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The Fekete and Szegö problem on bounded starlike circular domain in $\mathbb{C}^{n*}$

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Abstract: In this paper, we establish the Fekete and Szegö inequality for a class of holomorphic mappings on the bounded starlike circular domain in $\mathbb{C}^n$, which is natural extension to higher dimensions of some classical Fekete and Szegö inequalities for various subclasses of the normalized univalent functions in the unit disk.
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1. Introduction

Let $\mathcal{S}$ be the class of normalized univalent functions on the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}^*$, $\mathcal{S}^*_\alpha$ and $\mathcal{SS}^*_\beta$ denote the subclasses of $\mathcal{S}$ consisting respectively of the starlike functions, starlike functions of order α (see Definition 1) and strongly starlike functions of order β (see Definition 2) on $\mathbb{U}$.

In [1], Fekete and Szegö obtained the following classical result.

If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}$, then

$$\max_{f\in\mathcal{S}} |a_3 - \lambda a_2^2| = 1 + 2e^{-\frac{2\lambda}{1-\lambda}}$$

for $\lambda \in [0, 1]$.

The above relation is known as the Fekete and Szegö inequality. After that, there were many papers to consider the corresponding problems for various subclasses of the class $\mathcal{S}$, and many interesting results were obtained.

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For more details we refer the reader to survey papers [2], [3], [4] and [5]. The following results are well known.

Theorem A (See [6]). *Suppose that* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}^*$. *Then*

$$|a_3 - \lambda a_2^2| \le \max\{1, |3 - 4\lambda|\}, \quad \lambda \in \mathbb{C}.$$

The above estimation is sharp for the function $f(z) = \frac{z}{(1-z)^2}$ *if* $|\frac{3}{2} - 2\lambda| \ge \frac{1}{2}$, *and for* $f(z) = \frac{z}{1-z^2}$ *if* $|\frac{3}{2} - 2\lambda| \le \frac{1}{2}$.

Theorem B (See [7]). *Suppose that* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_\alpha^*$, $\alpha \in [0, 1)$. *Then*

$$|a_3 - \lambda a_2^2| \le (1 - \alpha)\max\{1,\ |3 - 2\alpha - 4\lambda(1 - \alpha)|\},\ \lambda \in \mathbb{C}.$$

The above estimation is sharp for the function $f(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ *if* $|\frac{3-2\alpha}{2(1-\alpha)} - 2\lambda| \ge \frac{1}{2(1-\alpha)}$, *and for* $f(z) = \frac{z}{(1-z^2)^{(1-\alpha)}}$ *if* $\left|\frac{3-2\alpha}{2(1-\alpha)} - 2\lambda\right| \le \frac{1}{2(1-\alpha)}$.

Theorem C (See [8]). *Suppose that* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{SS}_\beta^*$, $\beta \in (0, 1]$. *Then*

$$|a_3 - \lambda a_2^2| \le \beta\max\{1,\ |3 - 4\lambda|\beta\},\ \lambda \in \mathbb{C}.$$

The above estimation is sharp for the function $f(z) = z\exp\int_0^z\left[\left(\frac{1+t}{1-t}\right)^\beta - 1\right]\frac{1}{t}dt$ *if* $|\frac{3}{2} - 2\lambda| \ge \frac{1}{2\beta}$, *and for* $f(z) = z\exp\int_0^z\left[\left(\frac{1+t^2}{1-t^2}\right)^\beta - 1\right]\frac{1}{t}dt$ *if* $|\frac{3}{2} - 2\lambda| \le \frac{1}{2\beta}$.

In fact, the Fekete and Szegö inequality is closely related to the Bieberbach conjecture [9], which was settled by de Branges [10], who proved that if a function $f(z) = z + \sum_{k=2}^{\infty} a_kz^k$ belongs to the class $\mathcal{S}$, then the estimates $|a_k| \le k$, for $k = 2, 3, \ldots$ hold.

However, Cartan [11] stated that the Bieberbach conjecture does not hold in several complex variables. Therefore, it is necessary to require some additional properties of mappings of a family in order to obtain some positive results, for instance, the convexity, the starlikeness and so on.

Some best-possible results concerning the coefficient estimates for subclasses of holomorphic mappings in several variables were obtained in the works of Bracci [12], Bracci et al. [13], Graham, Hamada and Kohr [14], Graham et al. [15], Graham et al. [16], Graham et al. [17, 18], Hamada et al. [19], Hamada and Honda [20], Kohr [21], Liu and Liu [22], and Xu and Liu [23].

In contrast, although the Fekete and Szegö inequalities for various subclasses of the class $\mathcal{S}$ were established, only a few results are known for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in several complex variables.

