

On Classifying Spaces for the Family of Virtually Cyclic Subgroups in Mapping Class Groups

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Abstract: We give a bound for the geometric dimension for the family of virtually cyclic groups in mapping class groups of an orientable compact surface with punctures, possibly with nonempty boundary and negative Euler characteristic.

Keywords: Mapping class groups, geometric dimension, classifying spaces.

1. Introduction

Let S be an orientable compact surface with finitely many punctures (possibly zero punctures) and negative Euler characteristic. The mapping class group, $\Gamma(S)$, of S is the group of isotopy classes of orientation preserving diffeomorphisms of S that fix point-wise the boundary. Let $m > 2$ be an integer and $\Gamma_m(S)$ be the congruence subgroup of $\Gamma(S)$, this is the subgroup of those elements that act trivially on $H_1(S, \mathbb{Z}/m)$.

Classifying spaces, \underline{EG} , for the family of finite subgroups of a group G , have been extensively studied. For the mapping class group, it is well-known that the Teichmüller space $\mathcal{T}(S)$ is a model for $\underline{E}\Gamma(S)$ by results of Kerckhoff given in [11]. Later, J. Aramayona and C. Martínez proved in [1] that the minimal dimension for which there exists a model for $\underline{E}\Gamma(S)$ coincides with the virtual cohomological dimension $vcd(\Gamma(S))$, this has been computed by J. L. Harer in [8].

Finite dimensional models for classifying spaces \underline{EG} for the family of virtually cyclics have been constructed for word-hyperbolic groups (Juan-Pineda, Leary [10]), for groups acting in CAT(0) spaces (Farley [7], Lück [14], Degrijse and Petrosyan [4]), and many other groups.

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A method to construct a model for \underline{EG} is to start with a model for \underline{EG} and then try to extend this to obtain a model for \underline{EG} . In [15] Lück and Weiermann gave a general construction using this idea.

The smallest possible dimension of a model of \underline{EG} is denoted by $\underline{\text{gd}}G$ and is called the geometric dimension of G for the family of virtually cyclic subgroups. We prove that $\underline{\text{gd}}\Gamma(S)$ is finite. Degrijse and Petrosyan proved this fact in [4] for closed surfaces of genus at least 2, although our method produces a larger bound, it is more general since it includes surfaces with boundary and our techniques are different from theirs.

This paper is organized as follows: we review the fundamental material about classifying spaces and mapping class groups in sections 2 and 3 respectively. In section 4, we develop the analysis of commensurators of infinite virtually cyclic subgroups in the mapping class groups and we prove:

Proposition 1. *Let S be an orientable closed surface with finitely many punctures and $\chi(S) < 0$. Let $m \geq 3$ be fixed. Let $C = \langle g \rangle \subset \Gamma(S)$ be infinite cyclic and $n \in \mathbb{N}$ such that $D = \langle g^n \rangle \subset \Gamma_m(S)$. Then*

$$N_{\Gamma(S)}[C] = N_{\Gamma(S)}(D)$$

where $N_{\Gamma(S)}[C]$ is the commensurator of C and $N_{\Gamma(S)}(D)$ is the normalizer of D . Furthermore, the subgroup D may be chosen to be maximal in $\Gamma_m(S)$.

From Proposition 1 and a description of normalizers of infinite cyclic subgroups that we develop in section 4, we prove the following:

Theorem 1. *Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. Then $\underline{\text{gd}}\Gamma(S) < \infty$, that is, the mapping class group $\Gamma(S)$ admits a finite dimensional model for $\underline{E}\Gamma(S)$.*

Our main result is the following:

Main Theorem. *Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. Let $m \geq 3$, then*

- (1) $\underline{\text{gd}}\Gamma_m(S) \leq \text{vcd}(\Gamma(S)) + 1$;
- (2) Let $[\Gamma(S) : \Gamma_m(S)]$ be the index of $\Gamma_m(S)$ in $\Gamma(S)$, then

$$\begin{aligned} \underline{\text{gd}}\Gamma(S) &\leq [\Gamma(S) : \Gamma_m(S)] \cdot \underline{\text{gd}}\Gamma_m(S) \\ &\leq [\Gamma(S) : \Gamma_m(S)] \cdot (\text{vcd}(\Gamma(S)) + 1). \end{aligned}$$

Where $\text{vcd}(\Gamma(S))$ is the virtual cohomological dimension of $\Gamma(S)$. We point out that the bound in (1) is sharp.

A key result in finding the above is the following Proposition which we could not find in the literature and we think is one of the most interesting contributions of this work, see Proposition 5.11.

Proposition 2. *Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. Let $m \geq 3$, then the group $\Gamma_m(S)$ satisfies the property that every infinite virtually cyclic subgroup is contained in a unique maximal virtually cyclic subgroup.*

We emphasize that Degrijse and Petrosyan proved in [4] the finiteness of $\text{gd}\Gamma(S)$ by other methods. In this work we outline a method that gives a precise description of a model for the classifying space for virtually cyclics of mapping class groups. It depends on certain Teichmüller spaces and mapping class groups of subsurfaces of S .

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2. Classifying Spaces for Families

Let G be a group. A *family* \mathcal{F} of subgroups of G is a set of subgroups of G which is closed under conjugation and taking subgroups. The following are natural examples of families of G :

$$\begin{aligned} \{1\} &= \text{the trivial subgroup;} \\ \mathcal{FIN}_G &= \text{finite subgroups of } G; \\ \mathcal{VCY}_G &= \text{virtually cyclic subgroups of } G; \\ \mathcal{ALL}_G &= \text{all subgroups of } G. \end{aligned}$$

Definition 2.1. Let \mathcal{F} be a family of subgroups of G . A model of the *classifying space* $E_{\mathcal{F}}G$ for the family \mathcal{F} is a G -CW-complex X , such that all of its isotropy groups belong to \mathcal{F} and if Y is a G -CW-complex with isotropy groups belonging to \mathcal{F} , there is precisely one G -map $Y \rightarrow X$ up to G -homotopy.

In other words, X is a terminal object in the category of G -CW complexes with isotropy groups belonging to \mathcal{F} . In particular, two models for $E_{\mathcal{F}}G$ are G -homotopy equivalent.

Definition 2.2. Let G be a group, $H \subseteq G$ and X a G -set. The H -fixed point set X^H is defined as

$$X^H = \{x \in X \mid \text{for all } h \in H, h \cdot x = x\}.$$

Theorem 2.3. [13, Thm. 1.9] *A G -CW-complex X is a model of $E_{\mathcal{F}}G$ if and only if the H -fixed point set X^H is contractible for $H \in \mathcal{F}$ and is empty for $H \notin \mathcal{F}$.*

The smallest possible dimension of a model of $E_{\mathcal{F}}G$ is called the *geometric dimension of G for the family \mathcal{F}* and is usually denoted as $\text{gd}_{\mathcal{F}}G$. When a finite dimensional model of $E_{\mathcal{F}}G$ does not exist, then $\text{gd}_{\mathcal{F}}G = \infty$. We abbreviate $\underline{E}G := E_{\mathcal{FIN}_G}G$ and call it the *universal G -CW-complex for proper G -actions*, and we abbreviate $\underline{E}G := E_{\mathcal{VCY}_G}G$. We denote by $\underline{\text{gd}}G = \text{gd}_{\mathcal{FIN}_G}G$ and $\underline{\underline{\text{gd}}}G = \text{gd}_{\mathcal{VCY}_G}G$. It is known that for any groups H_1, H_2 ,

$$(1) \quad \underline{\text{gd}}(H_1 \times H_2) \leq \underline{\text{gd}}H_1 + \underline{\text{gd}}H_2.$$

For a subgroup $H \subseteq G$, and a family \mathcal{F} of G , denote by

$$\mathcal{F} \cap H = \{\text{subgroups of } H \text{ belonging to } \mathcal{F}\},$$

the family of subgroups of H induced from \mathcal{F} . A model of $E_{\mathcal{F} \cap H}H$ is given by restricting the action of G to H in a model of $E_{\mathcal{F}}G$. Then

$$(2) \quad \text{gd}_{\mathcal{F} \cap H}H \leq \text{gd}_{\mathcal{F}}G.$$

2.1. Constructing models from models for smaller families

We will use the construction given by W. Lück and M. Weiermann in [15]. In particular for the families $\mathcal{FIN} \subset \mathcal{VCY}$, it is as follows: Let $\mathcal{VCY}_G^\infty = \mathcal{VCY}_G - \mathcal{FIN}_G$ be the collection of infinite virtually cyclic subgroups of G . Define an equivalence relation, \sim , on \mathcal{VCY}_G^∞ as

$$(3) \quad V \sim W \iff |V \cap W| = \infty,$$

for V and W in \mathcal{VCY}_G^∞ , where $|\star|$ denotes the cardinality of the set \star . Let $[\mathcal{VCY}_G^\infty]$ denote the set of equivalence classes under the above relation and let

$[H] \in [\mathcal{VC}_G^\infty]$ be the equivalence class of H . Define

$$(4) \quad N_G[H] = \{g \in G \mid |g^{-1}Hg \cap H| = \infty\},$$

this is the isotropy group of $[H]$ under the G -action on $[\mathcal{VC}_G^\infty]$ induced by conjugation. Observe that $N_G[H]$ is the commensurator of the subgroup $H \subseteq G$. Define a family of subgroups of $N_G[H]$ by

$$(5) \quad \mathcal{G}[H] := \{K \in \mathcal{VC}_{N_G[H]}^\infty \mid |K \cap H| = \infty\} \cup \mathcal{FLN}_{N_G[H]}.$$

The method to build a model of \underline{EG} from one of \underline{EG} is given in the following theorem.

Theorem 2.4. [15, Thm.2.3] *Let \mathcal{VC}_G^∞ and \sim be as above. Let I be a complete system of representatives, $[H]$, of the G -orbits in $[\mathcal{VC}_G^\infty]$ under the G -action coming from conjugation. Choose arbitrary $N_G[H]$ -CW-models for $\underline{E}(N_G[H])$, $E_{\mathcal{G}[H]}(N_G[H])$ and an arbitrary G -CW-model for $\underline{E}(G)$. Define X a G -CW-complex by the cellular G -pushout*

$$\begin{array}{ccc} \coprod_{[H] \in I} G \times_{N_G[H]} \underline{E}(N_G[H]) & \xrightarrow{i} & \underline{E}(G) \\ \downarrow \coprod_{[H] \in I} id_G \times_{N_G[H]} f_{[H]} & & \downarrow \\ \coprod_{[H] \in I} G \times_{N_G[H]} E_{\mathcal{G}[H]}(N_G[H]) & \longrightarrow & X \end{array}$$

such that $f_{[H]}$ is a cellular $N_G[H]$ -map for every $[H] \in I$ and i is an inclusion of G -CW-complexes, or such that every map $f_{[H]}$ is an inclusion of $N_G[H]$ -CW-complexes for every $[H] \in I$ and i is a cellular G -map. Then X is a model for $E_G(G)$.

The maps in Theorem 2.4 are given by the universal property of classifying spaces for families and inclusions of families of subgroups. Observe that if $\underline{gd}G$ is finite and both $\underline{gd}N_G[H]$ and $\underline{gd}_{\mathcal{G}[H]}N_G[H]$ are uniformly bounded, then $\underline{gd}G$ is finite.

