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Chern scalar curvature and symmetric products of compact Riemann surfaces

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Abstract: Let X be a compact connected Riemann surface of genus $g \geq 0$, and let $\mathrm{Sym}^d(X)$, $d \geq 1$, denote the d–fold symmetric product of X. We show that $\mathrm{Sym}^d(X)$ admits a Hermitian metric with

(1) negative Chern scalar curvature if and only if $g \geq 2$, and
(2) positive Chern scalar curvature if and only if $d > g$.

Keywords: Gauduchon metric, Chern scalar curvature, symmetric product, pseudo-effectiveness.

1. Introduction

The existence of Riemannian metrics with scalar curvature having a fixed sign on a compact manifold has been extensively studied over the last few decades. In particular, one knows that a metric with negative scalar curvature (which can, in fact, assumed to be a constant) exists on any compact smooth manifold while there are topological obstructions to the existence of metrics with positive scalar curvature. In the Kähler setting, much of the work has focused on finding an algebraic geometric characterization of projective varieties which admit extremal Kähler metrics, in particular, Kähler-Einstein and constant scalar curvature metrics.

The main result of this paper is a complete characterization of symmetric products of a compact Riemann surface, in terms of the number of factors in the product and the genus, which admit Hermitian metrics with Chern scalar curvature of a fixed sign:

Theorem 1.1. *Let X be a compact Riemann surface of genus $g \geq 0$. For $d \geq 1$, the d–fold symmetric product $\mathrm{Sym}^d(X)$ of X admits a Hermitian metric with*

(1) negative Chern scalar curvature if and only if $g \geq 2$, and

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(2) positive Chern scalar curvature if and only if $d > g$.

This result is a continuation of our earlier works on the existence of Kähler metrics with holomorphic bisectional curvature [BS1] and Ricci curvature [BS2] with fixed sign on symmetric products, and, more generally, Quot schemes associated to Riemann surfaces.

Theorem 1.1 raises some interesting questions which may have to be tackled by methods different from those in this paper:

Question 1: If $d > g \geq 0$, does $\mathrm{Sym}^d(X)$ admit a *Kähler* metric with positive scalar curvature?

Question 2: If $g \geq 2$ and $d \leq g$, can $\mathrm{Sym}^d(X)$ admit a *Riemannian* metric with positive scalar curvature?

The proof of Theorem 1.1 is based on a recent result of X. Yang, [Ya], giving criteria for the existence of positive or negative Chern scalar curvature Hermitian metrics on a compact complex manifold. These criteria are in terms of the pseudo-effectivity of the canonical or anti-canonical line bundles. In effect, we give necessary and sufficient conditions for the pseudo-effectivity (or the lack thereof) of the canonical and anti-canonical bundles of a symmetric product of Riemann surfaces.

2. Pseudo-effectivity of the canonical and anti-canonical bundles

2.1. The canonical line bundle of a symmetric product

Let X be a compact connected Riemann surface. The genus of X will be denoted by g. The holomorphic cotangent bundle of X will be denoted by K_X.

For any positive integer d, let P_d denote the group of permutations of $\{1, \cdots, d\}$. Let $\mathrm{Sym}^d(X)$ be the quotient of X^d by the action of P_d that permutes the factors of X^d. This $\mathrm{Sym}^d(X)$ is a complex projective manifold of complex dimension d. For convenience, $\mathrm{Sym}^0(X)$ will denote a point. The holomorphic cotangent bundle of $\mathrm{Sym}^d(X)$ will be denoted by $\Omega^1_{\mathrm{Sym}^d(X)}$. For any $1 \leq i \leq d$, let

$$p_i : X^d \longrightarrow X \tag{2.1}$$

be the projection to the i-th factor. The Néron–Severi group $\mathrm{NS}(\mathrm{Sym}^d(X))$ of $\mathrm{Sym}^d(X)$ is the subgroup of $H^2(\mathrm{Sym}^d(X), \mathbb{Z})$ of Hodge type $(1,1)$. We note that $\mathrm{NS}(\mathrm{Sym}^d(X))$ is torsion-free because $H^2(\mathrm{Sym}^d(X), \mathbb{Z})$ is torsion-free [Ma, p. 329, (12.3)].

