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Globally asymptotical stability and existence of limit cycle for a generalized predator-prey model with prey refuge

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Abstract: The stability property of the positive equilibrium and the existence of limit cycles for the Lotka-Volterra predator-prey system incorporating prey refuge with a generalized functional response are investigated. On the one hand, by constructing a suitable Lyapunov function and an auxiliary system, a new set of sufficient conditions which guarantee the global asymptotical stability of the positive equilibrium are obtained. On the other hand, a set of sufficient conditions which guarantee the existence of limit cycles are produced by modifying the theorem of Hesaaraki and Moghadas. Our results complement and supplement some known ones, and some published conclusions become the special cases of ours.

Keywords: Predator-prey system, generalized response function, prey refuge, globally asymptotical stability, existence of limit cycle.

1. Introduction

The prey refuges which are employed by prey populations exist generally in nature, such as Ephestia spp, Balanus glandula [1], and so on. Based on theoretical and/or empirical aspects, the general conclusions of prey refuge on predation systems were the stabilizing effect and the prevention of prey extinction [2–19]. The seminal work on this subject was based on qualitative analysis and ecological meanings [2]. González-Olivares and Ramos-Jiliberto [2] considered the stability properties of a certain predation system with prey refuge and the dynamical consequences of applying mathematical analysis method. They proposed the following predator-prey system with Holling II functional response and the effect of prey refuge:

$$(1)\qquad \begin{cases} \dfrac{dx}{dt} = & r\Big(1-\dfrac{x}{K}\Big)x - \dfrac{q(x-x_r)y}{x-x_r+a}, \\ \dfrac{dy}{dt} = & b\Big(\dfrac{p(x-x_r)y}{x-x_r+a} - c\Big)y. \end{cases}$$

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They studied the dynamic consequences of system (1) and obtained that the community equilibrium was stabilized by the addition of refuge for a prey population; therefore, prey extinction was prevented. Furthermore, Ma et al. [3] proposed a predator-prey system with a generalized functional response which is more interesting and comprehensive than those studied by many researchers [8–12]. They presented the following system:

$$\begin{cases} \dfrac{dx}{dt} = & r\left(1 - \dfrac{x}{K}\right)x - p\varphi(x - x_r)y, \\ \dfrac{dy}{dt} = & (q\varphi(x - x_r) - d)y. \end{cases} \tag{2}$$

with the following assumption:

$$\varphi(0) = 0, \quad \varphi'(x) > 0 \quad (x > 0). \tag{3}$$

Assuming that $x_r = \beta x$ and using the following change of variables:

$$\Phi: \ (R_0^+)^2 \to (R_0^+)^2, \quad \Phi(x, y) = \left(\frac{x}{1-\beta}, (1-\beta)y\right),$$

system (2) with assumption (3) becomes the following form:

$$\begin{cases} \dfrac{dx}{dt} = & r\left(1 - \dfrac{x}{(1-\beta)K}\right)x - p\varphi(x)y, \\ \dfrac{dy}{dt} = & (q\varphi(x) - d)y. \end{cases} \tag{4}$$

with the following assumption:

$$\varphi(0) = 0, \quad \varphi'(x) > 0 \quad (x > 0), \quad \varphi''(x) < 0 \quad (x > 0). \tag{5}$$

By simple computation, one can obtain the positive equilibrium point of system (4) which is denoted as (x^*, y^*), where $x^* = \varphi^{-1}(d/q)$, $y^* = \frac{qrx^*}{pd}(1 - \frac{x^*}{(1-\beta)K})$.

They considered only the local stability of system (4) and mainly focused on the dynamical consequences induced by the effect of prey refuge. In fact, the global stability is more robust in ecological systems than the local property for some types of predation systems, and hence is an interesting issue [20–25]. The limit cycle means the interacting populations periodically oscillate and is important to the harvest rule for the economic populations [20, 21]. Recently, Chen et al. [4] constructed a suitable Lyapunov function and found a set of sufficient conditions which ensure the global asymptotical stability of the positive equilibrium. Unfortunately, they investigated only the predation system with Holling I functional response. However, the global stability of the system with a generalized functional response and prey refuge is more interesting and important than that with Holling type one. Therefore, the global stability and the existence of limit cycle of system (4) with the initial conditions (5) are still an open issue. Motivated by these, we focus on the global stability of the positive equilibrium point and existence of limit cycle of system (4) with the initial conditions (5) in this paper. Our work complements and supplements some known ones, such as González-Olivares and Ramos-Jiliberto [2], Ma et al. [3], Chen et al. [4] and so all.