Remark 2.5. Observe that if $H, K \in \mathcal{VC}_G^\infty$ and both C_H and C_K are infinite cyclic subgroups of H and K respectively, let \sim as in (3)

$$(6) \quad H \sim K \quad \text{if only if} \quad C_H \sim C_K, \quad \text{and}$$

$$N_G[H] = \{g \in G \mid |g^{-1}Hg \cap H| = \infty\}$$

$$(7) \quad = \{g \in G \mid |g^{-1}C_Hg \cap C_H| = \infty\}.$$

Let \mathcal{C}_G^∞ be the set of infinite cyclic subgroups C of G . Then the equivalence relation \sim given in (3) can be defined in \mathcal{C}_G^∞ , we will denote by $[\mathcal{C}_G^\infty]$ the set of equivalence classes.

3. Mapping Class groups

Let S be an orientable compact surface with a finite set \mathcal{P} , called punctures, of points removed from the interior. We will assume that the surface has negative Euler characteristic. References for this section are [5], [9] and [6].

Let $\text{Diff}^+(S, \partial S)$ denote the group of orientation preserving diffeomorphisms of S that restrict to the identity on the boundary ∂S . We endow this group with the compact-open topology.

Definition 3.1. The *mapping class group* of S , denoted $\Gamma(S)$, is the group

$$\Gamma(S) = \pi_0(\text{Diff}^+(S, \partial S)),$$

that is, the group of (smooth) isotopy classes of elements of $\text{Diff}^+(S, \partial S)$ where isotopies are required to fix the boundary pointwise.

Congruence subgroups. Let $m \in \mathbb{Z}$, $m > 1$. We denote by $\Gamma_m(S)$ the kernel of the natural homomorphism

$$\Gamma(S) \rightarrow \text{Aut}(H_1(S, \mathbb{Z}/m\mathbb{Z}))$$

defined by the action of diffeomorphisms on the homology group, $\Gamma_m(S)$ is called the congruence subgroup of $\Gamma(S)$. Note that this subgroup has finite index in $\Gamma(S)$.

Complex of curves. An *essential* curve is a simple closed curve of S that is not homotopic to a point, a puncture, or a boundary component. The *complex of curves*, denoted by $\mathcal{C}(S)$, is the abstract simplicial complex associated to S such that, (i) Vertices are isotopy classes of essential curves, we denote by $V(S)$ the set of vertices; (ii) $\mathcal{C}(S)$ has a k -simplex for each $(k + 1)$ -tuple of vertices, where each pair of corresponding isotopy classes have disjoint representants. The *realization* of a simplex is the union of mutually disjoint curves that represent its vertices. The mapping class group acts on $V(S)$: if $f \in \Gamma(S)$, $\alpha \in V(S)$, the action is given by $f \cdot \alpha = f(\alpha)$. Then $\Gamma(S)$ acts on $\mathcal{C}(S)$, since this action sends simplicies into simplicies.

Dehn twists. Let α, β be isotopy classes of simple closed curves in S . We will denote the Dehn twist about α as T_α . Let $f \in \Gamma(S)$ and $j, k \in \mathbb{Z} - \{0\}$. We have the following properties of Dehn twists, see [5, Sec. 3.3],

- 1) $T_\alpha^j = T_\beta^k$ iff $\alpha = \beta$ and $j = k$;
- 2) $fT_\alpha^j f^{-1} = T_{f(\alpha)}^j$;
- 3) $T_\alpha^j T_\beta^k = T_\beta^k T_\alpha^j$ iff $i(\alpha, \beta) = 0$.

3.1. Classification of elements in $\Gamma(S)$

We will first assume that S has empty boundary. We assume the reader is familiar with the theory of transverse singular foliations. See [5] or [6] for references.

Pseudo-Anosov diffeomorphisms. A diffeomorphism $\phi \in \text{Diff}^+(S)$ is called *pseudo-Anosov* if there exists a pair of transverse measured foliations (\mathcal{F}^s, μ^s) , (\mathcal{F}^u, μ^u) of S and a real number $\lambda > 1$ such that

- i. $\phi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s)$; and $\phi(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$.
- ii. the 1-prongs singularities of these foliations belong to the set of punctures.

The measure foliation (\mathcal{F}^s, μ^s) is called the *stable foliation* for ϕ and (\mathcal{F}^u, μ^u) is called the *unstable foliation* for ϕ , and λ is the dilatation of ϕ .

Definition 3.2. An element $f \in \Gamma(S)$ is called *pseudo-Anosov* if it is represented by a pseudo-Anosov diffeomorphism.

Definition 3.3. An element f is called *reducible* if f fixes some simplex of $\mathcal{C}(S)$ and *irreducible* otherwise.

Among the irreducible elements, those of finite order are periodic and those of infinite order are pseudo-Anosov. There is the following classification theorem for elements of the mapping class group, see [5, Thm. 13.2].

Theorem 3.4. (NIELSEN-THURSTON CLASSIFICATION) *Let $g, n \geq 0$. Let S be an orientable surface of genus g and n punctures. Each $f \in \Gamma(S)$ is either periodic, reducible, or pseudo-Anosov. Further, pseudo-Anosov mapping classes are neither periodic nor reducible. A periodic element is represented by a finite order diffeomorphism.*

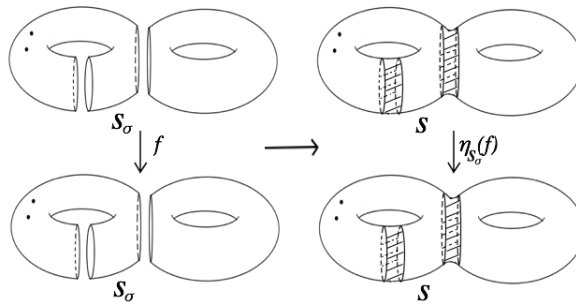


Figure 1: The homomorphism $\eta_{S_\sigma}(f)$ is the identity on the shaded regions and coincides with f on the unshaded regions.

Surfaces with boundary. If S has non-empty boundary, we define $f \in \Gamma(S)$ to be pseudo-Anosov if f restricts to a pseudo-Anosov diffeomorphism on the punctured surface obtained by removing the boundary ∂S .

3.2. Induced homomorphisms from inclusions

Let S be an orientable closed surface with finitely many punctures. Let S' be an orientable compact subsurface of S , the inclusion $S' \rightarrow S$ induces a natural homomorphism

$$(8) \quad \eta: \Gamma(S') \rightarrow \Gamma(S),$$

let $g \in \Gamma(S')$ and $\psi \in \text{Diff}^+(S', \partial S')$ be a representative diffeomorphism of g . Then $\eta(g)$ is defined as the isotopy class of the diffeomorphism which coincides with ψ in S' and is the identity in $S - S'$.

Let $\sigma \in \mathcal{C}(S)$ with vertices $\alpha_1, \dots, \alpha_r$ and C be its realization in S . Let N_σ be an open regular neighborhood of C in S and denote by $S_\sigma = S - N_\sigma = S_1 \cup \dots \cup S_k$, where each S_i is a connected component. Let β_i and γ_i denote the two boundary components of N_σ that are isotopic to α_i in S .

Remark 3.5. Suppose that the surface S has genus g and n punctures. Note that $\chi(S_\sigma) = \chi(S)$ and $\chi(S_i) \leq -1$, therefore $k \leq -\chi(S)$ and $r \leq \frac{-3\chi(S)-n}{2}$, see [5] page 249.

See [5, Thm. 3.18] as a reference for the following homomorphisms.

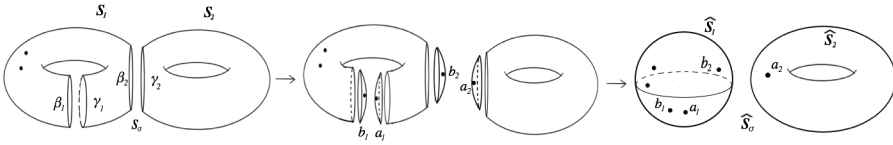


Figure 2: Corking each boundary component with a 1-punctured disc.

Cutting the surface. From the inclusions $S_\sigma \rightarrow S$ and $S_i \rightarrow S$ denote the induced homomorphisms by

$$(9) \quad \eta_{S_\sigma} : \Gamma(S_\sigma) \rightarrow \Gamma(S), \quad \eta_{S_i} : \Gamma(S_i) \rightarrow \Gamma(S),$$

with $\ker(\eta_{S_\sigma}) = \langle T_{\beta_1} T_{\gamma_1}^{-1}, \dots, T_{\beta_r} T_{\gamma_r}^{-1} \rangle$. Observe that $\Gamma(S_\sigma) = \prod_{i=1}^k \Gamma(S_i)$ and any element in the image $\eta_{S_\sigma}(\Gamma(S_\sigma))$ leaves each subsurface S_i of S invariant.

Corking. We glue a 1-punctured disk in each boundary component of S_σ , that is, we corked all boundary components, and denote this surface by $\widehat{S}_\sigma = \widehat{S}_1 \cup \dots \cup \widehat{S}_k$, where the \widehat{S}_i are the connected components of \widehat{S}_σ (see Figure 2). Note that the surface S_σ has the same Euler characteristic as S .

From the inclusion $S_\sigma \rightarrow \widehat{S}_\sigma$, the induced homomorphism defined as (8) is called the *corking homomorphism of S_σ* , and it is denoted by

$$(10) \quad \theta_{S_\sigma} : \Gamma(S_\sigma) \rightarrow \Gamma(\widehat{S}_\sigma),$$

with $\ker(\theta_{S_\sigma}) = \langle T_{\beta_1}, \dots, T_{\beta_r}, T_{\gamma_1}, \dots, T_{\gamma_r} \rangle$. Let \mathcal{Q}_i be the set of punctures in \widehat{S}_i coming from boundary components of S_i . Note that $\theta_{S_i}(\Gamma(S_i)) = \Gamma(\widehat{S}_i, \mathcal{Q}_i)$ is the subgroup of $\Gamma(\widehat{S}_i)$ that fixes pointwise all $p \in \mathcal{Q}_i$. Since $\Gamma(S_\sigma) = \prod_{i=1}^k \Gamma(S_i)$ and $\theta_{S_\sigma} = \prod_{i=1}^k \theta_{S_i}$, then

$$(11) \quad \theta_{S_\sigma} : \Gamma(S_\sigma) \rightarrow \prod_{i=1}^k \Gamma(\widehat{S}_i, \mathcal{Q}_i).$$

Note that $\prod_{i=1}^k \Gamma(\widehat{S}_i, \mathcal{Q}_i)$ can be seen as the subgroup of $\Gamma(\widehat{S}_\sigma)$ that fixes each subsurface \widehat{S}_i and fixes pointwise the punctures in $\mathcal{Q} = \cup_{i=1}^k \mathcal{Q}_i$.