Recently, in [24], Xu and Liu extended Theorem A to the case of a subclass of starlike mappings defined on the unit ball in a complex Banach space or on the unit polydisk in $\mathbb{C}^n$. In [25] and [26], using some restrictive assumptions, Theorems B and C were also extended to higher dimensions.

In this paper, we will first establish the Fekete and Szegö inequality for a class of holomorphic mappings on the bounded starlike circular domain in $\mathbb{C}^n$, which is natural extension to higher dimensions of some classical Fekete and Szegö inequalities for various subclasses of the class $\mathcal{S}$.

Throughout this paper, let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, z_2, \cdots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum\limits_{j=1}^{n} z_j \overline{w}_j$ and the norm $\|z\| = \langle z, z\rangle^{\frac{1}{2}}, z \in \mathbb{C}^n$. Let B^n be the Euclidean unit ball in $\mathbb{C}^n$, $\Omega \subset \mathbb{C}^n$ be a bounded starlike circular domain with $0 \in \Omega$, and its Minkowski functional $\rho(z) \in C^1$ in $\mathbb{C}^n \setminus \{0\}$. Let $\partial\Omega$ be the boundary of Ω, let $H(\Omega)$ be the set of all holomorphic mappings from Ω into $\mathbb{C}^n$, and let $H(\Omega, \Omega)$ be the set of all holomorphic mappings from Ω into Ω. As is known to us, if $f \in H(\Omega)$, then

$$f(w) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(z)((w-z)^k),$$

for all w in some neighborhood of $z \in \Omega$, where $D^k f(z)$ is the kth-Fréchet derivative of f at z, and for $k \geq 1$,

$$D^k f(z)((w-z)^k) = D^k f(z)\underbrace{(w-z, \cdots, w-z)}_{k}.$$

Let $J_f(z)$ be the Jacobian matrix of f at $z \in \Omega$, $\det J_f(z)$ be the Jacobian determinant of f at $z \in \Omega$. A holomorphic mapping $f : \Omega \longrightarrow \mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(\Omega)$. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic if $\det J_f(z) \neq 0$ for each $z \in \Omega$. If $f : \Omega \longrightarrow \mathbb{C}^n$ is a holomorphic mapping, we say that f is normalized if $f(0) = 0$ and $J_f(0) = I$, where I represents the identity matrix. Let $\mathcal{S}^*(\Omega)$ denote the class of starlike mappings on Ω. When $n = 1$, $\Omega = \mathbb{U}$, the class $\mathcal{S}^*(\mathbb{U})$ is denoted by $\mathcal{S}^*$.

Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded circular domain. The $m(m \geqslant 2)$-th Fréchet derivative of a mapping $f \in H(\Omega)$ at a point $z \in \Omega$ is written by

$D^m f(z)(a^{m-1}, .)$. The matrix representation is

$$D^m f(z)(a^{m-1}, .) = \left(\sum_{l_1, l_2, \cdots, l_{m-1}=1}^{n} \frac{\partial^m f_p(z)}{\partial z_k \partial z_{l_1} \cdots \partial z_{l_{m-1}}} a_{l_1} \cdots a_{l_{m-1}} \right)_{1 \leqslant p,k \leqslant n},$$

where $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))', a = (a_1, a_2, \cdots, a_n)' \in \mathbb{C}^n$.

The following definition is due to Liu and Lu [27] (see also [28]).

Definition 1. Suppose that $0 < \alpha < 1$. Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike circular domain with $0 \in \Omega$, and its Minkowski functional $\rho(z) \in C^1$ in $\mathbb{C}^n \setminus \{0\}$. A normalized locally biholomorphic mapping $f : \Omega \longrightarrow \mathbb{C}^n$ is called a starlike mapping of order α if

$$\left| \frac{2}{\rho(z)} \frac{\partial \rho(z)}{\partial z} J_f^{-1}(z) f(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad \forall z \in \Omega \setminus \{0\}.$$

Equivalently, we may express this as

$$\Re e \left\{ \frac{\rho(z)}{2 \frac{\partial \rho(z)}{\partial z} J_f^{-1}(z) f(z)} \right\} > \alpha, \qquad \forall z \in \Omega \setminus \{0\}.$$

When $\Omega = B^n$, obviously, the above inequality is equivalent to the following relation.

$$\Re e \left\{ \frac{\|z\|^2}{\langle J_f^{-1}(z) f(z), z \rangle} \right\} > \alpha, \qquad \forall z \in B^n \setminus \{0\}. \tag{1}$$

When $n = 1,\ \Omega = \mathbb{U}$, the relation (1) is equivalent to

$$\Re e \frac{z f'(z)}{f(z)} > \alpha, \quad \forall z \in \mathbb{U}.$$

Let $\mathcal{S}_\alpha^*(\Omega)$ denote the class of starlike mappings of order α on Ω. When $n = 1,\ \Omega = \mathbb{U}$, the class $\mathcal{S}_\alpha^*(\mathbb{U})$ is denoted by $\mathcal{S}_\alpha^*$.