2. Global stability property

2.1. Main results

Theorem 2.1.1. *Assuming that* $2\varphi'(x_B) + x_B\varphi''(x_B) > 0$, *then we have*

- *If*

$$\max\Big\{1 - \frac{dx_A + q\varphi(x_A)x^*}{d + q\varphi(x_A)K}, (1/K)[K - x_B - x^* + \sqrt{K^2 + (x_B + x^*)^2 + 2x^8K + 2x_BK - 4x_Bx^*}]\Big\} < \beta < 1 - \frac{\varphi^{-1}(d/q)}{K},$$

then the positive equilibrium point of system (4) is globally asymptotically stable.

in which $x^* = \varphi^{-1}(d/q)$, $x_A = (\varphi')^{-1}(\frac{d}{q(1-\beta)K})$ *and* $x_B = \varphi^{-1}(\frac{dx^*}{q((1-\beta)K-x^*)})$.

Theorem 2.1.2. *Assuming that* $2\varphi'(x_B) + x_B\varphi''(x_B) < 0$, *then we have*

- *If*

$$1 - \frac{dx_A + q\varphi(x_A)x^*}{d + q\varphi(x_A)K} < \beta < (1/K)\Big\{K - x_B - x^* + \sqrt{K^2 + (x_B + x^*)^2 + 2x^8K + 2x_BK - 4x_Bx^*}\Big\},$$

then the positive equilibrium point of system (4) is globally asymptotically stable.

in which $x^* = \varphi^{-1}(d/q)$, $x_A = (\varphi')^{-1}(\frac{d}{q(1-\beta)K})$ *and* $x_B = \varphi^{-1}(\frac{dx^*}{q((1-\beta)K-x^*)})$.

2.2. Proof of the main results

In this section, we will prove Theorem 2.1.1 and Theorem 2.1.2. This is completed according to the following three steps:

Step 1. Considering the following auxiliary system

$$\begin{cases} \dfrac{dx}{dt} = \left(\dfrac{x^* g(x^*)}{p\varphi(x^*)} - y\right)\varphi(x), \\ \dfrac{dy}{dt} = (q\varphi(x) - d)y. \end{cases} \tag{6}$$

and proving the global stability of the positive equilibrium point (x^*, y^*) of system (6) in which $g(x) = r(1 - \frac{x}{(1-\beta)K})$.

Now, choosing a Lyapunov function defined as follows

$$W(x(t), y(t)) = \int_{x^*}^{x} \frac{u - x^*}{u} du + a \int_{y^*}^{y} \frac{v - y^*}{v} dv.$$

By simple computation, we obtain that

$$\begin{aligned} \frac{dW}{dt} &= \frac{x - x^*}{x}\frac{dx}{dt} + a\frac{y - y^*}{y}\frac{dy}{dt} \\ &= (x - x^*)\left(\frac{x^* g(x^*)}{p\varphi(x^*)} - y\right)\varphi(x)/x + a(y - y^*)(q\varphi(x) - d) \\ &= -(x - x^*)(y - y^*)\varphi(x)/x + aq\varphi'(x^*)(x - x^*)(y - y^*) \\ &\quad + \frac{aq\varphi''(x^*)}{2}(x - x^*)^2(y - y^*). \end{aligned}$$

For the function $\varphi(x)/x$, the unique extremum point $\bar{x}$ exists when $\phi'(0) = \frac{\varphi((1-\beta)K)}{(1-\beta)K}$ by Roll Mean Theorem.

