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On Yau Rigidity Theorem for Submanifolds in Pinched Manifolds

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Abstract: In this paper, we investigate Yau's rigidity problem for compact submanifolds with parallel mean curvature in pinched Riemannian manifolds. Firstly, we prove that if M^n is an oriented closed minimal submanifold in an $(n+p)$-dimensional complete simply connected Riemannian manifold N^{n+p}, then there exists a constant $\delta_0(n,p) \in (0,1)$ such that if the sectional curvature of N satisfies $\overline{K}_N \in [\delta_0(n,p), 1]$, and if M has a lower bound for the sectional curvature and an upper bound for the normalized scalar curvature, then N is isometric to S^{n+p}. Moreover, M is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ in S^{n+1} for $k = 1, \cdots, n-1$, or the Veronese submanifold in S^{n+d}, where $d = \frac{1}{2}n(n+1) - 1$. We then generalize the above theorem to the case where M ia a compact submanifold with parallel mean curvature in a pinched Riemannian manifold.
Keywords: Submanifolds, Rigidity theorem, Sectional curvature, Mean curvature, Pinched Riemannian manifold.

1. Introduction

Rigidity of submanifolds with parallel mean curvature plays an important role in submanifold geometry. Let M^n be an $n(\geq 2)$-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. Denote by H and S the mean curvature and the squared length of the second fundamental form of M, respectively. After the pioneering rigidity theorem for minimal submanifolds in a sphere proved by Simons [25], Lawson [12] and Chern-do Carmo-Kobayashi [3] obtained a famous classification theorem for oriented compact minimal submanifolds in S^{n+p} satisfies $S \leq n/(2-1/p)$. It was partially extended to compact submanifolds with parallel mean curvature

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in a sphere by Okumura [20, 21], Yau [32] and others. In 1990, the first named author [27] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

Theorem A. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere S^{n+p}. If $S \le C(n,p,H)$, then M is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, a Clifford hypersurface in an $(n+1)$-sphere, or the Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Here the constant $C(n,p,H)$ is defined by*

$$C(n,p,H) = \begin{cases} \alpha(n,H), & for\ p=1, \\ & or\ p=2\ and\ H\neq 0, \\ \frac{n}{2-\frac{1}{p}}, & for\ p\geq 2\ and\ H=0, \\ \min\left\{\alpha(n,H), \frac{n+nH^2}{2-\frac{1}{p-1}}+nH^2\right\}, & for\ p\geq 3\ and\ H\neq 0, \end{cases}$$

$$\alpha(n,H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}.$$

Later, Li-Li [15] improved Simons' pinching constant for n-dimensional compact minimal submanifolds in S^{n+p} to $\max\{\frac{n}{2-1/p}, \frac{2}{3}n\}$. Chen-Xu [2] also gave a proof for the rigidity result by using a different argument. Using Li-Li's matrix inequality, Xu [28] improved the above pinching constant $C(n,p,H)$ to

$$C'(n,p,H) = \begin{cases} \alpha(n,H), & \text{for } p=1, \\ & \text{or } p=2 \text{ and } H\neq 0, \\ \min\left\{\alpha(n,H), \frac{1}{3}(2n+5nH^2)\right\}, & \text{otherwise.} \end{cases}$$

When N is a pinched Riemannian manifold, we denote $\overline{K}_N$ the sectional curvature of N. Making use of 1-forms and geometric inequalities, Shiohama and Xu [23, 30] proved the following rigidity theorem.

Theorem B. *For given positive integers $n(\geq 2)$, p and a nonnegative constant H, there exists a number $\rho(n,p)$ such that $0<\rho(n,p)<1$ with the following properties: If M^n is an oriented compact submanifold with parallel mean curvature normal field with its norm H in a complete and simply connected $(n+p)$-dimensional Riemannian manifold with $\rho(n,p)\leq \overline{K}_N \leq 1$,* *and if*

$$nH^2 + C_1(n,p)(1-c) + C_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4} \le S$$
$$\le C'(n,p,H) - D_1(n,p)(1-c) - D_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4},$$

where $c := \inf \overline{K}_N$, then N^{n+p} is isometric to S^{n+p}. Moreover, M is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, a Clifford hypersurface in an $(n+1)$-sphere, or the Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Here $C_1(n,p)$, $C_2(n,p)$, $D_1(n,p)$, $D_2(n,p)$ are nonnegative constants depending on n and p; $C'(n,p,H)$ is defined as above.

In 1975, Yau [32] proved that if M is an n-dimensional oriented compact minimal submanifold in the unit sphere S^{n+p} whose sectional curvature satisfies $K_M \ge \frac{p-1}{2p-1}$, then either M is the totally geodesic sphere, a Clifford hypersurface in S^{n+1}, or the Veronese surface in S^4. The pinching constant above is the best possible in the case where $p=1$, or $n=2$ and $p=2$. Later, Itoh [11] proved that if M^n is an oriented compact minimal submanifold in S^{n+p} whose sectional curvature satisfies $K_M \ge \frac{n}{2(n+1)}$, then M is the totally geodesic sphere or the Veronese submanifold. In 2012, Gu and Xu [7] improved the pinching constants above to $\min\{\frac{p\cdot sgn(p-1)}{2(p+1)}, \frac{n}{2(n+1)}\}$, where $sgn(\cdot)$ is the standard sign function. Combining the rigidity results above and their extensions [7, 11, 24, 31, 32], Gu and Xu presented the following generalized Yau's rigidity theorem.

Theorem C. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature in $F^{n+p}(c)$, where $c+H^2>0$. Set $\tau(m,n) = \min\{m \cdot sgn(m-1), n\}$. Then we have*

(1) *if $H=0$ and*

$$K_M \ge \frac{\tau(p,n)c}{2[\tau(p,n)+1]},$$

then M is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces $S^k(\sqrt{\frac{k}{nc}}) \times S^{n-k}(\sqrt{\frac{n-k}{nc}})$ in $F^{n+1}(c)$ for $k=1,\cdots,n-1$, or the Veronese submanifold in $F^{n+d}(c)$, where $d=\frac{1}{2}n(n+1)-1$;

(2) *if $H \ne 0$ and*

$$K_M \ge \frac{\tau(p-1,n)(c+H^2)}{2[\tau(p-1,n)+1]},$$

then M is congruent to one of the following:

(i) $S^n(\frac{1}{\sqrt{c+H^2}})$;
(ii) *one of the Clifford hypersurfaces* $S^k(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}})\times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ *in* $F^{n+1}(c)$ *with* $c>0$, *where* $\lambda(k,n,H,c)=\frac{1}{2k}[nH+\sqrt{n^2H^2+4k(n-k)c}],\ k=1,\cdots,n-1$;
(iii) *one of the Clifford minimal hypersurfaces* $S^k(\sqrt{\frac{k}{n(c+H^2)}})\times S^{n-k}(\sqrt{\frac{n-k}{n(c+H^2)}})$ *in* $F^{n+1}(c+H^2),\ k=1,\cdots,n-1$;
(iv) *the Clifford torus* $S^1(r_1)\times S^1(r_2)$ *in* $F^3(c+H^2-H_0^2)$ *with constant mean curvature* H_0, *where* $r_1,r_2=[2(c+H^2)\pm 2H_0(c+H^2)^{1/2}]^{-1/2}, 0\le H_0\le H$, *and* $c+H^2-H_0^2>0$;
(v) *the product of three spheres* $S^{k_1}(\sqrt{\frac{k_1}{k(c+\lambda^2(k,n,H,c))}})\times S^{k-k_1}(\sqrt{\frac{k-k_1}{k(c+\lambda^2(k,n,H,c))}})\times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ *in* $F^{n+2}(c)$ *with* $c>0$, *where* $\lambda(k,n,H,c)=\frac{1}{2k}[nH+\sqrt{n^2H^2+4k(n-k)c}]$, $1\le k_1<k\le n-1$;
(vi) *the Veronese submanifold in* $F^{n+d}(c+H^2)$, *where* $d=\frac{1}{2}n(n+1)-1$.

Remark 1.1. The first part of Theorem C is the unified version of the rigidity results due to Yau [32], Itoh [11] and Gu-Xu [7]. Since Yau is the first one who investigated the sectional curvature pinching problem on minimal submanifolds in spheres and his rigidity result is sharp, the first part of Theorem C was called the Yau rigidity theorem by Gu and Xu [7]. Correspondingly, Theorem C was called the generalized Yau rigidity theorem (see [7]). The second sectional curvature pinching problem on minimal surfaces in spheres has been studied by Kozlowski and Simon [9].

Motivated by Theorems B and C, we investigate the rigidity problem of compact submanifolds with parallel mean curvature in pinched Riemannian manifolds. We first prove the following rigidity theorem for minimal submanifolds under sectional curvature pinching condition.

Theorem 1.1. *Let M be an n-dimensional oriented closed minimal submanifold in an $(n+p)$-dimensional complete simply connected Riemannian manifold N^{n+p}. Denote by R_0 the normalized scalar curvature of M. Then* *there exists a constant* $\delta_0(n,p) \in (0,1)$ *such that if* $\overline{K}_N \in [\delta_0(n,p),1]$*, and if*

$$\begin{aligned} K_M &\geq \frac{\tau(p,n)}{2(\tau(p,n)+1)} + A_0(n,p)(1-c), \\ R_0 &\leq 1 - B_0(n,p)(1-c), \end{aligned}$$

where $c := \inf \overline{K}_N$*, then* N *is isometric to* S^{n+p}*. Moreover,* M *is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces* $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ *in* S^{n+1} *for* $k = 1, \cdots, n-1$*, or the Veronese submanifold in* S^{n+d}*, where* $d = \frac{1}{2}n(n+1)-1$*. Here* $\delta_0(n,p)$*,* $A_0(n,p)$*,* $B_0(n,p)$ *will be given in the proof, which are nonnegative constants depending on* n *and* p*,* $\tau(m,n)$ *is defined in Theorem C.*

More generally, we consider compact submanifolds with parallel mean curvature ($H \neq 0$), and prove the following rigidity theorem.