Definition 2 (See [29])**.** Suppose $0 < \beta \leq 1$. Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike circular domain with $0 \in \Omega$, and its Minkowski functional $\rho(z) \in C^1$ in $\mathbb{C}^n \setminus \{0\}$. A normalized locally biholomorphic mapping $f : \Omega \longrightarrow \mathbb{C}^n$ is called a strongly starlike mapping of order β if

$$\left| \arg \frac{2}{\rho(z)} \frac{\partial \rho(z)}{\partial z} J_f^{-1}(z) f(z) \right| < \frac{\pi}{2} \beta, \quad \forall z \in \Omega \setminus \{0\}.$$

When $\Omega = B^n$, obviously, the above inequality is equivalent to the following relation.

$$\left|\arg\langle J_f^{-1}(z)f(z), z\rangle\right| < \frac{\pi}{2}\beta, \quad \forall z \in B^n \setminus \{0\}\,. \tag{2}$$

When $n = 1,\ \Omega = \mathbb{U}$, the relation (2) is equivalent to

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\beta, \quad \forall z \in \mathbb{U}.$$

Let $\mathcal{SS}^*_\beta(\Omega)$ denote the class of strongly starlike mappings of order β on Ω. When $n = 1,\ \Omega = \mathbb{U}$, the class $\mathcal{SS}^*_\beta(\mathbb{U})$ is denoted by $\mathcal{SS}^*_\beta$.

Definition 2 was first introduced on B^n by Kohr [30] (see also [31]).

We next recall a class of mappings $\mathcal{M}$ which plays an important role in the study of the Loewner chains and the Loewner differential equation in several complex variables (see [29], [32] and [33]).

$$\mathcal{M} = \{h \in H(\Omega) : h(0) = 0, J_h(0) = I,\ \Re e\frac{\partial\rho(z)}{\partial z}h(z) > 0,\ z \in \Omega\backslash\{0\}\},$$

where $\frac{\partial\rho(z)}{\partial z} = \left(\frac{\partial\rho(z)}{\partial z_1}, \cdots, \frac{\partial\rho(z)}{\partial z_n}\right)$.

Now, we introduce the following class $\mathcal{M}_g$ on $\Omega \subset \mathbb{C}^n$, which has been introduced by Kohr [21] on B^n and studied by Graham, Hamada and Kohr [14] (also see [16]).

Definition 3. Let $g \in H(\mathbb{U})$ be a biholomorphic function such that $g(0) = 1$, $\Re e\ g(\xi) > 0$ on $\xi \in \mathbb{U}$. We define $\mathcal{M}_g$ as the class of mappings given by

$$\mathcal{M}_g = \left\{h \in H(\Omega) : h(0) = 0, J_h(0) = I,\ \frac{\rho(z)}{2\frac{\partial\rho(z)}{\partial z}h(z)} \in g(\mathbb{U}),\ z \in \Omega\backslash\{0\}\right\}.$$

Clearly, if $g(\xi) = \frac{1+\xi}{1-\xi}$, $\xi \in \mathbb{U}$, then $\mathcal{M}_g$ coincides with the class $\mathcal{M}$. Especially, if $\Omega = B^n$, then

$$\mathcal{M}_g = \left\{h \in H(B^n) : h(0) = 0, J_h(0) = I,\ \frac{\|z\|^2}{\langle h(z), z\rangle} \in g(\mathbb{U}),\ z \in B^n\backslash\{0\}\right\}.$$

Remark 1. Let $f \in H(\Omega)$ be a normalized locally biholomorphic mapping. If $J_f^{-1}(z)f(z) \in \mathcal{M}_g$, then there are many choices of the function g which would provide interesting subclasses of $\mathcal{S}(\Omega)$. For example, if we let $g(\xi) = \frac{1+\xi}{1-\xi}$,

$g(\xi) = \frac{1+(1-2\alpha)\xi}{1-\xi}$ and $g(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^{\beta}$ (the branch of the power function is chosen such that $\left(\frac{1+\xi}{1-\xi}\right)^{\beta}|_{\xi=0} = 1$) in Definition 3, then we easily obtain $f \in \mathcal{S}^*(\Omega)$, $f \in \mathcal{S}^*_{\alpha}(\Omega)$ and $f \in \mathcal{SS}^*_{\beta}(\Omega)$, respectively.

