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Integral of Distance Function on Compact Riemannian Manifolds

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Abstract: In this paper we show that, under some curvature assumptions the integral of distance function on a compact Riemannian manifold is bounded below by the product of diameter, volume and a constant only depending on the dimension.
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1. introduction

Let M be a compact Riemannian manifold. Let $d(p,q)$ be the distance between points p and q. If we fix a point $p \in M$, then we obtain a distance function $d(p,x), x \in M$. It is a continuous function and differentiable almost everywhere. The distance function plays a basic role in Riemannian geometry. In this paper we consider the integral of $d(p,x)$ on M. This gives a function

$$f(p) = \int_M d(p,x)dv, p \in M. \tag{1.1}$$

Obviously it has an upper bound $d(M)V(M)$. Here $d(M)$ denotes the diameter of M and $V(M)$ is the volume of M.

By the mean value theorem, for every $p \in M$ we can find a point $\xi_p \in M$ such that

$$f(p) = d(p, \xi_p)V(M).$$

So we can ask a natural question: For any compact Riemannian manifold M of dimension n, do we have

$$f(p) \geq c(n)d(M)V(M) \tag{1.2}$$

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for all $p \in M$? The $c(n)$ is a positive constant only depending on the dimension n. Unfortunately, the answer is negative. In fact we can construct examples such that $\frac{f(p)}{d(M)V(M)}$ is arbitrarily small for some point p (example 2.4). But if we add some curvature conditions, the answer is positive. The first result of this paper is

Theorem 1.1. *Let M be an n-dimensional compact Riemannian manifold with nonnegatively Ricci curvature. Then*

$$f(p) > c(n)d(M)V(M)$$

for all $p \in M$. The $c(n)$ can be chosen to equal $(1 - \frac{1}{n+1})^n \cdot \frac{1}{2^{n+1}(n+1)}$.

A natural problem is to determine the sharp value for $c(n)$. Examples in section 2 show that $c(n)$ can achieve $\frac{1}{2}$.

The Bishop-Gromov's volume comparison plays an essential role in the proof of the theorem.

In last section we will prove two similar results on complete noncompact Riemannian manifolds.

2. examples and properties

Following the triangle inequality, one obviously has

- $f(p) + f(q) \geq d(p,q)V(M)$,
- $|f(p) - f(q)| \leq d(p,q)V(M)$.

Let $p, q \in M$ satisfy $d(p,q) = d(M)$. One can see that either $f(p)$ or $f(q)$ is greater than or equal to $\frac{1}{2}d(M)V(M)$. So

$$\max_{y \in M} f(y) \geq \frac{1}{2}d(M)V(M).$$

We present three examples such that $f(y) \geq \frac{1}{2}d(M)V(M)$ holds for all $y \in M$.

Example 2.1. Let γ be a closed curve with length l. $d(\gamma) = \frac{l}{2}$ and $V(\gamma) = l$. Then $f = \frac{1}{2}d(\gamma)V(\gamma)$. In fact, if we assume that $\gamma(t)$ has arc length parameter, then

$$f = \int_0^{l/2} tdt + \int_{l/2}^{l} (l-t)dt = \frac{l^2}{4} = \frac{1}{2}d(\gamma)V(\gamma).$$

Example 2.2. If M is a compact symmetric space, then for any points $p, q \in M$ there exists an isometric mapping p to q. Hence f is a constant. Choose p, q such that $d(p, q) = d(M)$. Then $2f = f(p) + f(q) \geq d(p, q)d(M)$. So we have

$$f \geq \frac{1}{2} d(M)V(M).$$

For a special case when M is sphere space form S_k^n (k is the sectional curvature), let $p \in S_k^n$ and q is the antipodal point of p. Then for any $x \in S_k^n$, $d(p, x) + d(q, x) = d(S_k^n)$. Hence we have

$$f = \frac{1}{2} d(S_k^n)V(S_k^n).$$

Example 2.3. Let T^2 be the flat 2-torus of area 1. Then

$$f = \int_0^{\frac{1}{2}} r \cdot 2\pi r dr + 4 \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} r \cdot (\frac{\pi}{2} - 2 \arccos \frac{1}{2r}) r dr$$

$$= \frac{1}{6}(\sqrt{2} + \ln(\sqrt{2} + 1)) \doteq 0.3826.$$

$f > \frac{1}{2} diam(T^2)V(T^2) = \frac{\sqrt{2}}{4} \doteq 0.3535.$

However the following example shows that $\frac{f(y)}{d(M)V(M)}$ can achieves every value in $(0, 1)$.