Theorem 1.2. *Let* M *be an* n*-dimensional oriented compact submanifold with parallel mean curvature* ($H \neq 0$) *in an* $(n+p)$*-dimensional complete simply connected Riemannian manifold* N^{n+p}*. Denote by* R_0 *the normalized scalar curvature of* M*. Then there exists a constant* $\delta_1(n,p) \in (0,1)$ *such that if* $\overline{K}_N \in [\delta_1(n,p),1]$*, and if*

$$\begin{aligned} K_M &\geq \frac{\tau(p-1,n)}{2(\tau(p-1,n)+1)}(1+H^2) + A_1(n,p)(1-c) \\ &\quad + A_2(n,p)[H(1+H^2)]^{1/2}(1-c)^{1/4}, \\ R_0 &\leq 1 + H^2 - B_1(n,p)(1-c) - B_2(n,p)[H(1+H^2)]^{1/2}(1-c)^{1/4}, \end{aligned}$$

where $c := \inf \overline{K}_N$*, then* N *is isometric to* S^{n+p}*. Moreover,* M *is congruent to one of the following:*

(*i*) $S^n(\frac{1}{\sqrt{1+H^2}})$;

(*ii*) *one of the Clifford hypersurfaces* $S^k(\frac{1}{\sqrt{1+\mu^2(k,n,H)}}) \times S^{n-k}(\frac{\mu(k,n,H)}{\sqrt{1+\mu^2(k,n,H)}})$ *in* $F^{n+1}(1)$*, where* $\mu(k,n,H) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)}]$, $k = 1, \cdots, n-1$;

(*iii*) *one of the Clifford minimal hypersurfaces* $S^k(\sqrt{\frac{k}{n(1+H^2)}}) \times S^{n-k}(\sqrt{\frac{n-k}{n(1+H^2)}})$ *in* $F^{n+1}(1+H^2)$, $k = 1, \cdots, n-1$;

(*iv*) *the Clifford torus* $S^1(r_1)\times S^1(r_2)$ *in* $F^3(1+H^2-H_0^2)$ *with constant mean curvature* H_0, *where* $r_1, r_2 = [2(1+H^2)\pm 2H_0(1+H^2)^{1/2}]^{-1/2}, 0\le H_0\le H$, *and* $1+H^2-H_0^2>0$;

(*v*) *the product of three spheres* $S^{k_1}(\sqrt{\frac{k_1}{k(1+\mu^2(k,n,H))}})\times S^{k-k_1}(\sqrt{\frac{k-k_1}{k(1+\mu^2(k,n,H))}})\times S^{n-k}(\frac{\mu(k,n,H)}{\sqrt{1+\mu^2(k,n,H)}})$ *in* $F^{n+2}(1)$, *where* $\mu(k,n,H)=\frac{1}{2k}[nH+\sqrt{n^2H^2+4k(n-k)}], 1\le k_1<k\le n-1$;

(*vi*) *the Veronese submanifold in* $F^{n+d}(1+H^2)$, *where* $d=\frac{1}{2}n(n+1)-1$.

Here $\delta_1(n,p)$, $A_1(n,p)$, $A_2(n,p)$, $B_1(n,p)$, $B_2(n,p)$ *will be given in the proof, which are nonnegative constants depending on* n *and* p, $\tau(m,n)$ *is defined in Theorem C.*

Remark 1.2. In particular, if $c=1$, then the conditions on the upper bound for normalized scalar curvature in Theorems 1.1 and 1.2 are automatically satisfied. In this case, Theorems 1.1 and 1.2 are reduced to the first and second parts of Theorem C, respectively. Therefore, the unification of Theorems 1.1 and 1.2 is the most general version of the Yau rigidity theorem up to date.

2. Notation and lemmas

Throughout this paper let M^n be an n-dimensional compact Riemannian manifold isometrically immersed into an $(n+p)$-dimensional Riemannian manifold N^{n+p}. The following convention of indices are used throughout:

$$1\le A,B,C,\ldots\le n+p;\ 1\le i,j,k,\ldots\le n;\ n+1\le\alpha,\beta,\gamma,\ldots\le n+p.$$

Choose a local orthonormal frame field $\{e_A\}$ in N^{n+p} such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-form of N^{n+p} respectively. Restricting these forms to M, we have

$$\omega_{\alpha i}=\sum_j h_{ij}^\alpha\omega_j,\ h_{ij}^\alpha=h_{ji}^\alpha.$$

The curvature tensors of N, M are denoted by $\overline{R}_{ABCD}$, R_{ijkl}, and the normal curvature tensor of M by $R_{\alpha\beta kl}$ respectively. The second fundamental form of M is denoted by h and the mean curvature field by ξ. We then have

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,\ \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha.$$

Moreover,

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.1}$$

$$R_{\alpha\beta kl} = \overline{R}_{\alpha\beta kl} + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \tag{2.2}$$

Denote by $K(\pi)$ the sectional curvature of M for tangent 2-plane $\pi(\subset T_xM)$ at point $x \in M$. Denote by $\overline{K}(\pi)$ the sectional curvature of N for tangent 2-plane $\pi(\subset T_xN)$ at point $x \in N$. Set $K_{\min} := \min_{\pi\subset T_xM} K(\pi)$. Let $a(x)$, $b(x)$ for $x \in N$ be the minimum and maximum of $\overline{K}_N$ at that point. Then by Berger's inequality we have

$$|\overline{R}_{ABCD}| \le \frac{2}{3}(b-a), \tag{2.3}$$

for all distinct indices A, B, C, D.

$$|\overline{R}_{ACBC}| \le \frac{1}{2}(b-a), \tag{2.4}$$

for all distinct indices A, B, C. We define $S = |h|^2,\ H = |\xi|,\ H_\alpha = (h^\alpha_{ij})_{n\times n}$. Then the normalized scalar curvature R_0 of M is given by

$$n(n-1)R_0 = \sum_{i,j} \overline{R}_{ijij} + n^2H^2 - S. \tag{2.5}$$

For a matrix $A = (a_{ij})$, we denote by $N(A)$ the square of the norm of A, i.e.,

$$N(A) = tr(AA^T) = \sum a^2_{ij}.$$

Definition 2.1. M is called a submanifold with parallel mean curvature if ξ is parallel in the normal bundle of M. In particular, M is called minimal if $\xi = 0$.

We assume that M admits a parallel mean curvature normal field and $H \neq 0$. Then we choose e_{n+1} such that $e_{n+1} \parallel \xi$ and $\mathrm{tr}H_{n+1} = nH$ and $\mathrm{tr}H_\beta = 0$,

$n+2 \le \beta \le n+p$. Set

$$S_H = \sum_{i,j}(h_{ij}^{n+1})^2, \quad S_I = \sum_{i,j,\beta\neq n+1}(h_{ij}^{\beta})^2.$$

Denoting the first and second covariant derivatives of h_{ij}^{α} by h_{ijk}^{α} and h_{ijkl}^{α} respectively, we have

$$\sum_k h_{ijk}^{\alpha}\omega_k = dh_{ij}^{\alpha} - \sum_k h_{ik}^{\alpha}\omega_{kj} - \sum_k h_{kj}^{\alpha}\omega_{ki} - \sum_{\beta} h_{ij}^{\beta}\omega_{\beta\alpha},$$

$$\sum_l h_{ijkl}^{\alpha}\omega_l = dh_{ijk}^{\alpha} - \sum_l h_{ijl}^{\alpha}\omega_{lk} - \sum_l h_{ilk}^{\alpha}\omega_{lj} - \sum_l h_{ljk}^{\alpha}\omega_{li} - \sum_{\beta} h_{ijk}^{\beta}\omega_{\beta\alpha}.$$

Hence,

$$\begin{aligned} & h_{ijk}^{\alpha} = h_{ikj}^{\alpha} - \overline{R}_{\alpha ijk}, \\ (2.6) \qquad & h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_m h_{im}^{\alpha}R_{mjkl} + \sum_m h_{mj}^{\alpha}R_{mikl} - \sum_{\beta} h_{ij}^{\beta}R_{\alpha\beta kl}, \end{aligned}$$

If M^n is a submanifold with parallel mean curvature of N^{n+p}, then trH_{α} is constant, i.e., $\sum_i h_{iikl}^{\alpha} = 0$. Therefore,

$$\begin{aligned} \triangle h_{ij}^{\alpha} &= -\sum_k(\overline{R}_{\alpha kikj} + \overline{R}_{\alpha ijkk}) \\ (2.7) \qquad & \quad + \sum_{k,m} h_{km}^{\alpha}R_{mijk} + \sum_{k,m} h_{mi}^{\alpha}R_{mkjk} - \sum_{k,\beta} h_{ki}^{\beta}R_{\alpha\beta jk}. \end{aligned}$$

The following lemmas will be used in the proof of our results.

Lemma 2.2([32]). *If M^n is a submanifold with parallel mean curvature in N^{n+p}, then either $H \equiv 0$ or H is a non-zero constant and $H_{\alpha}H_{n+1} = H_{n+1}H_{\alpha} + (\overline{R}_{n+1\alpha ij})_{n\times n}$ for $\alpha \neq n+1$.*

Lemma 2.3([10]). *Let M be an n-dimensional submanifold in N^{n+p}, then*

$$\sum_{\alpha,\beta}[tr(H_{\alpha}^2H_{\beta}^2) - tr(H_{\alpha}H_{\beta})^2] \le \frac{n}{2}\sum_{\alpha,\beta}[tr(H_{\alpha}H_{\beta})]^2.$$

The DDVV inequality proved by Lu, Ge-Tang [6, 18] is stated as follows. For further discussions in this area, we refer to see [4, 5, 17].