2. Some lemmas

In order to prove the desired results, we give some lemmas.

Lemma 1 ([29]). *Let* $s(\xi) = 1 + \sum_{k=1}^{\infty} b_k \xi^k \in H(\mathbb{U})$, *and* $\Re e\ s(\xi) > 0$, $\xi \in \mathbb{U}$. *Then*

$$|b_2 - \frac{1}{2}b_1^2| \leq 2 - \frac{1}{2}|b_1|^2.$$

Lemma 2. *Suppose that* $s \in H(\mathbb{U})$, h *is a biholomorphic function on* $\mathbb{U}$, *and* $s(0) = h(0)$, $s(\xi) \in h(\mathbb{U})$, $\forall \xi \in \mathbb{U}$. *Then*

$$\left|\frac{s''(0)}{2} - \frac{1}{2}\frac{h''(0)}{(h'(0))^2}(s'(0))^2\right| \leq |h'(0)| - \frac{|s'(0)|^2}{|h'(0)|}. \tag{3}$$

Proof. The condition of Lemma 2 yields that $s \prec h$. So, there exists $\varphi \in H(\mathbb{U}, \mathbb{U})$ such that

$$\varphi(0) = 0 \text{ and } s(\xi) = h(\varphi(\xi)), \quad \xi \in \mathbb{U}.$$

A simple computation shows that

$$s'(\xi) = h'(\varphi(\xi))\varphi'(\xi), \quad s''(\xi) = h''(\varphi(\xi))(\varphi'(\xi))^2 + h'(\varphi(\xi))\varphi''(\xi).$$

Thus, from the above relation, we find that

$$\varphi'(0) = \frac{s'(0)}{h'(0)}, \quad \varphi''(0) = \frac{s''(0)(h'(0))^2 - h''(0)(s'(0))^2}{(h'(0))^3}. \tag{4}$$

Also consider the function k defined by

$$k(\xi) = \frac{1+\varphi(\xi)}{1-\varphi(\xi)}, \quad \xi \in \mathbb{U}.$$

Then it is easy to deduce that

$$k(\xi) = 1 + 2\varphi(\xi) + 2\varphi^2(\xi) + \cdots, \text{ and } \quad \Re e\ k(\xi) > 0, \quad \xi \in \mathbb{U}.$$

Consequently, we have

$$k'(0) = 2\varphi'(0), \quad \frac{k''(0)}{2} = \varphi''(0) + 2(\varphi'(0))^2. \tag{5}$$

By Lemma 1, (4) and (5), we obtain (3), as claimed. This completes the proof. ☐

Lemma 3 ([34]). *$\Omega \subset \mathbb{C}^n$ is a bounded starlike circular domain if and only if there exists a unique real continuous function $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$, called the Minkowski functional of Ω, such that*

(i) $\rho(z) \geq 0, z \in \mathbb{C}^n; \rho(z) = 0 \Leftrightarrow z = 0;$
(ii) $\rho(tz) = |t|\rho(z), t \in \mathbb{C}, z \in \mathbb{C}^n;$
(iii) $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 1\}.$

Furthermore, if $\rho(z) \in C^1$ in $\mathbb{C}^n \setminus \{0\}$, then the function $\rho(z)$ has the following properties.

$$\begin{aligned} 2\frac{\partial \rho(z)}{\partial z} z &= \rho(z), \quad z \in \mathbb{C}^n, \\ 2\frac{\partial \rho(z_0)}{\partial z} z_0 &= 1, \quad z_0 \in \partial\Omega, \\ \frac{\partial \rho(\lambda z)}{\partial z} &= \frac{\partial \rho(z)}{\partial z}, \quad \lambda \in (0, \infty), \\ \frac{\partial \rho(e^{i\theta} z)}{\partial z} &= e^{-i\theta}\frac{\partial \rho(z)}{\partial z}, \quad \theta \in \mathbb{R}, \end{aligned} \tag{6}$$

where $\frac{\partial \rho(z)}{\partial z} = \left(\frac{\partial \rho(z)}{\partial z_1}, \cdots, \frac{\partial \rho(z)}{\partial z_n}\right)$.

3. Main results

In this section, we state and prove the main results of our present investigation.

Theorem 1. *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy the conditions of Definition 3, $f \in H(\Omega, \mathbb{C})$, $f(0) = 1$, $F(z) = zf(z)$ and suppose that $J_F^{-1}(z)F(z) \in \mathcal{M}_g$. Then*

$$\left| 2\frac{\partial \rho(z)}{\partial z} \frac{D^3F(0)(z^3)}{3!\rho^3(z)} - \lambda \left(2\frac{\partial \rho(z)}{\partial z} \frac{D^2F(0)(z^2)}{2!\rho^2(z)} \right)^2 \right|$$
$$\leq \frac{|g'(0)|}{2} \max\left\{1, \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1 - 2\lambda)g'(0)\right|\right\}, \ \lambda \in \mathbb{C}, \ z \in \Omega \setminus \{0\}. \tag{7}$$

The above estimation is sharp.

