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On the Complexity of Isometric Immersions of Hyperbolic Spaces in Any Codimension

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Abstract: Although the Nash theorem solves the isometric embedding problem, matters are inherently more involved if one is further seeking an embedding that is well-behaved from the standpoint of submanifold geometry. More generally, consider a Lipschitz map $F : M^m \to \mathbb{R}^n$, where M^m is a Hadamard manifold whose curvature lies between negative constants. The main result of this paper is that F must perform a substantial compression: *For every* $r > 0$, $\epsilon > 0$ *and integer* $k \geq 2$ *there exist* k *geodesic balls of radius* r *in* M^m *that are at least* ϵ^{-1} *apart, but whose images under* F *are* ϵ*-close in the Hausdorff sense of* $\mathbb{R}^n$. In particular, any isometric embedding $\mathbb{H}^m \to \mathbb{R}^n$ of hyperbolic space, proper or not, must have a rather complex asymptotic behavior, no matter how high the codimension $n - m$ is allowed to be.
Keywords: Isometric embeddings of hyperbolic spaces, Lipschitz map, Hadamard manifold, Nash theorem.

1. Introduction.

The Nash embedding theorem ([13], [16]), to the effect that any Riemannian manifold (M^m, g) can be isometrically embedded as a bounded subset of some high dimensional Euclidean space, represents a landmark in Riemannian geometry. Questions about the smoothness of the isometric immersion, connections with topology and partial differential equations, as well as the smallest dimension of the receiving space, have also attracted considerable attention over the years ([6], [7], [8]).

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Examining the issue further in the non-compact case, one would like to establish the existence of an isometric embedding that, from the perspective of submanifold geometry, is as well-behaved globally as the intrinsic geometry permits.

For instance, when (M^m, g) is non-compact but *complete,* one might aim for the existence of a *proper* isometric embedding $F : M^m \to \mathbb{R}^n$, for some sufficiently large n, whose behavior at infinity is as tame as possible.

A suitable testing ground for these ideas is $\mathbb{H}^m$, the complete simply-connected space of constant sectional curvature -1. For reasons that are not entirely clear, it has been rather difficult to realize Nash's theorem in the case of $\mathbb{H}^m$, namely to produce explicit isometric embeddings of $\mathbb{H}^m$ into *some* Euclidean space.

The earliest example of Euclidean embeddings of hyperbolic m-space that we are aware of is the 1955 paper of Blanusa [1]. Subsequent works are by Henke [9] in 1981, and Henke-Nettekoven [10] in 1987. In the latter, $\mathbb{H}^m$ is properly isometrically embedded in $\mathbb{R}^{6m-6}$ as a smooth complete graph over an m-dimensional subspace.

How can the existence of such a proper isometric embedding be reconciled with the fact that hyperbolic space $\mathbb{H}^m$ is much larger at infinity than any Euclidean space? Are there properties, related to but stronger than properness, that are never satisfied for any isometric embedding $\mathbb{H}^m \hookrightarrow \mathbb{R}^n$, regardless of how high the codimension is?

The main geometric finding of this paper is that, in order to accommodate the different orders of growth at infinity, any isometric embedding $F : \mathbb{H}^m \to \mathbb{R}^n$ - regardless of regularity, dimension or codimension - must exhibit a high degree of asymptotic complexity, which is expressed in a precise quantitative fashion. In particular, we provide an affirmative answer to the last question posed in the previous paragraph (see the notion of strong properness introduced below, together with Corollary 1.3).

It turns out that our arguments are not restricted to isometric immersions, being valid in the broader context of Lipschitz maps $M^m \to \mathbb{R}^n$, where M^m is a Hadamard manifold with curvature bounded away from zero.

But before stating the main result in its greatest generality, it is worthwhile to explain its geometric meaning in the special case of an isometric immersion $F : \mathbb{H}^m \to \mathbb{R}^n$:

(†) *For every $r > 0$, $\epsilon > 0$, and integer $k \geq 2$, there exist k geodesic balls of radius r in $\mathbb{H}^m$ that are at least ϵ^{-1} apart but whose images by F are ϵ-close in the Hausdorff sense.*

When F is the lift to $\mathbb{H}^m$ of an isometric immersion of a compact hyperbolic manifold and r is large enough, the images of the balls actually coincide with $F(\mathbb{H}^m)$.

