Pure and Applied Mathematics Quarterly
Volume 12, Number 4, 603–619, 2016

A global pinching theorem for complete translating solitons of mean curvature flow*

Huijuan Wang, Hongwei Xu and Entao Zhao

Abstract: In the present paper, we prove that for a smooth complete translating soliton $M^n (n \geq 3)$ with the mean curvature vector H satisfying $H = V^N$ for a unit constant vector V in the Euclidean space $\mathbb{R}^{n+p}$, if the trace-free second fundamental form $\mathring{A}$ satisfies $(\int_M |\mathring{A}|^n \mathrm{d}\mu)^{1/n} < K(n)$, $\int_M |\mathring{A}|^n e^{\langle V,X \rangle} \mathrm{d}\mu < \infty$, where $K(n)$ is an explicit positive constant depending only on n, then M is a linear subspace.

Keywords: Rigidity theorem, translating soliton, integral curvature pinching.

1. Introduction

Let $X_0 : M \to \mathbb{R}^{n+p}$ be an n-dimensional smooth submanifold isometrically immersed in an $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. The mean curvature flow with initial value X_0 is a smooth family of immersions $X : M \times [0,T) \to \mathbb{R}^{n+p}$ satisfying

$$\begin{cases} \dfrac{d}{dt} X(x,t) = H(x,t), \\ X(x,0) = X_0(x), \end{cases} \tag{1}$$

for $x \in M$ and $t \in [0,T)$. Here $H(x,t)$ is the mean curvature vector of $M_t = X_t(M)$ at $X(x,t)$ in $\mathbb{R}^{n+p}$ where $X_t(\cdot) = X(\cdot,t)$.

In the theory of the mean curvature flow, one of the most important fields is the singularity analysis. According to the blow-up rate of the second fundamental form A, singularities of the mean curvature flow are divided into two types called Type-I singularity and Type-II singularity. It is well-known that self-shrinkers describe the Type-I singularity models of the mean curvature

Received 30 January 2018.
2010 Mathematics Subject Classification: 53C44, 53C42.
*Research supported by the National Natural Science Foundation of China, Grant Nos. 11531012, 11371315, 11771394.

flow, see [8, 19]. For the study of self-shrinkers and their generalizations, see [3, 4, 5, 6, 12, 16, 17, 22].

A very important example of Type-II singularities is the translating soliton. A submanifold $X : M^n \to \mathbb{R}^{n+p}$ is said to be a translating soliton (translator for short) if there exists a constant vector V with unit length in $\mathbb{R}^{n+p}$ such that

$$H = V^N, \tag{2}$$

where $(\ )^N$ denotes the normal part of a vector field on $\mathbb{R}^{n+p}$. Let V^T be the tangent component of vector V, then we have

$$H + V^T = V. \tag{3}$$

Translating solitons often occur as Type-II singularities of a mean curvature flow after a rescaling. For instance, Huisken and Sinestrari [7] proved that if the initial hypersurface is mean convex, then the limit hypersurface at Type-II singularity is a convex translating soliton. On the other hand, every translating soliton gives a translating solution M_t defined by $M_t = M + tV$ for $t \in \mathbb{R}$ to the mean curvature flow. That is, it does not change the shape during the evolution, it's just moving by translation in the direction of V. Similar to self-shrinkers, translating solitons can be regarded as a minimal submanifolds in $(\mathbb{R}^{n+p}, \bar{g})$, where $\bar{g}$ is a conformally flat Riemannian metric due to [9].

There are few examples of translating solitons even in the hypersurface case. The well-known grim reaper Γ is a one-dimensional translating soliton in $\mathbb{R}^2$ defined by

$$y = -\log\cos x, \quad x \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

A trivial generalization is the euclidean product $\Gamma \times \mathbb{R}^{n-1}$ in $\mathbb{R}^{n+1}$, which is called the grim reaper cylinder.

Since the geometry of the solution near the Type-II singularity cannot be controlled well, the study of Type-II singularities is more complicated than Type-I. There are some results about the translating solitons, see [1, 10, 14, 15, 18]. For instance, Wang [18] studied the classification of Type-II singularities and proved that for $n = 2$ any entire convex translator must be rotationally symmetric in an appropriate coordinate system. Martin, Savas-Halilaj and Smoczyk [14] obtained classification results and topological obstructions for the existence of translating solitons. Xin [20] showed that a smooth complete translating soliton in $\mathbb{R}^{n+p}$ satisfying $\left(\int_M |A|^n e^{\langle V,X\rangle}\right) < \infty$

and $(\int_M |A|^n \mathrm{d}\mu)^{1/n} < C$ for certain positive constant C is a linear space. Here A denotes the second fundamental form of a submanifold.

Define the trace-free second fundamental form $\mathring{A}$ of a submanifold by $\mathring{A} = A - \frac{1}{n} g \otimes H$. In the present paper, we will prove a rigidity theorem for translating solitons under integral curvature pinching conditions of the trace-free second fundamental form.

Theorem 1. *Let $M^n (n \geq 3)$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+p}$. If the trace-free second fundamental form $\mathring{A}$ of M satisfies*

$$\Big(\int_M |\mathring{A}|^n \mathrm{d}\mu \Big)^{1/n} < K(n) \quad \text{and} \quad \int_M |\mathring{A}|^n e^{\langle V, X \rangle} < \infty,$$

where $K(n)$ is an explicit positive constant depending only on n, then M is a linear subspace.