Lemma 2.4(DDVV Inequality). *Let $B_1, ..., B_m$ be symmetric $(n \times n)$-matrices, then*

$$\sum_{r,s=1}^{m} N(B_r B_s - B_s B_r) \leq \Big[\sum_{r=1}^{m} N(B_r)\Big]^2,$$

where the equality holds if and only if under some rotation all B_r's are zero except two matrices which can be written as

$$\tilde{B}_r = P \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t, \quad \tilde{B}_s = P \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t,$$

where P is an orthogonal $(n \times n)$-matrix.

We also need the following lemma, which can be found in [22, 23] (see also [29]).

Lemma 2.5. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers which satisfy $\sum_i a_i = \sum_i b_i = 0$, $\sum_i a_i^2 = a$ and $\sum_i b_i^2 = b$, then*

$$|\sum_i a_i b_i^2| \leq (n-2)[n(n-1)]^{-1/2} a^{1/2} b,$$

where equality holds if and only if either $ab = 0$, or at least $n-1$ pairs of numbers of $(a_i, b_i)'s$ are the same.

3. Minimal submanifolds

If M^n is a compact oriented minimal submanifold in N^{n+p}, then we get from (2.7) that

$$\begin{aligned} \sum_{i,j,\alpha} h_{ij}^{\alpha} \triangle h_{ij}^{\alpha} = & -\sum_{i,j,k,\alpha} h_{ij}^{\alpha} (\overline{R}_{\alpha kikj} + \overline{R}_{\alpha ijkk}) - \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} \\ & + \sum_{i,j,k,m,\alpha} (h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} + h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk}). \end{aligned} \tag{3.1}$$

Using (2.1) and (2.2), we have

$$\begin{aligned}
&\sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \\
= &\sum_{i,j,k,m,\alpha} (h_{ij}^\alpha h_{mk}^\alpha \overline{R}_{mijk} + h_{ij}^\alpha h_{im}^\alpha \overline{R}_{mkjk}) + \sum_{\alpha,\beta} trH_\beta \cdot tr(H_\alpha^2 H_\beta) \\
&- \sum_{\alpha,\beta}[tr(H_\alpha H_\beta)]^2 - \sum_{\alpha,\beta}[tr(H_\alpha^2 H_\beta^2) - tr(H_\alpha H_\beta)^2],
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} = &\sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta \overline{R}_{\alpha\beta jk} \\
&+ \sum_{\alpha,\beta}[tr(H_\alpha^2 H_\beta^2) - tr(H_\alpha H_\beta)^2].
\end{aligned} \tag{3.3}$$

Since $(tr(H_\alpha H_\beta))$ is a symmetric $(p \times p)$-matrix, we can choose the normal frame field $\{e_\alpha\}$ such that

$$tr(H_\alpha H_\beta) = trH_\alpha^2 \cdot \delta_{\alpha\beta},$$

which implies

$$\sum_{\alpha,\beta}[tr(H_\alpha H_\beta)]^2 = \sum_\alpha (trH_\alpha^2)^2. \tag{3.4}$$

From the above equalities, we obtain

$$\begin{aligned}
\sum_{i,j,\alpha} h_{ij}^\alpha \triangle h_{ij}^\alpha = &- \sum_{i,j,k,\alpha} h_{ij}^\alpha (\overline{R}_{\alpha kikj} + \overline{R}_{\alpha ijkk}) - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta \overline{R}_{\alpha\beta jk} \\
&- t \sum_{i,j,k,m,\alpha} (h_{ij}^\alpha h_{mk}^\alpha \overline{R}_{mijk} + h_{ij}^\alpha h_{im}^\alpha \overline{R}_{mkjk}) \\
&+ (1+t) \sum_{i,j,k,m,\alpha} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}) \\
&+ (t-1) \sum_{\alpha,\beta}[tr(H_\alpha^2 H_\beta^2) - tr(H_\alpha H_\beta)^2] + t \sum_\alpha (trH_\alpha^2)^2,
\end{aligned} \tag{3.5}$$

where $t \in \mathbb{R}$. Setting

$$\begin{aligned} X_1 &:= (1+t)\sum_{i,j,k,m,\alpha}(h^\alpha_{ij}h^\alpha_{km}R_{mijk}+h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}) \\ &\quad +(t-1)\sum_{\alpha,\beta}[tr(H^2_\alpha H^2_\beta)-tr(H_\alpha H_\beta)^2]+t\sum_{\alpha,\beta}(trH^2_\alpha)^2, \\ Y_1 &:= -t\sum_{i,j,k,m,\alpha}(h^\alpha_{ij}h^\alpha_{jm}\overline{R}_{mkik}+h^\alpha_{mk}h^\alpha_{ij}\overline{R}_{mijk})-\sum_{i,j,k,\alpha,\beta}h^\alpha_{ij}h^\beta_{ki}\overline{R}_{\alpha\beta jk}, \\ Z_1 &:= \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2-\sum_{i,j,k,\alpha}(h^\alpha_{ij}\overline{R}_{\alpha kikj}+h^\alpha_{ij}\overline{R}_{\alpha ijkk}), \end{aligned}$$

we get

$$\frac{1}{2}\triangle S=\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2+\sum_{i,j,\alpha}h^\alpha_{ij}\Delta h^\alpha_{ij}=X_1+Y_1+Z_1.$$

For fixed α, we choose the orthonormal frame fields $\{e_i\}$ such that $h^\alpha_{ij}=\lambda^\alpha_i\delta_{ij}$. Then we get the following lemma.

Lemma 3.1. *If* $0\le t<1$, *then*
(i) $X_1\ge(1+t)nK_{\min}S+[\frac{t}{p}+\frac{sgn(p-1)}{2}(t-1)]S^2$;
(ii) $X_1\ge(1+t)nK_{\min}S+[t+\frac{n}{2}(t-1)]\sum_\alpha(trH^2_\alpha)^2$.

Proof. Since

$$\begin{aligned} &\sum_{i,j,k,m}h^\alpha_{ij}h^\alpha_{km}R_{mijk}+\sum_{i,j,k,m}h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} \\ =\ &\sum_{i,k}\lambda^\alpha_i\lambda^\alpha_kR_{kiik}+\sum_{i,k}\lambda^\alpha_i\lambda^\alpha_iR_{ikik} \\ \ge\ &\frac{1}{2}K_{\min}\sum_{i,j}(\lambda^\alpha_i-\lambda^\alpha_j)^2 \\ =\ &nK_{\min}\cdot trH^2_\alpha, \end{aligned} \tag{3.6}$$

we have

$$\sum_{i,j,k,m,\alpha}h^\alpha_{ij}h^\alpha_{km}R_{mijk}+\sum_{i,j,k,m,\alpha}h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}\ge nK_{\min}S. \tag{3.7}$$

(i) By a direct computation and Lemma 2.4, we obtain

$$\begin{aligned}\sum_{\alpha,\beta}[tr(H_\alpha^2H_\beta^2)-tr(H_\alpha H_\beta)^2] &= \frac{1}{2}\sum_{\alpha,\beta}N(H_\alpha H_\beta - H_\beta H_\alpha)\\ &\le \frac{1}{2}sgn(p-1)\Big(\sum_\alpha trH_\alpha^2\Big)^2\\ &= \frac{1}{2}sgn(p-1)S^2, \qquad (3.8)\end{aligned}$$

where $sgn(\cdot)$ is the standard sign function. Combing (3.7), (3.8) and the Schwarz inequality, we get the conclusion.

(ii) From Lemma 2.3 and (3.4), we have

$$\sum_{\alpha,\beta}[tr(H_\alpha^2H_\beta^2)-tr(H_\alpha H_\beta)^2] \le \frac{n}{2}\sum_\alpha (trH_\alpha^2)^2. \qquad (3.9)$$

This together with (3.7) implies the conclusion.

This proves the lemma.

Lemma 3.2. *If* $t \ge 0$, *then* $Y_1 \ge -ntbS - \frac{2}{3}(p-1)(b-a)S$.

Proof. It's easy to see that

$$\begin{aligned}\sum_{i,j,k,m}(h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik} + h_{mk}^\alpha h_{ij}^\alpha \overline{R}_{mijk}) &= \sum_{i,k}(h_{ii}^\alpha)^2\overline{R}_{ikik} + \sum_{i,k}h_{kk}^\alpha h_{ii}^\alpha \overline{R}_{kiik}\\ &= \frac{1}{2}\sum_{i,k}(h_{ii}^\alpha - h_{kk}^\alpha)^2\overline{R}_{ikik}\\ &\le nb\cdot \mathrm{tr}H_\alpha^2. \qquad (3.10)\end{aligned}$$

From Berger's inequality (2.3), we obtain

$$\begin{aligned}-\sum_{i,j,k,\alpha,\beta}h_{ij}^\alpha h_{ki}^\beta \overline{R}_{\alpha\beta jk} &\ge -\sum_{i,\alpha,\beta}|\overline{R}(e_\alpha, e_\beta, \sum_j h_{ij}^\alpha e_j, \sum_k h_{ik}^\beta e_k)|\\ &\ge -\sum_{i,\alpha\ne\beta}\frac{2}{3}(b-a)\sqrt{\sum_j (h_{ij}^\alpha)^2}\sqrt{\sum_k (h_{ik}^\beta)^2}\\ &\ge -\frac{1}{3}(b-a)\sum_{i,j,\alpha\ne\beta}[(h_{ij}^\alpha)^2+(h_{ij}^\beta)^2]\\ &= -\frac{2}{3}(p-1)(b-a)S. \qquad (3.11)\end{aligned}$$

Then the assertion follows from (3.10) and (3.11). This proves the lemma.

