sigma} \\ \sigma^\circ \subseteq \sigma_\circ^\Theta}} R(\sigma^\circ, m, u) \\ &= \sum_{\Theta \preceq \Delta} E(\Theta, u) \cdot (1-u)^{d-\dim \Theta} R(\sigma^\Theta, -m, u). \quad \square \end{aligned}$$

Theorem 4.11. *Let $\Delta \subset M_{\mathbb{R}}$ be an arbitrary d -dimensional almost pseudoreflexive polytope. Denote by X a canonical Calabi-Yau model of a Δ -nondegenerate affine hypersurface $Z_{\Delta} \subset \mathbb{T}_d$ in the d -dimensional algebraic torus \mathbb{T}_d . Then*

$$e_{\text{st}}(X) = \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta \geq 1}} (-1)^{\dim \Theta - 1} v(\Theta) \cdot v(\sigma^{\Theta} \cap \Delta^*),$$

where σ^{Θ} is the cone in the normal fan of the polytope Δ and $\Delta^* \subset N_{\mathbb{R}}$ is the polar polytope of Δ .

Proof. One has

$$e_{\text{st}}(X) = \lim_{u \rightarrow 1} E_{\text{st}}(X; u, 1) = \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta \geq 1}} E(\Theta, 1) \lim_{u \rightarrow 1} R(\sigma^{\Theta}, u) \cdot (1 - u)^{d - \dim \Theta}.$$

It remains to apply Corollary 4.7

$$E(\Theta, 1) = (-1)^{\dim \Theta - 1} v(\Theta)$$

and Proposition 4.5

$$\lim_{u \rightarrow 1} R(\sigma^{\Theta}, u) \cdot (1 - u)^{d - \dim \Theta} = v(\sigma^{\Theta} \cap \Delta^*). \quad \square$$

Remark 4.12. It is easy to see that the formula for the stringy Euler number in Theorem 4.11 is a generalization of the formula (1.1) in the case when Δ is a reflexive polytope. If $\Theta \prec \Delta$ is a $(d - k)$ -dimensional face of reflexive polytope Δ then the k -dimensional polytope $\sigma^{\Theta} \cap \Delta^*$ is a lattice pyramid with height 1 over the $(k - 1)$ -dimensional dual face Θ^* of the polar reflexive polytope Δ^* . Therefore, $v(\sigma^{\Theta} \cap \Delta^*) = v(\Theta^*)$. On the other hand, the d -dimensional reflexive polytope Δ is the union of d -dimensional pyramids over all $(d - 1)$ -dimensional faces $\Theta \prec \Delta$. So we have

$$v(\Delta) = \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta = d - 1}} v(\Theta).$$

Thus, we obtain

$$\sum_{\substack{\Theta \prec \Delta \\ \dim \Theta \geq 1}} (-1)^{\dim \Theta - 1} v(\Theta) \cdot v(\sigma^{\Theta} \cap \Delta^*) = \sum_{k=1}^{d-2} (-1)^{k-1} \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta = k}} v(\Theta) \cdot v(\Theta^*).$$

We consider below several examples illustrating applications of our formula to non-reflexive polytopes Δ .

Example 4.13. The Newton polytope Δ of a general 3-dimensional quintic X in \mathbb{P}^4 containing the point $(1 : 0 : 0 : 0 : 0) \in \mathbb{P}^4$ is the almost reflexive polytope

$$\Delta = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}_{\geq 0}^4 : 1 \leq x_1 + x_2 + x_3 + x_4 \leq 5\},$$

which is a set-theoretic difference of two 4-dimensional simplices. The Fine interior of Δ consists of the single lattice point $p = (1, 1, 1, 1)$. The integral distance between p and the 3-dimensional face Θ defined by the equation $x_1 + x_2 + x_3 + x_4 = 1$ equals 3. Therefore, Δ is not a reflexive polytope. One has $v(\Delta) = 5^4 - 1^4 = 624$. The polytope Δ has 6 faces Θ of codimension 1 (4 faces Θ with $v(\Theta) = 5^3 - 1^3 = 124$, one face Θ with $v(\Theta) = 5^3$ and one face Θ with $v(\Theta) = 1^3$). There are also 14 faces of dimension 2 and 16 faces of dimension 1 in Δ . Our formula for the stringy Euler number gives:

$$\begin{aligned} e_{\text{str}}(X) &= -(5^4 - 1^4) + 4(5^3 - 1^3) + 5^3 + 1^3 \cdot \frac{1}{3} \\ &\quad - 4 \cdot 5^2 - 6 \cdot (5^2 - 1^2) - 4 \cdot 1^2 \cdot \frac{1}{3} + 6 \cdot 5 \\ &\quad + 4 \cdot (5 - 1) + 6 \cdot 1 \cdot \frac{1}{3} = -200. \end{aligned}$$

This is a well-known fact, since X is a smooth quintic 3-fold.