It is obvious that the curvature condition in Theorem 1 is weaker than that in the rigidity theorem of Xin [20].

2. Preliminaries

Let $X : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional immersed submanifold. Denote by g the induced metric on M. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n + p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p.$$

Choose a local field of orthonormal frame field $\{e_A\}$ in $\mathbb{R}^{n+p}$ such that, restricted to M, the e_i's are tangent to M^n. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of $\mathbb{R}^{n+p}$, respectively. Restricting these forms to M, we have

$$\begin{aligned}
\omega_{\alpha i} &= \sum_j h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \\
A &= \sum_{\alpha, i, j} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha = \sum_{ij} h_{ij} \omega_i \otimes \omega_j, \\
H &= \sum_{\alpha, i} h^\alpha_{ii} e_\alpha = \sum_\alpha H^\alpha e_\alpha, \\
R_{ijkl} &= \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \\
R_{\alpha\beta kl} &= \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}),
\end{aligned}$$

where $A, H, R_{ijkl}, R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the Riemannian curvature tensor, the normal curvature tensor of M, respectively. The trace-free second fundamental form is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$. We have the relations $|\mathring{A}|^2 = |A|^2 - \frac{1}{n}|H|^2$ and $|\nabla\mathring{A}|^2 = |\nabla A|^2 - \frac{1}{n}\nabla|H|^2$.

Denoting the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl} respectively, we have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha},$$
$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

Then we have

$$h^\alpha_{ijk} = h^\alpha_{ikj},$$
$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl} - \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}.$$

Hence

$$\begin{aligned}\Delta h^\alpha_{ij} &= \sum_k h^\alpha_{ijkk} \\ &= \sum_k h^\alpha_{kkij} + \sum_k \Big(\sum_m h^\alpha_{km}R_{mijk} + \sum_m h^\alpha_{mi}R_{mkjk} - \sum_\beta h^\beta_{ki}R_{\alpha\beta jk}\Big).\end{aligned} \tag{4}$$

As in [20], we need a linear operator $\mathcal{L}_{II}$ on M

$$\mathcal{L}_{II} = \Delta + \langle V, \nabla(\cdot)\rangle = e^{-\langle V,X\rangle}\mathrm{div}(e^{\langle V,X\rangle}\nabla(\cdot)),$$

where Δ, div and ∇ denote the Laplacian, divergence and the gradient operator on M, respectively. It can be shown that $\mathcal{L}_{II}$ is self-adjoint respect to the measure $e^{\langle V,X\rangle}\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the volume form of M. We denote $\varrho = e^{\langle V,X\rangle}$ and $\mathrm{d}\mu$ might be omitted in the integrations for notational simplicity.

In order to prove our theorem, we need the following Simons type identities. The first equality has been proved by Xin [20]. For the convention of readers, we also include this part of the proof here.

Lemma 1. *On a translating soliton M^n in $\mathbb{R}^{n+p}$, we have*

(5)
$$\mathcal{L}_{II}|A|^2 = 2|\nabla A|^2 - 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\Big)^2 - 2\sum_{i,j,\alpha,\beta}\Big(\sum_p (h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{pi})\Big)^2,$$

$$\mathcal{L}_{II}|H|^2 = 2|\nabla H|^2 - 2\sum_{i,j}\Big(\sum_{\alpha} H^\alpha h_{ij}^\alpha\Big)^2, \tag{6}$$

where $H^\alpha = \sum_i h_{ii}^\alpha$.

Proof. From the translating soliton equation $H = V^N$, we derive

$$\nabla_i H^\alpha = -\sum_k \langle V, e_k\rangle h_{ik}^\alpha,$$

and

$$\nabla_j\nabla_i H^\alpha = -\langle H, h_{jk}\rangle h_{ik}^\alpha - \sum_k \langle V, e_k\rangle h_{ikj}^\alpha. \tag{7}$$

Combining (4) and (7), we obtain that

$$\begin{aligned}\sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha =& \sum_{i,j,\alpha} h_{ij}^\alpha \nabla_j\nabla_i H^\alpha \\ &+ \sum_{i,j,k,\alpha} h_{ij}^\alpha\Big(\sum_m h_{km}^\alpha R_{mijk} + \sum_m h_{mi}^\alpha R_{mkjk} - \sum_\beta h_{ki}^\beta R_{\alpha\beta jk}\Big)\\ =& -\sum_k \langle V, e_k\rangle h_{ikj}^\alpha h_{ij}^\alpha - \sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta\Big)^2 \\ &- \sum_{i,j,\alpha,\beta}\Big(\sum_p (h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{pi}^\beta)\Big)^2.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}_{II}|A|^2 =& \Delta|A|^2 + \langle V, \nabla|A|^2\rangle \\ =& 2\sum_{i,j,\alpha} h_{ij}^\alpha\Delta h_{ij}^\alpha + 2|\nabla A|^2 + 2\sum_k \langle V, e_k\rangle h_{ij}^\alpha h_{ijk}^\alpha \\ =& 2|\nabla A|^2 - 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta\Big)^2 - 2\sum_{i,j,\alpha,\beta}\Big(\sum_p (h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{pi}^\beta)\Big)^2.\end{aligned}$$

On the other hand, from (7) one has

$$\Delta|H|^2 = 2|\nabla H|^2 - 2\sum_{i,j}(\sum_\alpha H^\alpha h_{ij}^\alpha)^2 - 2\sum_{\alpha,i} H^\alpha H_i^\alpha \langle V, e_i\rangle,$$

where $H^\alpha = \sum_i h_{ii}^\alpha$.