The estimate about Z_1 can be find in [30], and the proof is useful when we prove Theorem 1.1.

Lemma 3.3. $\int_M Z_1 dM \geq -\frac{1}{72}pn(n-1)(26n-25)\int_M (b-a)^2 dM.$

Proof. Note that

$$\begin{aligned} -\sum_{i,j,k,\alpha}(h^\alpha_{ik}\overline{R}_{\alpha jijk} + h^\alpha_{ij}\overline{R}_{\alpha ijkk}) &= -\sum_{i,j,k,\alpha}\nabla_k(h^\alpha_{ik}\overline{R}_{\alpha jij} + h^\alpha_{ij}\overline{R}_{\alpha ijk}) \\ &\quad + \sum_{i,j,k,\alpha}(h^\alpha_{ikk}\overline{R}_{\alpha jij} + h^\alpha_{ijk}\overline{R}_{\alpha ijk}). \end{aligned}$$

Define a differential 1-form as

$$\omega := \sum_{i,j,k,\alpha}(h^\alpha_{ik}\overline{R}_{\alpha jij} + h^\alpha_{ij}\overline{R}_{\alpha ijk})\omega_k.$$

It follows that

$$\mathrm{div}\omega = \sum_{i,j,k,\alpha}\nabla_k(h^\alpha_{ik}\overline{R}_{\alpha jij} + h^\alpha_{ij}\overline{R}_{\alpha ijk}).$$

Thus

$$Z_1 = \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \sum_{i,j,k,\alpha}(h^\alpha_{ikk}\overline{R}_{\alpha jij} + h^\alpha_{ijk}\overline{R}_{\alpha ijk}) - \mathrm{div}\omega.$$

Since M is minimal, we have $\sum_i h^\alpha_{iij} = 0$ for all j, α. Then we get

$$\begin{aligned} \sum_{i,j,k,\alpha} h^\alpha_{ikk}\overline{R}_{\alpha jij} &= \sum_{i,j,k,\alpha}(h^\alpha_{kki} - \overline{R}_{\alpha kik})\overline{R}_{\alpha jij} \\ &\geq -\frac{1}{4}pn(n-1)^2(b-a)^2. \end{aligned} \tag{3.12}$$

$$\begin{aligned} \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \sum_{i,j,k,\alpha} h^\alpha_{ijk}\overline{R}_{\alpha ijk} &\geq -\frac{1}{4}\sum_{\alpha,i,j,k}(\overline{R}_{\alpha ijk})^2 \\ &\geq -\frac{1}{9}pn(n-1)(n-2)(b-a)^2 \\ &\quad -\frac{1}{8}pn(n-1)(b-a)^2. \end{aligned} \tag{3.13}$$

Then the integral inequality holds by Green's divergence theorem.

We take $t = sgn(p-1)\frac{p}{p+2}$ when $p \le n$, and take $t = \frac{n}{n+2}$ when $p > n$ in Lemmas 3.1 and 3.2. Hence we get the following theorem.

Theorem 3.1. *If M^n is an oriented closed minimal submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}, then*

$$\int_M \Big\{ nS\Big[\frac{2(\tau(p,n)+1)}{\tau(p,n)+2}K_{\min} - \frac{\tau(p,n)b}{\tau(p,n)+2} - C(n,p)(b-a)\Big] \\ -D(n,p)(b-a)^2\Big\} dM \le 0.$$

Here

$$\begin{aligned}
&\tau(m,n) = \min\{sgn(m-1)m, n\},\\
&C(n,m) := \frac{2}{3n}(m-1),\\
&D(n,m) := \frac{1}{72}mn(n-1)(26n-25).
\end{aligned}$$

Denote by

$$\begin{aligned}
A_0(n,p) &:= \frac{(\tau(p,n)+2)(p-1)}{3n(\tau(p,n)+1)} + \frac{1}{12}\sqrt{\frac{p(\tau(p,n)+2)(26n-25)}{(\tau(p,n)+1)n}},\\
B_0(n,p) &:= 1 + \frac{1}{12}\sqrt{\frac{p(\tau(p,n)+2)(26n-25)}{(\tau(p,n)+1)n}},\\
\delta_0(n,p) &:= 1 - \frac{\tau(p,n)+2}{2(\tau(p,n)+1)[A_0(n,p)+B_0(n,p)]}.
\end{aligned}$$

We'll give the proof of Theorem 1.1.

The proof of Theorem 1.1. Since $c \le a(x) \le b(x) \le 1$, we get from Theorem 3.1 that

$$\int_M \Big\{ nS\Big[\Big(1+\frac{\tau(p,n)}{\tau(p,n)+2}\Big)K_{\min} - \frac{\tau(p,n)}{\tau(p,n)+2} - C(n,p)(1-c)\Big] \\ -D(n,p)(1-c)^2\Big\} dM \le 0. \tag{3.14}$$

From the assumption, we get

$$\delta_0(n,p) = 1 - \frac{\tau(p,n)+2}{2(\tau(p,n)+1)[A_0(n,p)+B_0(n,p)]}.$$

Therefore,

$$\begin{aligned} 1-c &\le 1-\delta_0(n,p) \\ &= \frac{\tau(p,n)+2}{2(\tau(p,n)+1)[A_0(n,p)+B_0(n,p)]}. \end{aligned} \tag{3.15}$$

Then we have

$$\frac{\tau(p,n)}{2(\tau(p,n)+1)} + A_0(n,p)(1-c) \le 1 - B_0(n,p)(1-c), \tag{3.16}$$

and the assumption is not trivial. From (2.5) and the assumption about R_0, we get

$$S \ge [B_0(n,p)-1]n(n-1)(1-c). \tag{3.17}$$

This together with the definitions of $A_0(n,p)$, $B_0(n,p)$ and the assumption

$$K_M \ge \frac{\tau(p,n)}{2(\tau(p,n)+1)} + A_0(n,p)(1-c)$$

implies that

$$\begin{aligned} & nS\Big[\Big(1+\frac{\tau(p,n)}{\tau(p,n)+2}\Big)K_{\min} - \frac{\tau(p,n)}{\tau(p,n)+2} - C(n,p)(1-c)\Big] \\ \ge\ & n^2(n-1)(B_0(n,p)-1)(1-c) \times \\ & \Big[\Big(1+\frac{\tau(p,n)}{\tau(p,n)+2}\Big)\Big(\frac{\tau(p,n)}{2(\tau(p,n)+1)} + A_0(n,p)(1-c)\Big) \\ & -\frac{\tau(p,n)}{\tau(p,n)+2} - C(n,p)(1-c)\Big] \\ =\ & D(n,p)(1-c)^2. \end{aligned} \tag{3.18}$$

Therefore, we obtain

$$\begin{aligned} \int_M \Big\{ & nS\Big[\Big(1+\frac{\tau(p,n)}{\tau(p,n)+2}\Big)K_{\min} - \frac{\tau(p,n)}{\tau(p,n)+2} - C(n,p)(1-c)\Big] \\ & -D(n,p)(1-c)^2\Big\} dM \ge 0. \end{aligned} \tag{3.19}$$

From (3.14) and (3.19), we know that the left side of (3.19) is equal to zero. This together with $c \le a \le b \le 1$ implies that $a \equiv c$ and $b \equiv 1$. Moreover, it follows from (3.12), (3.13) that

$$\sum_{i,j,k,\alpha} h^\alpha_{ikk}\overline{R}_{\alpha jij} = -\frac{1}{4}pn(n-1)^2(1-c)^2,$$

and

$$0 = h^\alpha_{ikk} + \frac{1}{2}\overline{R}_{\alpha ikk} = h^\alpha_{ikk}.$$

This implies $c = 1$. Since N is complete and simply connected, N is isometric to S^{n+p}. Moreover, it follows from Theorem C that M is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ in S^{n+1} for $k = 1, \cdots, n-1$, or the Veronese submanifold in S^{n+d}, where $d = \frac{1}{2}n(n+1) - 1$. This proves Theorem 1.1.

4. Submanifolds with parallel mean curvature

If M^n is an oriented compact submanifold with parallel mean curvature in N^{n+p}, then we get from (2.7) that

$$\begin{aligned}\sum_{i,j} h^{n+1}_{ij}\triangle h^{n+1}_{ij} &= -\sum_{i,j,k} h^{n+1}_{ij}(\overline{R}_{n+1kikj} + \overline{R}_{n+1ijkk}) \\ &\quad + \sum_{i,j,k,m}(h^{n+1}_{ij}h^{n+1}_{km}R_{mijk} + h^{n+1}_{ij}h^{n+1}_{mi}R_{mkjk}).\end{aligned} \tag{4.1}$$

Hence,

$$\begin{aligned}\frac{1}{2}\triangle S_H &= (h^{n+1}_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij}\triangle h^{n+1}_{ij} \\ &= (h^{n+1}_{ijk})^2 - \sum_{i,j,k} h^{n+1}_{ij}(\overline{R}_{n+1kikj} + \overline{R}_{n+1ijkk}) \\ &\quad + \sum_{i,j,k,m}(h^{n+1}_{ij}h^{n+1}_{km}R_{mijk} + h^{n+1}_{ij}h^{n+1}_{mi}R_{mkjk}).\end{aligned} \tag{4.2}$$

We define a differential 1-form ω as follows:

$$\omega := \sum_{i,j,k}(h^{n+1}_{ik}\overline{R}_{n+1jij} + h^{n+1}_{ij}\overline{R}_{n+1ijk})\omega_k.$$