Let

$$K_{X,d} := K_{\mathrm{Sym}^d(X)} = \bigwedge^d \Omega^1_{\mathrm{Sym}^d(X)}$$

be the canonical line bundle of $\mathrm{Sym}^d(X)$. Let

$$K^{-1}_{X,d} := (K_{X,d})^*$$

be the anti-canonical line bundle of $\mathrm{Sym}^d(X)$. We will now recall a description of the Néron–Severi class of $K_{X,d}$. Since $\mathrm{Sym}^1(X) = X$, the class of $K_{X,d}$ is $2g-2 \in \mathbb{Z} = \mathrm{NS}(\mathrm{Sym}^1(X))$. Next assume that $d \geq 2$. Let Δ' denote the image of the morphism

$$\mathrm{Sym}^{d-2}(X) \times X \longrightarrow \mathrm{Sym}^d(X)\,,$$
$$((z_1, \cdots, z_{d-2}), z) \longmapsto (z_1, \cdots, z_{d-2}, z, z)\,.$$

The Néron–Severi class of the holomorphic line bundle

$$\mathcal{O}_{\mathrm{Sym}^d(X)}(\Delta') \longrightarrow \mathrm{Sym}^d(X)$$

on $\mathrm{Sym}^d(X)$ defined by the divisor Δ' will be denoted by Δ. For any holomorphic line bundle ξ on X, consider the holomorphic line bundle

$$\widehat{\xi} := \bigotimes_{i=1}^{d} p_i^* \xi$$

on X^d, where p_i is the projection in (2.1). The action P_d on X^d lifts to $\widehat{\xi}$; for any $z \in X^d$, the action on the fiber $\widehat{\xi}_z$ of the isotropy subgroup for z is trivial. Hence $\widehat{\xi}$ descends to a holomorphic line bundle on $\mathrm{Sym}^d(X)$. The element in $\mathrm{NS}(\mathrm{Sym}^d(X))$ corresponding this holomorphic line bundle will be denoted by $\mathcal{L}_\xi$. This class $\mathcal{L}_\xi$ can also be constructed as follows. For any $y \in X$, let $X_y \subset \mathrm{Sym}^d(X)$ be the image of the morphism

$$\mathrm{Sym}^{d-1}(X) \longrightarrow \mathrm{Sym}^d(X)\,,$$
$$(z_1, \cdots, z_{d-1}) \longmapsto (z_1, \cdots, z_{d-1}, y)\,.$$

Now define

$$\mathcal{L}_{\mathcal{O}(y)} := [\mathcal{O}_{\mathrm{Sym}^d(X)}(X_y)] \in \mathrm{NS}(\mathrm{Sym}^d(X)) \tag{2.2}$$

(see [Ko2, p. 18]). Note that $\mathcal{L}_{\mathcal{O}(y)}$ is independent of the choice of y.

For an effective divisor $D = \sum_{i=1}^{n} y_i$ on X, define

$$\mathcal{L}_{\mathcal{O}(D)} = \sum_{i=1}^{n} L_{\mathcal{O}(y_i)} .$$

Finally, for effective divisors D, D' on X, define

$$\mathcal{L}_{\mathcal{O}(D-D')} = \mathcal{L}_{\mathcal{O}(D)} - \mathcal{L}_{\mathcal{O}(D')}$$

(see [Ko2, p. 18]). Now we have

$$K_{X,d} = \mathcal{L}_{K_X} - \frac{1}{2}\Delta \in \mathrm{NS}(\mathrm{Sym}^d(X)) \tag{2.3}$$

[Ko2, p. 19, Proposition 2.6].

A divisor D on $\mathrm{Sym}^d(X)$ is called effective if $H^0(\mathrm{Sym}^d(X), \mathcal{O}_{\mathrm{Sym}^d(X)}(D)) \neq 0$. The pseudo-effective cone of $\mathrm{Sym}^d(X)$ is the closure of the convex cone in $\mathrm{NS}(\mathrm{Sym}^d(X))_{\mathbb{R}} = \mathrm{NS}(\mathrm{Sym}^d(X)) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the effective divisors.