Example 2.4. Let $M_1 = \{(x, y, z)|x^2 + y^2 + z^2 = 1, -1 + \epsilon \leq z \leq 1\}$ and $M_2 = \{(x, y, z)|x^2 + y^2 = 2\epsilon - \epsilon^2, -1 + \varepsilon - L \leq z \leq -1 + \varepsilon\} \cup \{(x, y, z)|x^2 + y^2 \leq 2\epsilon - \epsilon^2, z = -1 + \varepsilon - L\}$. Let $M = M_1 \cup M_2$ be M_1 glued to M_2. We write $C = 2\pi\sqrt{2\epsilon - \epsilon^2}$. Let $p = (0, 0, 1)$ and $q = (0, 0, -1 + \varepsilon - L)$. When C is very small,

$$\begin{aligned} f(p) &\doteq \int_{S_1^2} d(p, x)dS + \int_0^L (\pi + t)Cdt \\ &= \frac{1}{2}\pi V(S_1^2) + C(\pi L + \frac{L^2}{2}) \end{aligned}$$

and

$$d(M)V(M) \doteq (L + \pi)(V(S_1^2) + LC).$$

Set $C = \frac{1}{L^3}$. We can see that $\frac{f(p)}{d(M)V(M)} \to 0$ as $L \to \infty$. We also have $\frac{f(q)}{d(M)V(M)} \to 1$ as $L \to \infty$.

The following proposition is a consequence of Bishop-Gromov volume comparison.

Proposition 2.5. *If* $Ric_M \geq (n-1)k > 0$, *then* $f \leq \frac{1}{2}d(S_k^n)V(S_k^n)$. *The equality holds if and only if* M *is isometric to* S_k^n.

Proof. By the Fubini theorem and Bishop-Gromov volume comparison,

$$\begin{aligned} f(p) &= \int_M d(p,x)dv = \int_{-\infty}^{+\infty} d\lambda \int_{S_\lambda} d(p,x)dv_\lambda \\ &= \int_0^{d(M)} d\lambda \int_{S_\lambda} \lambda dv_\lambda = \int_0^{d(M)} \lambda V(S_\lambda)d\lambda \\ &\leq \int_0^{d(S_k^n)} \lambda V(S_{k\lambda})d\lambda \\ &= \frac{1}{2}d(S_k^n)V(S_k^n). \end{aligned}$$

The S_λ denotes the sphere center at p with radius λ and $V(S_\lambda)$ is the induced volume of S_λ. If the equality holds, we have $d(M) = d(S_k^n)$ and $V(S_\lambda) = V(S_{k\lambda})$. So M must be isometric to S_k^n. □

3. a proof of theorem 1.1

Let $B_p(r)$ (respectively $B_o(r)$) denote the ball center at p of radius r in M (respectively ball center at origin of radius r in $\mathbb{R}^n$). The $V_p(r)$ (respectively $V_o(r)$) denotes the volume of $B_p(r)$ (respectively $B_o(r)$).

Lemma 3.1. *For any* $p \in M$, *we have* $\max_{x\in M} d(p,x) \geq \frac{1}{2}d(M)$.

Proof. If on the contrary, $\max_{x\in M} d(p,x) < \frac{1}{2}d(M)$. Choosing $q_1, q_2 \in M$ such that $d(q_1,q_2) = d(M)$, thus

$$d(M) \leq d(p,q_1) + d(p,q_2) < \frac{1}{2}d(M) + \frac{1}{2}d(M) = d(M).$$

This is a contradiction. □

Because the Ricci curvature of M is nonnegative. The Bishop-Gromov's volume comparison implies that

$$\frac{V_p(r)}{V_o(r)} \geq \frac{V_p(R)}{V_o(R)}$$

for $r \le R$. Hence

$$V_p(r) \ge \frac{V_o(r)}{V_o(R)} V_p(R) = \frac{r^n}{R^n} V_p(R).$$

Let $R \to d = d(M)$. We get

$$V_p(r) \ge \frac{r^n}{d^n} V(M)$$

for all $r \le d$.

By the lemma, for any $p \in M$, we can choose $q \in M$ such that $d(p,q) \ge \frac{1}{2} d(M)$. Thus

$$\begin{aligned} f(p) &= \int_{B_p(\frac{1}{2}d-r)} d(p,x)dv + \int_{M \setminus B_p(\frac{1}{2}d-r)} d(p,x)dv \\ &> \int_{M \setminus B_p(\frac{1}{2}d-r)} d(p,x)dv \\ &> (\frac{1}{2}d - r) V(M \setminus B_p(\frac{1}{2}d - r)) \\ &> (\frac{1}{2}d - r) V_q(r) \\ &\ge V(M)(\frac{1}{2}d - r)\frac{r^n}{d^n}. \end{aligned}$$

Let $g(r) = (\frac{1}{2}d - r)\frac{r^n}{d^n}, 0 \le r \le \frac{1}{2}d$. When

$$g'(r) = \frac{1}{d^n}[\frac{nd}{2} r^{n-1} - (n+1)r^n] = 0,$$

$r = \frac{n}{2(n+1)}d$, $g(r)$ achieves the maximal value $(1 - \frac{1}{n+1})^n \cdot \frac{1}{2^{n+1}(n+1)} d$. Hence we get

$$f(p) > (1 - \frac{1}{n+1})^n \cdot \frac{1}{2^{n+1}(n+1)} d(M)V(M).$$

4. noncompact analogues of theorem 1.1

In this section we consider the noncompact version of theorem 1.1. Let M be a complete noncompact Riemannian manifold of dimension n. For a point

$p \in M$ and $d > 0$, we write

$$f(p,d) = \int_{B_p(d)} d(p,x)dv$$

Then we have

Theorem 4.1. *If M is a Cartan-Hadamard manifold, then*

$$f(p,d) > \frac{n}{n+1} \cdot \frac{1}{\sqrt[n]{n+1}} dV_p(d),$$

for any $p \in M$ and all $d > 0$.