3.3. Pure elements

Let S be an orientable compact surface with finitely many punctures. A diffeomorphism $\psi \in \text{Diff}^+(S)$ is called *pure* if there exists a 1-submanifold C (possibly empty) of S such that the following conditions are satisfied:

- (P) C is the realization of an element $\sigma \in \mathcal{C}(S)$ or C is empty; if $C \neq \emptyset$, ψ fixes C , it does not rearrange the components of $S - C$, and it induces on each component of $S - C$ a diffeomorphism isotopic to either a pseudo-Anosov or the identity diffeomorphism.

We call an element $f \in \Gamma(S)$ *pure* if the isotopy class of f contains a pure diffeomorphism. We call $H \subseteq \Gamma(S)$ *pure* if H consists of pure elements.

Theorem 3.6. [9, Cor. 1.5 and 1.8] *If $m \geq 3$, then $\Gamma_m(S)$ is a torsion free group and a pure subgroup of $\Gamma(S)$.*

From now on, we will consider the subgroup $\Gamma_m(S)$ with $m \geq 3$.

Canonical reduction system. Let $G \subseteq \Gamma_m(S)$. An isotopy class $\alpha \in V(S)$ is called an essential reduction class for G if the following conditions are satisfied (i) $g(\alpha) = \alpha$ for all $g \in G$; (ii) if $\beta \in \mathcal{C}(S)$ with $i(\alpha, \beta) \neq 0$,* then $h(\beta) \neq \beta$ for some $h \in G$. The set of essential reduction classes of G is a simplex of $\mathcal{C}(S)$ and it is called a *canonical reduction system* of G and it is denoted by $\sigma(G)$. In general, for a subgroup $H \subseteq \Gamma(S)$ define $\sigma(H) := \sigma(H \cap \Gamma_m(S))$ and for $f \in \Gamma(S)$ define $\sigma(f) := \sigma(\langle f \rangle)$.

Lemma 3.7. [9, Sec. 7.2, 7.3] Let $f \in \Gamma(S)$, and $G \subseteq \Gamma(S)$.

- (i) If $H \trianglelefteq G$ has finite index, then $\sigma(G) = \sigma(H)$,
(ii) $\sigma(fGf^{-1}) = f\sigma(G)$.

Lemma 3.8. Let S be an orientable compact surface with finitely many punctures. If $f, g \in \Gamma(S)$ are such that $f^q = gf^p g^{-1}$ for some $p, q \in \mathbb{Z} - \{0\}$, then $g\sigma(f) = \sigma(f)$.

Proof. The hypothesis entails that $\sigma(f^q) = \sigma(gf^p g^{-1})$, by Lemma 3.7

$$\sigma(f^q) = \sigma(f) \quad \text{and} \quad \sigma(gf^p g^{-1}) = \sigma(gfg^{-1}),$$

then $\sigma(gfg^{-1}) = \sigma(f)$ and by Lemma 3.7 $\sigma(gfg^{-1}) = g\sigma(f)$, therefore $g\sigma(f) = \sigma(f)$. \square

*The geometric intersection number of α and β is denoted by $i(\alpha, \beta)$.

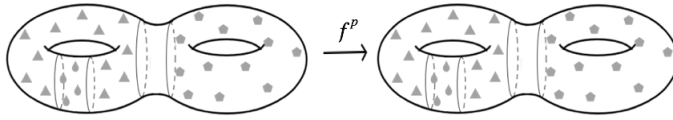


Figure 3: The canonical form of f^k , where each subsurface is fixed. A shaded region indicates a pseudo-Anosov component or a Dehn Twist and a unshade region indicates an identity component.

Canonical Form. By cutting S along a reduction system σ' of an element in $\Gamma(S)$ and applying the Nielsen-Thurston classification Theorem to each subsurface of $S_{\sigma'}$ we can obtain a decomposition of the element as follows, see Figure 3.

Theorem 3.9. [5, Cor. 13.3] *Let $f \in \Gamma(S)$ and $\sigma = \sigma(f)$ be its canonical reduction system with vertices $\alpha_1, \dots, \alpha_r$. Let $S_{\sigma} = S_1 \cup \dots \cup S_k$ be as before and let $\overline{N_{\sigma}} = S_{k+1} \cup \dots \cup S_{k+r}$ be the union of pairwise disjoint closed neighborhoods S_{k+i} of curves representatives of the α_i . Then there is a representative ϕ of f that permutes the S_i , so that some power of ϕ leaves invariant each S_i . Moreover, there exists an integer $p > 0$ so that $\phi^p(S_i) = S_i$ for all i and*

$$(12) \quad f^p = \prod_{i=1}^k \eta_{S_i}(\bar{f}_i) \prod_{j=1}^r T_{\alpha_j}^{n_j},$$

where each $\bar{f}_i \in \Gamma(S_i)$ is either pseudo-Anosov or the identity and $n_j \in \mathbb{Z}$ for $1 \leq j \leq r$.

When $f \in \Gamma_m(S)$ and $m \geq 3$ the integer p can always be taken to be one.

3.4. Stabilizers

In this section, we shall consider S with empty boundary. The stabilizers $\Gamma(S)_{\sigma}$ will be used to understand the normalizers of reducible elements of $\Gamma(S)$.

Let $\sigma \in \mathcal{C}(S)$ and $S_\sigma = S_1 \cup \dots \cup S_k$ be as in Section 3.1. Denote the stabilizer subgroup of σ in $\Gamma(S)$ by

$$\Gamma(S)_\sigma = \{g \in \Gamma(S) \mid g(\sigma) = \sigma\}.$$

Proposition 3.10. [5, Prop. 3.20] *Let $\sigma \in \mathcal{C}(S)$ with vertices $\alpha_1, \dots, \alpha_r$ and $S_\sigma = S_1 \cup \dots \cup S_k$. Then there is a well defined homomorphism*

$$(13) \quad \rho_\sigma : \Gamma(S)_\sigma \rightarrow \Gamma(\widehat{S}_\sigma).$$

where $\ker(\rho_\sigma) = \langle T_{\alpha_1}, \dots, T_{\alpha_r} \rangle$ is the free abelian subgroup generated by Dehn twists about the curves $\alpha_1, \dots, \alpha_r$.

Let $\Gamma(S)_\sigma^0$ be the finite index subgroup of $\Gamma(S)_\sigma$ that fixes each α_i with orientation, since elements in $\Gamma(S)_\sigma^0$ are orientation-preserving, it follows that they also preserve the sides of each curve α_i in S , thus they fix each subsurface S_i . Denote the restriction $\rho_{\sigma,0} = \rho_\sigma|_{\Gamma(S)_\sigma^0}$, then $\rho_{\sigma,0} = \theta_{S_\sigma} \eta_{S_\sigma}^{-1}$. Therefore we have that

$$(14) \quad \rho_{\sigma,0} : \Gamma(S)_\sigma^0 \rightarrow \prod_{i=1}^k \Gamma(\widehat{S}_i, \mathcal{Q}_i),$$

is surjective and $\ker(\rho_{\sigma,0}) = \ker(\rho_\sigma)$.

Remark 3.11. By [9, Thm. 1.2], elements in $\Gamma_m(S)_\sigma$ do not rearrange the components of S_σ and fix each curve of σ , so $\Gamma_m(S)_\sigma \subseteq \Gamma(S)_\sigma^0$. Let $f \in \Gamma_m(S)$, $\sigma = \sigma(f)$ with vertices $\alpha_1, \dots, \alpha_r$, and let the canonical form of f be as follows

$$f = \Pi_{i=1}^k \eta_{S_i}(\bar{f}_i) \Pi_{j=1}^r T_{\alpha_j}^{n_j}.$$

Moreover, assume that $\rho_{\sigma,0}(f) = (f_1, \dots, f_k)$. Note that $f \in \Gamma_m(S)_\sigma$ and

$$(\bar{f}_1, \dots, \bar{f}_k) \in \eta_{S_\sigma}^{-1}(f).$$

Since $\rho_{\sigma,0}(f) = \theta_{S_\sigma} \eta_{S_\sigma}^{-1}(f)$ and $\rho_{\sigma,0}$ is well defined, then

$$\begin{aligned} (f_1, \dots, f_k) &= \theta_{S_\sigma}(\bar{f}_1, \dots, \bar{f}_k) \\ &= (\theta_{S_1}(\bar{f}_1), \dots, \theta_{S_k}(\bar{f}_k)), \end{aligned}$$

hence for each i , $f_i = \theta_{S_i}(\bar{f}_i)$, therefore f_i is pseudo-Anosov or the identity.

4. Commensurators in $\Gamma(S)$

In order to build a model $\underline{E}\Gamma(S)$ as mentioned in Sec. 2.1, we need to know the commensurators of the infinite virtually cyclic subgroups in $\Gamma(S)$.

4.1. Condition (C) for $\Gamma(S)$

Using the following condition, we may give a description of the commensurators of infinite virtually cyclic subgroups.

(C) For every $g, h \in G$, with $|h| = \infty$, and $k, l \in \mathbb{Z}$,

$$\text{if } gh^k g^{-1} = h^l \text{ then } |k| = |l|.$$

Proposition 4.1. *Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. The group $\Gamma(S)$ satisfies condition (C).*

Proof. Case I. The proof when S has empty boundary is given in Propositions 4.2 and 4.3, since elements with infinite order in $\Gamma(S)$ are reducible or pseudo-Anosov.

Case II. Assume $\partial S \neq \emptyset$ and it has b connected components. Let $f, g \in \Gamma(S)$, with $|f| = \infty$ such that $gf^p g^{-1} = f^q$ for some $p, q \in \mathbb{Z} - \{0\}$. Let

$$\theta_S: \Gamma(S) \rightarrow \Gamma(\widehat{S})$$

be the corking homomorphism of S as in (10), then $\ker(\theta_S) \simeq \mathbb{Z}^b$ is the free abelian group generated by Dehn twists about curves isotopic to the boundary components of S . Since $\ker(\theta_S)$ is central in $\Gamma(S)$, if $f \in \ker(\theta_S)$ we conclude that $p = q$.

On the other hand, if $f \notin \ker(\theta_S)$, applying θ_S , we have

$$\theta_S(g)\theta_S(f)^p\theta_S(g)^{-1} = \theta_S(f)^q,$$

and $|\theta_S(f)| = \infty$, applying Case I we conclude that $|p| = |q|$. □

About powers of a pseudo-Anosov diffeomorphism we have the following: Suppose that S has empty boundary. Let $f \in \Gamma(S)$ be a pseudo-Anosov element and ϕ be a pseudo-Anosov diffeomorphism in its class, with $(\mathcal{F}_\phi^s, \mu_\phi^s)$,

$(\mathcal{F}_\phi^u, \mu_\phi^u)$ the stable and unstable foliations of ϕ respectively and λ_ϕ its dilatation. For $n \in \mathbb{Z} - \{0\}$,

$$(15) \quad \begin{aligned} \phi^n(\mathcal{F}_\phi^s, \mu_\phi^s) &= (\mathcal{F}_\phi^s, \lambda_\phi^{-n} \mu_\phi^s), \\ \phi^n(\mathcal{F}_\phi^u, \mu_\phi^u) &= (\mathcal{F}_\phi^u, \lambda_\phi^n \mu_\phi^u). \end{aligned}$$

The reader may consult [6] and [17] as references.