Example 4.14. The Newton polytope Δ of a general 3-dimensional quintic in \mathbb{P}^4 having an isolated quadratic (conifold) singularity at point $x = (1 : 0 : 0 : 0 : 0) \in \mathbb{P}^4$ is the almost pseudoreflexive polytope

$$\Delta = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}_{\geq 0}^4 : 2 \leq x_1 + x_2 + x_3 + x_4 \leq 5\},$$

which is again a set-theoretic difference of two 4-dimensional simplices. The Fine interior of Δ consists of the single lattice point $p = (1, 1, 1, 1)$. The integral distance between p and the 3-dimensional face Θ defined by the equation $x_1 + x_2 + x_3 + x_4 = 2$ equals 2. Therefore, Δ is not a reflexive polytope. One has $v(\Delta) = 5^4 - 2^4 = 609$. The polytope Δ has 6 faces Θ of codimension 1 (4 faces Θ with $v(\Theta) = 5^3 - 2^3 = 117$, one face Θ with $v(\Theta) = 5^3$ and one face Θ with $v(\Theta) = 2^3$). There are also 14 faces of dimension 2 and 16 faces of dimension 1 in Δ . Our formula for the stringy Euler number gives:

$$\begin{aligned} e_{\text{str}}(X) &= -(5^4 - 2^4) + 4(5^3 - 2^3) + 5^3 + 2^3 \cdot \frac{1}{2} \\ &\quad - 4 \cdot 5^2 - 6 \cdot (5^2 - 2^2) - 4 \cdot 2^2 \cdot \frac{1}{2} + 6 \cdot 5 \\ &\quad + 4 \cdot (5 - 2) + 6 \cdot 2 \cdot \frac{1}{2} = -198. \end{aligned}$$

Unfortunately, this singular Calabi-Yau 3-fold X does not have a projective small resolution of its singularity. So X has no a smooth projective birational Calabi-Yau model.

Remark 4.15. If a log-desingularization $\rho : Y \rightarrow X$ of a projective variety X contains only one smooth exceptional divisor D such that $K_Y = \rho^*K_X + aD$, then

$$e_{\text{st}}(X) = e(Y) + e(D) \left(\frac{-a}{a+1} \right).$$

In particular, $e_{\text{st}}(X)$ is not an integer, if $(a+1)$ does not divide $e(D)$.

Example 4.16. One can generalize Example 4.14 and compute the stringy Euler number of a general d -dimensional Calabi-Yau hypersurface $X' \subset \mathbb{P}^{d+1}$ of degree $d+2$ with a single quadratic singularity at point $x = (1 : 0 : \dots : 0)$. The blow up of this singular point is a desingularization $\rho : \widehat{X}' \rightarrow X'$ such that the exceptional divisor $D \subset \widehat{X}'$ is isomorphic to a $(d-1)$ -dimensional quadric. One has

$$K_{\widehat{X}'} = K_{X'} + (d-2)D.$$

By 4.15, we obtain

$$e_{\text{str}}(X') = e(\widehat{X}') - \frac{d-2}{d-1}e(D).$$

Therefore, the local stringy Euler number of the singular point $x \in X'$ equals

$$e_{\text{str}}(X', x) = e(D) - \frac{d-2}{d-1}e(D) = \frac{e(D)}{d-1}$$

If the dimension $d \geq 4$ is an even number then $e(D) = d$ and $e_{\text{str}}(X') = \frac{c}{d-1} \in \mathbb{Q} \setminus \mathbb{Z}$ for some coprime numbers $c, d-1$. In particular, $e_{\text{str}}(X')$ is not an integer.

So far no mirror manifolds have been known for singular Calabi-Yau varieties X with non-integral stringy Euler number $e_{\text{st}}(X) \in \mathbb{Q} \setminus \mathbb{Z}$.

5. Calabi-Yau hypersurfaces in $\mathbb{P}(a, 1, \dots, 1)$

Let $a, b \in \mathbb{N}$ be two integers $a, b \geq 2$. We put $d := ab + l$ for some integer $1 \leq l \leq a-1$ and consider Calabi-Yau hypersurfaces of degree $a+d$ in the weighted projective space

$$\mathbb{P}(a, 1^d) := \mathbb{P}(a, \underbrace{1, \dots, 1}_d)$$

of dimension $d \geq 5$. The space of quasihomogeneous polynomials of degree $a + d$ in $d + 1$ variables z_0, z_1, \dots, z_d ($\deg z_0 = a$, $\deg z_i = 1$, $1 \leq i \leq d$) has the monomial basis $z_0^{m_0} z_1^{m_1} \cdots z_d^{m_d}$ determined by the lattice points $(m_0, m_1, \dots, m_d) \in \mathbb{Z}_{\geq 0}^{d+1}$ satisfying the condition

$$am_0 + m_1 + \cdots + m_d = a + d.$$

The convex set

$$S_d := \{(x_0, x_1, \dots, x_d) \in \mathbb{R}_{\geq 0}^{d+1} : ax_0 + x_1 + \cdots + x_d = a + d\}.$$

is a d -dimensional simplex having the unique interior lattice point $p := (1, \dots, 1) \in \mathbb{Z}^{n+1}$, d integral vertices $\nu_i = (0, \dots, \underbrace{a+d}_{i}, \dots, 0)$ $1 \leq i \leq d$ and