Let now $F : \mathbb{H}^m \to \mathbb{R}^n$ be a proper isometric embedding (say, as in [10]), so that we can think of F as the inclusion $\mathbb{H}^m \hookrightarrow \mathbb{R}^n$. Fix $r > 0$ and an integer $k \geq 2$, both arbitrarily large, and let $\epsilon = \epsilon_j$ in (†) be a sequence of positive numbers converging to zero. One obtains a dynamical picture for the asymptotic behavior of the embedding $\mathbb{H}^m \hookrightarrow \mathbb{R}^n$ that can be expressed in words as follows:

(††) *There exists a sequence of configurations $\mathcal{C}_j = \{B_{1j}, B_{2j}, \cdots, B_{kj}\}$, $1 \leq j < \infty$, of k disjoint geodesic balls B_{ij} of radius r in $\mathbb{H}^m \subset \mathbb{R}^n$, the intrinsic distance between any two balls in $\mathcal{C}_j$ going to infinity as $j \to \infty$, such that the union $\mathcal{K}_j$ of the balls in $\mathcal{C}_j$ gives a sequence of disjoint "stacks" with the property that $\mathcal{K}_j$ tends to infinity in $\mathbb{R}^n$ while their Euclidean "thickness" tends to zero.*

The complexity of the phenomenon described above is perhaps an indication of why, historically, isometric embeddings of hyperbolic spaces into Euclidean spaces have been so hard to produce explicitly. Indeed, any ansatz that one writes down, i.e. a multi-parametric family of potential candidates for an isometric embedding $\mathbb{H}^m \hookrightarrow \mathbb{R}^n$, has to be *a priori* sufficiently rich in order to accommodate the intricate behavior in (††).

There are of course classical simple models of hyperbolic geometry that retain some features of Euclidean geometry. Unfortunately, the complexity exhibited in (††) shows that there is no simple realization of hyperbolic m-space inside Euclidean spaces, no matter how high the codimension is allowed to be.

As mentioned before, our main result holds for maps that are more general than isometric immersions $\mathbb{H}^m \to \mathbb{R}^n$. Its formal statement reads as follows:

Theorem 1.1. *Let $m \geq 2$, $n \geq 1$ be integers, M^m a Hadamard manifold whose curvature is bounded above by a negative constant, and $F : M^m \to \mathbb{R}^n$ a Lipschitz map. Then, for every $r > 0$, $\epsilon \in (0, 1)$ and integer $k \geq 2$, there are points $p_1, \ldots, p_k \in M^m$ for which the geodesic balls $B(p_i, r)$ satisfy, for all distinct $i, j \in \{1, \ldots, k\}$:*

i) The Riemannian distance between $B(p_i, r)$ and $B(p_j, r)$ is $\geq \epsilon^{-1}$.

ii) The Euclidean distance between $F(B(p_i, r))$ and $F(B(p_j, r))$ is $\leq \epsilon$.

If the curvature of M^m lies between negative constants, then i) and iii) below hold:

iii) The Hausdorff distance between $F(B(p_i, r))$ and $F(B(p_j, r))$ is $\leq \epsilon$.

Besides its application to isometric immersions, Theorem 1.1 can also be regarded as a weak extension to Hadamard manifolds of some geometric aspects behind the classical Fatou theorem on the boundary behavior of bounded holomorphic functions on the unit disc (see Section 2).

The proof of Theorem 1.1, to be given in Section 3, is based on a careful study of the interplay between the asymptotic growth of some specially defined packings of geodesic balls in the strongly curved Hadamard manifold M^m, and the massive compression they must undergo under the action of a Lipschitz map that takes values in some Euclidean space.

The following remarks are meant to shed light on Theorem 1.1 , vis-a-vis the nature of the known examples of isometric embeddings $\mathbb{H}^m \to \mathbb{R}^n$. Since these embeddings can be proper, we single out a property stronger than properness that is never satisfied by these embeddings.

Call a map $G : M \to N$ between non-compact complete Riemannian manifolds *strongly proper* if, for every pair of sequences (x_n), (y_n) in M, the following condition is fulfilled:

$$d(x_n, y_n) \to \infty \implies \liminf d(G(x_n), G(y_n)) > 0.$$

As the terminology indicates, a strongly proper map can easily be seen to be proper in the usual sense. On the other hand, the following elementary example shows that not every proper immersion is strongly proper.

Consider $A \subset \mathbb{R}$ given by the disjoint union, over all integers $k \geq 1$, of the open intervals $(k - 1/(k+1), k + 1/(k+1))$, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $\lim_{k\to\infty} f(k) = \infty$, $f(x) > 0$ for $x \in A$, $f(x) = 0$ for $x \in \mathbb{R} - A$.

Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = f(x)$. It is easy to see that the (complete, properly embedded) graph $S \subset \mathbb{R}^3$ of g is not strongly proper.

Indeed, the points $p_k = (k - 1/(k+1), 0, 0)$ and $q_k = (k + 1/(k+1), 0, 0)$ lie in S and satisfy $||p_k - q_k|| = 2/(k+1) \to 0$. But, on the other hand, one can argue using the symmetries of S that $d_S(p_k, q_k) \geq 2f(k) \to \infty$.

The following result is a direct consequence of Theorem 1.1:

Corollary 1.2. *Let M^m be a Hadamard manifold whose curvature is bounded above by a negative constant, and $F : M^m \to \mathbb{R}^n$ a Lipschitz map. Then, for every integer $k \geq 2$ there are k sequences $(x_l^{(i)})$ in M^m, $1 \leq i \leq k$,* *such that, for all distinct $i, j \in \{1, \dots, k\}$, one has*

$$\lim_{l\to\infty} d(x_l^{(i)}, x_l^{(j)}) \to \infty \quad \textit{and} \quad \lim_{l\to\infty} ||F(x_l^{(i)}) - F(x_l^{(j)})|| = 0.$$

In particular, F is not strongly proper ($k = 2$).