Then it follows that

$$\mathcal{L}_{II}|H|^2 = \Delta|H|^2 + \langle V, \nabla|H|^2\rangle = 2|\nabla H|^2 - 2\sum_{i,j}(\sum_{\alpha} H^\alpha h_{ij}^\alpha)^2.$$

□

The following Sobolev inequality for submanifolds in the Euclidean space is very useful in the proof of our theorem.

Lemma 2 ([21])**.** *Let $M^n (n \geq 3)$ be a complete submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Let f be a nonnegative C^1 function with compact support. Then for all $s \in \mathbb{R}^+$, we have*

$$\|f\|^2_{\frac{2n}{n-2}} \leq D^2(n)\Big[\frac{4(n-1)^2(1+s)}{(n-2)^2}\|\nabla f\|_2^2 + \Big(1+\frac{1}{s}\Big)\frac{1}{n^2}\||H|f\|_2^2\Big],$$

where $D(n) = 2^n(1+n)^{\frac{n+1}{n}}(n-1)^{-1}\sigma_n^{-\frac{1}{n}}$, and σ_n denotes the volume of the unit ball in $\mathbb{R}^n$.

3. Proof of Theorem 1

In this section, we will give several lemmas first to prove Theorem 1.

Lemma 3. *On a translating soliton M^n in $\mathbb{R}^{n+p}$, we have*

$$\mathcal{L}_{II}|\mathring{A}|^2 \geq 2|\nabla|\mathring{A}||^2 - \iota|\mathring{A}|^4 - \frac{2}{n}|H|^2|\mathring{A}|^2, \tag{8}$$

where

$$\iota = \begin{cases} 2, & \text{if} \quad p = 1, \\ 4, & \text{if} \quad p \geq 2. \end{cases}$$

Proof. Combining (5) and (6), we have

$$\begin{aligned}\mathcal{L}_{II}|\mathring{A}|^2 =& \mathcal{L}_{II}|A|^2 - \frac{1}{n}\mathcal{L}_{II}|H|^2 \\ =& 2|\nabla\mathring{A}|^2 + \frac{2}{n}\sum_{i,j}\Big(\sum_{\alpha} H^\alpha h_{ij}^\alpha\Big)^2 - 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta\Big)^2 \\ & - 2\sum_{i,j,\alpha,\beta}\Big(\sum_{p}(h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{pi}^\beta)\Big)^2.\end{aligned} \tag{9}$$

When the codimension is one, it can be easily obtained that

$$\mathcal{L}_{II}|\mathring{A}|^2 \geq 2|\nabla|\mathring{A}||^2 - 2|\mathring{A}|^4 - \frac{2}{n}|H|^2|\mathring{A}|^2,$$

where we have used the inequality $|\nabla\mathring{A}|^2 \geq |\nabla|\mathring{A}||^2$, which is an easy consequence of the Schwartz inequality.

In the codimension $p \geq 2$ case, we need the following estimates. At the point where the mean curvature vector is zero, we have

$$\begin{aligned}
&\frac{2}{n}\sum_{i,j}\Big(\sum_{\alpha} H^{\alpha}h_{ij}^{\alpha}\Big)^2 - 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2 \\
&\quad - 2\sum_{i,j,\alpha,\beta}\Big(\sum_{p}(h_{ip}^{\alpha}h_{pj}^{\beta} - h_{jp}^{\alpha}h_{pi}^{\beta})\Big)^2 \\
&= -2\sum_{\alpha,\beta} N(A^{\alpha}A^{\beta} - A^{\beta}A^{\alpha}) - 2\sum_{\alpha,\beta}[\mathrm{tr}(A^{\alpha}A^{\beta})]^2 \\
&\geq -3|A|^4,
\end{aligned} \tag{10}$$

where $A^{\alpha} = (h_{ij}^{\alpha})_{n\times n}$ and we have used Theorem 1 in [13] to get the inequality.

At the point where the mean curvature vector is nonzero, we choose $e_{n+1} = \frac{H}{|H|}$. The second fundamental form can be written as $A = \sum_{\alpha} h^{\alpha}e_{\alpha}$, where $h^{\alpha}, n+1 \leq \alpha \leq n+p$, are symmetric 2-tensors.

By the choice of e_{n+1}, we see that $\mathrm{tr}h^{n+1} = |H|$ and $\mathrm{tr}h^{\alpha} = 0$ for $\alpha \geq n+2$. The trace-free second fundamental form may be rewritten as $\mathring{A} = \sum_{\alpha}\mathring{h}^{\alpha}e_{\alpha}$, where $\mathring{h}^{n+1} = h^{n+1} - \frac{|H|}{n}\mathrm{Id}$ and $\mathring{h}^{\alpha} = h^{\alpha}$ for $\alpha \geq n+2$. We set

$$A_H = h^{n+1}e_{n+1}, \quad A_I = \sum_{\alpha\geq n+2} h^{\alpha}e_{\alpha},$$
$$\mathring{A}_H = \mathring{h}^{n+1}e_{n+1}, \quad \mathring{A}_I = \sum_{\alpha\geq n+2} \mathring{h}^{\alpha}e_{\alpha}.$$