one rational vertex $\nu_0 := (\frac{a+d}{a}, 0, \dots, 0)$. The convex hull of the set $S_d \cap \mathbb{Z}^{d+1}$ is the lattice polytope Δ which is the intersection of the simplex S_d with the half-space define by the inequality $x_0 \leq b + 1$. The lattice polytope Δ has $2d$ vertices, first d vertices of Δ belong to the hyperplane $x_0 = 0$ and the remaining d vertices of Δ belong to the hyperplane $x_0 = b + 1$. The lattice polytop Δ is not reflexive, because the integral distance between its single interior lattice point p and the $(d - 1)$ -dimensional face in the hyperplane $x_0 = b + 1$ is equal to $b \geq 2$. However, it is easy to show that Δ is a pseudoreflexive polytope. Its dual pseudoreflexive polytope $\Delta^\vee = [\Delta^*]$ is a d -dimensional lattice simplex whose vertices $v_0, v_1, \dots, v_d \in N$ satisfy the relation $av_0 + \sum_{i=1}^d v_i = 0$. The polar polytope $(\Delta^\vee)^*$ can be identified with the rational d -dimensional simplex $S_d \subset \mathbb{R}_{\geq 0}^{d+1}$ in the hyperplane $ax_0 + \sum_{i=1}^d x_i = a + d$. The lattice vectors $v_0, v_1, \dots, v_d \in N$ are exactly the generators of all 1-dimensional cones in the d -dimensional fan describing the weighted projective space $\mathbb{P}(a, 1^d)$ as a d -dimensional toric variety.

The combinatorial structure of the pseudoreflexive polytope Δ is rather simple, because Δ is combinatorially equivalent to the product of $(d - 1)$ -dimensional and 1-dimensional simplices. Its polar polytope Δ^* is simplicial and it has $d + 2$ vertices: the lattice vertices v_0, v_1, \dots, v_d and the rational vertex $v_{d+1} := -\frac{1}{b}v_0 = \frac{1}{ab} \sum_{i=1}^d v_i$. The vertices v_0, v_1, \dots, v_{d-1} can be chosen as a basis of the lattice N .

Remark 5.1. If $l = 1$, i.e., $d = ab + 1$ then the weighted projective space $\mathbb{P}(a, 1^d)$ contains d different quasi-smooth Calabi-Yau hypersurfaces $X_i \subset \mathbb{P}(a, 1^d)$ ($1 \leq i \leq d$) defined respectively by the invertible polynomials

$$F_i(z_0, z_1, \dots, z_d) := z_1^{a+d} + \cdots + z_d^{a+d} + z_0^{b+1} z_i, \quad 1 \leq i \leq d,$$

such that Berglund-Hübsch-Krawitz mirror construction can be applied to every hypersurface X_i ($1 \leq i \leq d$) [BHü93, Kra09].

Proposition 5.2. *Assume that $l = 1$. Then the stringy Euler number $e_{\text{str}}(X)$ of a general quasi-smooth Calabi-Yau hypersurface X in $\mathbb{P}(a, 1^d)$ equals*

$$a - \frac{1}{a} + (-1)^{d-2} \sum_{i=0}^{d-2} (-1)^i \binom{d}{i} (a+d)^{d-1-i} + (-1)^{d-1} \sum_{i=0}^{d-1} (-1)^i \binom{n}{i} \frac{(a+d)^{d-i}}{a}.$$

Proof. The polytope Δ is the difference of two d -dimensional simplices. Therefore, all proper faces Θ of Δ are either simplices or differences of two simplices. For any $1 \leq k \leq d-1$ there exist exactly $2\binom{d}{k-1}$ simplicial $(d-k)$ -dimensional faces of Δ : $\binom{d}{k-1}$ of these $(d-k)$ -dimensional simplicial faces are contained in the hyperplane $x_0 = b+1$ and the other $\binom{d}{k-1}$ simplicial $(d-k)$ -dimensional faces of Δ contained in the hyperplane $x_0 = 0$. There exist exactly $\binom{d}{k}$ nonsimplicial $(d-k)$ -dimensional faces Θ of Δ which are differences of two simplices. A $(d-k)$ -dimensional face $\Theta \prec \Delta$ is singular if and only if it is contained in the hyperplane $x_0 = b+1$, and in this case Θ^* is a simplex with the rational vertex v_{d+1} and for the corresponding rational polytope $\sigma^\Theta \cap \Delta^*$ one has $v(\sigma^\Theta \cap \Delta^*) = 1/b$. For all regular $(d-k)$ -dimensional faces $\Theta \prec_{\text{reg}} \Delta$ one has $v(\sigma^\Theta \cap \Delta^*) = 1$.

Now we can apply the formula (4.11) for computing the stringy Euler number of a generic Calabi-Yau hypersurface $X \subset \mathbb{P}(a, 1^d)$:

$$\begin{aligned} e_{\text{st}}(X) &= (-1)^{d-1} v(\Delta) + \sum_{k=1}^{d-1} (-1)^{d-1-k} \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta = d-k}} v(\Theta) \cdot v(\sigma^\Theta \cap \Delta^*) \\ &= (-1)^{d-1} \left(\frac{(a+d)^d}{a} - \frac{1}{a} \right) \\ &\quad + (-1)^{d-2} \left(\binom{d}{0} \frac{1}{b} + \binom{d}{0} (a+d)^{d-1} + \binom{d}{1} \left(\frac{(a+d)^{d-1}}{a} - \frac{1}{a} \right) \right) \\ &\quad \dots + (-1)^{d-1-k} \left(\binom{d}{k-1} \frac{1}{b} + \binom{d}{k-1} (a+d)^{d-k} \right. \\ &\quad \left. + \binom{d}{k} \left(\frac{(a+d)^{d-k}}{a} - \frac{1}{a} \right) \right) + \dots \\ &\quad + (-1)^0 \left(\binom{d}{d-2} \frac{1}{b} + \binom{d}{d-2} (a+d) + \binom{d}{d-1} \left(\frac{a+d}{a} - \frac{1}{a} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= (-1)^{d-2} \frac{1}{b} \sum_{i=0}^{d-2} (-1)^i \binom{d}{i} + (-1)^d \frac{1}{a} \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} + \\
&\quad + (-1)^{d-2} \sum_{i=0}^{d-2} (-1)^i \binom{d}{i} (a+d)^{d-1-i} \\
&\quad + (-1)^{d-1} \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} \frac{(a+d)^{d-i}}{a} \\
&= a - \frac{1}{a} + (-1)^{d-2} \sum_{i=0}^{d-2} (-1)^i \binom{d}{i} (a+d)^{d-1-i} \\
&\quad + (-1)^{d-1} \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} \frac{(a+d)^{d-i}}{a}. \quad \square
\end{aligned}$$

Proposition 5.3. *For any l ($1 \leq l \leq a-1$) the stringy Euler number $e_{\text{str}}(X^\vee)$ of a canonical Calabi-Yau model X^\vee of a Δ^\vee -nondegenerated affine hypersurface in \mathbb{T}_a defined by the Laurent polynomial*

$$f(x) = x_d^{-a} \prod_{i=1}^{d-1} x_i^{-1} + \sum_{i=1}^d x_i$$

equals

$$(-1)^{d-1} \left(a - \frac{1}{a} \right) - \sum_{i=1}^{d-2} (-1)^i \binom{d}{i} (a+d)^{d-i-1} + \sum_{i=2}^{d-1} (-1)^i \binom{d}{i} \frac{(a+d)^{d-i}}{a}.$$

Proof. For the dual pseudoreflexive polytope Δ^\vee one has $v(\Delta^\vee) = a+d$. All faces $\Theta \preceq \Delta^\vee$ are lattice simplices. A $(n-k)$ -dimensional face $\Theta \prec \Delta^\vee$ is singular if and only if it is contained in the $(d-1)$ -simplex with vertices v_1, \dots, v_d .

Now we apply the formula (4.11) for computing the stringy Euler number of the canonical Calabi-Yau model X^\vee :

$$\begin{aligned}
e_{\text{st}}(X^\vee) &= (-1)^{d-1} v(\Delta^\vee) + \sum_{k=1}^{d-1} (-1)^{d-1-k} \sum_{\substack{\Theta \prec \Delta^\vee \\ \dim \Theta = d-k}} v(\Theta) \cdot v(\sigma^\Theta \cap (\Delta^\vee)^*) = \\
&\quad (-1)^{d-1} (a+d) + (-1)^{d-2} \left(d + \frac{1}{a} \right) + \\
&\quad + (-1)^{d-3} \left(\binom{d}{d-2} (a+d) + \binom{d}{d-1} \frac{(a+d)}{a} \right) +
\end{aligned}$$

$$\begin{aligned}
& +(-1)^{d-4} \left(\binom{d}{d-3} (a+d)^2 + \binom{d}{d-2} \frac{(a+d)^2}{a} \right) + \\
& +(-1)^{d-5} \left(\binom{d}{d-4} (a+d)^3 + \binom{d}{d-3} \frac{(a+d)^3}{a} \right) + \dots \\
& +(-1)^0 \left(\binom{d}{1} (a+d)^{d-2} + \binom{d}{2} \frac{(a+d)^{d-2}}{a} \right) = \\
& =(-1)^{d-1} \left(a - \frac{1}{a} \right) - \\
& - \sum_{i=1}^{d-2} (-1)^i \binom{d}{i} (a+d)^{d-i-1} + \sum_{i=2}^{d-1} (-1)^i \binom{d}{i} \frac{(a+d)^{d-i}}{a}. \quad \square
\end{aligned}$$

Corollary 5.4. *If $l = 1$ then one has*

$$e_{\text{st}}(X) = (-1)^{d-1} e_{\text{st}}(X^\vee).$$

Proof. Comparing the formulas for $e_{\text{st}}(X)$ and $e_{\text{st}}(X^\vee)$ in 5.2 and 5.3, we see that

$$(-1)^{d-1} e_{\text{st}}(X^\vee) - (d+a)^{d-1} + \frac{(a+d)^d}{a} - \binom{d}{1} \frac{(d+a)^{d-1}}{a} = e_{\text{st}}(X).$$