Since [10] provides examples of proper isometric embeddings $\mathbb{H}^m \hookrightarrow \mathbb{R}^{6m-6}$, Corollary 1.2 implies:

Corollary 1.3. *There are examples of proper isometric immersions $F : \mathbb{H}^m \to \mathbb{R}^n$, but no such F can be strongly proper.*

Needless to say, Theorem 1.1 contains much more information about isometric immersions $\mathbb{H}^m \to \mathbb{R}^n$ than Corollary 1.3.

For the sake of completeness, we mention the well-known problem that $\mathbb{H}^m$ cannot be C^2 isometrically immersed in $\mathbb{R}^{2m-1}$, although this conjecture is not the focus of the present work (indeed, our results are valid in arbitrary codimension). For background on this problem, as well as related works, see [4], [5], [11], [15], [17] - [21]).

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2. Theorem 1.1 and complex analysis on the unit disc.

A nice illustration of Theorem 1.1, unrelated to isometric embeddings, is provided by the classical Fatou theorem [12] about the boundary behavior of bounded holomorphic functions on the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

Recall that Fatou's theorem asserts that if $f : D \to \mathbb{C}$ is a bounded holomorphic function, then there exists a subset E of the unite circle, of full measure, such that for every $\zeta \in E$ the non-tangential (angular) limit

$$\hat{f}(\zeta) := \lim_{z\to\zeta, z\in T_\zeta} f(z) \tag{2.1}$$

exists. Here, T_ζ stands for any open triangle contained in D and having ζ as a vertex. Since the holomorphic function f is bounded on D, it satisfies an estimate of the type

$$|f'(z)| \leq \frac{C}{1 - |z|}, \; |z| < 1. \tag{2.2}$$

The geometric content of (2.2) is that $f : D \to \mathbb{C}$ can be regarded as a Lipschitz map from the Poincaré disc into the flat plane $\mathbb{R}^2$. In particular, Theorem 1.1 should apply to the map f. This can be seen in a rather concrete way as follows.

Let $R > 0$ be arbitrary and $\zeta \in E$ as before. Taking the angle of T_ζ at ζ to be sufficiently close to π, one can be sure that T_ζ contains an infinite sequence of disjoint hyperbolic balls B_n of radius R, centered at suitable points in the radius $\{t\zeta, 0 < t < 1\}$.

Since the limit of f exists within T_ζ, the sets $f(B_n)$ shrink, as $n \to \infty$, to the singleton given by the non-tangential limit $\hat{f}(\zeta) \in \mathbb{C}$. In particular, for every $N \in \mathbb{N}$ there are N hyperbolic balls of radius R which are arbitrarily far from each other in the hyperbolic sense, but whose images under f are arbitrarily Hausdorff-close in $\mathbb{R}^2$, as predicted by Theorem 1.1.

3. Lipschitz maps and asymptotic densities.

This section contains the proof of Theorem 1.1, presented after some preparatory material. Given a complete non-compact m-dimensional Riemannian manifold M^m, $R \in (0, \infty)$, $C \in (0, R)$, and $p \in M^m$, denote by $\#(p, C, R; M^m)$ the maximum number of disjoint metric balls of radius C that are contained in the open ball $B(p, R)$.

Lemma 3.1. *Let M^m be a Hadamard manifold with curvature bounded away from zero. Then, for all $p \in M^m$ and $C > 0$, $\#(p, C, R; M^m)$ grows exponentially as $R \to \infty$.*

In the special case $M^m = \mathbb{H}^m$, a non-computational proof of this result can be given using the fact that there are compact hyperbolic manifolds P^m with an arbitrarily large injectivity radius, together with the exponential growth of the fundamental group of P^m. Since we were unable to locate a reference for Lemma 3.1 in the case of variable curvature, a detailed proof will be provided.

The following standard result ([2], [3]) will be used in Lemmas 3.1 and 3.4:

Lemma 3.2. *Let M^n and $\widetilde{M}^n$ be Riemannian manifolds and suppose that $\widetilde{K}_{\widetilde{x}}(\widetilde{\sigma}) \geq K_x(\sigma)$, for all $x \in M$, $\widetilde{x} \in \widetilde{M}$, $\sigma \subset T_xM$, $\widetilde{\sigma} \subset T_{\widetilde{x}}\widetilde{M}$. Let $p \in M$, $\widetilde{p} \in \widetilde{M}$ and fix a linear isometry $\iota : T_pM \to T_{\widetilde{p}}\widetilde{M}$. Let $r > 0$ such that $\exp_p|_{B_r(0)}$ is a diffeomorphism and $\exp_{\widetilde{p}}|_{\widetilde{B}_r(0)}$ is non-singular. Let $c : [0, a] \to$*

$\exp_p\big(B_r(0)\big) \subset M$ *be smooth and define* $\widetilde{c} : [0, a] \to \exp_{\tilde{p}}\big(\widetilde{B}_r(0)\big) \subset \widetilde{M}$ *by* $\widetilde{c}(s) = \exp_{\tilde{p}} \circ \iota \circ \exp_p^{-1}\big(c(s)\big)$, $s \in [0, a]$. *Then* $l(c) \geq l(\widetilde{c})$.