Then we have

$$|A_I|^2 = \sum_{\alpha\geq n+2}|h^{\alpha}|^2 = |A|^2 - |A_H|^2,$$
$$|\mathring{A}_I|^2 = \sum_{\alpha\geq n+2}|\mathring{h}^{\alpha}|^2 = |\mathring{A}|^2 - |\mathring{A}_H|^2.$$

Note that $|\mathring{A}_H|^2 = |A_H|^2 - \frac{|H|^2}{n}$ and $|\mathring{A}_I|^2 = |A_I|^2$. Since e_{n+1} is chosen globally, $|A_H|^2, |\mathring{A}_H|^2$ and $|A_I|^2$ are defined globally and independent of the choice of e_i. Then we have

$$\sum_{\alpha,\beta}\Big(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2 =|\mathring{A}_H|^4+\frac{2}{n}|H|^2|\mathring{A}_H|^2+\frac{1}{n^2}|H|^4 + 2\sum_{\alpha\neq n+1}\Big(\sum_{i,j}\mathring{h}_{ij}^{n+1}\mathring{h}_{ij}^{\alpha}\Big)^2+\sum_{\alpha,\beta\neq n+1}\Big(\sum_{i,j}\mathring{h}_{ij}^{\alpha}\mathring{h}_{ij}^{\beta}\Big)^2, \tag{11}$$

$$\sum_{i,j,\alpha,\beta}\Big(\sum_{p}(h_{ip}^{\alpha}h_{pj}^{\beta}-h_{jp}^{\alpha}h_{pi}^{\beta})\Big)^2 =2\sum_{\alpha\neq n+1}\sum_{i,j}\Big(\sum_{p}(h_{ip}^{n+1}\mathring{h}_{pj}^{\alpha}-h_{jp}^{n+1}\mathring{h}_{pi}^{\alpha})\Big)^2 + \sum_{\alpha,\beta\neq n+1}\sum_{i,j}\Big(\sum_{p}(\mathring{h}_{ip}^{\alpha}\mathring{h}_{pj}^{\beta}-\mathring{h}_{jp}^{\alpha}\mathring{h}_{pi}^{\beta})\Big)^2, \tag{12}$$

and

$$\sum_{i,j}\Big(\sum_{\alpha}H^{\alpha}h_{ij}^{\alpha}\Big)^2=|H|^2|\mathring{A}_H|^2+\frac{1}{n}|H|^4. \tag{13}$$

From (11), (12) and (13), we obtain the following

$$\begin{aligned}
&2\sum_{\alpha,\beta}\Big(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2+2\sum_{i,j,\alpha,\beta}\Big(\sum_{p}(h_{ip}^{\alpha}h_{pj}^{\beta}-h_{jp}^{\alpha}h_{pi}^{\beta})\Big)^2-\frac{2}{n}\sum_{i,j}\Big(\sum_{\alpha}H^{\alpha}h_{ij}^{\alpha}\Big)^2\\
=&2|\mathring{A}_H|^4+\frac{2}{n}|H|^2|\mathring{A}_H|^2\\
&+4\sum_{\alpha\neq n+1}\Big(\sum_{i,j}\mathring{h}_{ij}^{n+1}\mathring{h}_{ij}^{\alpha}\Big)^2+4\sum_{\alpha\neq n+1}\sum_{i,j}\Big(\sum_{p}(h_{ip}^{n+1}\mathring{h}_{pj}^{\alpha}-h_{jp}^{n+1}\mathring{h}_{pi}^{\alpha})\Big)^2\\
&+2\sum_{\alpha,\beta\neq n+1}\Big(\sum_{i,j}\mathring{h}_{ij}^{\alpha}\mathring{h}_{ij}^{\beta}\Big)^2+2\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\Big(\sum_{p}(\mathring{h}_{ip}^{\alpha}\mathring{h}_{pj}^{\beta}-\mathring{h}_{jp}^{\alpha}\mathring{h}_{pi}^{\beta})\Big)^2.
\end{aligned} \tag{14}$$

Choose $\{e_i\}$ such that $h_{ij}^{n+1}=\lambda_i\delta_{ij}$. Then $\mathring{h}_{ij}^{n+1}=\mathring{\lambda}_i\delta_{ij}$, where $\mathring{\lambda}_i=\lambda_i-\frac{|H|}{n}$. We have the following estimates.

$$\begin{aligned}
&4\sum_{\alpha\neq n+1}\Big(\sum_{i,j}\mathring{h}_{ij}^{n+1}\mathring{h}_{ij}^{\alpha}\Big)^2\\
=&4\sum_{\alpha\neq n+1}\Big(\sum_{i}\mathring{\lambda}_i\mathring{h}_{ii}^{\alpha}\Big)^2\\
\leq&4\Big(\sum_{i}\mathring{\lambda}_i^2\Big)\Big(\sum_{\alpha\neq n+1}\sum_{i}(\mathring{h}_{ii}^{\alpha})^2\Big)
\end{aligned}$$