Proof. Fix $z \in \Omega\backslash\{0\}$, and denote $z_0 = \frac{z}{\rho(z)}$. Note that $J_F^{-1}(z)$ exists, we have $f(z) \neq 0,\ z \in \Omega$. Let $p : \mathbb{U} \longrightarrow \mathbb{C}$ be given by

$$p(\xi) = \begin{cases} \frac{\xi}{2\frac{\partial \rho(z_0)}{\partial z} J_F^{-1}(\xi z_0) F(\xi z_0)}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Using Lemma 3 $\left(2\frac{\partial\rho(z_0)}{\partial z} z_0 = 1,\ z_0 \in \partial\Omega\right)$, we obtain $p \in H(\mathbb{U})$, and since $J_F^{-1}(z)F(z) \in \mathcal{M}_g$, we deduce that

$$\begin{aligned} p(\xi) &= \frac{\xi}{2\frac{\partial\rho(z_0)}{\partial z} J_F^{-1}(\xi z_0)F(\xi z_0)} \\ &= \frac{\rho(\xi z_0)}{2\frac{\partial\rho(\xi z_0)}{\partial z} J_F^{-1}(\xi z_0)F(\xi z_0)} \in g(\mathbb{U}), \quad \xi \in \mathbb{U}\setminus\{0\}. \end{aligned}$$

By Lemma 2, we obtain

$$\left|\frac{p''(0)}{2} - \frac{1}{2}\frac{g''(0)}{(g'(0))^2}(p'(0))^2\right| \leq |g'(0)| - \frac{|p'(0)|^2}{|g'(0)|}. \tag{8}$$

Using an argument similar to that in [35] (also see [29, Theorem 7.1.14]), we have

$$J_F^{-1}(z) = \frac{1}{f(z)}\left(I - \frac{\frac{zJ_f(z)}{f(z)}}{1 + \frac{J_f(z)z}{f(z)}}\right).$$

Therefore, we obtain

$$J_F^{-1}(z)F(z) = z\left(\frac{1}{1 + \frac{J_f(z)z}{f(z)}}\right) = \frac{zf(z)}{f(z) + J_f(z)z}, \quad z \in \Omega\setminus\{0\}, \tag{9}$$

which implies that

$$\frac{\rho(z)}{2\frac{\partial\rho(z)}{\partial z} J_F^{-1}(z)F(z)} = 1 + \frac{J_f(z)z}{f(z)}, \quad z \in \Omega\setminus\{0\}. \tag{10}$$

In view of (10), we have

$$p(\xi) = \frac{\rho(\xi z_0)}{2\frac{\partial\rho(\xi z_0)}{\partial z} J_F^{-1}(\xi z_0)F(\xi z_0)} = 1 + \frac{J_f(\xi z_0)\xi z_0}{f(\xi z_0)}.$$

From this we can conclude that

$$p(\xi)f(\xi z_0) = f(\xi z_0) + J_f(\xi z_0)\xi z_0.$$

Using Taylor series expansions of p in ξ, we obtain

$$\begin{aligned}&\left(1 + p'(0)\xi + \frac{p''(0)}{2}\xi^2 + \cdots\right)\left(1 + J_f(0)(z_0)\xi + \frac{D^2 f(0)(z_0^2)}{2}\xi^2 + \cdots\right)\\ &= \left(1 + J_f(0)(z_0)\xi + \frac{D^2 f(0)(z_0^2)}{2}\xi^2 + \cdots\right)\\ &\quad + (J_f(0)(z_0)\xi + D^2 f(0)(z_0^2)\xi^2 + \cdots).\end{aligned}$$

Comparing the homogeneous expansions of two sides of the above equality, we deduce that

$$p'(0) = J_f(0)(z_0), \quad \frac{p''(0)}{2} = D^2 f(0)(z_0^2) - (J_f(0)(z_0))^2.$$

That is

$$p'(0)\rho(z) = J_f(0)(z), \quad \frac{p''(0)}{2}\rho^2(z) = D^2 f(0)(z^2) - (J_f(0)(z))^2. \tag{11}$$

Also, since $F(z) = zf(z)$, we have

$$\frac{D^3 F(0)(z^3)}{3!} = \frac{D^2 f(0)(z^2)}{2!} z, \quad \frac{D^2 F(0)(z^2)}{2!} = J_f(0)(z)z. \tag{12}$$