Proposition 4.2. *Let S be an orientable closed surface with finitely many punctures and $\chi(S) < 0$. Let $r \in \mathbb{N}$ and $h_1, \dots, h_r \in \Gamma(S)$ be pseudo-Anosov mapping classes, suppose that there exist $g_1, \dots, g_r \in \Gamma(S)$ and a permutation $\gamma \in \Sigma_r$ such that*

$$(16) \quad g_i h_i^p g_i^{-1} = h_{\gamma(i)}^q, \quad \text{for all } i \in \{1, \dots, r\},$$

for some $p, q \in \mathbb{Z} - \{0\}$, then $|p| = |q|$.

Proof. Suppose that $p > 0$ and let $I = \{1, \dots, r\}$. For each $i \in I$, let ϕ_i be pseudo-Anosov diffeomorphisms in the class h_i , for each i there exists $G_i \in \text{Diff}^+(S)$ in the class g_i , such that $G_i \phi_i^p G_i^{-1} = \phi_{\gamma(i)}^q$, this follows by the uniqueness of pseudo-Anosovs [6, Exp. 12].

Let $(\mathcal{F}_i^s, \mu_i^s)$ and $(\mathcal{F}_i^u, \mu_i^u)$ be the stable and unstable foliations, respectively, of ϕ_i , with dilatation $\lambda_i > 1$, $i \in I$. By (15), if $n > 0$ (or $n < 0$), $(\mathcal{F}_i^s, \mu_i^s)$ and $(\mathcal{F}_i^u, \mu_i^u)$ are the stable and unstable (unstable and stable) foliations respectively of ϕ_i^n with dilatation λ_i^n (or λ_i^{-n}).

Suppose that $q > 0$. Then G_i sends the stable and unstable foliations of ϕ_i^p to the stable and unstable foliations of $\phi_{\gamma(i)}^q$ respectively ([6, Lem. 16, Exp. 12]). Since the foliations are uniquely ergodic, the measure is up to a constant, that is,

$$G(\mathcal{F}_i^s, \mu_i^s) = (\mathcal{F}_{\gamma(i)}^s, a\mu_{\gamma(i)}^s), \quad G(\mathcal{F}_i^u, \mu_i^u) = (\mathcal{F}_{\gamma(i)}^u, b\mu_{\gamma(i)}^u),$$

with $ab = 1$. Thus

$$\begin{aligned} \phi_{\gamma(i)}^q(\mathcal{F}_{\gamma(i)}^s, a\mu_{\gamma(i)}^s) &= \phi_{\gamma(i)}^q(G(\mathcal{F}_i^s, \mu_i^s)) \\ &= G\phi_i^p G^{-1}(G(\mathcal{F}_i^s, \mu_i^s)) \\ &= G\phi_i^p(\mathcal{F}_i^s, \mu_i^s) \\ &= G(\mathcal{F}_i^s, \lambda_i^{-p} \mu_i^s) \\ &= (\mathcal{F}_\phi^s, \lambda_i^{-p} a\mu_\phi^s), \end{aligned}$$

then the diffeomorphisms ϕ_i^p and $\phi_{\gamma(i)}^q$ have the same dilatation. Then for each i , we have $\lambda_i^p = \lambda_{\gamma(i)}^q$. Since $\lambda_i > 1$, for all i , λ_i^p has only one real positive q -root, then we can conclude that

$$\lambda_i^{\frac{p}{q}} = \lambda_{\gamma(i)}, \quad \text{for all } i.$$

If for some i , $\gamma(i) = i$, then we conclude that $p = q$. If $\gamma(i) \neq i$ for all i , let $n \geq 2$ be the minimum positive integer such that $\gamma^n(1) = 1$. Since $\lambda_i^{\frac{p}{q}} = \lambda_{\gamma(i)}$ for all i , then

$$(17) \quad \lambda_{\gamma^a(1)}^{\frac{p}{q}} = \lambda_{\gamma^{a+1}(1)}, \quad a \in \{0, \dots, n\},$$

thus $\lambda_1 = \lambda_1^{(\frac{p}{q})^{n-1}}$. Since $\lambda_1 > 1$, we conclude that $p = q$. On the other hand, if $q < 0$, we have that $\lambda_i^p = \lambda_{\gamma(i)}^{-q}$, and in a similar way, we conclude $p = -q$. □

Proposition 4.3. *Let S be an orientable closed surface with finitely many punctures and $\chi(S) < 0$. Let $f \in \Gamma(S)$ be a reducible element such that $f^q = gf^pg^{-1}$ for some $g \in \Gamma(S)$ and $p, q \in \mathbb{Z} - \{0\}$, then $|q| = |p|$.*

Proof. Let $\sigma = \sigma(f)$, suppose that σ has vertices $\alpha_1, \dots, \alpha_r$. By hypothesis and by Lemma 3.8, $g\sigma = \sigma$, so g lies in the stabilizer $\Gamma(S)_\sigma$. Let ρ_σ as in Theorem 3.10,

$$\rho_\sigma : \Gamma(S)_\sigma \rightarrow \Gamma(\widehat{S}_\sigma).$$

Case 1. Suppose that $f \in \ker(\rho_\sigma)$, that is, $f = \prod_{i=1}^r T_{\alpha_i}^{n_i}$; without loss of generality we can suppose that $\gcd\{n_1, \dots, n_r\} = 1$. Note that g may permute $\alpha_1, \dots, \alpha_r$, let $\delta \in \Sigma_r$ such that $g(\alpha_i) = \alpha_{\delta(i)}$ for all $i \in I$. Since the α_i 's have disjoint realizations in S , the Dehn twists T_{α_i} commute, thus by hypothesis and because $gT_{\alpha_i}g^{-1} = T_{g(\alpha_i)} = T_{\alpha_{\delta(i)}}$ (see [5, Sec. 3.3]), we have the following,

$$\begin{aligned} \prod_{i=1}^r T_{\alpha_i}^{qn_i} &= g \prod_{i=1}^r T_{\alpha_i}^{pn_i} g^{-1} \\ &= \prod_{i=1}^r T_{\alpha_{\delta(i)}}^{pn_i} \end{aligned}$$

then we conclude that $qn_i = pn_{\delta^{-1}(i)}$ for all $i \in I$, we can regard these as vectors $v_1 = (n_1, \dots, n_r)$ and $v_2 = (n_{\delta^{-1}(1)}, \dots, n_{\delta^{-1}(r)})$ in \mathbb{Z}^r and $pv_1 = qv_2$. Then v_1 and v_2 are in the same line of \mathbb{R}^r , and since v_2 is obtained by a permutation of the coordinates of v_1 , we conclude that $v_1 = v_2$ or $v_1 = -v_2$,

therefore $|p| = |q|$.

Case 2. Since $\Gamma_m(S) \trianglelefteq \Gamma(S)$ is of finite index, we may assume that $f \in \Gamma_m(S)$. Suppose that $f \notin \ker(\rho_\sigma)$. Since $f \in \Gamma_m(S)$, $\rho_\sigma(f)$ fixes each subsurface \widehat{S}_j of \widehat{S}_σ . Let $J = \{1, \dots, k\}$, for every $j \in J$, let $\tilde{f}_j = \rho_\sigma(f)|_{\widehat{S}_j} : \widehat{S}_j \rightarrow \widehat{S}_j$. Note that $\rho_\sigma(g)$ may permute the subsurfaces \widehat{S}_i of \widehat{S}_σ , let $\tilde{g}_j = \rho_\sigma(g)|_{\widehat{S}_j}$ for each $j \in J$ and let $\gamma \in \Sigma_k$ such that $\tilde{g}_j : \widehat{S}_j \rightarrow \widehat{S}_{\gamma(j)}$, for all $j \in J$. By hypothesis we have that $\rho_\sigma(f)^q = \rho_\sigma(g)\rho_\sigma(f)^p\rho_\sigma(g)^{-1}$, then

$$(18) \quad \tilde{g}_j \tilde{f}_j^p \tilde{g}_j^{-1} = \tilde{f}_{\gamma(j)}^q, \quad \text{for all } j \in J.$$

Since $f \in \Gamma_m(S)$ and $\rho_\sigma(f) \neq Id$, for some $l \in J$, \tilde{f}_l is pseudo-Anosov (see Remark 3.11). Let $x > 0$ be the minimum positive integer such that $\gamma^x(l) = l$, from (18) it follows that

$$\tilde{g}_{\gamma^i(l)} \tilde{f}_{\gamma^i(l)}^p \tilde{g}_{\gamma^i(l)}^{-1} = \tilde{f}_{\gamma^{i+1}(l)}^q, \quad \text{for all } i \in \{0, 1, \dots, x\}.$$

Then $\tilde{f}_{\gamma^i(l)}$ is pseudo-Anosov for each $i \in \{0, 1, \dots, x-1\}$. Observe that \widehat{S}_l and $\widehat{S}_{\gamma^i(l)}$ are homeomorphic for all $i \in \{0, 1, \dots, x-1\}$, then we can apply Proposition 4.2, therefore $|p| = |q|$. \square

4.2. Description of commensurators

We denote by $C_G(f)$ and $N_G(f)$ the centralizer and normalizer respectively of the subgroup $\langle f \rangle$ in G .

Theorem 4.4. [3, Thm. 6.1] *Let S be an orientable compact surface with finitely many punctures. Let $G \subseteq \Gamma(S)$ be a pure subgroup. If $f, g \in G$ are such that $f^t = g^t$ for some $t \geq 1$, then $f = g$.*

Lemma 4.5. Let S be an orientable compact surface with finitely many punctures. Let $f \in \Gamma_m(S)$, $t \in \mathbb{Z} - \{0\}$ and let Γ be either $\Gamma(S)$ or $\Gamma_m(S)$, then

$$C_\Gamma(f) = C_\Gamma(f^t) \quad \text{and} \quad N_\Gamma(f) = N_\Gamma(f^t).$$

Proof. Suppose that $t \geq 1$. Since $N_{\Gamma_m(S)}(f) \subseteq N_{\Gamma_m(S)}(f^t)$, we need to prove that $N_{\Gamma_m(S)}(f^t) \subseteq N_{\Gamma_m(S)}(f)$. If $h \in N_{\Gamma_m(S)}(f^t)$, we have

$$(hfh^{-1})^t = (f^t)^t$$

for some $i \in \{1, -1\}$. Since $f^i, hfh^{-1} \in \Gamma_m(S)$ and $\Gamma_m(S)$ is pure (Thm. 3.6), we can apply Theorem 4.4 and we conclude that

$$hfh^{-1} = f^i,$$

therefore, $h \in N_{\Gamma_m(S)}(f)$. Thus $\Gamma_m(S)$ is a normal subgroup of $\Gamma(S)$, we can use Theorem 4.4 in similar way to prove that $N_{\Gamma(S)}(f) = N_{\Gamma(S)}(f^t)$. We have the proof for the centralizers by taking $i = 1$. \square