Therefore, we get

$$(-1)^{d-1} e_{\text{st}}(X^\vee) = e_{\text{st}}(X). \quad \square$$

The weighted projective space $V = \mathbb{P}(a, 1^d)$ is a toric variety defined by a simplicial d -dimensional fan whose 1-dimensional cones are generated by lattice vectors v_0, v_1, \dots, v_d satisfying the relation $av_0 + \sum_{i=1}^d v_i = 0$. It is easy to show that again the convex hull of $\{v_0, v_1, \dots, v_d\}$ is a pseudoreflexive simplex Δ^\vee which is dual to Δ . There is a toric desingularization $\rho : V' \rightarrow V$ having one exceptional divisor $E \cong \mathbb{P}^{d-1}$ that corresponds to the lattice point $-v_0$ so that the set of lattice vectors $\{-v_0, v_0, v_1, \dots, v_d\}$ can be identified with the set of inner normal vectors to facets of Δ . The toric desingularization $\rho : V' \rightarrow V$ induces a desingularization $\rho : X' \rightarrow X$ of the generic Calabi-Yau hypersurface $X \subset V$ such that the fiber $D := \rho^{-1}(p)$ over the unique singular point $x \in X$ is isomorphic to a generic hypersurface of degree k in \mathbb{P}^{d-1} . One has

$$K_{X'} = \rho^* K_X + bD,$$

because the linear function φ on the cone $\sum_{i=1}^d \mathbb{R}v_i$ with $\varphi(v_1) = \dots = \varphi(v_d) = 1$ has value $b+1$ on the lattice vector $-v_0 = 1/a \sum_{i=1}^d v_i$.

Theorem 5.5. *Let X be a generic Δ -nondegenerated Calabi-Yau hypersurface in the d -dimensional weighted projective space $\mathbb{P}(a, 1^d)$ ($d = ab + l$, $l \geq 2$) and let X^\vee be a canonical Calabi-Yau model of the Δ^\vee -nondegenerated affine hypersurface $Z \subset \mathbb{T}_d$ defined by the Laurent polynomial*

$$f(x) = x_d^{-a} \prod_{i=1}^{d-1} x_i^{-1} + \sum_{i=1}^d x_i.$$

Then $e_{\text{str}}(X) \in \frac{1}{b}\mathbb{Z}$ and $e_{\text{str}}(X^\vee) \in \frac{1}{a}\mathbb{Z}$. Moreover, if $l = 2$ then $e_{\text{str}}(X)$ is not an integer. In particular, the equality

$$e_{\text{str}}(X) = (-1)^{d-1} e_{\text{str}}(X^\vee)$$

can not be satisfied if $l = 2$ and (a, b) is a pair of distinct odd prime numbers.

Proof. We compute the stringy Euler number of a generic Calabi-Yau hypersurface $X \subset \mathbb{P}(a, 1^d)$ as in 5.2:

$$\begin{aligned} e_{\text{str}}(X) &= (-1)^{d-1} \left(\frac{(a+d)^d}{a} - \frac{l^d}{a} \right) + \\ &+ (-1)^{d-2} \left(\binom{n}{0} l^{d-1} \frac{1}{b} + \binom{d}{0} (a+d)^{d-1} + \binom{d}{1} \left(\frac{(d+a)^{d-1}}{a} - \frac{l^{d-1}}{a} \right) \right) \\ &+ (-1)^{d-3} \left(\binom{n}{1} l^{d-2} \frac{1}{b} + \binom{d}{1} (d+a)^{d-2} + \binom{d}{2} \left(\frac{(d+a)^{d-2}}{a} - \frac{l^{d-2}}{a} \right) \right) + \\ &+ (-1)^{d-4} \left(\binom{d}{2} l^{d-3} \frac{1}{b} + \binom{d}{2} (d+a)^{d-3} + \binom{d}{3} \left(\frac{(d+a)^{d-3}}{a} - \frac{l^{d-3}}{a} \right) \right) + \\ &\quad \dots \\ &+ (-1)^0 \left(\binom{d}{d-2} l \frac{1}{b} + \binom{d}{d-2} (d+a) + \binom{d}{d-1} \left(\frac{d+a}{a} - \frac{l}{a} \right) \right). \end{aligned}$$

Since a divides $(d+a)^i - l^i = (ab+a+l)^i - l^i$ for any $i \in \mathbb{N}$, we obtain that $e_{\text{str}}(X) \in \frac{1}{b}\mathbb{Z}$. The terms in $e_{\text{str}}(X)$ having the denominator b sum up to

$$\begin{aligned} A &:= (-1)^{d-2} \frac{1}{lb} \sum_{i=0}^{d-2} (-1)^i \binom{d}{i} l^{d-i} \\ &= (-1)^{d-2} \frac{1}{lb} \left((l-1)^d - (-1)^d - (-1)^{d-1} dl \right). \end{aligned}$$

In particular, for $l = 2$ and odd integers a, b the dimension $d = ab + 2$ is odd,

$$A = \frac{1-d}{b} \notin \mathbb{Z}$$

and $e_{\text{str}}(X)$ is not an integer.

On the other hand, we did already the computation for $e_{\text{str}}(X^\vee)$ in 5.3 and obtained

$$\begin{aligned} e_{\text{str}}(X^\vee) &= (-1)^{d-1} \left(a - \frac{1}{a} \right) - \sum_{i=1}^{d-2} (-1)^i \binom{d}{i} (a+d)^{d-i-1} \\ &\quad + \sum_{i=2}^{d-1} (-1)^i \binom{d}{i} \frac{(a+d)^{d-i}}{a}. \end{aligned}$$

This shows that $e_{\text{str}}(X^\vee) \in \frac{1}{a}\mathbb{Z}$. Therefore, if a and b two distinct odd prime numbers the equality $e_{\text{str}}(X) = (-1)^{d-1}e_{\text{str}}(X^\vee)$ can hold only if the stringy Euler numbers are integers, but for $l = 2$ this is not the case. \square