$$=4|\mathring{A}_H|^2\sum_{\alpha\neq n+1}\sum_i(\mathring{h}^\alpha_{ii})^2,$$

where we have used the Cauchy-Schwarz inequality. We also have

$$\begin{aligned}&4\sum_{\alpha\neq n+1}\sum_{i,j}\Big(\sum_p(h^{n+1}_{ip}\mathring{h}^\alpha_{pj}-h^{n+1}_{jp}\mathring{h}^\alpha_{pi})\Big)^2\\ =&4\sum_{\alpha\neq n+1}\sum_{i\neq j}(\lambda_i-\lambda_j)^2(\mathring{h}^\alpha_{ij})^2\\ =&4\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{\lambda}_i-\mathring{\lambda}_j)^2(\mathring{h}^\alpha_{ij})^2\\ \leq&8\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{\lambda}_i^2+\mathring{\lambda}_j^2)(\mathring{h}^\alpha_{ij})^2\\ \leq&8|\mathring{A}_H|^2\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{h}^\alpha_{ij})^2\\ =&8|\mathring{A}_H|^2\Big(|\mathring{A}_I|^2-\sum_{\alpha\neq n+1}\sum_i(\mathring{h}^\alpha_{ii})^2\Big).\end{aligned}$$

By using Theorem 1 in [13], we obtain that

$$2\sum_{\alpha,\beta\neq n+1}\Big(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\Big)^2+2\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\Big(\sum_p(\mathring{h}^\alpha_{ip}\mathring{h}^\beta_{pj}-\mathring{h}^\alpha_{jp}\mathring{h}^\beta_{pi})\Big)^2\leq 3|\mathring{A}_I|^4.$$

Hence, we have the following estimate

$$\begin{aligned}&\frac{2}{n}\sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2-2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2\\ &-2\sum_{i,j,\alpha,\beta}\Big(\sum_p(h^\alpha_{ip}h^\beta_{pj}-h^\alpha_{jp}h^\beta_{pi})\Big)^2\\ \geq&-4|\mathring{A}|^4-\frac{2}{n}|H|^2|\mathring{A}|^2.\end{aligned}\tag{15}$$

Combining (10) and (15), we have

$$\begin{aligned}&\frac{2}{n}\sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2-2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2\\ &-2\sum_{i,j,\alpha,\beta}\Big(\sum_p(h^\alpha_{ip}h^\beta_{pj}-h^\alpha_{jp}h^\beta_{pi})\Big)^2\\ \geq&-4|\mathring{A}|^4-\frac{2}{n}|H|^2|\mathring{A}|^2.\end{aligned}\tag{16}$$

Substituting (16) into (9), we obtain that

$$\mathcal{L}_{II}|\mathring{A}|^2 \geq 2|\nabla|\mathring{A}||^2 - 4|\mathring{A}|^4 - \frac{2}{n}|H|^2|\mathring{A}|^2.$$

Thus, we complete the proof. □

Lemma 4. *For any smooth function η with compact support on M and any $0 < \varepsilon < n-1$, we have*

$$\begin{aligned}\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\eta^2\varrho \leq & \frac{1}{n-1-\varepsilon}\Big(\frac{\iota}{2}\int_M |\mathring{A}|^{n+2}\eta^2\varrho \\ & + \frac{1}{n}\int_M |\mathring{A}|^n|H|^2\eta^2\varrho + \frac{1}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho\Big).\end{aligned} \tag{17}$$

Proof. Multiplying $|\mathring{A}|^{n-2}\eta^2$ on both sides of the (8) and integrating by parts with respect to the measure $\varrho d\mu$ on M yield

$$\begin{aligned}0 \geq & 2\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\eta^2\varrho - \iota\int_M |\mathring{A}|^{n+2}\eta^2\varrho - \frac{2}{n}\int_M |\mathring{A}|^n|H|^2\eta^2\varrho \\ & - \int_M |\mathring{A}|^{n-2}\eta^2\mathcal{L}_{II}|\mathring{A}|^2\varrho.\end{aligned} \tag{18}$$

Since η has compact support on M, by the Stokes theorem, we obtain that

$$\begin{aligned} & -\int_M |\mathring{A}|^{n-2}\eta^2\mathcal{L}_{II}|\mathring{A}|^2\varrho \\ = & -\int_M |\mathring{A}|^{n-2}\eta^2 \mathrm{div}(\varrho\cdot\nabla|\mathring{A}|^2) \\ = & 2\int_M \varrho|\mathring{A}|\nabla|\mathring{A}|\cdot\nabla(|\mathring{A}|^{n-2}\eta^2) \\ = & 2(n-2)\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\eta^2\varrho + 4\int_M (\nabla|\mathring{A}|\cdot\nabla\eta)|\mathring{A}|^{n-1}\eta\varrho.\end{aligned} \tag{19}$$

Combining (18) and (19), we get

$$\begin{aligned}0 \geq & 2(n-1)\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\eta^2\varrho - \iota\int_M |\mathring{A}|^{n+2}\eta^2\varrho - \frac{2}{n}\int_M |\mathring{A}|^n|H|^2\eta^2\varrho \\ & + 4\int_M (\nabla|\mathring{A}|\cdot\nabla\eta)|\mathring{A}|^{n-1}\eta\varrho.\end{aligned}$$

By the Cauchy inequality, for any $0 < \varepsilon < n-1$, we obtain that

$$\begin{aligned}&\iota\int_M |\mathring{A}|^{n+2}\eta^2\varrho + \frac{2}{n}\int_M |\mathring{A}|^n|H|^2\eta^2\varrho + \frac{2}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho\\ &\geq 2(n-1-\varepsilon)\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\eta^2\varrho.\end{aligned}$$

□

Lemma 5. *Setting $f = |\mathring{A}|^{n/2}\varrho^{1/2}\eta$, we have*

$$\int_M |\nabla f|^2 = \int_M |\nabla(|\mathring{A}|^{n/2}\eta)|^2\varrho - \frac{1}{2}\int_M |\mathring{A}|^n\eta^2\varrho + \frac{1}{4}\int_M |\mathring{A}|^n|V^T|^2\eta^2\varrho. \tag{20}$$

where η is a smooth function with compact support on M and V^T is the tangent component of vector V.