Defining $M = \max\{\phi'(0), \frac{\varphi(\bar{x})}{\bar{x}}\}$ and $a = \frac{M}{q\varphi'(x^*)}$, then

$$\begin{aligned} &-(x - x^*)(y - y^*)\varphi(x)/x + aq\varphi'(x^*)(x - x^*)(y - y^*) \\ &\quad + \frac{aq\varphi''(x^*)}{2}(x - x^*)^2(y - y^*) < \frac{aq\varphi''(x^*)}{2}(x - x^*)^2(y - y^*). \end{aligned}$$

It is clear that $\frac{dW}{dt} < 0$ in $\Sigma_1 = \{(x, y) \mid y > y^*\}$ if $\varphi''(x) < 0$ or $\Sigma_1 = \{(x, y) \mid y > y^*\}$ if $\varphi''(x) > 0$.

Hence, the positive equilibrium point (x^*, y^*) of system (6) is globally stable if $\varphi''(x) \neq 0$. In fact, the condition $\varphi''(x) \neq 0$ is satisfied by most response functions, such as Holling type functional response.

Step 2. We will prove that the flow of system (4) is always directed inwards with respect to the flow of system (6).

According to $(\frac{xg(x)}{p\varphi(x)} - y^*)(x - x^*) \leq 0$, then we have

- $\frac{xg(x)}{p\varphi(x)} \geq y^* \ \ (0 < x < x^*), \ \ \frac{xg(x)}{p\varphi(x)} \leq y^* \ \ (x^* < x < (1-\beta)K)$.

Defining

$$\overrightarrow{S_1} = \left(\left(\frac{x^*g(x^*)}{p\varphi(x^*)} - y^*\right)\varphi(x), (q\varphi(x) - d)y, 0\right),$$
$$\overrightarrow{S_2} = \left(\left(\frac{xg(x)}{p\varphi(x)} - y^*\right)\varphi(x), (q\varphi(x) - d)y, 0\right).$$

Then, we get

$$\overrightarrow{S_1} \times \overrightarrow{S_2} = (0, 0, \varphi(x)y(q\varphi(x) - d)\left(\frac{x^*g(x^*)}{p\varphi(x^*)} - \frac{xg(x)}{p\varphi(x)}\right)).$$

Hence, we have

- $q\varphi(x) - d \leq 0 \ \ (0 < x < x^*), \quad q\varphi(x) - d \geq 0 \ \ (x^* < x < (1-\beta)K)$,

since x^* is the uniquely positive solution and the function $q\varphi(x) - d$ increases monotonously.

Combining the above case with the previous assumption

- $\frac{xg(x)}{p\varphi(x)} \geq y^* \ \ (0 < x < x^*), \ \ \frac{xg(x)}{p\varphi(x)} \leq y^* \ \ (x^* < x < (1-\beta)K)$.

We have

$$\varphi(x)y(q\varphi(x) - d)\left(\frac{x^*g(x^*)}{p\varphi(x^*)} - \frac{xg(x)}{p\varphi(x)}\right) \geq 0.$$

Hence, the flow of system (4) is always directed inwards with respect to the flow of system (6).

Therefore, the positive equilibrium point (x^*, y^*) of system (4) is globally stabile since system (4) and system (6) have the same positive equilibrium point and the flow of system (4) is always directed inwards with respect to the flow of system (6) which is globally asymptotically stabile.

According to the above analysis, the following conclusion is obtained:

- If $(\frac{xg(x)}{p\varphi(x)} - y^*)(x - x^*) \leq 0$, then the positive equilibrium point (x^*, y^*) of system (4) is globally asymptotically stable.

Step 3. We will propose the sufficient conditions which can guarantee the inequality $(\frac{xg(x)}{p\varphi(x)} - y^*)(x - x^*) \le 0$.