6. An additional condition on singular facets

Let $\Delta \subset M_{\mathbb{R}}$ be a d -dimensional pseudoreflexive polytope and let $\Delta^\vee := [\Delta^*]$ be the Mavlyutov dual pseudoreflexive polytope. We consider also two additional d -dimensional almost pseudoreflexive polytopes $\Delta_1 \subseteq \Delta$ and $\Delta_2 \subseteq \Delta^\vee$ such that one has the inclusions

$$\Delta_1 \subseteq [\Delta_1^{\text{can}}] = \Delta,$$

$$\Delta_2 \subseteq [\Delta_2^{\text{can}}] = \Delta^\vee.$$

A generalization of Berglund-Hübsch-Krawitz mirror construction suggested by Artebani, Comparin and Guilbot [ACG16] needs an additional condition that guarantees that the Zariski closure of an affine Δ_1 -nondegenerated hypersurface Z_1 in the toric variety $\mathbb{P}_{\Delta_2^*}$ associated with the rational polar polytope Δ_2^* will be quasi-smooth. The same condition is demanded for the Zariski closure of a Δ_2 -nondegenerated affine hypersurface Z_2 in the toric variety $\mathbb{P}_{\Delta_1^*}$ associated with the rational polar polytope Δ_1^* . The quasi-smoothness condition implies that the singularities of Calabi-Yau hypersurfaces are locally quotient singularities. In particular, the stringy Euler number of such Calabi-Yau hypersurfaces is always an integer.

In [Bor13, Def. 7.1.1, Prop. 7.1.3] Borisov suggested to generalize the quasi-smoothness condition using some versions of Jacobian rings. It is not

quite clear how Borisov's condition can be described by purely combinatorial properties of convex polytopes, but it is satisfied in two cases: 1) for reflexive polytopes and 2) for almost pseudoreflexive simplices Δ_1 and Δ_2 that appear in the Berglund-Hübsch-Krawitz mirror construction.

Our purpose is to describe a new another condition on Calabi-Yau varieties X and X^\vee that must be added to the Mavlyutov duality for pairs of d -dimensional pseudoreflexive polytopes Δ and Δ^\vee such that the stringy Euler numbers $e_{\text{str}}(X)$ and $e_{\text{str}}(X^\vee)$ will be integers satisfying the equation

$$e_{\text{str}}(X) = (-1)^{d-1} e_{\text{str}}(X^\vee).$$

We remark that a pseudoreflexive lattice polytope Δ is not reflexive if and only if there exists at least one singular facet $\Theta \prec_{\text{sing}} \Delta$. Our additional condition on a pseudoreflexive polytope Δ is exactly an additional condition on its singular facets of Δ . By 3.17, there exist a natural bijection between singular facets Θ of pseudoreflexive polytope Δ and non-integral vertices ν_Θ of the polar polytope Δ^* .

Let $Z \subset \mathbb{T}_d$ be a Δ^\vee -nondegenerated hypersurface and let X^\vee be its canonical Zariski closure in the toric \mathbb{Q} -Fano variety \mathbb{P}_{Δ^*} corresponding to the polar polytope Δ^* . Then for any singular facet $\Theta \prec_{\text{sing}} \Delta$ the Calabi-Yau hypersurface $X^\vee \subset \mathbb{P}_{\Delta^*}$ contains the torus fixed point $x_\Theta \in X^\vee$ corresponding to the rational vertex $\nu_\Theta \in \Delta^*$.

Definition 6.1. Let $\Theta \prec_{\text{sing}} \Delta$ be a singular facet of a d -dimensional pseudoreflexive polytope Δ . Denote by n_Θ ($n_\Theta \geq 2$) the integral distance from $0 \in M$ to the facet Θ . We call the facet Θ *quasi-regular* if the local stringy Euler number $e_{\text{str}}(X^\vee, x_\Theta)$ is an integer that can be computed by the formula:

$$e_{\text{str}}(X^\vee, x_\Theta) = n_\Theta \cdot v(\Theta).$$

We illustrate this definition with the examples of the Mavlyutov pairs (Δ, Δ^\vee) from the previous section.

Example 6.2. We consider Δ^\vee to be a d -dimensional pseudoreflexive simplex which is the convex hull of a basis e_1, \dots, e_d of the lattice M and a point $e_0 = -ae_d + \sum_{i=1}^{d-1} e_i$, where a does not divide d and $a \leq d/2$. Let $d = ab + l$ for some integers $1 \leq l < a$ and $b \geq 2$. Then the dual pseudoreflexive polytope $\Delta = (\Delta^\vee)^\vee$ is the Newton polytope of a Calabi-Yau hypersurface X of degree $a + d$ in the d -dimensional weighted projective space $\mathbb{P}(a, 1^d)$ that contains a torus fixed point $x := (1 : 0 : \dots : 0)$. A desingularization of $\mathbb{P}(a, 1^d)$

at x contains a single exceptional divisor isomorphic to \mathbb{P}^{d-1} . The induced birational morphism $\rho : Y \rightarrow X$ has a single exceptional divisor $D \subset \mathbb{P}^{d-1}$ which is a hypersurface of degree l and one has $K_Y = \rho^*K_X + (b-1)D$. By 4.15, we obtain $e_{\text{str}}(X, x) = e(D)/b$. On the other hand, we have $n_{\Theta} \cdot v(\Theta) = a$, where Θ is a singular facet with vertices v_1, \dots, v_d of Δ^\vee corresponding to the point x . The equality $e_{\text{str}}(X, x) = n_{\Theta} \cdot v(\Theta)$ is equivalent to $e(D) = ab = d-l$. This can happen for a smooth $(d-2)$ -dimensional hypersurface D of degree l in \mathbb{P}^{d-1} if and only if $l = 1$, i.e., only if X is a quasi-smooth hypersurface in $\mathbb{P}(a, 1^d)$.