2.2. Hyperbolic Riemann surfaces

Proposition 2.1. *Assume that $g \geq 2$. Then $K_{X,d}^{-1}$ is not pseudo-effective for any d.*

Proof. Let

$$\phi : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^d(X) \tag{2.4}$$

be the morphism defined by $(z_1, \cdots, z_d) \longmapsto \mathcal{O}_X(\sum_{i=1}^{d} z_i)$. For a theta line bundle θ_d on $\mathrm{Pic}^d(X)$, let

$$\Theta \in \mathrm{NS}(\mathrm{Sym}^d(X))$$

be the Néron–Severi class of the pullback $\phi^*\theta_d$. We have

$$\frac{1}{2}\Delta = (d+g-1)\mathcal{L}_{\mathcal{O}(y)} - \Theta , \tag{2.5}$$

where $\mathcal{L}_{\mathcal{O}(y)}$ is the class in (2.2) [Pa, p. 1119, Lemma 2.1(i)]; we note that $\mathcal{L}_{\mathcal{O}(y)}$ is denoted by x in both [Pa] and [Ko1].

Note that $\mathcal{L}_{K_X} = (2g-2)\mathcal{L}_{\mathcal{O}(y)}$, because degree$(K_X) = 2g-2$. Therefore, combining (2.3) and (2.5), we have

$$K_{X,d} = (g-d-1)\mathcal{L}_{\mathcal{O}(y)} + \Theta\,.$$

This implies that

$$K_{X,d}^{-1} = (d-g+1)\mathcal{L}_{\mathcal{O}(y)} - \Theta\,. \tag{2.6}$$

Since $d-g+1 < d+g-1$, the point $(-1,\, d-g+1) \in \mathbb{R}^2$ lies in the open sector determined by the two half-lines $t\cdot(-1,\, d+g-1)$ and $t\cdot(0,\, -1)$, $t \geq 0$, going anti-clockwise from $t\cdot(-1,\, d+g-1)$. Now from [Ko1, p. 124, Theorem 3] and (2.6) it follows that $K_{X,d}^{-1}$ is not pseudo-effective. To see this consider the plane in $\mathrm{NS}(\mathrm{Sym}^d(X))_{\mathbb{R}}$ generated by Θ and $\mathcal{L}_{\mathcal{O}(y)}$. First note that $\Theta \in \mathrm{NS}(\mathrm{Sym}^d(X))_{\mathbb{R}}$, which corresponds to the point $(1,\, 0) \in \mathbb{R}^2$, lies in the complement of the closure of the above sector bounded by $t\cdot(-1,\, d+g-1)$ and $t\cdot(0,\, -1)$, $t \geq 0$, while Θ is pseudo-effective because a theta line bundle on $\mathrm{Pic}^d(X)$ is ample. The intersection of the pseudo-effective cone with the plane in $\mathrm{NS}(\mathrm{Sym}^d(X))_{\mathbb{R}}$ generated by Θ and $\mathcal{L}_{\mathcal{O}(y)}$ is the cone generated by the half-line $t\cdot(-1,\, d+g-1)$, $t \geq 0$, and a half-line in the fourth quadrant (where coefficient of Θ is positive and the coefficient of $\mathcal{L}_{\mathcal{O}(y)}$ is negative) (see [Ko1, p. 124, Theorem 3] and [Ko1, Section 7]). Hence from (2.6) it follows that $K_{X,d}^{-1}$ is not pseudo-effective. □

Proposition 2.2. *Assume that $g \geq 2$. Then $K_{X,d}$ is pseudo-effective if and only if $d \leq g$.*

Proof. First assume that $d \leq g$. In this case, the map ϕ in (2.4) is birational onto its image. The canonical line bundle $\Omega^d_{\mathrm{Pic}^d(X)}$ is generated by one global section because it is trivial. Hence it follows that a nonzero section of $\Omega^d_{\mathrm{Pic}^d(X)}$ pulls back to a nonzero section of $K_{X,d}$ by the natural homomorphism $\phi^*\Omega^d_{\mathrm{Pic}^d(X)} \longrightarrow K_{X,d}$ constructed using the differential of ϕ. Therefore, we conclude that $K_{X,d}$ is pseudo-effective.