Proof. Integrating by parts, one obtain

$$\begin{aligned}\int_M |\nabla f|^2 &= \int_M |\nabla(|\mathring{A}|^{n/2}\eta)|^2\varrho + \frac{1}{2}\int_M \nabla(|\mathring{A}|^n\eta^2)\nabla\varrho + \int_M |\mathring{A}|^n\eta^2|\nabla\varrho^{\frac{1}{2}}|^2\\ &= \int_M |\nabla(|\mathring{A}|^{n/2}\eta)|^2\varrho - \frac{1}{2}\int_M |\mathring{A}|^n\eta^2\Delta\varrho + \int_M |\mathring{A}|^n\eta^2|\nabla\varrho^{\frac{1}{2}}|^2.\end{aligned}$$

By direct computations, we have

$$\nabla\varrho = \nabla e^{\langle V,X\rangle} = \varrho V^T,$$

and

$$\nabla\varrho^{\frac{1}{2}} = \frac{1}{2}\varrho^{-\frac{1}{2}}\nabla\varrho = \frac{1}{2}\varrho^{\frac{1}{2}}V^T.$$

By the translating soliton equation $H = V^N$, we get

$$\Delta\varrho = \sum_i \nabla_i\varrho\langle V, e_i\rangle + \sum_i \varrho\langle V, \nabla_i e_i\rangle = \varrho(|V^T|^2 + |V^N|^2) = \varrho.$$

Hence, it follows that

$$\int_M |\nabla f|^2 = \int_M |\nabla(|\mathring{A}|^{n/2}\eta)|^2\varrho - \frac{1}{2}\int_M |\mathring{A}|^n\eta^2\varrho + \frac{1}{4}\int_M |\mathring{A}|^n|V^T|^2\eta^2\varrho.$$

□

Now we will give the proof of Theorem 1.

Proof. Combining the Sobolev inequality in Lemma 2 and (20) in Lemma 5, we have

$$\begin{aligned}
&\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\leq& D^2(n)\cdot\Big\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\int_M|\nabla f|^2+\Big(1+\frac{1}{s}\Big)\cdot\frac{1}{n^2}\int_M|H|^2f^2\Big\}\\
=&D^2(n)\cdot\Big\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\Big(\int_M|\nabla(|\mathring{A}|^{n/2}\eta)|^2\varrho-\frac{1}{2}\int_M|\mathring{A}|^n\eta^2\varrho\\
&+\frac{1}{4}\int_M|\mathring{A}|^n|V^T|^2\eta^2\varrho\Big)+\Big(1+\frac{1}{s}\Big)\cdot\frac{1}{n^2}\int_M|\mathring{A}|^n|H|^2\eta^2\varrho\Big\}.
\end{aligned}$$

Note that

$$|V^T|^2+|V^N|^2=|V^T|^2+|H|^2=1.$$

We deduce that

$$\begin{aligned}
&\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\leq& D^2(n)\cdot\Big\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\Big(\int_M|\nabla(|\mathring{A}|^{n/2}\eta)|^2\varrho-\frac{1}{4}\int_M|\mathring{A}|^n|V^T|^2\eta^2\varrho\\
&-\frac{1}{2}\int_M|\mathring{A}|^n|H|^2\eta^2\varrho\Big)+\Big(1+\frac{1}{s}\Big)\cdot\frac{1}{n^2}\int_M|\mathring{A}|^n|H|^2\eta^2\varrho\Big\}\\
=&D^2(n)\cdot\Big\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\Big(\int_M\frac{n^2}{4}|\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\eta^2\varrho\\
&+\int_M n|\mathring{A}|^{n-1}\eta\nabla|\mathring{A}|\cdot\nabla\eta\varrho+\int_M|\mathring{A}|^n|\nabla\eta|^2\varrho-\frac{1}{4}\int_M|\mathring{A}|^n|V^T|^2\eta^2\varrho\\
&-\frac{1}{2}\int_M|\mathring{A}|^n|H|^2\eta^2\varrho\Big)+\Big(1+\frac{1}{s}\Big)\cdot\frac{1}{n^2}\int_M|\mathring{A}|^n|H|^2\eta^2\varrho\Big\}.
\end{aligned}$$

By the Cauchy inequality, we have for any $\delta>0$

(21)

$$\begin{aligned}
&\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\leq&\frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2}\Big\{(1+\delta)\frac{n^2}{4}\int_M|\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\eta^2\varrho\\
&+\Big(1+\frac{1}{\delta}\Big)\int_M|\mathring{A}|^n|\nabla\eta|^2\varrho-\frac{1}{4}\int_M|\mathring{A}|^n|V^T|^2\eta^2\varrho-\frac{1}{2}\int_M|\mathring{A}|^n|H|^2\eta^2\varrho\Big\}\\
&+D^2(n)\Big(1+\frac{1}{s}\Big)\cdot\frac{1}{n^2}\int_M|\mathring{A}|^n|H|^2\eta^2\varrho.
\end{aligned}$$