Now, we compute the term $(\frac{xg(x)}{p\varphi(x)} - y^*)(x - x^*)$.

$$
\begin{aligned}
&\Big(\frac{xg(x)}{p\varphi(x)} - y^*\Big)(x - x^*)\\
&= \Big[\frac{rx(1-\frac{x}{(1-\beta)K})}{p\varphi(x)} - \frac{qrx^*}{pd}\Big(1 - \frac{x^*}{(1-\beta)K}\Big)\Big](x - x^*)\\
&= \Big[dx\Big(1 - \frac{x}{(1-\beta)K}\Big) - q\varphi(x)x^*\Big(1 - \frac{x^*}{(1-\beta)K}\Big)\Big](x - x^*)\\
&= \Big[dx - q\varphi(x)x^* - \frac{1}{(1-\beta)K}(dx^2 - q\varphi(x)xx^*)\Big](x - x^*)\\
&= (d + q\varphi(x))(x - x^*)^2 + (dx^* - q\varphi(x)x)(x - x^*)\\
&\quad + \frac{1}{(1-\beta)K}(\sqrt{q\varphi(x)}x^* + \sqrt{d}x)(\sqrt{q\varphi(x)}x^* - \sqrt{d}x)(x - x^*)\\
&= (d + q\varphi(x))(x - x^*)^2 + (dx^* - q\varphi(x)x)(x - x^*)\\
&\quad + \frac{\sqrt{q\varphi(x)}x^* + \sqrt{d}x}{(1-\beta)K}[(\sqrt{q\varphi(x)}x^* - \sqrt{d}x)(x - x^*)\\
&\quad - (\sqrt{q\varphi(x)} + \sqrt{d})(x - x^*)^2]\\
&= [d + q\varphi(x) - \sqrt{dq\varphi(x)}(x + x^*)](x - x^*)^2\\
&\quad - \Big[\frac{1}{(1-\beta)K}(\sqrt{q\varphi(x)}x^* + \sqrt{d}x)(\sqrt{q\varphi(x)} + \sqrt{d})\Big](x - x^*)^2\\
&\quad + \Big[dx^* - q\varphi(x)x + \frac{1}{(1-\beta)K}(q\varphi(x) - d)x^*x\Big](x - x^*)\\
&= \frac{1}{(1-\beta)K}[d((1-\beta)K - x) + q\varphi(x)((1-\beta)K - x^*)](x - x^*)^2\\
&\quad - [((1-\beta)K + 1)\sqrt{dq\varphi(x)}(x + x^*)](x - x^*)^2\\
&\quad + [d(1-\beta)K - x)x^* + q\varphi(x))((1-\beta)K - x^*)](x - x^*).
\end{aligned}
$$

Defining the following labels:

$$
\begin{aligned}
A(x) &= d((1-\beta)K - x) + q\varphi(x)((1-\beta)K - x^*),\\
B(x) &= d(1-\beta)K - x)x^* + q\varphi(x))((1-\beta)K - x^*).
\end{aligned}
$$

Therefore, it can be obtained that

$$\left(\frac{xg(x)}{p\varphi(x)} - y^*\right)(x - x^*) \leq 0$$

if

$$A(x) \leq 0 \quad (0 < x < K),$$

and

$$B \geq 0 \quad (0 < x < x^*); B \leq 0 \quad (x^* < x < K).$$

On the one hand, supposing that

$$\frac{dA(x)}{dx} = q(1-\beta)K\varphi'(x) - d = 0,$$

then, it is obtained that

$$x_A = (\varphi')^{-1}\left(\frac{d}{q(1-\beta)K}\right).$$

Again, assuming that $\varphi''(x) < 0$, then we have

$$\frac{d^2A(x)}{dx^2} = q(1-\beta)K\varphi''(x) < 0 \quad .$$

Thus, x_A is the maximum value of $A(x)$ in the interval $(0, K)$.
Based on the above analyses, we have the following conclusion:

- if $\max A(x) < 0$, then $A(x) < 0$ in $(0, K)$.