From (12), we obtain

$$2\frac{\partial \rho(z)}{\partial z}\frac{D^3 F(0)(z^3)}{3!} = \frac{D^2 f(0)(z^2)\rho(z)}{2!} \tag{13}$$

and

$$2\frac{\partial \rho(z)}{\partial z}\frac{D^2 F(0)(z^2)}{2!} = J_f(0)(z)\rho(z). \tag{14}$$

Thus, from (8), (11), (13) and (14), we have

$$\left|2\frac{\partial \rho(z)}{\partial z}\frac{D^3 F(0)(z^3)\rho(z)}{3!} - \lambda\left(2\frac{\partial \rho(z)}{\partial z}\frac{D^2 F(0)(z^2)}{2!}\right)^2\right|$$

$$
\begin{aligned}
&= \left|\rho^2(z)\frac{D^2 f(0)(z^2)}{2!} - \lambda\rho^2(z)(J_f(0)(z))^2\right| \\
&= \frac{1}{2}\left|\rho^2(z)D^2 f(0)(z^2) - 2\lambda\rho^2(z)(J_f(0)(z))^2\right| \\
&= \frac{1}{2}\left|\rho^2(z)D^2 f(0)(z^2) - \rho^2(z)(J_f(0)(z))^2 + (1-2\lambda)\rho^2(z)(J_f(0)(z))^2\right| \\
&= \frac{1}{2}\rho^4(z)\left|\frac{p''(0)}{2} + (1-2\lambda)(p'(0))^2\right| \\
&= \frac{1}{2}\rho^4(z)\left|\frac{p''(0)}{2} - \frac{1}{2}\frac{g''(0)}{(g'(0))^2}(p'(0))^2 + \left(\frac{1}{2}\frac{g''(0)}{(g'(0))^2} + 1 - 2\lambda\right)(p'(0))^2\right| \\
&\le \frac{1}{2}\rho^4(z)\left(|g'(0)| - \frac{|p'(0)|^2}{|g'(0)|} + \left|\frac{1}{2}\frac{g''(0)}{(g'(0))^2} + 1 - 2\lambda\right||p'(0)|^2\right) \\
&= \frac{1}{2}\rho^4(z)\left(|g'(0)| - \frac{|p'(0)|^2}{|g'(0)|} + \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right|\frac{|p'(0)|^2}{|g'(0)|}\right).
\end{aligned}
$$

We consider the following two cases.

Case I. If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right| \le 1$, then

$$
\begin{aligned}
&\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3 F(0)(z^3)\rho(z)}{3!} - \lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2 F(0)(z^2)}{2!}\right)^2\right| \\
&\quad\le \frac{1}{2}\rho^4(z)\left(|g'(0)| - \frac{|p'(0)|^2}{|g'(0)|} + \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right|\frac{|p'(0)|^2}{|g'(0)|}\right) \\
&\quad\le \frac{1}{2}|g'(0)|\rho^4(z). \qquad (15)
\end{aligned}
$$

Case II. If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right| \ge 1$, then

$$
\begin{aligned}
&\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3 F(0)(z^3)\rho(z)}{3!} - \lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2 F(0)(z^2)}{2!}\right)^2\right| \\
&\le \frac{1}{2}\rho^4(z)\left(|g'(0)| - \frac{|p'(0)|^2}{|g'(0)|} + \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right|\frac{|p'(0)|^2}{|g'(0)|}\right) \\
&= \frac{1}{2}|g'(0)|\rho^4(z) + \frac{1}{2}\rho^4(z)\left(\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right| - 1\right)\frac{|p'(0)|^2}{|g'(0)|}.
\end{aligned}
$$

Since $|p'(0)| \le |g'(0)|$, we obtain

$$\begin{aligned}
&\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)\rho(z)}{3!} - \lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!}\right)^2\right| \\
&\le \frac{1}{2}|g'(0)|\rho^4(z) + \frac{1}{2}\rho^4(z)\left(\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right| - 1\right)\frac{|p'(0)|^2}{|g'(0)|} \\
&\le \frac{1}{2}|g'(0)|\rho^4(z) + \frac{1}{2}\rho^4(z)\left(\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right| - 1\right)\frac{|g'(0)|^2}{|g'(0)|} \\
&= \frac{1}{2}|g'(0)|\rho^4(z)\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right|. \qquad (16)
\end{aligned}$$

From (15) and (16), we deduce (7), as desired.

To see that the estimation of Theorem 1 is sharp, it suffices to consider the following examples.