Then we have

$$
\begin{aligned}
&-\sum_{i,j,k}(h_{ij}^{n+1}\overline{R}_{n+1kikj}+h_{ij}^{n+1}\overline{R}_{n+1ijkk})\\
(4.3)\qquad =\;&\sum_{i,j,k}(h_{ijj}^{n+1}\overline{R}_{n+1kik}+h_{ijk}^{n+1}\overline{R}_{n+1ijk})-\mathrm{div}\omega.
\end{aligned}
$$

From $\sum_{i,j,k}(h_{ijk}^{n+1}+\frac{1}{2}\overline{R}_{n+1ijk})^2\geq 0$, we get

$$
\begin{aligned}
\sum_{i,j,k}(h_{ijk}^{n+1})^2+\sum_{i,j,k}h_{ijk}^{n+1}\overline{R}_{n+1ijk} &\geq -\frac{1}{4}\sum_{i,j,k}\overline{R}^2_{n+1ijk}\\
&= -\frac{1}{4}\sum_{i\neq j\neq k\neq i}\overline{R}^2_{n+1ijk}-\frac{1}{2}\sum_{i\neq j}\overline{R}^2_{n+1iji}\\
(4.4)\qquad &\geq -\frac{1}{72}n(n-1)(8n-7)(b-a)^2.
\end{aligned}
$$

Since M has parallel mean curvature,

$$
\sum_i h_{iij}^{\alpha}=0,\ \text{for all } j.
$$

This together with Berger's inequality (2.4) and the Codazzi equation (2.2) implies that

$$
(4.5)\qquad \sum_{i,j,k}h_{ijj}^{n+1}\overline{R}_{n+1kik}=-\sum_i(\sum_j\overline{R}_{n+1jij})^2\geq -\frac{1}{4}n(n-1)^2(b-a)^2.
$$

On the other hand, we set $h_{ij}^{n+1}=\delta_{ij}\lambda_{ij}^{n+1}$ and obtain

$$
\begin{aligned}
&\sum_{i,j,k,m}h_{ij}^{n+1}h_{km}^{n+1}R_{mijk}+\sum_{i,j,k,m}h_{ij}^{n+1}h_{mi}^{n+1}R_{mkjk}\\
=\;&\sum_{i,k}\lambda_i^{n+1}\lambda_k^{n+1}R_{kiik}+\sum_{i,k}\lambda_i^{n+1}\lambda_i^{n+1}R_{ikik}\\
\geq\;&\frac{1}{2}K_{\min}\sum_{i,j}(\lambda_i^{n+1}-\lambda_j^{n+1})^2\\
(4.6)\qquad =\;&nK_{\min}(S_H-nH^2).
\end{aligned}
$$

From (4.2)-(4.6), we get the following theorem.

Theorem 4.1. *If M^n is an oriented compact submanifold with parallel* *mean curvature in an $(n+p)$-dimensional Riemannian manifold N^{n+p} and $H \neq 0$, then*

$$\int_M [(S_H - nH^2)nK_{\min} - D(n,1)(b-a)^2]dM \leq 0.$$

Here

$$D(n,m) := \frac{1}{72}mn(n-1)(26n-25).$$

If $p \geq 2$, then

$$\begin{aligned} \sum_{i,j,\alpha\neq n+1} h^\alpha_{ij}\triangle h^\alpha_{ij} &= -\sum_{i,j,k,\alpha\neq n+1} h^\alpha_{ij}(\overline{R}_{\alpha kikj} + \overline{R}_{\alpha ijkk}) \\ &\quad + \sum_{i,j,k,m,\alpha\neq n+1}(h^\alpha_{ij}h^\alpha_{km}R_{mijk} + h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}) \\ &\quad - \sum_{i,j,k,\alpha\neq n+1,\beta} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}. \end{aligned} \tag{4.7}$$

From (2.1), (2.2) and a direct computation, we get

$$\begin{aligned} &\sum_{i,j,k,m,\alpha\neq n+1}(h^\alpha_{ij}h^\alpha_{km}R_{mijk} + h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}) \\ =& \sum_{i,j,k,m,\alpha\neq n+1}(h^\alpha_{ij}h^\alpha_{mk}\overline{R}_{mijk} + h^\alpha_{ij}h^\alpha_{im}\overline{R}_{mkjk}) \\ &+ \sum_{\alpha\neq n+1,\beta} trH_\beta \cdot tr(H^2_\alpha H_\beta) - \sum_{\alpha\neq n+1,\beta}[tr(H_\alpha H_\beta)]^2 \\ &- \sum_{\alpha\neq n+1,\beta}[tr(H^2_\alpha H^2_\beta) - tr(H_\alpha H_\beta)^2], \end{aligned}$$

$$\begin{aligned} \sum_{i,j,k,\alpha\neq n+1,\beta} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk} &= \sum_{i,j,k,\alpha,\beta\neq n+1} h^\alpha_{ij}h^\beta_{ki}\overline{R}_{\alpha\beta jk} \\ &\quad + \sum_{\alpha,\beta\neq n+1}[tr(H^2_\alpha H^2_\beta) - tr(H_\alpha H_\beta)^2]. \end{aligned}$$

From the equalities above, we obtain

$$
\begin{aligned}
&\sum_{i,j,\alpha\neq n+1} h_{ij}^{\alpha}\triangle h_{ij}^{\alpha}\\
= \quad & -\sum_{i,j,k,\alpha\neq n+1} h_{ij}^{\alpha}(\overline{R}_{\alpha kikj}+\overline{R}_{\alpha ijkk}) - \sum_{i,j,k,\alpha,\beta\neq n+1} h_{ij}^{\alpha}h_{ki}^{\beta}\overline{R}_{\alpha\beta jk}\\
& -t\sum_{i,j,k,m,\alpha\neq n+1}(h_{ij}^{\alpha}h_{mk}^{\alpha}\overline{R}_{mijk}+h_{ij}^{\alpha}h_{im}^{\alpha}\overline{R}_{mkjk})\\
& +(1+t)\sum_{i,j,k,m,\alpha\neq n+1}(h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}+h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk})\\
& +(t-1)\sum_{\alpha,\beta\neq n+1}[tr(H_\alpha^2H_\beta^2)-tr(H_\alpha H_\beta)^2]+t\sum_{\alpha,\beta\neq n+1}[tr(H_\alpha H_\beta)]^2\\
& +t\sum_{\alpha\neq n+1}[tr(H_\alpha^2H_{n+1}^2)-tr(H_\alpha H_{n+1})^2]\\
& +t\Big\{-\sum_{\alpha\neq n+1} tr(H_\alpha^2 H_{n+1})\cdot trH_{n+1}+\sum_{\alpha\neq n+1}[tr(H_\alpha H_{n+1})]^2\Big\},
\end{aligned}
\tag{4.8}
$$

where $t\in\mathbb{R}$. If $\alpha,\beta\neq n+1$, then $(tr(H_\alpha H_\beta))$ ia a symmetric $(p-1)\times(p-1)$-matrix. We can choose the normal frame field $\{e_\alpha\}$ for $\alpha\neq n+1$ such that

$$tr(H_\alpha H_\beta)=trH_\alpha^2\cdot\delta_{\alpha\beta.}$$

Then we have

$$\sum_{\alpha,\beta\neq n+1}[tr(H_\alpha H_\beta)]^2=\sum_{\alpha\neq n+1}(trH_\alpha^2)^2. \tag{4.9}$$

Therefore,

$$\frac{1}{2}\triangle S_I=\sum_{i,j,k,\alpha\neq n+1}(h_{ijk}^{\alpha})^2+\sum_{i,j,\alpha\neq n+1}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}=W_2+X_2+Y_2+Z_2,$$

where

$$
\begin{aligned}
W_2 \quad := \quad & (1+t)\sum_{i,j,k,m,\alpha\neq n+1}(h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}+h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk})\\
& +(t-1)\sum_{\alpha,\beta\neq n+1}[tr(H_\alpha^2H_\beta^2)-tr(H_\alpha H_\beta)^2]+t\sum_{\alpha\neq n+1}(trH_\alpha^2)^2,
\end{aligned}
$$

$$\begin{aligned} X_2 &:= -t \sum_{i,j,k,m,\alpha\neq n+1} (h^\alpha_{ij} h^\alpha_{jm} \overline{R}_{mkik} + h^\alpha_{mk} h^\alpha_{ij} \overline{R}_{mijk}) \\ &\quad - \sum_{i,j,k,\alpha,\beta\neq n+1} h^\alpha_{ij} h^\beta_{ki} \overline{R}_{\alpha\beta jk}, \\ Y_2 &:= \sum_{i,j,k,\alpha\neq n+1} (h^\alpha_{ijk})^2 - \sum_{i,j,k,\alpha\neq n+1} (h^\alpha_{ij} \overline{R}_{\alpha kikj} + h^\alpha_{ij} \overline{R}_{\alpha ijkk}) \\ &\quad + t \sum_{\alpha\neq n+1} [tr(H^2_\alpha H^2_{n+1}) - tr(H_\alpha H_{n+1})^2], \\ Z_2 &:= t\Big\{ - \sum_{\alpha\neq n+1} tr(H^2_\alpha H_{n+1}) \cdot tr H_{n+1} + \sum_{\alpha\neq n+1} [tr(H_\alpha H_{n+1})]^2 \Big\}. \end{aligned}$$

For fixed $\alpha \neq n+1$, we choose the orthonormal frame fields $\{e_i\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$.

Lemma 4.1. *If* $0 \leq t < 1$, *then:*
(i) $W_2 \geq (1+t)nK_{\min} S_I + [\frac{t}{p-1} + \frac{sgn(p-2)}{2}(t-1)]S^2_I$;
(ii) $W_2 \geq (1+t)nK_{\min} S_I + [t + \frac{n}{2}(t-1)] \sum_{\alpha\neq n+1} (tr H^2_\alpha)^2$.