Now, we will compute the inequality $\max A(x) < 0$

$$\begin{aligned} \max A(x) < 0 \quad &\Leftrightarrow \quad d((1-\beta)K - x_A) + q\varphi(x_A)((1-\beta)K - x^*) \\ &\Leftrightarrow \quad (1-\beta)(d + q\varphi(x_A)K) < dx_A + q\varphi(x_A)x^* \\ &\Leftrightarrow \quad \beta > 1 - \frac{dx_A + q\varphi(x_A)x^*}{d + q\varphi(x_A)K}. \end{aligned}$$

Again, supposing that

$$\frac{dB(x)}{dx} = q((1-\beta)K - x^*)(\varphi(x) + x\varphi'(x)) - dx^* = 0,$$

then, it is obtained that

$$\varphi(x) = \frac{dx^*}{q((1-\beta)K - x^*)},$$

or

$$x_B = \varphi^{-1}\Big(\frac{dx^*}{q((1-\beta)K - x^*)}\Big).$$

By simple computation

$$\frac{d^2B(x)}{dx^2} = q((1-\beta)K - x^*)(2\varphi'(x) + x\varphi''(x)).$$

Therefore, we can obtain the following conclusions:

- if $\min B(x) \geq 0$, then $B(x) \geq 0$ $(0 < x < x*)$.
  Assuming that $2\varphi'(x_B) + x_B\varphi''(x_B) > 0$, by simple computation, we have

  $$\min B(x) = d(1-\beta)K - x_B)x^* + q\varphi(x_B))((1-\beta)K - x^*).$$

  Hence, it is obtained that

  $$\begin{aligned} \min B(x) \geq 0 \quad &\Leftrightarrow \quad 2x^*x_B - (1-\beta)Kx_B - (1-\beta)K(\beta K + x^*) \geq 0 \\ &\Leftrightarrow \quad K^2\beta^2 + K(x_B + x^* - K)\beta + 2x^*x_B \\ &\qquad - Kx_B - Kx^* \geq 0. \\ &\Leftrightarrow \quad \beta \geq (1/K)[K - x_B - x^* \\ &\qquad + \sqrt{K^2 + (x_B + x^*)^2 + 2x^8K + 2x_BK - 4x_Bx^*}]. \end{aligned}$$

- if $\max B(x) \leq 0$, then $B(x) \leq 0$ $(x* < x < K)$.
  Assuming that $2\varphi'(x_B) + x_B\varphi''(x_B) < 0$, by simple computation, we get

  $$\begin{aligned} \max B(x) \leq 0 \quad &\Leftrightarrow \quad \beta \leq (1/K)[K - x_B - x^* \\ &\qquad + \sqrt{K^2 + (x_B + x^*)^2 + 2x^8K + 2x_BK - 4x_Bx^*}]. \end{aligned}$$

Hence the theorem is proved. □

3. Existence of limit cycle

3.1. Main results

Theorem 3.1.1. *Assuming that $\varphi'(x^*) < \frac{d}{qx^*}$, then system (4) with initial conditions (5) has at least one limit cycle if and only if*

$$0 < \beta < 1 - \frac{x^*}{K}\left[\frac{2d - qx^*\varphi'(x^*)}{d - qx^*\varphi'(x^*)}\right].$$

Theorem 3.1.2. *Assuming that $\varphi'(x^*) > \frac{2d}{qx^*}$, then the system (4) with initial conditions (5) has at least one limit cycle if and only if*

$$1 - \frac{x^*}{K}\left[\frac{2d - qx^*\varphi'(x^*)}{d - qx^*\varphi'(x^*)}\right] < \beta < 1 - \frac{x^*}{K}.$$

3.2. Proof of main results

In order to prove the above main results, we give the following Lemma by taking advantage from a simple modification of Theorem 1.1 in Hesaaraki and Moghadas [26].

Lemma 3.2.1. *The system (4) with initial conditions (5) has at least one limit cycle if and only if $\frac{2rx^*}{(1-\beta)K} + py^*\varphi'(x^*) - r < 0$.*

The proof of Lemma 3.2.1 is similar with Theorem 2.1.1 which is conducted by Hesaaraki and Moghadas [26] and is omitted.