Substituting (17) into (21), we get

$$\begin{aligned}
&\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\le&\frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2}\Big\{\frac{n^2(1+\delta)}{4(n-1-\varepsilon)}\Big(\frac{\iota}{2}\int_M |\mathring{A}|^{n+2}\eta^2\varrho\\
&+\frac{1}{n}\int_M |\mathring{A}|^n|H|^2\eta^2\varrho+\frac{1}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho\Big)\\
&+\Big(1+\frac{1}{\delta}\Big)\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho-\frac{1}{2}\int_M |\mathring{A}|^n|H|^2\eta^2\varrho\Big\}\\
&+D^2(n)\Big(1+\frac{1}{s}\Big)\cdot\frac{1}{n^2}\int_M |\mathring{A}|^n|H|^2\eta^2\varrho.
\end{aligned}$$

Put

$$\delta=\delta(s,\varepsilon)=\frac{[2sn^2(n-1)^2-(n-2)^2](n-1-\varepsilon)}{sn^3(n-1)^2}-1>0,$$

for some positive constant s satisfies

$$s>\frac{(n-2)^2(n-1-\varepsilon)}{n^2(n-1)^2(n-2-2\varepsilon)}\in\mathbb{R}^+$$

and some $\varepsilon\in(0,\frac{n-2}{2})$ to be defined later. Then we conclude that

$$\begin{aligned}
&\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\le&\frac{n^2(1+s)(1+\delta)}{4(n-1-\varepsilon)}\Big(\frac{\iota}{2}\int_M |\mathring{A}|^{n+2}\eta^2\varrho+\frac{1}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho\Big)\\
&+(1+s)\Big(1+\frac{1}{\delta}\Big)\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho\\
=&\frac{(1+s)\iota[2sn^2(n-1)^2-(n-2)^2]}{8sn(n-1)^2}\int_M |\mathring{A}|^{n+2}\eta^2\varrho\\
&+C(s,\varepsilon,n)\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho,
\end{aligned}\tag{22}$$

where $C(s,\varepsilon,n)$ is an explicit positive constant depending on s,ε and n, and

$$\kappa=\frac{4D^2(n)(n-1)^2}{(n-2)^2}.$$

By the Hölder inequality, we have

$$\int_M |\mathring{A}|^{n+2}\eta^2\varrho\le\Big(\int_M |\mathring{A}|^{2\cdot\frac{n}{2}}\Big)^{\frac{2}{n}}\cdot\Big(\int_M (|\mathring{A}|^n\eta^2\varrho)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}.$$

Hence

$$
\begin{aligned}
&\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\
\leq &\frac{(1+s)\iota[2sn^2(n-1)^2-(n-2)^2]}{8sn(n-1)^2}\Big(\int_M |\mathring{A}|^n\Big)^{\frac{2}{n}}\cdot\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\
&+C(s,\varepsilon,n)\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho.
\end{aligned}
\tag{23}
$$

Put

$$K(n,s)=\sqrt{\frac{8sn(n-1)^2}{(1+s)\iota[2sn^2(n-1)^2-(n-2)^2]\kappa}}.$$

For simplicity, we choose

$$s=s(\varepsilon)=\frac{(n-2)^2}{n^2(n-1)(n-2-2\varepsilon)}$$

such that

$$K(n,\varepsilon)=K(n,s(\varepsilon))=\sqrt{\frac{2n(n-2)^2}{\iota D^2(n)(n+2\varepsilon)[(n-2)^2/(n-2-2\varepsilon)+n^2(n-1)]}}.$$

Set

$$K(n)=\sup_{\varepsilon\in(0,\frac{n-2}{2})}K(n,\varepsilon)=\sqrt{\frac{2(n-2)^2}{\iota D^2(n)[n-2+n^2(n-1)]}},$$

where

$$\iota=\begin{cases}2, & \text{if} \quad p=1,\\ 4, & \text{if} \quad p\geq 2.\end{cases}$$

Since we have the assumption

$$\Big(\int_M |\mathring{A}|^n \mathrm{d}\mu\Big)^{1/n}<K(n),$$

there exists a positive constant $\check{K}$ such that

$$\Big(\int_M |\mathring{A}|^n \mathrm{d}\mu\Big)^{1/n}<\check{K}<K(n). \tag{24}$$

Thus, there exists $\varepsilon=\varepsilon_0>0$ such that

$$\check{K}<K(n,\varepsilon_0)<K(n).$$

That is to say

$$\frac{(1+s)\iota[2sn^2(n-1)^2-(n-2)^2]}{8sn(n-1)^2}=\kappa^{-1}\cdot K(n,\varepsilon_0)^{-2}, \tag{25}$$

where

$$s=s(\varepsilon_0)=\frac{(n-2)^2}{n^2(n-1)(n-2-2\varepsilon_0)}.$$

Combining (23), (24) and (25), it implies that there exists $0<\epsilon<1$ such that

$$\begin{aligned}&\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ \leq&\kappa^{-1}\cdot K(n,\varepsilon_0)^{-2}\cdot\check{K}^2\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+\tilde{C}(n,\varepsilon_0)\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho\\ \leq&\frac{1-\epsilon}{\kappa}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+\tilde{C}(n,\varepsilon_0)\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho,\end{aligned}$$

namely,

$$\frac{\epsilon}{\kappa}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\leq\tilde{C}(n,\varepsilon_0)\int_M |\mathring{A}|^n|\nabla\eta|^2\varrho. \tag{26}$$