Proof. It follows from (3.6) that

$$\sum_{i,j,k,m,\alpha\neq n+1} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{i,j,k,m,\alpha\neq n+1} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} \geq nK_{\min} S_I. \tag{4.10}$$

(i) By a direct computation and Lemma 2.4, we get

$$\begin{aligned} \sum_{\alpha,\beta\neq n+1} [tr(H^2_\alpha H^2_\beta) - tr(H_\alpha H_\beta)^2] &= \frac{1}{2} \sum_{\alpha,\beta\neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) \\ &\leq \frac{1}{2} sgn(p-2) S^2_I, \end{aligned} \tag{4.11}$$

where $sgn(\cdot)$ is the standard sign function. Then the assertion follows from (4.10), (4.11) and the Schwarz inequality.

(ii) It follows from Lemma 2.3 and (4.9) that

$$\sum_{\alpha,\beta\neq n+1} [tr(H^2_\alpha H^2_\beta) - tr(H_\alpha H_\beta)^2] \leq \frac{n}{2} \sum_{\alpha\neq n+1} (tr H^2_\alpha)^2. \tag{4.12}$$

This together with (4.10) implies the conclusion.

This proves the lemma.

Lemma 4.2. *If $t \geq 0$, then $X_2 \geq -ntbS_I - \frac{2}{3}(p-2)(b-a)S_I$.*

Proof. A similar argument as in (3.6) shows that

$$\begin{aligned}\sum_{i,j,k,m}(h^\alpha_{ij}h^\alpha_{jm}\overline{R}_{mkik} + h^\alpha_{mk}h^\alpha_{ij}\overline{R}_{mijk}) &= \sum_{i,k}(h^\alpha_{ii})^2\overline{R}_{ikik} + \sum_{i,k}h^\alpha_{kk}h^\alpha_{ii}\overline{R}_{kiik} \\ &= \frac{1}{2}\sum_{i,k}(h^\alpha_{ii} - h^\alpha_{kk})^2\overline{R}_{ikik} \\ &\leq nb\cdot \mathrm{tr}H^2_\alpha. \qquad (4.13)\end{aligned}$$

For $\alpha, \beta \neq n+1$, we get from Berger's inequality (2.3) that

$$\begin{aligned}-\sum_{i,j,k,\alpha,\beta}h^\alpha_{ij}h^\beta_{ki}\overline{R}_{\alpha\beta jk} &\geq -\sum_{i,\alpha,\beta}|\overline{R}(e_\alpha, e_\beta, \sum_j h^\alpha_{ij}e_j, \sum_k h^\beta_{ik}e_k)| \\ &\geq -\sum_{i,\alpha\neq\beta}\frac{2}{3}(b-a)\sqrt{\sum_j (h^\alpha_{ij})^2}\sqrt{\sum_k (h^\beta_{ik})^2} \\ &\geq -\frac{1}{3}(b-a)\sum_{i,j,\alpha\neq\beta}[(h^\alpha_{ij})^2 + (h^\beta_{ij})^2] \\ &= -\frac{2}{3}(p-2)(b-a)S_I. \qquad (4.14)\end{aligned}$$

Combing (4.13) and (4.14), we get the conclusion. This proves the lemma.

Now, we'll give the estimation about Y_2.

Lemma 4.3. $\int_M Y_2 dM \geq -\frac{1}{72}(p-1)n(n-1)(26n-25)\int_M (b-a)^2 dM$.

Proof. We define a differential 1-form ω as follows:

$$\omega := \sum_{i,j,k,\alpha\neq n+1}(h^\alpha_{ik}\overline{R}_{\alpha jij} + h^\alpha_{ij}\overline{R}_{\alpha ijk})\omega_k,$$

then

$$\begin{aligned}&-\sum_{i,j,k,\alpha\neq n+1}(h^\alpha_{ij}\overline{R}_{\alpha kikj} + h^\alpha_{ij}\overline{R}_{\alpha ijkk}) \\ =&\sum_{i,j,k,\alpha\neq n+1}(h^\alpha_{ijj}\overline{R}_{\alpha kik} + h^\alpha_{ijk}\overline{R}_{\alpha ijk}) - \mathrm{div}\omega.\end{aligned}$$

Since $\sum_{i,j,k}(h^{\alpha}_{ijk}+\frac{1}{2}\overline{R}_{\alpha ijk})^2\geq 0$, we get

$$\begin{aligned}
&\sum_{i,j,k,\alpha\neq n+1}(h^{\alpha}_{ijk})^2+\sum_{i,j,k,\alpha\neq n+1}h^{\alpha}_{ijk}\overline{R}_{\alpha ijk}\\
\geq\ & -\frac{1}{4}\sum_{i,j,k,\alpha\neq n+1}\overline{R}^2_{\alpha ijk}\\
=\ & -\frac{1}{4}\sum_{i\neq j\neq k\neq i,\alpha\neq n+1}\overline{R}^2_{\alpha ijk}-\frac{1}{2}\sum_{i\neq j,\alpha\neq n+1}\overline{R}^2_{\alpha iji}\\
\geq\ & -\frac{1}{72}(p-1)n(n-1)(8n-7)(b-a)^2. \qquad (4.15)
\end{aligned}$$

Because M admits parallel mean curvature, we have $\sum_i h^{\alpha}_{iij}=0$ for all j. This together with (2.2) and (2.4) implies that

$$\begin{aligned}
\sum_{i,j,k,\alpha\neq n+1}h^{\alpha}_{ijj}\overline{R}_{\alpha kik} &= -\sum_{i,\alpha\neq n+1}(\sum_j\overline{R}_{\alpha jij})^2\\
&\geq -\frac{1}{4}(p-1)n(n-1)^2(b-a)^2. \qquad (4.16)
\end{aligned}$$

It follows from (2.2) and Lemma 2.2 that

$$\begin{aligned}
&\sum_{\alpha\neq n+1}[tr(H^2_{\alpha}H^2_{n+1})-tr(H_{\alpha}H_{n+1})^2]\\
&=\frac{1}{2}\sum_{i,j,k,\alpha\neq n+1}(h^{\alpha}_{ij}h^{n+1}_{ik}+h^{\alpha}_{ik}h^{n+1}_{ij})\overline{R}_{n+1\alpha jk}+\frac{1}{2}\sum_{j,k,\alpha\neq n+1}\overline{R}^2_{n+1\alpha jk}\\
&\geq 0. \qquad (4.17)
\end{aligned}$$

Then we get from (4.15) and (4.16) that

$$Y_2\geq -\frac{1}{72}(p-1)n(n-1)(26n-25)(b-a)^2-\mathrm{div}\omega,$$

which implies the conclusion.

Lemma 4.4. *If $t\geq 0$, then*

$$Z_2\geq[-nH^2-(n-2)n^{1/2}(n-1)^{-1/2}H(S_H-nH^2)^{1/2}]tS_I.$$

Proof. We rewrite Z_2 as follows:

$$\begin{aligned} Z_2 &= t\sum_{\alpha\neq n+1}[tr(H_\alpha(H_{n+1}-HI))]^2 \\ &\quad -tnH\sum_{\alpha\neq n+1}tr[H_\alpha^2(H_{n+1}-HI)]-tnH^2\sum_{\alpha\neq n+1}trH_\alpha^2 \\ &\geq -tnH\sum_{\alpha\neq n+1}tr[H_\alpha^2(H_{n+1}-HI)]-tnH^2S_I. \end{aligned} \tag{4.18}$$

Here I is the unit $(n\times n)$-matrix. For fixed e_α, we let $\{e_i\}$ be a frame diagonalizing the matrix H_α such that $h_{ij}^\alpha=0$ for $i\neq j$. This together with Lemma 2.5 and the Schwarz inequality implies that

$$\begin{aligned} & nHtr[H_\alpha^2(H_{n+1}-HI)] \\ =\ & nH[\sum_i(h_{ii}^\alpha)^2(h_{ii}^{n+1}-H)] \\ \leq\ & (n-2)n^{1/2}(n-1)^{-1/2}H[\sum_i(h_{ii}^{n+1}-H)^2]^{1/2}[\sum_i(h_{ii}^\alpha)^2] \\ =\ & (n-2)n^{1/2}(n-1)^{-1/2}H(S_H-nH^2)^{1/2}trH_\alpha^2. \end{aligned} \tag{4.19}$$

The conclusion follows from (4.18) and (4.19).

Let $t=sgn(p-2)\frac{p-1}{p+1}$ when $p-1\leq n$, and let $t=\frac{n}{n+2}$ when $p-1>n$ in Lemmas 4.1, 4.2 and 4.4. Then we get following theorem.