In order to prove Lemma 3.1 we may assume, without loss of generality, that the sectional curvature K of M^m is at most -1. As before, denote by $\mathbb{H}^m$ the m-dimensional hyperbolic space and fix $p \in M^m$, $\tilde{p} \in \mathbb{H}^m$. Since $\exp_p : T_pM \to M$, $\exp_{\tilde{p}} : T_{\tilde{p}}\mathbb{H}^m \to \mathbb{H}^m$ are diffeomorphisms, for any fixed linear isometry $\iota : T_pM \to T_{\tilde{p}}\mathbb{H}^m$ the map $\phi := \exp_{\tilde{p}} \circ \iota \circ \exp_p^{-1} : M^m \to \mathbb{H}^m$ is also a diffeomorphism. Moreover, by Lemma 3.2,

$$d_M(x, y) \geq d_{\mathbb{H}^m}\big(\phi(x), \phi(y)\big), \quad x, y \in M^m. \tag{3.1}$$

Our strategy will be to identify a suitable two-dimensional surface with the property that the maximum number of disjoint balls of radius C that are contained in $B(p, R)$, and whose centers lie in the said surface, already grows exponentially as $R \to \infty$.

Given $R > 2C$, let $\alpha \in (0, \pi/2)$ be such that

$$\sin\alpha = \frac{\sinh C}{\sinh(R - C)}\,. \tag{3.2}$$

Let k be largest positive integer that satisfies $k\alpha \leq \pi - \alpha$, so that

$$k > \frac{\pi - \alpha}{\alpha} - 1. \tag{3.3}$$

Let $u, w \in T_pM$ be orthogonal unit vectors and, for each integer $0 \leq j \leq k$, set

$$v_j = \cos(2j\alpha)u + \sin(2j\alpha)w.$$

We claim that, for all $0 \leq i < j \leq k$, the smallest angle $\angle(v_i, v_j)$ between v_i and v_j is at least 2α. In fact, taking the inner product of v_i and v_j we obtain

$$\cos\angle(v_i, v_j) = \cos(2j\alpha - 2i\alpha) = \cos\big(2\pi - (2j\alpha - 2i\alpha)\big).$$

If $2j\alpha - 2i\alpha \leq \pi$, then $\angle(v_i, v_j) = (j - i)2\alpha \geq 2\alpha$. On the other hand, if $2j\alpha - 2i\alpha > \pi$ we have

$$\angle(v_i, v_j) = 2\pi - (j - i)2\alpha \geq 2\pi - 2k\alpha.$$

That $\angle(v_i, v_j) \geq 2\alpha$ is valid also in this case is an immediate consequence of the above inequality and our choice of k.

For $0 \leq j \leq k$, consider the geodesic in M given by $\gamma_j(t) = \exp_p(tv_j)$, and let $p_j = \gamma_j(R - C)$.

Lemma 3.3. *$d_M(p_i, p_j) \geq 2C$ for all distinct i, j in $\{0, 1, ..., k\}$.*

To prove Lemma 3.3, for each j such that $0 \leq j \leq k$ consider the geodesic in $\mathbb{H}^m$ defined by $\tilde{\gamma}_j(t) = \exp_{\tilde{p}}\big(t\,\iota(v_j)\big)$, and let $\tilde{p}_j = \tilde{\gamma}_j(R - C)$. Since $\phi(p_j) = \tilde{p}_j$, it follows from (3.1) that

$$d_M(p_i, p_j) \geq d_{\mathbb{H}^m}(\tilde{p}_i, \tilde{p}_j), \quad i, j \in \{0, 1, ..., k\},\ i \neq j. \tag{3.4}$$

We can assume that $\angle(v_i, v_j)$ is strictly less than π, otherwise $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ would be opposite geodesics, and so

$$d_{\mathbb{H}^m}(\tilde{p}_i, \tilde{p}_j) = 2(R - C) > 2C.$$

At this point in the proof we need to invoke a classical formula in hyperbolic trigonometry, the so-called hyperbolic law of sines [14, p. 432]. This formula states that if a triangle in the hyperbolic plane has sides of lengths a, b, c, and the corresponding opposite angles have measures λ, μ, ν, then

$$\frac{\sinh a}{\sin \lambda} = \frac{\sinh b}{\sin \mu} = \frac{\sinh c}{\sin \nu}.$$