Lemma 4.6. Let S be an orientable compact surface with finitely many punctures. Let $\sigma \in \mathcal{C}(S)$ with vertices $\alpha_1, \dots, \alpha_r$, and $f = \prod_{i=1}^r T_{\alpha_i}^{n_i}$, with $n_i \in \mathbb{Z} - \{0\}$, then for any $k \neq 0$,

$$N_{\Gamma(S)}(f) = N_{\Gamma(S)}(f^k).$$

Proof. Let $g \in N_{\Gamma(S)}(f^k)$, we will prove that $g \in N_{\Gamma(S)}(f)$. By Lemma 3.7, $g(\sigma) = \sigma$ and g may permute the classes $\alpha_1, \dots, \alpha_r$. Let $\delta \in \Sigma_r$ such that $g(\alpha_i) = \alpha_{\delta(i)}$ for all i . By the results about Dehn twists given in Section 3.1 and since $f^{jk} = g f^k g^{-1}$, for some $j \in \{1, -1\}$, we have

$$\begin{aligned} \prod_{i=1}^r T_{\alpha_i}^{jkn_i} &= g \prod_{i=1}^r T_{\alpha_i}^{kn_i} g^{-1} \\ &= \prod_{i=1}^r T_{\alpha_{\delta(i)}}^{kn_i}, \end{aligned}$$

then we conclude that $jkn_i = kn_{\delta(i)}$ for all i , so $jn_i = n_{\delta(i)}$ for all i . On the other hand, we have that

$$\begin{aligned} gfg^{-1} &= g \prod_{i=1}^r T_{\alpha_i}^{n_i} g^{-1} \\ &= \prod_{i=1}^r T_{\alpha_{\delta(i)}}^{n_i}, \end{aligned}$$

since $jn_i = n_{\delta(i)}$ then

$$gfg^{-1} = \prod_{i=1}^r T_{\alpha_{\delta(i)}}^{jn_{\delta(i)}} = f^j,$$

therefore $g \in N_{\Gamma(S)}(f)$. \square

Lemma 4.7. [14, Lem. 4.2] Suppose that G satisfies Condition (C). Then, for any $C \in \mathcal{C}_G^\infty$ there is a nested sequence of subgroups

$$N_G(C) \subseteq N_G(2!C) \subseteq N_G(3!C) \subseteq N_G(4!C) \subseteq \dots$$

where $k!C$ is the subgroup of C given by $\{h^{k!} | h \in C\}$, observe that

$$N_G[C] = \bigcup_{k \geq 1} N_G(k!C).$$

The subgroup $N_G(K!C)$ denotes the normalizer of $k!C$ in G .

We will follow the same notation as Section 2.1. Let \mathcal{C}_G^∞ be the set of infinite cyclic subgroups of G .

Proposition 4.8. *Let S be an orientable closed surface with finitely many punctures and $\chi(S) < 0$. Let $m \geq 3$ be fixed. Let $C = \langle g \rangle \in \mathcal{C}_{\Gamma(S)}^\infty$ and $n \in \mathbb{N}$ such that $g^n \in \Gamma_m(S)$. Then*

$$N_{\Gamma(S)}[C] = N_{\Gamma(S)}(g^n).$$

Furthermore, the subgroup $\langle g^n \rangle$ can be chosen maximal in $\mathcal{C}_{\Gamma_m(S)}^\infty$.

Proof. Let $C \in \mathcal{C}_{\Gamma(S)}^\infty$, $[C]$ its class and suppose that $C = \langle g \rangle$. From Lemma 4.7, we have that

$$N_{\Gamma(S)}(g) \subseteq N_{\Gamma(S)}(g^{2!}) \subseteq N_{\Gamma(S)}(g^{3!}) \subseteq \dots, \quad \text{and}$$

$$N_{\Gamma(S)}[C] = \bigcup_{k \geq 1} N_{\Gamma(S)}(g^{k!}).$$

Let $n \in \mathbb{N}$ such that $g^n \in \Gamma_m(S)$, by Lemma 4.5, we have that

$$N_{\Gamma(S)}(g) \subseteq N_{\Gamma(S)}(g^{2!}) \subseteq \dots \subseteq N_{\Gamma(S)}(g^{n!}) = N_{\Gamma(S)}(g^{(n+k)!})$$

for any $k \geq 1$, and $N_{\Gamma(S)}(g^{n!}) = N_{\Gamma(S)}(g^n)$, then

$$N_{\Gamma(S)}[C] = N_{\Gamma(S)}(g^n).$$

We will prove in Proposition 5.11 that $\langle g^n \rangle$ is contained in a unique maximal $\overline{C} \in \mathcal{C}_{\Gamma_m(S)}^\infty$, therefore

$$N_{\Gamma(S)}[C] = N_{\Gamma(S)}(g^n) = N_{\Gamma(S)}(\overline{C}).$$

□

By Lemma 4.6, we have:

Corollary 4.9. Let S be an orientable compact surface with finitely many punctures. Let $\sigma \in \mathcal{C}(S)$ with vertices $\alpha_1, \dots, \alpha_r$, and $f = \prod_{i=1}^r T_{\alpha_i}^{n_i}$, with $n_i \in \mathbb{Z} - \{0\}$. Then

$$N_{\Gamma(S)}[\langle f \rangle] = N_{\Gamma(S)}(f).$$

4.3. Description of normalizers

For surfaces with empty boundary, by the Nielsen-Thurston Classification Theorem, infinite order elements are pseudo-Anosov or reducible.

Theorem 4.10. [17, Thm. 1] *Let S be an orientable closed surface with finitely many punctures. Let $f \in \Gamma(S)$ be a pseudo-Anosov mapping class. The centralizer $C_{\Gamma(S)}(f)$ is a finite extension of an infinite cyclic group. The normalizer, $N_{\Gamma(S)}(f)$ is either equal to $C_{\Gamma(S)}(f)$ or contains $C_{\Gamma(S)}(f)$ as a normal subgroup of index 2.*

Since $\Gamma_m(S)$ is torsion free for $m \geq 3$, for $g \in \Gamma_m(S)$ a pseudo-Anosov mapping class, we know that $C_{\Gamma_m(S)}(g) = N_{\Gamma_m(S)}(g)$ is an infinite cyclic group.

Let $f \in \Gamma_m(S)$ be reducible and $\sigma = \sigma(f)$ with vertices $\alpha_1, \dots, \alpha_r$. By Lemma 3.8, $C_{\Gamma(S)}(f)$ and $N_{\Gamma(S)}(f)$ are subgroups of the stabilizer $\Gamma(S)_\sigma$. Recall that $\Gamma(S)_\sigma^0$ is the subgroup of $\Gamma(S)_\sigma$ that fixes each α_i with orientation. Denote by $C_{\Gamma(S)}(f)^0 = C_{\Gamma(S)}(f) \cap \Gamma(S)_\sigma^0$ and $N_{\Gamma(S)}(f)^0 = N_{\Gamma(S)}(f) \cap \Gamma(S)_\sigma^0$, the finite index subgroups of $C_{\Gamma(S)}(f)$ and $N_{\Gamma(S)}(f)$ respectively.

Proposition 4.11. *Let S be an orientable closed surface with finitely many punctures. Let $f \in \Gamma_m(S)$ with $\rho_{\sigma,0}(f) = (f_1, \dots, f_k)$, then*

$$(19) \quad 1 \longrightarrow \mathbb{Z}^r \longrightarrow C_{\Gamma(S)}(f)^0 \xrightarrow{\rho_{\sigma,0}} \prod_{i=1}^k C_{\Gamma(\hat{S}_i, \mathcal{Q}_i)}(f_i) \longrightarrow 1,$$

and $C_{\Gamma(S)}(f)^0$ has index $\leq 2^k$ in $N_{\Gamma(S)}(f)^0$.

Proof. Write f in its canonical form as in (12),

$$f = \prod_{i=1}^k \eta_{S_i}(\bar{f}_i) \prod_{j=1}^r T_{\alpha_j}^{n_j},$$

by Remark 3.11 we have

$$(20) \quad (f_1, \dots, f_k) = (\theta_{S_1}(\bar{f}_1), \dots, \theta_{S_k}(\bar{f}_k)).$$

Let $g \in \Gamma(S)_\sigma^0$, following the method of Theorem 3.9, since g fixes each subsurface S_i of S_σ and each α_i with orientation, g can be written as

$$g = \prod_{i=1}^k \eta_{S_i}(\bar{g}_i) \prod_{j=1}^r T_{\alpha_j}^{m_j},$$

and \bar{g}_i can be reducible, periodic or pseudo-Anosov, for each i . In a similar way as in Remark 3.11,

$$\rho_{\sigma,0}(g) = (g_1, \dots, g_k) = (\theta_{S_1}(\bar{g}_1), \dots, \theta_{S_k}(\bar{g}_k)).$$

Thus

$$(21) \quad gfg^{-1} = \prod_{i=1}^k \eta_{S_i}(\bar{g}_i) \eta_{S_i}(\bar{f}_i) \eta_{S_i}(\bar{g}_i)^{-1} \prod_{j=1}^r T_{\alpha_j}^{n_j},$$

then

$$\begin{aligned} f &= gfg^{-1} \\ \text{if and only if } \eta_{S_i}(\bar{f}_i) &= \eta_{S_i}(\bar{g}_i) \eta_{S_i}(\bar{f}_i) \eta_{S_i}(\bar{g}_i)^{-1}, \quad \text{for all } i, \\ \text{if and only if } \bar{f}_i &= \bar{g}_i \bar{f}_i \bar{g}_i^{-1}, \quad \text{for all } i. \end{aligned}$$

The result follows since by the definition of η_{S_i} , its kernel is generated by elements of the form $T_\beta T_\gamma^{-1}$ and any commutator has no Dehn twists about boundary components.

Now, if $\theta_{S_i}(\bar{f}_i) = \theta_{S_i}(\bar{g}_i) \theta_{S_i}(\bar{f}_i) \theta_{S_i}(\bar{g}_i)^{-1}$, then $\bar{g}_i \bar{f}_i \bar{g}_i^{-1} \bar{f}_i^{-1} \in \ker \theta_{S_i}$ which is generated by Dehn twists about boundary components of S_i , then we have $\bar{g}_i \bar{f}_i \bar{g}_i^{-1} \bar{f}_i^{-1} = Id$, therefore

$$\begin{aligned} \bar{f}_i &= \bar{g}_i \bar{f}_i \bar{g}_i^{-1}, \quad \text{for all } i, \\ \text{if and only if } \theta_{S_i}(\bar{f}_i) &= \theta_{S_i}(\bar{g}_i) \theta_{S_i}(\bar{f}_i) \theta_{S_i}(\bar{g}_i)^{-1} \quad \text{for all } i, \\ \text{if and only if } f_i &= g_i f_i g_i^{-1} \quad \text{for all } i, \end{aligned}$$

which follows by the equality given in (20).

Since there are no restrictions for m_j with $j \in \{1, \dots, r\}$, we conclude (19).

From the equality (21), if some $n_j \neq 0$, then $C_{\Gamma(S)}(f)^0 = N_{\Gamma(S)}(f)^0$.