Note that $\frac{n}{n+1} \cdot \frac{1}{\sqrt[n]{n+1}}$ tends to 1 as n goes to $+\infty$. On the other hand, we always have $f(p,d) < dV_p(d)$. So theorem 4.1 is more or less surprise.

Proof. Since the sectional curvature of M is nonpositive and M has no cut point. By the Bishop-Gromov's volume comparison (c.f. [1] page 169), one has

$$\frac{V_p(r)}{V_o(r)} \leq \frac{V_p(d)}{V_o(d)}$$

for $r \leq d$. Hence

$$V_p(r) \leq \frac{V_o(r)}{V_o(d)} V_p(d) = \frac{r^n}{d^n} V_p(d).$$

We estimate the lower bound of f.

$$\begin{aligned} f(p,d) &= \int_{B_p(r)} d(p,x)dv + \int_{B_p(d)\setminus B_p(r)} d(p,x)dv \\ &> \int_{B_p(d)\setminus B_p(r)} d(p,x)dv \\ &> rV(B_p(d) \setminus B_p(r)) = r(V_p(d) - V_p(r)) \\ &\geq V_p(d) r (1 - \frac{r^n}{d^n}). \end{aligned}$$

Let $g(r) = r(1 - \frac{r^n}{d^n})$, $0 \leq r \leq d$. When

$$g^{'}(r) = 1 - (n+1)\frac{r^n}{d^n} = 0,$$

$r = \frac{d}{\sqrt[n]{n+1}}$, $g(r)$ achieves the maximal value $\frac{n}{n+1} \cdot \frac{1}{\sqrt[n]{n+1}} \cdot d$. So we have

$$f(p,d) > \frac{n}{n+1} \cdot \frac{1}{\sqrt[n]{n+1}} dV_p(d).$$

□

Theorem 4.2. *If the Ricci curvature of M is nonnegative, then*

$$f(p,d) > c(n)dV_p(d),$$

for any $p \in M$ and all $d > 0$. The concrete value of $c(n)$ is given in the following proof.

The constant $c(n)$ is different to the one in Theorem 1.1.

Proof. Let $0 \le t \le \frac{d}{2}$. q is a point satisfying $d(p,q) = d - t$. Then

$$\begin{aligned} f(p,d) &= \int_{B_p(d-2t)} d(p,x)dv + \int_{B_p(d)\setminus B_p(d-2t)} d(p,x)dv \\ &> \int_{B_p(d)\setminus B_p(d-2t)} d(p,x)dv \\ &> (d-2t)V(B_p(d)\setminus B_p(d-2t)) \\ &> (d-2t)V_q(t) \\ &\ge (d-2t)\frac{t^n}{(2d-t)^n}V_q(2d-t) \\ &> (d-2t)\frac{t^n}{(2d-t)^n}V_p(d). \end{aligned}$$

Since $B_q(t) \subset B_p(d) \setminus B_p(d-2t)$, the third ">" holds. The last ">" follows from $B_q(2d-t) \supset B_p(d)$. Let $g(t) = (d-2t)\frac{t^n}{(2d-t)^n}, 0 \le t \le \frac{d}{2}$. When

$$g^{'}(t) = -2(\frac{t}{2d-t})^n + (d-2t)n(\frac{t}{2d-t})^{n-1}\frac{2d}{(2d-t)^2} = 0,$$

namely,

$$t^2 - 2d(n+1)t + nd^2 = 0,$$

$t = (n+1-\sqrt{n^2+n+1})d$. $g(t)$ achieves the maximal value

$$\frac{3}{2\sqrt{n^2+n+1}+2n+1}(\frac{n}{n+2+2\sqrt{n^2+n+1}})^n d.$$

Choose

$$c(n) = \frac{3}{2\sqrt{n^2+n+1}+2n+1}(\frac{n}{n+2+2\sqrt{n^2+n+1}})^n.$$

We obtain

$$f(p,d) \geq c(n) d V_p(d).$$

□

Recall a well-known theorem of Calabi and Yau [2]: Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any $p \in M$, we have $V_p(r) \geq c(n,p)r$. Consequently M has infinite volume. Combining this result with Theorem 4.2, we obtain

Corollary 4.3. *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Then*

$$\lim_{r\to\infty} \int_{B_p(r)} \frac{d(p,x)}{r} dv = +\infty,$$

for all $p \in M$.

References

1. S. Gallot, D. Hulin and J. Lafontaine, *Riemannian Geometry.* Springer-Verlag, 2004.
2. S.T. Yau, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry.* Indiana Univ. Math. J. 25 (1976), no. 7, 659-670.

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