Theorem 4.2. *If M^n is an oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional Riemannian manifold N^{n+p} and $H\neq 0$, then*

$$\begin{aligned} &\int_M\Big\{nS_I\Big[\frac{2(\tau(p-1,n)+1)}{\tau(p-1,n)+2}K_{\min}-\frac{\tau(p-1,n)}{\tau(p-1,n)+2}(b+H^2) \\ &-\frac{\tau(p-1,n)}{\tau(p-1,n)+2}\frac{n-2}{\sqrt{n(n-1)}}H(S_H-nH^2)^{1/2} \\ &-C(n,p-1)(b-a)\Big]-D(n,p-1)(b-a)^2\Big\}dM\leq 0. \end{aligned}$$

Here

$$\tau(m,n) := \min\{sgn(m-1)m, n\},$$
$$D(n,m) := \frac{1}{72}mn(n-1)(26n-25),$$
$$C(n,m) := \frac{2}{3n}(m-1).$$

Theorem 4.3. *Let M^n be an oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional complete simply connected Riemannian manifold N^{n+p}, $p \le 2$ and $H \neq 0$. Then there exists a constant $\theta_1(n,p) \in (0,1)$ such that if $\overline{K}_N \in [\theta_1(n,p), 1]$, and*

$$K_M \ge \beta_1(n,p)(1-c),$$
$$R_0 \le 1 + H^2 - \gamma_1(n,p)(1-c),$$

where $c := \inf \overline{K}_N$, then N^{n+p} is isometric to S^{n+p}. Moreover, M is isometric to (i) – (v) of Theorem 1.2. Here

$$\beta_1(n,p) = \frac{1}{12}\sqrt{\frac{p(52n-50)}{n}},$$
$$\gamma_1(n,p) = 1 + \frac{1}{12}\sqrt{\frac{p(52n-50)}{n}},$$
$$\theta_1(n,p) = 1 - [\beta_1(n,p) + \gamma_1(n,p)]^{-1}.$$

Proof. If $c \le a(x) \le b(x) \le 1$ and $p \le 2$, then we get from Theorems 4.1 and 4.2 that

$$\int_M [(S - nH^2)nK_{\min} - D(n,p)(1-c)^2]dM \le 0. \tag{4.20}$$

From the assumption $\theta_1(n,p) = 1 - [\beta_1(n,p) + \gamma_1(n,p)]^{-1}$, we have

$$1 - c \le 1 - \delta_1(n,p) = [\beta_1(n,p) + \gamma_1(n,p)]^{-1}. \tag{4.21}$$

Hence

$$\beta_1(n,p)(1-c) \le 1 - \gamma_1(n,p)(1-c) \le 1 + H^2 - \gamma_1(n,p)(1-c), \tag{4.22}$$

and the assumption is not trivial. It follows from (2.5) and the upper bound of R_0 that

$$S - nH^2 \geq [\gamma_1(n,p) - 1]n(n-1)(1-c). \tag{4.23}$$

This together with the definitions of $\beta_1(n,p)$, $\gamma_1(n,p)$ and the lower bound of K_M implies that

$$\begin{aligned} & (S - nH^2)nK_{\min} \\ \geq \; & n^2(n-1)[\gamma_1(n,p) - 1]\beta_1(n,p)(1-c)^2 \\ = \; & D(n,p)(1-c)^2. \end{aligned} \tag{4.24}$$

Therefore,

$$\int_M [(S - nH^2)nK_{\min} - D(n,p)(1-c)^2]dM \geq 0. \tag{4.25}$$

From (4.20) and (4.25), we obtain that the left side of (4.25) is equal to zero. This together with Theorems 4.1, 4.2 and the condition $c \leq a \leq b \leq 1$ implies $a \equiv c$ and $b \equiv 1$. Moreover, it follows from (4.4), (4.5) that

$$\sum_{i,j,k} h_{ikk}^{n+1}\overline{R}_{n+1jij} = -\frac{1}{4}n(n-1)^2(1-c)^2,$$

and

$$0 = h_{ikk}^{n+1} + \frac{1}{2}\overline{R}_{n+1ikk} = h_{ikk}^{n+1}.$$

This implies that $c = 1$. Since N is complete and simply connected, we know that N is isometric to S^{n+p}. Moreover, we get the classification of M from Theorem C. This proves the theorem.

For the case $p \geq 3$, we need the following lemma.

Lemma 4.5. *Let M^n be an oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional Riemannian manifold N^{n+p} and $H \neq 0$. If*

$$K_M \geq \beta_2(n,p)(d-c),$$

then $d = c$, or

$$\int_M (S_H - nH^2)dM \leq \alpha(n,p) \int_M (b-a)dM.$$

Here $c := \inf \overline{K}_N,\ d := \sup \overline{K}_N,\ \alpha(n,p) = \frac{D(n,1)}{n\beta_2(n,p)}$.

Proof. Since $c \le a \le b \le d$, it follows from Theorem 4.1 that

$$\int_M [(S_H - nH^2)nK_{\min} - D(n,1)(b-a)(d-c)]dM \le 0.$$

From the assumption, we get

$$\int_M [(S_H - nH^2)n\beta_2(n,p)(d-c) - D(n,1)(b-a)(d-c)]dM \le 0. \tag{4.26}$$

Then we have $d = c$, or

$$\int_M (S_H - nH^2)dM \le \alpha(n,p)\int_M (b-a)dM.$$

This completes the proof.

Theorem 4.4. *Let M^n be an oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional complete simply connected Riemannian manifold N^{n+p}, $p \ge 3$, and $H \ne 0$. Then there exists a constant $\theta_2(n,p) \in (0,1)$ such that if $\overline{K}_N \in [\theta_2(n,p), 1]$, and*

$$\begin{aligned} K_M \ge\ & \frac{\tau(p-1,n)}{2[\tau(p-1,n)+1]}(1+H^2) \\ & + \beta_2(n,p)(1-c) + \beta_3(n,p)[H(1+H^2)]^{1/2}(1-c)^{1/4}, \\ R_0 \le\ & 1 + H^2 - \gamma_2(n,p)(1-c) - \gamma_3(n,p)[H(1+H^2)]^{1/2}(1-c)^{1/4}, \end{aligned}$$

where $c := \inf \overline{K}_N$, then N^{n+p} is isometric to S^{n+p}. Moreover, M is isometric to $(i) - (vi)$ of Theorem 1.2. Here $\beta_2(n,p)$, $\beta_3(n,p)$, $\gamma_2(n,p)$, $\gamma_3(n,p)$ will be defined in the proof of the theorem, and

$$\theta_2(n,p) = 1 - \frac{[\tau(p-1,n)+2]^4}{16[\beta_2(n,p)+\gamma_2(n,p)+\sqrt{2}\beta_3(n,p)]^4(\tau(p-1,n)+1)]^4}.$$

Remark 4.1. From the choice of $\theta_2(n,p)$, we see that the pinching condition of M makes sense.

Proof. We assume $c \neq 1$. It follows from Theorem 4.2 that

$$\begin{aligned}&\int_M \Big\{nS_I\Big[E(n,p)K_{\min} - \frac{\tau(p-1,n)}{\tau(p-1,n)+2}(1+H^2)\\ &- \frac{\tau(p-1,n)}{\tau(p-1,n)+2}\frac{n-2}{\sqrt{n(n-1)}}H(S_H - nH^2)^{1/2}\\ &-C(n,p-1)(1-c)\Big] - D(n,p-1)(1-c)^2\Big\}dM \leq 0. \qquad (4.27)\end{aligned}$$

Here

$$E(n,p) := \frac{2(\tau(p-1,n)+1)}{\tau(p-1,n)+2}.$$

From the Gauss equation, the assumption $K_M \geq \frac{\tau(p-1,n)}{2[\tau(p-1,n)+1]}(1+H^2)$, and $S = S_H + S_I \geq nH^2 + S_I$, we obtain

$$S_I \leq S - nH^2 \leq E^{-1}(n,p)n(n-1)(1+H^2). \qquad (4.28)$$

On the other hand, we have

$$K_M \geq \frac{\tau(p-1,n)}{2[\tau(p-1,n)+1]}(1+H^2) + \beta_2(n,p)(1-c) \geq \beta_2(n,p)(1-c).$$

This together with Lemma 4.5 implies that

$$\int_M (S_H - nH^2)dM \leq \alpha(n,p)\int_M (1-c)dM. \qquad (4.29)$$

Then we have from the Schwarz inequality that

$$\begin{aligned}&\int_M HS_I(S_H - nH^2)^{1/2}dM\\ \leq\ & H\max S_I \cdot \mathrm{vol}^{1/2}(M)\Big[\int_M (S_H - nH^2)dM\Big]^{1/2}\\ \leq\ & \alpha^{1/2}(n,p)E^{-1}(n,p)H(1+H^2)n(n-1)(1-c)^{1/2}\mathrm{vol}(M). \qquad (4.30)\end{aligned}$$

Combing (4.27) and (4.30), we get

$$\begin{aligned}&\int_M \Big\{nS_I\Big[E(n,p)K_{\min} - \frac{\tau(p-1,n)}{\tau(p-1,n)+2}(1+H^2)\\ &-C(n,p-1)(1-c)\Big] - F(n,p)H(1+H^2)(1-c)^{1/2}\\ &-D(n,p-1)(1-c)^2\Big\}dM \leq 0. \qquad (4.31)\end{aligned}$$

Here, we set

$$F(n,p) := \frac{\tau(p-1,n)}{2[\tau(p-1,n)+1]} n^{3/2}(n-2)(n-1)^{1/2}\alpha^{1/2}(n,p).$$

Let

$$\beta_2(n,p) := C(n,p-1)E^{-1}(n,p) + D^{1/2}(n,p-1)E^{-1/2}(n,p)n^{-1}(n-1)^{-1/2},$$

$$\beta_3(n,p) := F^{1/2}(n,p)E^{-1/2}(n,p)n^{-1}(n-1)^{-1/2}.$$

Substituting the lower bound of K_M into (4.31), we have

$$\begin{aligned}\int_M S_I dM \leq & [D^{1/2}(n,p-1)E^{-1/2}(n,p)(n-1)^{1/2}(1-c) \\ & + F^{1/2}(n,p)E^{-1/2}(n,p)(n-1)^{1/2}[H(1+H^2)]^{1/2}(1-c)^{1/4}]\mathrm{vol}(M).\end{aligned}$$

This together with (4.29) implies that

$$\begin{aligned}\int_M (S-nH^2)dM \leq & \{[\alpha(n,p) + D^{1/2}(n,p-1)E^{-1/2}(n,p)(n-1)^{1/2}](1-c) \\ & + F^{1/2}(n,p)E^{-1/2}(n,p)(n-1)^{1/2}[H(1+H^2)]^{1/2}(1-c)^{1/4}\}\mathrm{vol}(M).\end{aligned} \tag{4.32}$$