Now assume that $d \geq g+1$. Then there is a nonempty Zariski open subset $U \subset \mathrm{Pic}^d(X)$ such that the restriction

$$\phi_0 := \phi|_V : V := \phi^{-1}(U) \xrightarrow{\phi} U$$

is an algebraic fiber bundle with fiber $\mathbb{CP}^{d-g}$. For any $z \in U$, the restriction of $K_{X,d}$ to $\phi_0^{-1}(z) = \mathbb{CP}^{d-g}$ is isomorphic to $\mathcal{O}_{\mathbb{CP}^{d-g}}(g-d-1)$. Since

$$g-d-1 < 0\,,$$

the restriction of $K_{X,d}$ to $\phi_0^{-1}(z)$ does not admit any nonzero holomorphic section. In fact, this restriction is the dual of an ample line bundle on $\phi_0^{-1}(z)$. From this it follows that $K_{X,d}$ is not pseudo-effective. □

2.3. Low genera

First, assume that $g = 0$. Then $\mathrm{Sym}^d(X) = \mathbb{CP}^d$, and hence $K_{X,d}^{-1}$ is ample. This implies that $K_{X,d}$ is not pseudo-effective, and $K_{X,d}^{-1}$ is pseudo-effective.

Now assume that $g = 1$.

Lemma 2.3. *The canonical line bundle $K_{X,d}$ is pseudo-effective if and only if $d = 1$.*

Proof. If $d = 1$, then $K_{X,d} = K_X$ is trivial, hence it is pseudo-effective. If $d > 1$, then ϕ in (2.4) is a holomorphic fiber bundle over $\mathrm{Pic}^d(X)$ with fiber $\mathbb{CP}^{d-1}$. Now, as in the proof of Proposition 2.2, it follows that $K_{X,d}$ is not pseudo-effective. □

Lemma 2.4. *Assume that $g = 1$. The anti-canonical line bundle $K_{X,d}^{-1}$ is pseudo-effective for all d.*

Proof. The point $(d-g+1)\mathcal{L}_{\mathcal{O}(y)} - \Theta \in \mathrm{NS}(\mathrm{Sym}^d(X))_{\mathbb{R}}$ coincides with the point $(d+g-1)\mathcal{L}_{\mathcal{O}(y)} - \Theta$ if $g = 1$. Hence from [Ko1, p. 124, Theorem 3] and (2.6) it follows immediately that $K_{X,d}^{-1}$ is pseudo-effective. □

3. Scalar curvature and symmetric products

A Gauduchon metric on $\mathrm{Sym}^d(X)$ is a Hermitian metric h on $\mathrm{Sym}^d(X)$ such that the associated $(1, 1)$–form ω_h satisfies the equation

$$\partial\overline{\partial}\omega_h^{d-1} = 0\,.$$

Take a Gauduchon metric h. The *Chern scalar curvature* of h is the real valued function

$$s_h := \mathrm{tr}_{\omega_h}\mathrm{Ric}(\omega_h)\,,$$

where $\mathrm{Ric}(\omega_h)$ is the Ricci curvature for ω_h. The *total scalar curvature* for h is the integral

$$\int_{\mathrm{Sym}^d(X)} s_h\omega_h^d = \int_{\mathrm{Sym}^d(X)} \mathrm{Ric}(\omega_h)\wedge\omega_h^{d-1}\,.$$

Let $\mathcal{W}(\mathrm{Sym}^d(X))$ denote the space of all Gauduchon metrics on $\mathrm{Sym}^d(X)$. Let

$$\mathcal{F} : \mathcal{W}(\mathrm{Sym}^d(X)) \longrightarrow \mathbb{R}, \quad h \longmapsto \int_{\mathrm{Sym}^d(X)} s_h \omega_h^d$$

be the map that assigns the total scalar curvature to a Gauduchon metric.