Let $\tilde{q}$ be the midpoint of the segment $\tilde{p}_i\tilde{p}_j$, so that the triangle $\tilde{p}\tilde{q}\tilde{p}_i$ has a right angle at $\tilde{q}$ and an angle of measure $\frac{1}{2}\angle(v_i, v_j)$ at $\tilde{p}$. Applying the hyperbolic law of sines to the triangle $\tilde{p}\tilde{q}\tilde{p}_i$, one obtains

$$\sinh\big(d_{\mathbb{H}^m}(\tilde{q}, \tilde{p}_i)\big) = \sinh(R - C)\sin\left(\frac{\angle(v_i, v_j)}{2}\right). \tag{3.5}$$

Since, by the previous claim, $\angle(v_i, v_j) \geq 2\alpha$, it follows from (3.2) and (3.5) that

$$\sinh\big(d_{\mathbb{H}^m}(\tilde{q}, \tilde{p}_i)\big) \geq \sinh(R - C)\sin\alpha = \sinh C. \tag{3.6}$$

Lemma 3.3 now follows from (3.4) and (3.6):

$$d_M(p_i, p_j) \geq d_{\mathbb{H}^m}(\tilde{p}_i, \tilde{p}_j) = 2d_{\mathbb{H}^m}(\tilde{q}, \tilde{p}_i) \geq 2C. \tag{3.7}$$

We now resume the proof of Lemma 3.1. As $d_M(p, p_j) = R - C$, it follows from the triangle inequality that $B(p_j, C) \subset B(p, R)$ for $0 \leq j \leq k$.

Moreover, by Lemma 3.3, $B(p_i,C)\cap B(p_j,C)=\emptyset$ for all distinct $i,j\in\{0,1,...,k\}$. From (3.2) and (3.3), we then obtain

$$\begin{aligned}\#(p,C,R;M^m) &\geq k+1>\frac{\pi-\alpha}{\alpha}\\ &= \frac{\sin\alpha}{\alpha}\,\frac{\pi-\alpha}{\sin\alpha}\\ &= \frac{\sin\alpha}{\alpha}\,\frac{\pi-\alpha}{\sinh C}\sinh(R-C)\\ &> \frac{1}{2}\,\frac{\sin\alpha}{\alpha}\,\frac{\pi}{\sinh C}\sinh(R-C). \end{aligned}\tag{3.8}$$

Since, by (3.2), $\alpha\to 0$ as $R\to\infty$, it follows from (3.8) that $\#(p,C,R;M^m)$ grows exponentially with R, for every $C>0$ fixed. This concludes the proof of Lemma 3.1. □

Lemma 3.4. *Let M^m be a Hadamard manifold whose sectional curvature K is bounded from below. Then, for all $\rho>0$ and $\delta>0$ there exist a positive integer $l=l(\rho,\delta)$ and maps $\sigma_1,...,\sigma_l:M\to M$ such that, for all $p\in M$,*

i) $\sigma_j(p)\in B(p,\rho)$, $1\leq j\leq l$,

ii) $B(p,\rho)\subset\bigcup_{j=1}^{l}B(\sigma_j(p),\delta)$.

To prove the lemma, set $b=\inf_M K>-\infty$ and let $\widetilde{M}$ be the complete simply-connected m-dimensional Riemannian manifold with constant sectional curvature b. Fix $\widetilde{q}\in\widetilde{M}$ and let $\widetilde{q}_1,...,\widetilde{q}_l\in B(\widetilde{q},\rho)$ be such that

$$B(\widetilde{q},\rho)\subset\bigcup_{j=1}^{l}B(\widetilde{q}_j,\delta).\tag{3.9}$$

For $q\in M$ fixed, consider orthonormal bases $\{\widetilde{v}_1,...,\widetilde{v}_m\}$ and $\{v_1,...,v_m\}$ of $T_{\widetilde{q}}\widetilde{M}$ and T_qM, respectively. For each $p\in M$, $p\neq q$, let $\{V_1(p),...,V_m(p)\}$ be the (orthonormal) basis of T_pM obtained by the parallel transport of $v_1,...,v_m$ along the (unique) geodesic joining q to p. Consider also the linear isometry $\iota_p:T_{\widetilde{q}}\widetilde{M}\to T_pM$ satisfying $\iota_p(\widetilde{v}_i)=V_i(p)$, and define a diffeomorphism $\phi_p:\widetilde{M}\to M$ by $\phi_p=\exp_p\circ\,\iota_p\circ\exp_{\widetilde{q}}^{-1}$.