On the other hand, if $n_i = 0$, for all i , since each f_i is the identity or pseudo-Anosov, in case f_i is pseudo-Anosov, $C_{\Gamma(\widehat{S}_i)}(f_i)$ is a subgroup of index 1 or 2 in $N_{\Gamma(\widehat{S}_i)}(f_i)$ (Theorem 4.10), therefore we conclude that $C_{\Gamma(S)}(f)^0$ has index $\leq 2^k$ in $N_{\Gamma(S)}(f)^0$. □

By Proposition 4.11 and Theorem 4.10, if we rename the subsurfaces S_i as necessary, we have:

Proposition 4.12. *Let S be an orientable closed surface with finitely many punctures. Let $f \in \Gamma_m(S)$ with $\rho_{\sigma,0}(f) = (Id_{\widehat{S}_1}, \dots, Id_{\widehat{S}_a}, f_{a+1}, \dots, f_k)$ where f_{a+1}, \dots, f_k are pseudo-Anosov. Then*

$$(22) \quad 1 \longrightarrow \mathbb{Z}^r \longrightarrow C_{\Gamma(S)}(f)^0 \xrightarrow{\rho_\sigma} \prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i) \prod_{j=a+1}^k V_j \longrightarrow 1,$$

where $V_j = C_{\Gamma(\widehat{S}_j, \mathcal{Q}_j)}(f_j)$ is virtually cyclic for each $j \in \{a + 1, \dots, k\}$.

SURFACES WITH BOUNDARY. Suppose that S has $b \neq 0$ boundary components β_1, \dots, β_b , let the corking homomorphism $\theta_S: \Gamma(S) \rightarrow \Gamma(\widehat{S})$, with kernel $\ker(\theta_S) \simeq \mathbb{Z}^b$, generated by Dehn twists about curves isotopic to boundary components of S . Let \mathcal{R} be the set of punctures of \widehat{S} which comes from the boundary components of S . Then $\theta_S(\Gamma(S)) = \Gamma(\widehat{S}, \mathcal{R})$ which is the subgroup of $\Gamma(\widehat{S})$ that fixes pointwise the set \mathcal{R} .

Let $f, g \in \Gamma(S)$, suppose that $\theta_S(f)$ and $\theta_S(g)$ commute, then $gfg^{-1}f^{-1}$ is in $\ker \theta_S$, but $gfg^{-1}f^{-1}$ has no Dehn twists about boundary components, so $gfg^{-1}f^{-1} = Id$. Therefore

$$(23) \quad 1 \longrightarrow \mathbb{Z}^b \longrightarrow C_{\Gamma(S)}(f) \xrightarrow{\theta_S} C_{\Gamma(\widehat{S}, \mathcal{R})}(\theta_S(f)) \longrightarrow 1.$$

Observe that elements in $\Gamma(S)$ leave invariant a regular neighborhood of the boundary ∂S . Then, if f has a non-zero power of a Dehn twist T_{β_i} , for any $g \in \Gamma(S)$, gfg^{-1} has the same power of T_{β_i} , then

$$(24) \quad N_{\Gamma(S)}(f) = C_{\Gamma(S)}(f).$$

Moreover, if f has no Dehn twist about curves β_1, \dots, β_b , then

$$gfg^{-1} = f^{\pm 1} \quad \text{iff} \quad \theta_S(g)\theta_S(f)\theta_S(g)^{-1} = \theta_S(f)^{\pm 1}.$$

Then

$$(25) \quad 1 \longrightarrow \mathbb{Z}^b \longrightarrow N_{\Gamma(S)}(f) \xrightarrow{\theta_S} N_{\Gamma(\widehat{S}, \mathcal{R})}(\theta_S(f)) \longrightarrow 1.$$

5. Geometric dimension for the family \mathcal{VCY}

In Section 5.1 we prove that $\text{gd}\Gamma(S) < \infty$, and in Section 5.2, we will give bounds for $\text{gd}\Gamma_m(S)$ and $\text{gd}\overline{\Gamma(S)}$.

Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. It is well-known that the Teichmüller space $\mathcal{T}(S)$ is a finite

dimensional space which is contractible, on which $\Gamma(S)$ acts properly and it is a model for $\underline{E}\Gamma(S)$ by results of Kerckhoff given in [11]. On the other hand, J. Aramayona and C. Martínez proved in [1] the following:

Theorem 5.1. *[1, Cor. 1.3] Let S be an orientable compact surface with finitely many punctures. Then there exist a cocompact model for $\underline{E}\Gamma(S)$ of dimension equal to the virtual cohomological dimension $vcd(\Gamma(S))$.*

And Harer computed $vcd(\Gamma(S))$ in [8].

Theorem 5.2. *[8, Thm. 4.1] Let S be an orientable surface surface with genus g , b boundary components and n punctures. If $2g + b + n > 2$, then*

$$vcd(\Gamma(S)) = \begin{cases} 4g + 2b + n - 4 & \text{if } g > 0, b + n > 0, \\ 4g - 5 & \text{if } n, b = 0, \\ 2b + n - 3 & \text{if } g = 0. \end{cases}$$

5.1. Geometric dimension for $\Gamma(S)$

We will use the same notation of Section 2.1. We will prove that there exist finite dimensional models for $\underline{E}N_{\Gamma(S)}[C]$ and $E_{\mathcal{G}[C]}N_{\Gamma(S)}[C]$ and a uniform bound on $\text{gd}_{\mathcal{G}[C]}N_{\Gamma(S)}[C]$ for any $[C] \in [\mathcal{C}_{\Gamma(S)}^\infty]$, with these results, Theorem 2.4 and the fact that $\underline{\text{gd}}\Gamma(S)$ is finite, we have:

Theorem 5.3. *Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. Then $\underline{\text{gd}}\Gamma(S) < \infty$, that is, the mapping class group $\Gamma(S)$ admits a finite dimensional model for $\underline{E}\Gamma(S)$.*

For the proof of Theorem 5.3 we need the following results.

Proposition 5.4. *[10, Prop. 4] Let G be an infinite virtually cyclic group, then there is model for $\underline{E}G$ with finitely many orbits of cells which is homeomorphic to the real line.*

Theorem 5.5. *[13, Thm. 5.16] Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence of groups. Suppose that H has the property that for any group \tilde{H} which contains H as subgroup of finite index, $\underline{\text{gd}}\tilde{H} \leq n$. If $\underline{\text{gd}}K \leq k$, then $\underline{\text{gd}}G \leq n + k$.*

In [13, Ex. 5.26] Lück shows that virtually poly-cyclic groups satisfies the condition about H in Theorem 5.5, in particular \mathbb{Z}^n satisfies such condition.

Theorem 5.6. [16, Thm. 2.4] *Suppose $H \subseteq G$ is a subgroup of finite index n , then $\underline{\text{gd}}G \leq \underline{\text{gd}}H \cdot n$ and $\underline{\underline{\text{gd}}}G \leq \underline{\underline{\text{gd}}}H \cdot n$.*

Proposition 5.7. [12, Lem. 4.3] *Let $1 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow F$ be an exact sequence of groups with F finite. Then G admits an n -dimensional cocompact $\underline{E}G$ homeomorphic to \mathbb{R}^n , with G acting by affine maps.*

Remark 5.8. Let $[C] \in [\mathcal{C}_{\Gamma(S)}^\infty]$. From Proposition 4.8, we could assume that C is a cyclic maximal subgroup in $\mathcal{C}_{\Gamma_m(S)}^\infty$ and $N_{\Gamma(S)}[C] = N_{\Gamma(S)}(C)$. Let $W_{\Gamma(S)}(C) = N_{\Gamma(S)}(C)/C$ and $p: N_{\Gamma(S)}(C) \rightarrow W_{\Gamma(S)}(C)$, the projection. From Theorem 2.3, a model for $\underline{E}W_{\Gamma(S)}(C)$ with the $N_{\Gamma(S)}[C]$ -action induced from the projection p is a model for $E_{\mathcal{G}[C]}N_{\Gamma(S)}[C]$. Then it is sufficient to consider models for $\underline{E}N_{\Gamma(S)}(C)$ and $\underline{E}W_{\Gamma(S)}(C)$, of maximal infinite cyclic subgroups C in $\mathcal{C}_{\Gamma_m(S)}^\infty$.

Proof of Theorem 5.3: Suppose that the surface S has genus g , b boundary components and n punctures.

Part 1. For any $[C] \in [\mathcal{C}_{\Gamma(S)}^\infty]$, $\underline{\text{gd}}N_{\Gamma(S)}[C]$ is finite. We may assume that $C \in \mathcal{C}_{\Gamma_m(S)}^\infty$, it follows that $N_{\Gamma(S)}[C] = N_{\Gamma(S)}(C)$. Since $N_{\Gamma(S)}(C) \leq \Gamma(S)$ from the properties given in (2) and Theorem 5.1, we conclude that $\underline{\text{gd}}N_{\Gamma(S)}[C] \leq \underline{\text{gd}}\Gamma(S) = \text{vcd}\Gamma(S)$, which is finite.

Part 2. We will prove that there exist $z \in \mathbb{Z}$, such that for any $[C] \in [\mathcal{C}_{\Gamma(S)}^\infty]$, $\underline{\text{gd}}_{\mathcal{G}[C]}N_{\Gamma(S)}[C] \leq z$.

(I) *The surface S has empty boundary.* Let $[C] \in \mathcal{C}_{\Gamma(S)}^\infty$, with $C = \langle f \rangle$. By the Nielsen-Thurston classification Theorem, f is either, a pseudo-Anosov class or a reducible element.

(a) If f is pseudo-Anosov, then $N_{\Gamma(S)}[C] = N_{\Gamma(S)}(f)$ is virtually cyclic and $\mathcal{G}[C]$ is the family of all subgroups of $N_{\Gamma(S)}(f)$, hence a point is a model for $E_{\mathcal{G}[C]}N_{\Gamma(S)}[C]$, therefore $\underline{\text{gd}}_{\mathcal{G}[C]}N_{\Gamma(S)}[C] = 0$.

(b) If f is reducible and $f = \prod_{i=1}^r T_{\alpha_i}^{n_i}$, with $n_i \in \mathbb{Z} - \{0\}$, where $\alpha_1, \dots, \alpha_r$ are the vertices of $\sigma = \sigma(f)$. By Corollary 4.9, $N_{\Gamma(S)}[C] = N_{\Gamma(S)}(f)$ and we can suppose that $\text{gcd}\{n_1, \dots, n_k\} = 1$.

Following the same idea as in Remark 5.8, a model for $\underline{E}W_{\Gamma(S)}(f)$ with the induced action of the projection $N_{\Gamma(S)}(f) \rightarrow W_{\Gamma(S)}(f)$ is a model for $E_{\mathcal{G}[C]}N_{\Gamma(S)}[C]$. We will prove that $\underline{\text{gd}}W_{\Gamma(S)}(f)$ is finite.