Theorem 6.3. *Let (Δ, Δ^\vee) be a Mavlyutov pair of d -dimensional pseudoreflexive polytopes Δ and Δ^\vee . Assume all singular facets $\Theta' \prec_{\text{sing}} \Delta^\vee$ are quasi-regular. Then the stringy Euler number $e_{\text{str}}(X)$ of a canonical Calabi-Yau model X of a Δ -nondegenerate hypersurface can be computed by the following formula:*

$$\begin{aligned} & \sum_{\substack{\Theta' \prec_{\text{sing}} \Delta^\vee \\ \dim \Theta' = d-1}} n_{\Theta'} \cdot v(\Theta') \\ & + \sum_{\substack{\Theta \prec_{\text{reg}} \Delta \\ 1 \leq \dim \Theta \leq d-2}} (-1)^{\dim \Theta - 1} v(\Theta) \cdot v(\Theta^\vee) + (-1)^{d-1} \sum_{\substack{\Theta \prec_{\text{sing}} \Delta \\ \dim \Theta = d-1}} n_{\Theta} \cdot v(\Theta). \end{aligned}$$

In particular, if all singular facets of Δ are also quasi-regular then for canonical Calabi-Yau models X^\vee of a Δ^\vee -nondegenerate hypersurface one obtains the equality

$$e_{\text{str}}(X) = (-1)^{d-1} e_{\text{str}}(X^\vee).$$

Proof. Let $Z \subset \mathbb{T}_d$ be a Δ -nondegenerate affine hypersurface. There are two projective closures of Z : the closure \overline{Z} in the toric variety \mathbb{P}_Δ and the canonical model X obtained as the Zariski closure of Z in the toric \mathbb{Q} -Fano variety corresponding to the rational polytope $\Delta^{\text{can}} = (\Delta^\vee)^* \subset M_{\mathbb{R}}$.

We choose a regular simplicial fan $\widehat{\Sigma}$ which is a common subdivision of two rational polyhedral fans: the normal fan Σ^Δ and the normal fan $\Sigma^{\Delta^{\text{can}}}$. So we obtain two birational toric morphisms ρ_1 and ρ_2 :

$$\begin{array}{ccc} & \mathbb{P}_{\widehat{\Sigma}} & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ \mathbb{P}_\Delta & \overset{f}{\dashrightarrow} & \mathbb{P}_{\Delta^{\text{can}}} \end{array}$$

together with the induced birational morphisms

$$\begin{array}{ccc}
 & \widehat{Z} & \\
 \rho_1 \swarrow & & \searrow \rho_2 \\
 \overline{Z} & \overset{f}{\dashrightarrow} & X
 \end{array}$$

The canonical Calabi-Yau hypersurface $X \subset \mathbb{P}_{\Delta^{can}}$ is a disjoint union of locally closed strata $X_F := X \cap \mathbb{T}_F$ where \mathbb{T}_F is a torus orbit in the projective toric variety $\mathbb{P}_{\Delta^{can}}$ and F runs over all faces $F \preceq \Delta^{can}$ of the rational polytope Δ^{can} :

$$X = \bigcup_{F \preceq \Delta^{can}} X_F.$$

Let v_1, \dots, v_s be the set of primitive lattice generators of 1-dimensional cones in the fan $\widehat{\Sigma}$. We set $I := \{1, \dots, s\}$. Then k -dimensional cone $\sigma \in \widehat{\Sigma}$ is determined by a subset $J \subset I$ such that $|J| = k$ and σ is generated by v_j ($j \in J$).

For any face $F \preceq \Delta^{can}$ we define the stringy Euler number

$$e_{\text{str}}(X, X_F) := \sum_{\emptyset \subseteq J \subseteq I} e(D_J^\circ \cap \rho_2^{-1}(\mathbb{T}_F)) \prod_{j \in J} \frac{1}{a_j + 1},$$

where D_J° are either empty or a locally closed stratum on the smooth projective hypersurface \widehat{Z} in the toric variety $\mathbb{P}_{\widehat{\Sigma}}$ corresponding to a cone $\sigma \in \widehat{\Sigma}$ of dimension $|J|$. By additivity of the Euler number, we obtain

$$e_{\text{str}}(X) = \sum_{F \preceq \Delta^{can}} e_{\text{str}}(X, X_F).$$

So it remains to compute $e_{\text{str}}(X, X_F)$ for any face $F \preceq \Delta^{can}$.