For all $p\in M$ and $j\in\{1,...,l\}$, set $\sigma_j(p)=\phi_p(\widetilde{q}_j)\in M$. Since $\phi_p(B(\widetilde{q},\rho))=B(p,\rho)$, we have $\sigma_j(p)\in B(p,\rho)$ whenever $1\leq j\leq l$. Given $x\in B(p,\rho)$, we obtain from (3.9) that $\phi_p^{-1}(x)\in B(\widetilde{q}_j,\delta)$ for some j, $1\leq j\leq$

l. Applying Lemma 3.2 with the roles of M and $\widetilde{M}$ interchanged, one has

$$d_{\widetilde{M}}\big(\phi_p^{-1}(x), \phi_p^{-1}(y)\big) \geq d_M(x, y),$$

and so $d_M(x, \sigma_j(p)) \leq d_{\widetilde{M}}(\phi_p^{-1}(x), \widetilde{q}_j) < \delta$, which establishes ii). □

With these preliminaries out of the way, we are ready to begin the proof of Theorem 1.1. To this end, fix $p_0 \in M^m$ and let C be a positive number to be specified later. For each $R > C$, take a collection $\widehat{\mathcal{C}}_{R,C}$ of disjoint balls of radius C inside the ball $B(p_0, R) \subset M^m$ such that $|\widehat{\mathcal{C}}_{R,C}| = \#(p_0, C, R; M^m)$.

According to Lemma 3.1, the cardinality $|\widehat{\mathcal{C}}_{R,C}|$ grows exponentially as $R \to \infty$. In particular, one has

$$\lim_{R\to\infty} \frac{|\widehat{\mathcal{C}}_{R,C}|}{R^n} = \infty. \tag{3.10}$$

Consider, for each $R > C$, a subfamily $\mathcal{C}_{R,C}$ of $\widehat{\mathcal{C}}_{R,C}$ satisfying:

a) If $|\mathcal{C}_{R,C}| > 1$ and p, q are centers of distinct balls in $\mathcal{C}_{R,C}$, then $||F(p) - F(q)|| \geq \frac{1}{C}$.

b) $\mathcal{C}_{R,C}$ is maximal with respect to property a).

Writing $D(q, t)$ for the Euclidean ball in $\mathbb{R}^n$ of radius t and center q, we observe that if $|\mathcal{C}_{R,C}| > 1$ and $B(q_i, C)$, $B(q_j, C)$ are distinct balls in $\mathcal{C}_{R,C}$, then

$$D\left(F(q_i), \frac{1}{3C}\right) \cap D\left(F(q_j), \frac{1}{3C}\right) = \emptyset. \tag{3.11}$$

Indeed, if (3.11) were to fail, the triangle inequality would imply $||F(q_i) - F(q_j)|| < \frac{2}{3C}$, contradicting a) above.

Denote by L the Lipschitz constant of F. Since

$$||F(q_i) - F(p_0)|| \leq Ld(q_i, p_0) < LR$$

we have, for all $x \in D\big(F(q_i), 1/3C\big)$,

$$||x - F(p_0)|| \leq ||x - F(q_i)|| + ||F(q_i) - F(p_0)|| < \frac{1}{3C} + LR.$$

As a consequence,

$$\bigcup_{q_i} D\left(F(q_i), \frac{1}{3C}\right) \subset D\left(F(p_0), LR + \frac{1}{3C}\right), \tag{3.12}$$

where q_i runs over the centers of all balls in $\mathcal{C}_{R,C}$.

An individual ball $D(q,t)$ in $\mathbb{R}^n$ has volume $c(n)t^n$, the explicit value of the constant $c(n)$ being unimportant for our current purposes. The volume of each ball $D\big(F(q_i), 1/3C\big)$ is a fixed constant, say λ_0. There are $|\mathcal{C}_{R,C}|$ such balls in $\mathbb{R}^n$ and, as observed in (3.11), they are pairwise disjoint. Thus, the volume of the union in (3.12) is $\lambda_0|\mathcal{C}_{R,C}|$ and, furthermore,

$$\lambda_0|\mathcal{C}_{R,C}| \leq c(n)\big(LR + 1/3C\big)^n.$$

In particular,

$$\limsup_{R\to\infty} \frac{|\mathcal{C}_{R,C}|}{R^n} < \infty. \tag{3.13}$$

It follows from (3.10) and (3.13) that, for all sufficiently large R, say $R > R_0$, the inclusion $\mathcal{C}_{R,C} \subset \widehat{\mathcal{C}}_{R,C}$ is proper.

Consider any ball $B(p,C)$ from $\widehat{\mathcal{C}}_{R,C} - \mathcal{C}_{R,C}$. The family $\{B(p,C)\} \cup \mathcal{C}_{R,C}$ consists of disjoint balls of radius C and so, by a) and the maximality of $\mathcal{C}_{R,C}$ that was stipulated in b), one can select a (not necessarily unique) ball $B(q,C)$ in $\mathcal{C}_{R,C}$ such that

$$||F(p) - F(q)|| < \frac{1}{C}. \tag{3.14}$$

Any such assignment $B(p,C) \rightsquigarrow B(q,C)$ gives rise to a map

$$\Theta_{R,C} : \widehat{\mathcal{C}}_{R,C} - \mathcal{C}_{R,C} \to \mathcal{C}_{R,C}, \quad R > R_0.$$

We claim that when R tends to infinity, the cardinality of *some* fiber of $\Theta_{R,C}$ becomes larger than any specified integer j.