Note that any element $g \in \Gamma(S)_\sigma^0$ commutes with f , because g fixes each class α_i , then $\Gamma(S)_\sigma^0 \subseteq N_{\Gamma(S)}(f)$. On the other hand, we have $N_{\Gamma(S)}(f) \subseteq \Gamma(S)_\sigma$, therefore $\Gamma(S)_\sigma^0$ is a finite index subgroup of $N_{\Gamma(S)}(f)$ and it is normal, then

$$1 \longrightarrow \Gamma(S)_\sigma^0 \longrightarrow N_{\Gamma(S)}(f) \longrightarrow B \longrightarrow 1,$$

where B is finite, we will obtain a uniform bound for the order of B . Since $f \in \Gamma(S)_\sigma^0$, we have

$$1 \longrightarrow \Gamma(S)_\sigma^0 / \langle f \rangle \longrightarrow N_{\Gamma(S)}(f) / \langle f \rangle \longrightarrow B \longrightarrow 1.$$

Since the index $[\Gamma(S)_\sigma : \Gamma(S)_\sigma^0] \leq (2r)!$, then $|B| \leq (2r)!$. By Theorem 5.6,

$$(26) \quad \begin{aligned} \underline{\text{gd}}W_{\Gamma(S)}(f) &\leq \underline{\text{gd}}(\Gamma(S)_\sigma^0 / \langle f \rangle) \cdot |B| \\ &\leq \underline{\text{gd}}(\Gamma(S)_\sigma^0 / \langle f \rangle) \cdot (2r)!. \end{aligned}$$

From (14), since $f \in \ker(\rho_{\sigma,0})$ we have

$$1 \longrightarrow \langle T_{\alpha_1}, \dots, T_{\alpha_r} \rangle / \langle f \rangle \longrightarrow \Gamma(S)_\sigma^0 / \langle f \rangle \longrightarrow \prod_{i=1}^k \Gamma(\widehat{S}_i, \mathcal{Q}_i) \longrightarrow 1,$$

since $\langle T_{\alpha_1}, \dots, T_{\alpha_r} \rangle \simeq \mathbb{Z}^r$ and f is identified with the point $(n_1, \dots, n_r) \in \mathbb{Z}^r$ via that isomorphism, then the quotient $\mathbb{Z}^r / \langle (n_1, \dots, n_r) \rangle \simeq \mathbb{Z}^{r-1}$, because $\text{gcd}\{n_1, \dots, n_r\} = 1$. Then

$$1 \longrightarrow \mathbb{Z}^{r-1} \longrightarrow \Gamma(S)_\sigma^0 / \langle f \rangle \longrightarrow \prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i) \longrightarrow 1,$$

we apply Theorem 5.5, the properties given in (1) and (2), and Theorem 5.1 to conclude that

$$(27) \quad \begin{aligned} \underline{\text{gd}}(\Gamma(S)_\sigma^0 / \langle f \rangle) &\leq (r-1) + \underline{\text{gd}}(\prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i)) \\ &\leq (r-1) + \sum_{i=1}^a \underline{\text{gd}}(\Gamma(\widehat{S}_i, \mathcal{Q}_i)) \\ &\leq (r-1) + \sum_{i=1}^a \text{vcd}(\Gamma(\widehat{S}_i)). \end{aligned}$$

Observe that each \widehat{S}_i has at least one puncture, no boundary components and negative Euler characteristic, thus from Theorem 5.2 we can see that $\text{vcd}(\Gamma(\widehat{S}_i)) \leq -2\chi(\widehat{S}_i)$, then

$$(28) \quad \begin{aligned} \underline{\text{gd}}(\Gamma(S)_\sigma^0 / \langle f \rangle) &\leq (r-1) + \sum_{i=1}^a (-2\chi(\widehat{S}_i)) \\ &\leq (r-1) + (-2\chi(S)), \end{aligned}$$

from inequalities (26) and (28), we conclude

$$(29) \quad \underline{\text{gd}}W_{\Gamma(S)}(f) \leq (-2\chi(S) + r - 1) \cdot (2r)!.$$

By Remark 3.5 $r \leq \frac{-3\chi(S)-n}{2}$, therefore

$$(30) \quad \underline{\text{gd}}W_{\Gamma(S)}(f) \leq (-5\chi(S) - n)(-3\chi(S) - n)!$$

Note that the bound only depends on the surface.

(c) Now suppose that f is reducible, $\sigma = \sigma(f)$ has vertices $\alpha_1, \dots, \alpha_r$, and $\rho(f)$ is not trivial. Since $N_{\Gamma(S)}[C] = N_{\Gamma(S)}(f^n)$, for some $n \neq 0$ such that $f^n \in \Gamma_m(S)$, then we may assume that $f \in \Gamma_m(S)$ and $C = \langle f \rangle$ is maximal in $C_{\Gamma_m(S)}^\infty$. We will apply Remark 5.8 again.

Note that $C_{\Gamma(S)}(f)^0 \trianglelefteq N_{\Gamma(S)}(f)$ is of finite index, then

$$1 \longrightarrow C_{\Gamma(S)}(f)^0 \longrightarrow N_{\Gamma(S)}(f) \longrightarrow F \longrightarrow 1,$$

with $|F| \leq 2^k((2r)!)$, this follows by Proposition 4.11 and because the index $[N_{\Gamma(S)}(f) : C_{\Gamma(S)}(f)^0] \leq (2r)!$. Since $f \in \Gamma_m(S)$, then $f \in C_{\Gamma(S)}(f)^0$, and so,

$$1 \longrightarrow C_{\Gamma(S)}(f)^0 / \langle f \rangle \longrightarrow N_{\Gamma(S)}(f) / \langle f \rangle \longrightarrow F \longrightarrow 1.$$

By Theorem 5.6

$$(31) \quad \underline{\text{gd}}(W_{\Gamma(S)}(C)) \leq \underline{\text{gd}}(C_{\Gamma(S)}(f)^0 / \langle f \rangle) \cdot 2^k((2r)!).$$

Let $\rho_{\sigma,0}(f) = (f_1, \dots, f_k)$ and we rename the \widehat{S}_i such that $f_i = Id_{\Gamma(\widehat{S}_i)}$ for $i \in \{1, \dots, a\}$ and $f_j \in \Gamma(\widehat{S}_j, \mathcal{Q}_j)$ is pseudo-Anosov for $j \in \{a+1, \dots, k\}$, then by Proposition 4.12,

$$1 \longrightarrow \langle T_{\alpha_1}, \dots, T_{\alpha_r} \rangle \longrightarrow C_{\Gamma(S)}(f)^0 \xrightarrow{\rho_{\sigma,0}} \prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i) \prod_{j=a+1}^k V_j \longrightarrow 1,$$

where $V_j = C_{\Gamma(\widehat{S}_j, \mathcal{Q}_j)}(f_j)$ is virtually cyclic for each j .

Denote the group $\prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i) \prod_{j=a+1}^k V_j$ by Δ , then we have the following homomorphism

$$\begin{aligned} \psi: C_{\Gamma(S)}(f)^0 / \langle f \rangle &\rightarrow \Delta / \langle \rho_{\sigma,0}(f) \rangle \\ g \langle f \rangle &\mapsto \rho_{\sigma,0}(g) \langle \rho_{\sigma,0}(f) \rangle, \end{aligned}$$

which is well-defined because $\rho_{\sigma,0}(\langle f \rangle) = \langle \rho_{\sigma,0}(f) \rangle$, ψ is a homomorphism and since $\rho_{\sigma,0}$ is onto, then ψ is onto too. Thus

$$\ker \psi = \rho_{\sigma,0}^{-1}(\langle \rho_{\sigma,0}(f) \rangle) / \langle f \rangle,$$

is a free abelian subgroup isomorphic to \mathbb{Z}^r . Moreover,

$$\begin{aligned} \frac{\Delta}{\langle \rho_{\sigma,0}(f) \rangle} &= \frac{\prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i) \prod_{j=a+1}^k V_j}{\langle (Id_{\Gamma(\widehat{S}_1)}, \dots, Id_{\Gamma(\widehat{S}_a)}, f_{a+1}, \dots, f_k) \rangle} \\ &= \prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i) \times \frac{\prod_{j=a+1}^k V_j}{\langle (f_{a+1}, \dots, f_k) \rangle}. \end{aligned}$$

Since $\langle f_j \rangle \leq V_j$ is of finite index, for all j , if $\check{f} = (f_{a+1}, \dots, f_k)$,

$$(32) \quad 1 \longrightarrow \prod_{j=a+1}^k \langle f_j \rangle / \langle \check{f} \rangle \longrightarrow \prod_{j=a+1}^k V_j / \langle \check{f} \rangle \longrightarrow \prod_{j=a+1}^k F_j \longrightarrow 1,$$

with F_j finite for all j , thus from Proposition 5.7 and (32) we have that $\underline{\text{gd}}(\prod_{j=a+1}^k V_j / \langle \check{f} \rangle) \leq k - a - 1$. Thus applying Theorem 5.5 to the exact sequence given by ψ , the properties given in (1) and (2), and Theorem 5.1, we have:

$$\begin{aligned} \underline{\text{gd}}(C_{\Gamma(S)}^0 / \langle f \rangle) &\leq r + [\underline{\text{gd}}(\prod_{i=1}^a \Gamma(\widehat{S}_i, \mathcal{Q}_i)) + (k - a - 1)] \\ &\leq r + [\underline{\text{gd}}(\prod_{i=1}^a \Gamma(\widehat{S}_i)) + (k - a - 1)] \\ &\leq r + [\sum_{i=1}^a \underline{\text{gd}}(\Gamma(\widehat{S}_i)) + (k - a - 1)] \\ &\leq r + [\sum_{i=1}^a \text{vcd}(\Gamma(\widehat{S}_i)) + (k - a - 1)] \\ &\leq r + [\sum_{i=1}^a (-2\chi(\widehat{S}_i)) + (k - a - 1)] \\ &\leq r + [(-2\chi(S)) + (k - a - 1)] \\ (33) \quad &\leq \frac{-3\chi(S) - n}{2} + (-3\chi(S) - 1), \end{aligned}$$

the last inequality (33) holds because $k \leq -\chi(S)$ (see Remark 3.5) and $a \geq 0$. From (31) and (33) we conclude that

$$(34) \quad \underline{\text{gd}}(W_{\Gamma(S)}(C)) \leq (-6\chi(S) - n)((-3\chi(S) - n)!).$$

Note that the bound only depends on the surface. From (a) and the inequalities (34) and (30), for any $[C] \in [C_{\Gamma(S)}^\infty]$,

$$(35) \quad \text{gd}_{\mathcal{G}[C]} N_{\Gamma(S)}[C] \leq (-6\chi(S) - n)((-3\chi(S) - n)!).$$

(II) *The surface S has non-empty boundary.* Following in a similar way as in Part (I), from (23),(24) and (25), we conclude that there exist $z \in \mathbb{Z}$, such that for any $[C] \in [\mathcal{C}_{\Gamma(S)}^\infty]$, $\text{gd}_{[C]} N_{\Gamma(S)}[C] \leq z$. \square

5.2. Bounds for geometric dimension

We will prove that $\Gamma_m(S)$ satisfies the following property, which we will use to give a bound for $\underline{\text{gd}}\Gamma_m(S)$.

Definition 5.9. A group G satisfies $\text{Max}\mathcal{V}\mathcal{C}_G^\infty$ if every subgroup $H \in \mathcal{V}\mathcal{C}_G^\infty$ is contained in a unique $H_{\text{max}} \in \mathcal{V}\mathcal{C}_G^\infty$ which is maximal in $\mathcal{V}\mathcal{C}_G^\infty$.