Now, applying Lemma 3.2.1 in system (4), we have

$$\begin{aligned}
&\frac{2rx^*}{(1-\beta)K} + py^*\varphi'(x^*) - r < 0 \\
&\Leftrightarrow \frac{2rx^*}{(1-\beta)K} + \frac{qrx^*}{d}\left(1 - \frac{x^*}{(1-\beta)K}\right)\varphi'(x^*) - r < 0 \\
&\Leftrightarrow \frac{2rx^*}{(1-\beta)K} + \frac{qrx^*\varphi'(x^*)}{d} - \frac{qr(x^*)^2\varphi'(x^*)}{d(1-\beta)K} - r < 0 \\
&\Leftrightarrow (r/d)\left[\frac{2dx^*}{(1-\beta)K} + qx^*\varphi'(x^*) - \frac{q(x^*)^2\varphi'(x^*)}{(1-\beta)K} - d\right] < 0 \\
&\Leftrightarrow \left[\frac{x^*}{(1-\beta)K}(2d - qx^*\varphi'(x^*)) - (d - qx^*\varphi'(x^*))\right] < 0 \\
&\Leftrightarrow (1-\beta)(d - qx^*\varphi'(x^*)) > \frac{x^*}{K}(2d - qx^*\varphi'(x^*)).
\end{aligned}$$

Hence, system (4) with the initial conditions (5) has at least one limit cycle if and only if

$$(7) \qquad (1-\beta)(d - qx^*\varphi'(x^*)) > \frac{x^*}{K}(2d - qx^*\varphi'(x^*)).$$

Therefore, the above inequations can be considered as following two cases:
Case 1. If $d - qx^*\varphi'(x^*) > 0 \Leftrightarrow \varphi'(x^*) < \frac{d}{qx^*}$, then $2d - qx^*\varphi'(x^*) > 0$.

Thus, the inequality (7) is equivalent to the following inequality:

$$0 < \beta < 1 - \frac{x^*}{K}\left[\frac{2d - qx^*\varphi'(x^*)}{d - qx^*\varphi'(x^*)}\right].$$

Hence, system (4) with initial conditions (5) has at least one limit cycle if and only if

$$0 < \beta < 1 - \frac{x^*}{K}\left[\frac{2d - qx^*\varphi'(x^*)}{d - qx^*\varphi'(x^*)}\right].$$

Hence, the Theorem 3.1.1 is proved. □

Case 2. If $2d - qx^*\varphi'(x^*) < 0 \Leftrightarrow \varphi'(x^*) > \frac{2d}{qx^*}$, then $d - qx^*\varphi'(x^*) < 0$.

Thus, combining the existence of the positive equilibrium point if system (4), the inequality (7) is equivalent to the following inequality:

$$1 - \frac{x^*}{K}\left[\frac{2d - qx^*\varphi'(x^*)}{d - qx^*\varphi'(x^*)}\right] < \beta < 1 - \frac{x^*}{K}.$$

Hence, system (4) with initial conditions (5) has at least one limit cycle if and only if

$$1 - \frac{x^*}{K}\left[\frac{2d - qx^*\varphi'(x^*)}{d - qx^*\varphi'(x^*)}\right] < \beta < 1 - \frac{x^*}{K}.$$

Hence, the Theorem 3.1.2 is proved. □

4. Conclusion

Ma et al. [3] studied only the locally asymptotical stability of system (2) with initial conditions (3) and Chen et al. [4] investigated the globally asymptotical stability of system (2) with initial conditions (3) when the response function $\varphi(x) = x$. The most interesting things are the global stability and existence of limit cycles for the predation system with a generalized functional response. Motivated by these, this article obtains a set of sufficient conditions which guarantee the global asymptotical stability of the positive equilibrium and the existence of limit cycles by mathematical analysis. Unfortunately, the global stability property of the considered system (4) under the assumption $2\varphi'(x_B) + x_B\varphi''(x_B) = 0$ is still an open issue.

On the other hand, the unstable state of the positive equilibrium point of the considered predation systems, especially the existence of limit cycles have very important meanings from an ecological point of view, because these show the oscillatory behaviors of the interacting populations. In this paper, a set of necessary and sufficient conditions for the existence of limit cycles under some certain assumptions are obtained. According to Theorem 3.1.1 and 3.1.2, the conditions for the existence of limit cycles is very similar to those of the unstable conditions in the work of Ma et al. [3].

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