Indeed, if not,

$$\begin{aligned} |\widehat{\mathcal{C}}_{R,C}| &= |\mathcal{C}_{R,C}| + |\widehat{\mathcal{C}}_{R,C} - \mathcal{C}_{R,C}| \\ &\leq |\mathcal{C}_{R,C}| + j|\Theta_{R,C}(\widehat{\mathcal{C}}_{R,C} - \mathcal{C}_{R,C})| \\ &\leq (1+j)|\mathcal{C}_{R,C}|, \end{aligned}$$

contradicting (3.10) and (3.13).

Hence, by (3.14) and the previous assertion about the fibers of $\Theta_{R,C}$, there are points $q, p_1, \ldots, p_k \in M^m$ such that

$$\min_{1\leq i,j\leq k,\, i\neq j} d(p_i,p_j) \geq 2C \quad \text{and} \quad \max_{1\leq i\leq k} ||F(p_i)-F(q)|| < \frac{1}{C}\,\cdot$$

In particular,

$$\max_{1\leq i,j\leq k} ||F(p_i)-F(p_j)|| < \frac{2}{C}\,\cdot$$

Choosing $C = 2(2r\epsilon+1)/\epsilon$, we obtain (ii) in the statement of Theorem 1.1. To see that (i) is also valid with this choice of C, observe that, for all $x \in B(p_i, r)$ and $y \in B(p_j, r)$,

$$d(x,y) \geq d(p_i,p_j) - 2r \geq 2C - 2r > \frac{2r\epsilon+1}{\epsilon} - 2r = \frac{1}{\epsilon}\,.$$

We now move on to the second half of the theorem, and assume that the sectional curvature of M^m is bounded from above and below by negative constants.

By Lemma 3.4, with $\rho = r$ and $\delta = \frac{\epsilon}{2L}$, there exist a positive integer l and maps $\sigma_1, ..., \sigma_l : M \to M$ such that, for all $p \in M$,

$$\sigma_j(p) \in B(p,r), \;\; 1 \leq j \leq l, \tag{3.15}$$

$$B(p,r) \subset \bigcup_{j=1}^{l} B\big(\sigma_j(p), \epsilon/2L\big). \tag{3.16}$$

In order to control the Hausdorff distance, we introduce the following augmentation of the map F:

$$\widehat{F} : M^m \to \mathbb{R}^{nl}, \quad \widehat{F}(p) = \big(F(\sigma_1(p)), ..., F(\sigma_l(p))\big). \tag{3.17}$$

Fix $p_0 \in M^m$, and let C be a positive number to be specified later. As before, for each $R > C$ denote by $\widehat{\mathcal{C}}_{R,C}$ a collection of disjoint balls of radius C that are contained in the ball $B(p_0,R) \subset M^m$ and satisfy $|\widehat{\mathcal{C}}_{R,C}| = \#(p_0, C, R; M^m)$.

By Lemma 3.1, one has

$$\lim_{R\to\infty} \frac{|\widehat{\mathcal{C}}_{R,C}|}{R^{nl}} = \infty. \tag{3.18}$$

Consider, for each $R > C$, a subfamily $\widetilde{\mathcal{C}}_{R,C}$ of $\widehat{\mathcal{C}}_{R,C}$ satisfying:

a) If $|\widetilde{\mathcal{C}}_{R,C}| > 1$ and p, q are centers of distinct balls in $\widetilde{\mathcal{C}}_{R,C}$, then $||\widehat{F}(p) - \widehat{F}(q)|| \geq \frac{1}{C}$.

b) $\widetilde{\mathcal{C}}_{R,C}$ is maximal with respect to property a).

From this point on, we write $D(q,t)$ for the ball in $\mathbb{R}^{nl}$ of radius t and center q. If $|\widetilde{\mathcal{C}}_{R,C}| > 1$ and $B(q_i, C)$, $B(q_j, C)$ are distinct balls in $\widetilde{\mathcal{C}}_{R,C}$ then, by a) above,

$$D\left(\widehat{F}(q_i), \frac{1}{3C}\right) \cap D\left(\widehat{F}(q_j), \frac{1}{3C}\right) = \emptyset. \tag{3.19}$$

From

$$\begin{aligned} ||F(\sigma_s(p)) - F(\sigma_s(p_0))|| &\leq Ld(\sigma_s(p), \sigma_s(p_0)) \\ &\leq L\big[d(\sigma_s(p), p) + d(p, p_0) + d(p_0, \sigma_s(p_0))\big] \end{aligned}$$

and (3.15), one obtains

$$||F(\sigma_s(p)) - F(\sigma_s(p_0))|| < L\big(d(p, p_0) + 2r\big),$$

$$||\widehat{F}(p) - \widehat{F}(p_0)||^2 = \sum_{s=1}^{l} ||F(\sigma_s(p)) - F(\sigma_s(p_0))||^2 < lL^2\big(d(p, p_0) + 2r\big)^2.$$

Then, for all $x \in D(\widehat{F}(q_i), 1/3C))$ we have

$$||x - \widehat{F}(p_0)|| \leq ||x - \widehat{F}(q_i)|| + ||\widehat{F}(q_i) - \widehat{F}(p_0)|| < \frac{1}{3C} + \sqrt{l}L(R + 2r).$$

As a consequence,

$$\bigcup_{q_i} D\left(\widehat{F}(q_i), \frac{1}{3C}\right) \subset D\left(\widehat{F}(p_0), \sqrt{l}L(R + 2r) + \frac{1}{3C}\right), \tag{3.20}$$

where q_i runs over the centers of all balls in $\widetilde{\mathcal{C}}_{R,C}$.