Example. If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right| \ge 1$, we consider the following example

$$F(z) = z\exp\int_0^{\frac{z_1}{r}}(g(t)-1)\frac{dt}{t}, \quad z \in \Omega,$$

where $r = \sup\{|z_1| : z = (z_1, z_2, \cdots, z_n)' \in \Omega\}$. We deduce that $J_F^{-1}(z)F(z) \in \mathcal{M}_g$, and a short computation yields the relation

$$\frac{D^3F(0)(z^3)}{3!} = \left(\frac{g''(0)}{4} + \frac{(g'(0))^2}{2}\right)(\frac{z_1}{r})^2 z, \qquad \frac{D^2F(0)(z^2)}{2!} = g'(0)\frac{z_1}{r}z.$$

In view of the above relation, we obtain

$$\begin{aligned}
&\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)\rho(z)}{3!} - \lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!}\right)^2\right| \\
&= \left|\left(\frac{g''(0)}{4} + \frac{(g'(0))^2}{2}\right)(\frac{z_1}{r})^2\rho^2(z) - \lambda(g'(0))^2(\frac{z_1}{r})^2\rho^2(z)\right| \\
&= \frac{(\frac{z_1}{r})^2\rho^2(z)|g'(0)|}{2}\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right|. \qquad (17)
\end{aligned}$$

Setting $z = Ru$ $(0 < R < 1)$ in (17), where $u = (u_1, u_2, \cdots, u_n)' \in \partial\Omega, u_1 = r$, we have

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!\rho^3(z)}-\lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!\rho^2(z)}\right)^2\right|$$
$$=\frac{|g'(0)|}{2}\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+(1-2\lambda)g'(0)\right|.$$

If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+(1-2\lambda)g'(0)\right|\le 1$, we consider the following example

$$F(z)=z\exp\int_0^{\frac{z_1}{r}}(g(t^2)-1)\frac{dt}{t},\quad z\in\Omega, \tag{18}$$

where $r=\sup\{|z_1|:z=(z_1,z_2,\cdots,z_n)'\in\Omega\}$. It is elementary to verify that the mapping $F(z)$ defined in (18) satisfies $J_F^{-1}(z)F(z)\in\mathcal{M}_g$, and a simple computation shows that

$$\frac{D^3F(0)(z^3)}{3!}=\frac{g'(0)(\frac{z_1}{r})^2z}{2},\qquad \frac{D^2F(0)(z^2)}{2!}=0. \tag{19}$$

From (19), we have

(20)
$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)\rho(z)}{3!}-\lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!}\right)^2\right|=\frac{|g'(0)||\frac{z_1}{r}|^2\rho^2(z)}{2}.$$

Taking $z=Ru(0<R<1)$ in (20), where $u=(u_1,u_2,\cdots,u_n)'\in\partial\Omega, u_1=r$, we have

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!\rho^3(z)}-\lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!\rho^2(z)}\right)^2\right|=\frac{|g'(0)|}{2}.$$

This completes the proof of Theorem 1. □

When $\Omega=B^n$, we immediately obtain the following result, which we merely state here without proof.

Theorem 2. *Let $g:\mathbb{U}\to\mathbb{C}$ satisfy the conditions of Definition 3, $f\in H(B^n,\mathbb{C})$, $f(0)=1$, $F(z)=zf(z)$ and suppose that $J_F^{-1}(z)F(z)\in\mathcal{M}_g$. Then*

$$\left|\frac{1}{\|z\|^4}\frac{D^3F(0)(z^3)}{3!}\bar{z}-\lambda\left(\frac{1}{\|z\|^3}\frac{D^2F(0)(z^2)}{2!}\bar{z}\right)^2\right|$$

$$\le \frac{|g'(0)|}{2}\max\left\{1,\ \left|\frac{1}{2}\frac{g''(0)}{g'(0)}+(1-2\lambda)g'(0)\right|\right\},\ \lambda\in\mathbb{C},\ z\in B^n\setminus\{0\}.$$

The above estimation is sharp.

In view of Remark 1, if we set $g(\xi)=\frac{1+\xi}{1-\xi}$, $g(\xi)=\frac{1+(1-2\alpha)\xi}{1-\xi}$ and $g(\xi)=\left(\frac{1+\xi}{1-\xi}\right)^{\beta}$ (the branch of the power function is chosen such that $\left(\frac{1+\xi}{1-\xi}\right)^{\beta}|_{\xi=0}=1$) in Theorems 1 and 2, we can deduce Corollaries 1, 2 and 3, respectively, whose proofs we omit.