Here

$$\alpha(n,p) = D(n,1)n^{-1}\beta_2^{-1}(n,p) = D(n,p-1)n^{-1}(p-1)^{-1}\beta_2^{-1}(n,p).$$

From the assumption, we know that

$$\begin{aligned} S - nH^2 \geq & (\gamma_2(n,p)-1)n(n-1)(1-c) \\ & + \gamma_3(n,p)n(n-1)[H(1+H^2)]^{1/2}(1-c)^{1/4}.\end{aligned} \tag{4.33}$$

Let

$$\gamma_2(n,p) := 1 + [\alpha(n,p) + D^{1/2}(n,p-1)E^{-1/2}(n,p)(n-1)^{1/2}]n^{-1}(n-1)^{-1},$$

$$\gamma_3(n,p) := \beta_3(n,p),$$

then we obtain

$$\begin{aligned} S - nH^2 \equiv & [\alpha(n,p) + D^{1/2}(n,p-1)E^{-1/2}(n,p)(n-1)^{1/2}](1-c) \\ & + F^{1/2}(n,p)E^{-1/2}(n,p)(n-1)^{1/2}[H(1+H^2)]^{1/2}(1-c)^{1/4}.\end{aligned} \tag{4.34}$$

Therefore, the inequalities above all becomes equalities and $1-c = b-a$. Since $c \leq a \leq b \leq 1$, we have $a = c$, $b = 1$. Moreover, it follows from (4.15), (4.16) that

$$\sum_{i,j,k,\alpha\neq n+1} h^{\alpha}_{ikk}\overline{R}_{\alpha jij} = -\frac{1}{4}(p-1)n(n-1)^2(1-c)^2,$$

and

$$0 = h^{\alpha}_{ikk} + \frac{1}{2}\overline{R}_{\alpha ikk} = h^{\alpha}_{ikk}.$$

This implies that $c = 1$. This is contradict to the assumption. Because N is complete and simply connected, we know that N is isometric to S^{n+p}. Moreover, it follows from Theorems 4.1 and 4.2 that

$$S = nH^2, \text{ or } K_{\min} = \frac{\tau(p-1,n)}{2[\tau(p-1,n)+1]}(1+H^2). \tag{4.35}$$

Then the conclusion follows from Theorem C. This completes the proof.

Proof of Theorem 1.2. We first define the pinching constants in Theorem 1.2:

$$\delta_1(n,p) = \begin{cases} \theta_1(n,p), & \text{if } p \leq 2, \\ \theta_2(n,p), & \text{if } p \geq 3, \end{cases}$$

$$A_1(n,p) = \begin{cases} \beta_1(n,p), & \text{if } p \leq 2, \\ \beta_2(n,p), & \text{if } p \geq 3, \end{cases}$$

$$A_2(n,p) = \begin{cases} 0, & \text{if } p \leq 2, \\ \beta_3(n,p), & \text{if } p \geq 3, \end{cases}$$

$$B_1(n,p) = \begin{cases} \gamma_1(n,p), & \text{if } p \leq 2, \\ \gamma_2(n,p), & \text{if } p \geq 3, \end{cases}$$

$$B_2(n,p) = \begin{cases} 0, & \text{if } p \leq 2, \\ \gamma_3(n,p), & \text{if } p \geq 3. \end{cases}$$

Combing Theorems 4.3 and 4.4, we get the conclusion. This proves the theorem.

In 1986, Huisken [8] studied hypersurfaces moving along their mean curvature vector in general Riemannian manifold N^{n+1}, and proved a convergence theorem for compact hypersurfaces under curvature pinching condition of the second fundamental form. Motivated by Theorems 1.1, 1.2, and convergence results due to Huisken [8], Andrews-Baker [1], Liu-Xu-Zhao [16] and Lei-Xu [13, 14], we would like to propose the following problem on the mean curvature flow of arbitrary codimension in a pinched Riemannian manifold.

Open Problem. *Let $F_0 : M \to N^{n+p}$ be an $n(\geq 2)$-dimensional closed submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p} with $0 < \delta(n,p) \leq \overline{K}_N \leq 1$. Assume that*

$$\begin{aligned} K_M &> \frac{\tau(p,n)}{2(\tau(p,n)+1)}(1+H^2) + A_3(n,p)(1-c) \\ &\quad + A_4(n,p)[H(1+H)]^{1/2}(1-c)^{1/4}, \\ R_0 &\leq 1 + H^2 - B_3(n,p)(1-c) - B_4(n,p)[H(1+H)]^{1/2}(1-c)^{1/4}, \end{aligned}$$

where $c := \inf \overline{K}_N \geq 0$. Is it possible to prove that the mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) = n\xi(x,t), \ x \in M, \ t \geq 0, \\ F(\cdot,0) = F_0(\cdot), \end{cases}$$

exists smooth solution $F_t(\cdot)$, and $F_t(\cdot)$ converges to a round point in finite time, or $F_t(\cdot)$ converges to a totally geodesic submanifold as $t \to \infty$? Here $\delta(n,p)$, $A_3(n,p)$, $A_4(n,p)$, $B_3(n,p)$, $B_4(n,p)$ are nonnegative constants depending on n and p, $\tau(p,n)$ is defined in Theorem C.

References

[1] B. Andrews and C.Baker, Mean curvature flow of pinched submanifolds to spheres, *J. Differential Geom.*, **85**(2010), 357-396.

[2] Q. Chen and S. L. Xu, Rigidity of compact minimal submanifolds in a unit sphere, Geom. Dedicata, **45**(1993), 83-88.

[3] S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, in Functional Analysis and Related Fields, Springer-Verlag, New York(1970).

[4] T. Choi and Z. Lu, On the DDVV conjecture and the comass in calibrated geometry (I), *Math. Z.*, **260**(2008), 409-429.

[5] P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, A pointwise inequality in submanifold theory, *Arch. Math.*, **35**(1999), 115-128.

[6] J. Q. Ge and Z. Z. Tang, A proof of the DDVV conjecture and its equality case, *Pacific J. Math.*, **237**(2008), 87-95.

[7] J. R. Gu and H. W. Xu, On Yau rigidity theorem for minimal submanifolds in spheres, *Math. Res. Lett.*, **19**(2012), 511-523.

[8] G. Huisken, Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature, *Invent. Math.*, **84**(1986), 463-480.

[9] M. Kozlowski and U. Simon, Minimal immersions of 2-manifolds into spheres, *Math. Z.*, **186**(1984), 377-382.

[10] T. Itoh, On veronese manifolds, *J. Math. Soc. Japan.*, **27**(1975), 497-506.

[11] T. Itoh, Addendum to my paper "On veronese manifolds", *J. Math. Soc. Japan.*, **30**(1978), 73-74.

[12] B. Lawson, Local rigidity theorems for minimal hyperfaces, Ann. of Math. 89 (1969) 187-197.

[13] L. Lei and H. W. Xu, An optimal convergence theorem for mean curvature flow of arbitrary codimension in hyperbolic spaces, arXiv:1503.06747.

[14] L. Lei and H. W. Xu, Mean curvature flow of arbitrary codimension in spheres and sharp differentiable sphere theorem, arXiv:1506.06371.

[15] A. M. Li and J. M. Li, An intrinsic rigidity theorem for minimal submanifolds in a sphere, *Arch. Math.*, **58**(1992), 582-594.

[16] K. F. Liu, H. W. Xu and E.T. Zhao, Mean curvature flow of higher codimension in Riemannian manifolds, arXiv:1204.0107.

[17] Z. Lu, On the DDVV conjecture and the comass in calibrated geometry (II), arXiv:0708.2921.

[18] Z. Lu, Proof of the normal scalar curvature conjecture, arXiv:0711.3510.

[19] K. Nomizu and B. Smyth, A formula of Simons' type and hypersurfaces with constant mean curvature, *J. Differentiable Geom.*, **3**(1969), 367-377.

[20] M. Okumura, Submanifolds and a pinching problem on the second fundamental tensor, *Trans. Amer. Math. Soc.*, **178**(1973), 285-291.

[21] M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, *Amer. J. Math.*, **96**(1974), 207-213.

[22] W. Santos, Submanifolds with parallel mean curvature vector in spheres, *Tohoku Math. J.*, **46**(1994), 403-415.

[23] K. Shiohama and H. W. Xu, A general rigidity theorem for complete submanifolds, *Nagoya Math. J.*, **150**(1998), 105-134.

[24] Y. B. Shen, Submanifolds with nonnegative sectional curvature, *Chinese Ann. Math. SerB*, **5**(1984), 625-632.

[25] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. Math.*, **88**(1986), 62-105.

[26] R. Walter, Compact hypersurfaces with a constant higher mean curvature function, *Math. Ann.,* **270**(1985), 125-145.

[27] H. W. Xu, Pinching theorems, global pinching theorems and eigenvalues for Riemannian submanifolds, Ph.D. dissertation, Fudan University, 1990.

[28] H. W. Xu, A rigidity theorem for submanifolds with parallel mean curvature in a sphere, *Arch. Math.*, **61**(1993), 489-496.

[29] H. W. Xu, Annual Report, Graduate School of Mathematics, *Kyushu University*, March 1994, 91-92.

[30] H. W. Xu, On closed minimal submanifolds in pinched Riemannian manifolds, *Trans. Amer. Math. Soc.*, **347** (1995), 1743-1751.

[31] H. W. Xu and W. Han, Geometric rigidity theorem for submanifolds with positive curvature, *Appl. Math. J. Chinese Univ. Ser. B*, **20**(2005), 475-482.

[32] S. T. Yau, Submanifolds with constant mean curvature I, II, *Amer. J. Math.*, **96, 97**(1974, 1975), 346-366, 76-100.

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