We follow the same notation as in Section 3.1 and 3.4. The homomorphism ρ_σ is given in Section 3.4, and $\rho_{\sigma,m} = \rho_\sigma|_{\Gamma_m(S)}$. By [9, Thm. 1.2], elements in $\Gamma_m(S)_\sigma$ do not rearrange the components of S_σ and fix each curve of σ , then $\Gamma_m(S)_\sigma \subseteq \Gamma(S)_\sigma^0$. Note that if $g \in \Gamma_m(S)_\sigma$ and $\rho_{\sigma,m}(g) = (g_1, \dots, g_k)$, by definition of $\rho_{\sigma,m}$ and because g is pure, each g_i is pure, then the image $\rho_{\sigma,m}(\Gamma_m(S)_\sigma)$ is a pure subgroup of $\prod_{i=1}^k \Gamma(\widehat{S}_i, \mathcal{Q}_i)$. Let Γ_i be the projection of the image $\rho_{\sigma,m}(\Gamma_m(S)_\sigma)$ over $\Gamma(\widehat{S}_i, \mathcal{Q}_i)$ for each i , then each Γ_i is torsion free and we have

$$(36) \quad \rho_{\sigma,m}: \Gamma_m(S)_\sigma \rightarrow \prod_{i=1}^k \Gamma_i,$$

where $\ker(\rho_{\sigma,m})$ is a free abelian subgroup of $\ker(\rho_\sigma) \simeq \mathbb{Z}^r$, observe that $\ker(\rho_{\sigma,m}) \simeq \mathbb{Z}^r$ as $\Gamma_m(S)$ is of finite index and $\ker(\rho_{\sigma,m})$ is of finite index in $\ker(\rho_\sigma)$. As a reference, see [9, Sec. 7.5].

Lemma 5.10. [9, Lem. 8.7] Suppose that S has empty boundary. Let G be a subgroup of $\Gamma_m(S)$, $m \geq 3$. Let $\sigma = \sigma(G)$, and suppose that $\rho_\sigma(G) = \prod_{i=1}^k G_i$, where G_i denotes the projection of $\rho_{\sigma,m}(G)$ over $\Gamma(\widehat{S}_i, \mathcal{Q}_i)$ for each i . The group G is abelian if and only if each G_i is either trivial or an infinite cyclic group.

Proposition 5.11. *Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. Let $m \geq 3$, then the group $\Gamma_m(S)$ satisfies property $\text{Max}\mathcal{V}\mathcal{C}_{\Gamma_m(S)}^\infty$.*

Proof. Case I: Suppose that S has empty boundary. Since $\Gamma_m(S)$ is torsion free for $m \geq 3$, then $\mathcal{V}\mathcal{C}_{\Gamma_m(S)}^\infty = \mathcal{C}_{\Gamma(S)_m}^\infty$ is the set of infinite cyclic subgroups

of $\Gamma_m(S)$.

It is well-known that for surfaces S with empty boundary, periodic elements of $\Gamma(S)$ are of finite order. By the Nielsen-Thurston classification Theorem, each element of $\Gamma_m(S) - \{Id\}$ is either reducible or pseudo-Anosov.

Let $H = \langle f \rangle \in \mathcal{C}_{\Gamma_m(S)_m}^\infty$. Observe that if $\langle k \rangle = K \in \mathcal{C}_{\Gamma_m(S)_m}^\infty$ and $H \subseteq K$, then $f^n = k$ for some $n \in \mathbb{Z} - \{0\}$. By Lemma 4.5, $C_{\Gamma_m(S)}(K) = C_{\Gamma_m(S)}(H)$, thus $K \subseteq C_{\Gamma_m(S)}(H)$. Actually, we will prove that K lies in a free abelian subgroup of $\Gamma_m(S)$ (which does not depends on K but only on H), then H must be contained in an unique maximal subgroup of $\mathcal{C}_{\Gamma_m(S)}^\infty$. If f is pseudo-Anosov, by Theorem 4.10, $C_{\Gamma_m(S)}(H) \in \mathcal{C}_{\Gamma_m(S)}^\infty$ and it is the unique maximal subgroup of $\mathcal{C}_{\Gamma_m(S)}^\infty$ containing H .

On the other hand, suppose that f is reducible and let $\sigma = \sigma(f)$ be its canonical reduction system, by Lemma 3.8, $C_{\Gamma_m(S)}(H) \subseteq \Gamma_m(S)_\sigma$. Suppose that σ has vertices $\alpha_1, \dots, \alpha_r$, let $S_\sigma = S_1 \cup \dots \cup S_k$ and $\rho_{\sigma,m}$ as in (36),

$$\rho_{\sigma,m}: \Gamma_m(S)_\sigma \rightarrow \prod_{i=1}^k \Gamma_i,$$

where $\Gamma_i \subset \Gamma(\widehat{S}_i, \mathcal{Q}_i)$ is torsion free for each i and $\ker(\rho_{\sigma,m}) \simeq \mathbb{Z}^s$ is a free abelian subgroup of $\langle T_{\alpha_1}, \dots, T_{\alpha_r} \rangle \simeq \mathbb{Z}^r$. Note that each T_{α_i} commutes with f , if $\rho_{\sigma,m}(f) = (f_1, \dots, f_k)$, then

$$1 \longrightarrow \mathbb{Z}^s \longrightarrow C_{\Gamma_m(S)}(f) \xrightarrow{\rho_{\sigma,m}} \prod_{i=1}^k C_{\Gamma_i}(f_i) \longrightarrow 1,$$

furthermore each f_i is either the identity or pseudo-Anosov, see Remark 3.11. Let $\rho_{\sigma,m}(k) = (k_1, \dots, k_r)$, then we have that

- (a) for all i , $k_i f_i = f_i k_i$,
- (b) for all i , f_i is an n -th root of k_i because f is an n -th root of k ,
- (c) if $f_j = Id$ for some j , then $k_j = Id$, because each Γ_j is torsion free.

Let $L = \{l_1, \dots, l_d\} \subseteq \{1, \dots, k\}$, such that $l_i \in L$ if only if f_{l_i} is pseudo-Anosov. By Theorem 4.10, $C_{\Gamma_{l_i}}(f_{l_i})$ is an infinite cyclic subgroup for each $l_i \in L$. We regard $\prod_{j=1}^d C_{\Gamma_{l_j}}(f_{l_j})$ as subgroup of $\prod_{i=1}^k C_{\Gamma_i}(f_i)$, let

$$G = \rho_{\sigma,m}^{-1}(\prod_{j=1}^d C_{\Gamma_{l_j}}(f_{l_j})) \subseteq \Gamma_m(S)_\sigma,$$

then we have

$$1 \longrightarrow \mathbb{Z}^s \longrightarrow G \xrightarrow{\rho_{\sigma,m}} \prod_{j=1}^d C_{\Gamma_{l_j}}(f_{l_j}) \longrightarrow 1,$$

by Lemma 5.10, we conclude that G is a free abelian subgroup and by construction, we have that $K \subseteq G$. Since K was taken arbitrarily, we conclude that there exists a unique maximal subgroup $H_{max} \in \mathcal{C}_{\Gamma_m(S)}^\infty$ containing H . **Case II.** Suppose that S has non empty boundary components β_1, \dots, β_b . Let $\theta_S: \Gamma(S) \rightarrow \Gamma(\widehat{S})$, be the corking homomorphism as in (10), with

$$\ker(\theta_S) = \langle T_{\beta_1}, \dots, T_{\beta_b} \rangle \simeq \mathbb{Z}^b.$$

By definition of θ_S we have that $\theta_S(\Gamma_m(S)) \subseteq \Gamma_m(\widehat{S})$. Since each T_{β_i} acts trivially on $H_1(S, \mathbb{Z})$, $\ker(\theta_S) \subseteq \Gamma_m(S)$, then $\ker(\theta_S|_{\Gamma_m(S)}) = \ker(\theta_S)$. Then we have the following,

$$(37) \quad 1 \longrightarrow \mathbb{Z}^b \longrightarrow \Gamma_m(S) \xrightarrow{\theta_{S,m}} \Gamma_m(\widehat{S}).$$

It is well known that $\Gamma(S)$ is torsion free when S has non-empty boundary, then $\mathcal{C}_{\Gamma_m(S)}^\infty = \mathcal{VCY}_{\Gamma_m(S)} - \mathcal{FIN}_{\Gamma_m(S)}$ is the set of infinite virtually cyclic subgroups of $\Gamma_m(S)$.

Let $A = \langle x \rangle \in \mathcal{C}_{\Gamma_m(S)}^\infty$, if $B = \langle y \rangle \in \mathcal{C}_{\Gamma_m(S)}^\infty$ is such that $A \subseteq B$, then we have $\theta_S(A) = \langle \theta_S(x) \rangle \subseteq \langle \theta_S(y) \rangle = \theta_S(B)$. As in CASE I, we will prove that B lies in a free abelian subgroup of $\Gamma_m(S)$.

By CASE I, we have that there exists $\overline{G} \subset \Gamma_m(\widehat{S})$, a free abelian subgroup, such that if $K \in \mathcal{C}_{\Gamma_m(\widehat{S})}^\infty$ and $\theta_S(A) \subseteq K$, then $K \subseteq \overline{G}$. Let us consider the free abelian group $G = \theta_{S,m}(\Gamma_m(S)) \cap \overline{G}$, note $B \subseteq \theta_{S,m}^{-1}(G)$, because $y \in \theta_{S,m}^{-1}(\theta(y))$.

Let $g_1, g_2 \in G$, $\tilde{g}_1 \in \theta_S^{-1}(g_1)$ and $\tilde{g}_2 \in \theta_S^{-1}(g_2)$, since g_1 and g_2 commute, by definition of θ_S , it follows that \tilde{g}_1 and \tilde{g}_2 must commute, therefore $\theta^{-1}(G)$ is a free abelian subgroup. Since B was taken arbitrarily, we conclude that A is contained in an unique maximal subgroup $A_{max} \in \mathcal{C}_{\Gamma_m(S)}^\infty$.

Note that the free abelian subgroup does not depend on B but only on A . □

We will apply the following Theorem.

Theorem 5.12. [15, Thm. 5.8] *Let G be a group satisfying $MaxVC_\infty^\infty$. Suppose we know that $\underline{\underline{gd}}G < \infty$, then*

$$(38) \quad \underline{\underline{gd}}G \leq \underline{gd}G + 1.$$

Theorem 5.13. *Let S be an orientable compact surface with finitely many punctures and $\chi(S) < 0$. Let $m \geq 3$, then*

$$(1) \underline{\underline{\text{gd}}}\Gamma_m(S) \leq \text{vcd}(\Gamma(S)) + 1;$$

(2) Let $[\Gamma(S) : \Gamma_m(S)]$ be the index of $\Gamma_m(S)$ in $\Gamma(S)$, then

$$\begin{aligned} \underline{\underline{\text{gd}}}\Gamma(S) &\leq [\Gamma(S) : \Gamma_m(S)] \cdot \underline{\underline{\text{gd}}}\Gamma_m(S) \\ &\leq [\Gamma(S) : \Gamma_m(S)] \cdot (\text{vcd}(\Gamma(S)) + 1). \end{aligned}$$

Where $\text{vcd}(\Gamma(S))$ is the virtual cohomological dimension of $\Gamma(S)$.

Proof. By Theorem 5.12, Proposition 5.11 and Theorem 5.3, we conclude (1) and applying Theorem 5.6 we conclude (2). \square

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