Let $\eta(X)=\eta_r(X)=\phi(\frac{|X|}{r})$ for any $r>0$, where ϕ is a nonnegative function on $[0,+\infty)$ satisfying

$$\phi(x)=\begin{cases}1, & \text{if}\quad x\in[0,1),\\ 0, & \text{if}\quad x\in[2,+\infty),\end{cases} \tag{27}$$

and $|\phi'|\leq C$ for some absolute constant.

Since $\int_M |\mathring{A}|^n\varrho$ and the constant $\tilde{C}(n,\varepsilon_0)$ are bounded, the right hand side of (26) approaches to zero as $r\to+\infty$, which implies $|\mathring{A}|\equiv 0$. Therefore, M is a linear subspace. This completes the proof of Theorem 1. □

References

[1] C. Bao and Y. Shi, Gauss map of translating solitons of mean curvature flow, *Proc. Amer. Math. Soc.*, **142**(2014), 4333–4339.

[2] S. J. Cao, H. W. Xu and E. T. Zhao, Pinching theorems for self-shrinkers of higher codimension, preprint, 2014.

[3] Q. M. Cheng and G. Wei, A gap theorem of self-shrinkers, *Trans. Amer. Math. Soc.*, **367**(2015), 4895–4915.

[4] Q. M. Cheng and G. Wei, Complete λ-hypersurfaces of weighted volume-preserving mean curvature flow, *Calc. Var. Partial Differ. Equations*, (2018) 57:32, DOI: 10.1007/s00526-018-1303-4.

[5] T. H. Colding and W. P. Minicozzi II, Generic mean curvature flow I; generic singularities, *Ann. of Math.*, **175**(2012), 755–833.

[6] Q. Ding and Y. L. Xin, The rigidity theorems of self-shrinkers, *Trans. Amer. Math. Soc.*, **366**(2014), 5067–5085.

[7] G. Huisken and C. Sinestrari, Convexity estimates for mean curvature flow and singularities of mean convex surfaces, *Acta. Math.*, **183**(1999), 45–70.

[8] T. Ilmanen, Singularities of mean curvature flow of surfaces, preprint, 1995, available at https://people.math.ethz.ch/~ilmanen/papers/pub.html.

[9] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature, *Mem. Amer. Math. Soc.*, **108**(1994).

[10] D. Impera and M. Rimoldi, Rigidity results and topology at infinity of translating solitons of the mean curvature flow, *Commun. Contemp. Math.*, **19**(2017), 1750002 (21 pages).

[11] K. Kunikawa, Bernstein-type theorem of translating solitons in arbitrary codimension with flat normal bundle, *Calc. Var. Partial Differ. Equations*, **54**(2015), 1331–1344.

[12] L. Lei, H. W. Xu and Z. Y. Xu, A new pinching theorem for complete self-shrinkers and its generalization, arXiv:1712.01899.

[13] A. M. Li, J. M. Li, An intrinsic rigidity theorem for minimal submanifolds in a sphere, *Arch. Math.*, **58**(1992), 582–594.

[14] F. Martin, A. Savas-Halilaj and K. Smoczyk, On the topology of translating solitons of the mean curvature flow, *Calc. Var. Partial Differ. Equations*, **54**(2015), 2853–2882.

[15] X. H. Nguyen, Complete embedded self-translating surfaces under mean curvature flow, *J. Geom. Anal.*, **23**(2013), 1379–1426.

[16] H. J. Wang, H. W. Xu and E. T. Zhao, Gap theorems for complete λ-hypersurfaces, *Pacific. J. Math.*, **288**(2017), 453–474.

[17] H. J. Wang, H. W. Xu and E. T. Zhao, Submanifolds with parallel Gaussian mean curvature in Euclidean spaces, preprint, 2017.

[18] X. J. Wang, Convex solutions to mean curvature flow, *Ann. of Math.*, **173**(2011), 1185–1239.

[19] B. White, Stratification of minimal surfaces, mean curvature flows, and harmonic maps, *J. Reine Angew. Math.*, **488**(1997), 1–35.

[20] Y. L. Xin, Translating solitons of the mean curvature flow, *Calc. Var. Partial Differ. Equations*, **54**(2015), 1995–2016.

[21] H. W. Xu and J. R. Gu, A general gap theorem for submanifolds with parallel mean curvature in $\mathbb{R}^{n+p}$, *Comm. Anal. Geom.*, **15**(2007), 175–194.

[22] H. W. Xu and Z. Y. Xu, On Chern's conjecture for minimal hypersurfaces and rigidity of self-shrinkers, *J. Funct. Anal.*, **273**(2017), 3406–3425.

Huijuan Wang
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027
People's Republic of China
E-mail: whjuan@zju.edu.cn

Hongwei Xu
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027
People's Republic of China
E-mail: xuhw@zju.edu.cn

Entao Zhao
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027
People's Republic of China
E-mail: zhaoet@zju.edu.cn