Denote

$$\mathbb{R}^{>0} := \{c \in \mathbb{R} \mid c > 0\} \quad \text{and} \quad \mathbb{R}^{<0} = -\mathbb{R}^{>0}.$$

Theorem 3.1. *Let X be a compact connected Riemann surface of genus $g \geq 0$; take any $d \geq 1$. Then the following four hold:*

(1) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}$ if and only if $d > g \geq 2$.
(2) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}^{<0}$ if $g \geq 2$ and $d \leq g$.
(3) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}^{>0}$ if either $g = 0$ or $g = 1 < d$.
(4) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = 0$ if $g = 1 = d$.

Proof. In our context, Theorem 1.1 of [Ya] says the following:

(1) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}$ if and only if neither $K_{X,d}$ nor $K_{X,d}^{-1}$ is pseudo-effective;
(2) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}^{<0}$ if and only if $K_{X,d}$ is pseudo-effective but not unitary flat;
(3) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}^{>0}$ if and only if $K_{X,d}^{-1}$ is pseudo-effective but not unitary flat;
(4) $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = 0$ if and only if $K_{X,d}$ is unitary flat.

First, assume that $g \geq 2$. In view of the above result of [Ya], from Proposition 2.1 and Proposition 2.2 it follows that $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}$ if and only if $d > g$. It also follows from Proposition 2.1 and Proposition 2.2 that $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}^{<0}$ if $d \leq g$.

If $g = 0$, then $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}^{>0}$ for all d, because $K_{X,d}^{-1}$ is ample.

Now assume that $g = 1$. If $d = 1$, then $K_{X,d}$ is trivial, and hence from the fourth statement of the above theorem of [Ya] it follows that

$$\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = 0.$$

Next, we observe that $K_{X,d}^{-1}$ is not unitary flat when $d > g$. Indeed, in this case ϕ in (2.4) is a holomorphic fiber bundle with fibers $\mathbb{CP}^{d-1}$. The restriction of $K_{X,d}^{-1}$ to a fiber of ϕ is ample, so $K_{X,d}^{-1}$ is not unitary flat. Therefore, from Lemma 2.4 and the third statement of the above theorem of [Ya] we conclude that $\mathcal{F}(\mathcal{W}(\mathrm{Sym}^d(X))) = \mathbb{R}^{>0}$ if $d > g = 1$. □

Theorem 3.2. *Let X be a compact Riemann surface of genus $g \geq 0$. For $d \geq 1$, the d-fold symmetric product* $\mathrm{Sym}^d(X)$ *admits a Hermitian metric with*

(1) negative Chern scalar curvature if and only if $g \geq 2$, and
(2) positive Chern scalar curvature if and only if and $d > g$.

Proof. In our context, Theorem 1.3 of [Ya] says the following:

(1) $\mathrm{Sym}^d(X)$ admits a Hermitian metric with negative Chern scalar curvature if and only if $K_{X,d}^{-1}$ is not pseudo-effective;
(2) $\mathrm{Sym}^d(X)$ admits a Hermitian metric with positive Chern scalar curvature if and only if $K_{X,d}$ is not pseudo-effective.

First, assume that $g \geq 2$. Now, from Proposition 2.1, it follows that $\mathrm{Sym}^d(X)$ admits a Hermitian metric with negative Chern scalar curvature for every d. From Proposition 2.2 it follows that $\mathrm{Sym}^d(X)$ admits a Hermitian metric with positive Chern scalar curvature if and only if $d > g$.

If $g = 0$, then $\mathrm{Sym}^d(X)$ admits a Hermitian metric with positive Chern scalar curvature for all d.

Now assume that $g = 1$. From Lemma 2.3, it follows that for each $d \geq 2$, the symmetric product $\mathrm{Sym}^d(X)$ admits a Hermitian metric with positive Chern scalar curvature. On the other hand, from Lemma 2.4 it follows that that $\mathrm{Sym}^d(X)$ does not admit a Hermitian metric with negative Chern scalar curvature for all $d \geq 2$. If $d = 1 = g$, then there is no Hermitian metric with positive or negative Chern scalar curvature. □

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