We consider the following 4 possibilities for a face $F \preceq \Delta^{can}$:

- $\dim[F] = \dim F = k \geq 1$, i. e., $F = \Theta^*$ for some regular $(d - k - 1)$ -dimensional face $\Theta \preceq \Delta^\vee$
- $\dim[F] < \dim F = k \geq 1$, i. e., $F = \Theta^*$ for some singular $(d - k - 1)$ -dimensional face $\Theta \preceq \Delta^\vee$
- $\dim[F] = \dim F = 0$, i. e., $F = \Theta^*$ is a lattice vertex of Δ^{can} corresponding to some regular $(d - 1)$ -dimensional face $\Theta \preceq \Delta^\vee$
- $\dim F = 0$ and $[F] = \emptyset$, i. e., $F = \Theta^*$ is a rational vertex of Δ^{can} corresponding to some singular $(d - 1)$ -dimensional face $\Theta \preceq \Delta^\vee$.

If $\dim[F] = \dim F = k \geq 1$, then $[F] = \Theta$ for some k -dimensional face $\Theta \preceq \Delta$. For a generic Δ -nondegenerate hypersurface Z the affine hypersurface $X_F \subset \mathbb{T}_F$ is $[F]$ -nondegenerate and its Euler number equals $(-1)^{k-1}v([F])$ (see 4.7). Moreover, X has Gorenstein toroidal singularities along X_F corresponding to the $(d-k)$ -dimensional cone over the dual regular $(d-k-1)$ -dimensional face $|\Theta^\vee$ of Δ . So one has

$$e_{\text{str}}(X, X_F) = (-1)^{k-1}v(\Theta) \cdot v(\Theta^\vee).$$

If $\dim[F] < \dim F = k \geq 1$, then the affine hypersurface $X_F \subset \mathbb{T}_F$ is isomorphic to a product of $(\mathbb{C}^*)^{k-\dim[F]}$ and $[F]$ -nondegenerated affine hypersurface. Therefore $e(X_F) = 0$ and one has $e_{\text{str}}(X, X_F) = 0$.

If $\dim[F] = \dim F = 0$, then X_F is empty and one has $e_{\text{str}}(X, X_F) = 0$.

If $\dim F = 0$ and $[F] = \emptyset$, then X_F is a torus fixed point $x_\Theta \in \mathbb{P}_{\Delta^{\text{can}}}$. $e_{\text{str}}(X, X_F)$ equals to local stringy Euler number $e_{\text{str}}(X, x_\Theta) = n(\Theta) \cdot v(\Theta)$ for some singular facet $\Theta \prec_{\text{sing}} \Delta^\vee$.

Thus, we obtain

$$e_{\text{str}}(X) = \sum_{\substack{\Theta_{\text{sing}} \prec \Delta^\vee \\ \dim \Theta_{\text{sing}} = d-1}} n_\Theta \cdot v(\Theta) + \sum_{k=1}^d (-1)^{k-1} \sum_{\substack{\Theta_{\text{ord}} \prec \Delta \\ \dim \Theta_{\text{ord}} = k}} v(\Theta_{\text{ord}}) \cdot v(\Theta_{\text{ord}}^*).$$

Since the d -dimensional lattice polytope Δ is the union over all $(d-1)$ -dimensional faces $\Theta \prec \Delta$ of d -dimensional pyramids $\Pi_\Theta := \text{Conv}(0, \Theta)$ with vertex 0, one has

$$v(\Delta) = \sum_{\substack{\Theta \prec \Delta \\ \dim \Theta = d-1}} v(\Pi_\Theta).$$

On the other hand, $v(\Pi_\Theta) = v(\Theta) \cdot n_\Theta$, where n_Θ is the integral distance from Θ to $0 \in M$. The equality $n_\Theta = 1$ holds if and only if $\Theta \prec \Delta$ is a regular $(d-1)$ -dimensional face. This implies the equality

$$\begin{aligned} \sum_{k=d-1}^d (-1)^{k-1} \sum_{\substack{\Theta \prec_{\text{reg}} \Delta \\ \dim \Theta = k}} v(\Theta) \cdot v(\Theta^\vee) &= (-1)^{d-1} \left(v(\Delta) - \sum_{\substack{\Theta \prec_{\text{reg}} \Delta \\ \dim \Theta = d-1}} v(\Theta) \right) = \\ &= (-1)^{d-1} \left(\sum_{\substack{\Theta \prec_{\text{sing}} \Delta \\ \dim \Theta = d-1}} v(\Theta) \cdot n_\Theta \right) \end{aligned}$$

that proves the required formula for $e_{\text{str}}(X)$.

The equality $e_{\text{str}}(X) = (-1)^{d-1}e_{\text{str}}(X^\vee)$ follows now from the duality $\Theta \leftrightarrow \Theta^\vee$ between k -dimensional regular faces $\Theta \prec_{\text{reg}} \Delta$ and $(d - k - 1)$ -dimensional regular faces $\Theta^\vee \prec_{\text{reg}} \Delta^\vee$ and from the equality

$$\begin{aligned} & \sum_{k=1}^{d-2} (-1)^{k-1} \sum_{\substack{\Theta \prec_{\text{reg}} \Delta \\ \dim \Theta = k}} v(\Theta) \cdot v(\Theta^\vee) \\ &= (-1)^{d-1} \left(\sum_{k=1}^{d-2} (-1)^{k-1} \sum_{\substack{\Theta^\vee \prec_{\text{reg}} \Delta^\vee \\ \dim \Theta^\vee = k}} v(\Theta) \cdot v(\Theta^\vee) \right) \quad \square \end{aligned}$$

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