Denoting by $\overline{\lambda}$ the volume of each ball $D(\widehat{F}(q_i), 1/3C)$, it follows from (3.19) and (3.20) that

$$\overline{\lambda}|\widetilde{\mathcal{C}}_{R,C}| \leq c(nl)\big(\sqrt{l}L(R+2r) + 1/3C\big)^{nl},$$

where $c(nl)$ is the volume of the unit ball in $\mathbb{R}^{nl}$. In particular,

$$\limsup_{R\to\infty} \frac{|\widetilde{\mathcal{C}}_{R,C}|}{R^{nl}} < \infty. \tag{3.21}$$

It follows from (3.18) and (3.21) that, for all sufficiently large R, say $R > R_0$, the inclusion $\widetilde{\mathcal{C}}_{R,C} \subset \widehat{\mathcal{C}}_{R,C}$ is proper. Hence, by a) and the maximality of $\widetilde{\mathcal{C}}_{R,C}$ that was stipulated in b), for any ball $B(p, C)$ from $\widehat{\mathcal{C}}_{R,C} - \widetilde{\mathcal{C}}_{R,C}$ one can select a (not necessarily unique) ball $B(q, C)$ in $\widetilde{\mathcal{C}}_{R,C}$ such that

$$||\widehat{F}(p) - \widehat{F}(q)|| < \frac{1}{C}\,. \tag{3.22}$$

Any such assignment $B(p, C) \rightsquigarrow B(q, C)$ gives rise to a map

$$\widetilde{\Theta}_{R,C} : \widehat{\mathcal{C}}_{R,C} - \widetilde{\mathcal{C}}_{R,C} \to \widetilde{\mathcal{C}}_{R,C}, \quad R > R_0.$$

If all fibers of $\widetilde{\Theta}_{R,C}$ had size at most a fixed integer j, an argument similar to the discussion following (3.14) would show that $|\widehat{\mathcal{C}}_{R,C}| \leq (1+j)\widetilde{\mathcal{C}}_{R,C}$, contradicting (3.18) and (3.21).

Hence, by (3.22) there are points $q, p_1, \ldots, p_k \in M^m$ for which

$$\min_{1\leq i,j\leq k,\ i\neq j} d(p_i, p_j) \geq 2C \quad \text{and} \quad \max_{1\leq i\leq k} ||\widehat{F}(p_i) - \widehat{F}(q)|| < \frac{1}{C}\,.$$

In particular,

$$\max_{1\leq i,j\leq k} ||\widehat{F}(p_i) - \widehat{F}(p_j)|| < \frac{2}{C}\,. \tag{3.23}$$

For $x \in B(p_i, r)$ and $y \in B(p_j, r)$, one has

$$d(x, y) \geq d(p_i, p_j) - 2r \geq 2C - 2r,$$

and so, under the hypothesis that the curvature lies between negative constants, Theorem 1.1 i) follows by choosing any C satisfying

$$C \geq \frac{2r\epsilon + 1}{2\epsilon}.$$

Given $x \in B(p_i, r)$, (3.16) implies that there is $s \in \{1, \ldots, l\}$ with $d(x, \sigma_s(p_i)) < \epsilon/2L$, so that, by the Lipschitz condition,

$$||F(x) - F(\sigma_s(p_i))|| < \frac{\epsilon}{2}.$$

Since, by (3.17) and (3.23),

$$||F(\sigma_s(p_i)) - F(\sigma_s(p_j))|| \leq ||\widehat{F}(p_i) - \widehat{F}(p_j)|| < \frac{2}{C},$$

we have

$$||F(x) - F(\sigma_s(p_j))|| < \frac{\epsilon}{2} + \frac{2}{C}. \tag{3.24}$$

Finally, choosing

$$C \geq \max\left\{\frac{2r\epsilon + 1}{2\epsilon}, \frac{4}{\epsilon}\right\}$$

one sees from (3.24) that $F(x)$ lies in the ϵ-neighborhood of the set $F(B(p_j, r))$. As $x \in B(p_i, r)$ is arbitrary, $F(B(p_i, r))$ is contained in the ϵ-neighborhood of $F(B(p_j, r))$. Theorem 1.1 iii) now follows by reversing the roles of i and j in the argument above.

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