Corollary 1. *Let* $f:\Omega\to\mathbb{C},\ \ F(z)=zf(z)\in\mathcal{S}^*(\Omega)$. *Then*

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!\rho^3(z)}-\lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!\rho^2(z)}\right)^2\right|$$
$$\le \max\{1,\ |3-4\lambda|\},\ \lambda\in\mathbb{C},\ z\in\Omega\setminus\{0\}.$$

If $\Omega=B^n$, *then*

$$\left|\frac{1}{\|z\|^4}\frac{D^3F(0)(z^3)}{3!}\bar{z}-\lambda\left(\frac{1}{\|z\|^3}\frac{D^2F(0)(z^2)}{2!}\bar{z}\right)^2\right|$$
$$\le \max\{1,\ |3-4\lambda|\},\ \lambda\in\mathbb{C},\ z\in B^n\setminus\{0\}. \tag{21}$$

These estimates are sharp.

Especially, when $n=1,\ \Omega=\mathbb{U}$, *(21) reduces to the following*

$$\left|\frac{F^{(3)}(0)}{3!}-\lambda\left(\frac{F''(0)}{2!}\right)^2\right|\le \max\{1,\ |3-4\lambda|\},\ \lambda\in\mathbb{C},\ z\in\mathbb{U},$$

which is equivalent to Theorem A.

Corollary 2. *Let* $f:\Omega\to\mathbb{C},\ \ F(z)=zf(z)\in\mathcal{S}^*_{\alpha}(\Omega)$. *Then*

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!\rho^3(z)}-\lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!\rho^2(z)}\right)^2\right|$$
$$\le (1-\alpha)\max\{1,\ |3-2\alpha-4\lambda(1-\alpha)|\},\ \lambda\in\mathbb{C},\ z\in\Omega\setminus\{0\}.$$

If $\Omega=B^n$, *then*

$$\left|\frac{1}{\|z\|^4}\frac{D^3f(0)(z^3)}{3!}\bar{z}-\lambda\left(\frac{1}{\|z\|^3}\frac{D^2f(0)(z^2)}{2!}\bar{z}\right)^2\right|$$

$$\leq (1-\alpha)\max\{1,\ |3-2\alpha-4\lambda(1-\alpha)|\},\ \lambda\in\mathbb{C},\ z\in B^n\setminus\{0\}. \tag{22}$$

These estimates are sharp.

Especially, when $n=1$, $\Omega=\mathbb{U}$, (22) reduces to the following

$$\left|\frac{F^{(3)}(0)}{3!}-\lambda\left(\frac{F''(0)}{2!}\right)^2\right|$$
$$\leq (1-\alpha)\max\{1,\ |3-2\alpha-4\lambda(1-\alpha)|\},\ \lambda\in\mathbb{C},\ z\in\mathbb{U},$$

which is equivalent to Theorem B.

Corollary 3. *Let $f:\Omega\to\mathbb{C}$, $F(z)=zf(z)\in\mathcal{SS}^*_\beta(\Omega)$. Then*

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!\rho^3(z)}-\lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!\rho^2(z)}\right)^2\right|$$
$$\leq\beta\max\{1,\ |3-4\lambda|\beta\},\ \lambda\in\mathbb{C},\ z\in\Omega\setminus\{0\}.$$

If $\Omega=B^n$, then

$$\left|\frac{1}{\|z\|^4}\frac{D^3f(0)(z^3)}{3!}\bar{z}-\lambda\left(\frac{1}{\|z\|^3}\frac{D^2f(0)(z^2)}{2!}\bar{z}\right)^2\right|$$
$$\leq\beta\max\{1,\ |3-4\lambda|\beta\},\ \lambda\in\mathbb{C},\ z\in B^n\setminus\{0\}. \tag{23}$$

These estimates are sharp.

Especially, when $n=1$, $\Omega=\mathbb{U}$, (23) reduces to the following

$$\left|\frac{F^{(3)}(0)}{3!}-\lambda\left(\frac{F''(0)}{2!}\right)^2\right|\leq\beta\max\{1,\ |3-4\lambda|\beta\},\ \lambda\in\mathbb{C},\ z\in\mathbb{U},$$

which is equivalent to Theorem C.

According to Theorem 1, we naturally propose the following open problem.

Open Problem. *Let $g:\mathbb{U}\to\mathbb{C}$ satisfy the conditions of Definition 3, $F\in H(\Omega)$ be a normalized locally biholomorphic mapping. If $J_F^{-1}(z)F(z)\in\mathcal{M}_g$, then*

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^3F(0)(z^3)}{3!\rho^3(z)}-\lambda\left(2\frac{\partial\rho(z)}{\partial z}\frac{D^2F(0)(z^2)}{2!\rho^2(z)}\right)^2\right|$$

$$\leq \frac{|g'(0)|}{2} \max\left\{1,\ \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + (1-2\lambda)g'(0)\right|\right\},\ \lambda \in \mathbb{C},\ z \in \Omega \setminus \{0\}.$$

The above estimation is sharp.

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