Pure and Applied Mathematics Quarterly
Volume 12, Number 1, 33–74, 2016

Motivic Vanishing Cycles as a Motivic Measure

Valery A. Lunts and Olaf M. Schnürer

Abstract: We show that the motivic vanishing cycles introduced by J. Denef and F. Loeser give rise to a motivic measure on the Grothendieck ring of varieties over the affine line. We discuss the relation of this motivic measure to the motivic measure we constructed earlier using categories of matrix factorizations.
Keywords: Motivic vanishing cycles, motivic measure, Grothendieck ring of varieties, categorical measure, matrix factorizations, Thom-Sebastiani theorem.

1. Introduction

The motivic nearby fiber and the motivic vanishing cycles were introduced by J. Denef and F. Loeser (see [DL98, DL99, DL01, Loo02]). Let $V: X \to \mathbb{A}^1_{\mathsf{k}}$ be a morphism of k-varieties where k is an algebraically closed field of characteristic zero and X is smooth over k and connected. The motivic nearby

Received November 8, 2015.

fiber $\psi_{V,a}$ and the motivic vanishing cycles $\phi_{V,a}$ of V at a point $a \in \mathsf{k} = \mathbb{A}^1_{\mathsf{k}}(\mathsf{k})$ are elements of a localization $\mathcal{M}^{\hat{\mu}}_{|X_a|}$ of the equivariant Grothendieck ring $K_0(\mathrm{Var}^{\hat{\mu}}_{|X_a|})$ of varieties over the reduced fiber $|X_a|$ of V over a. We refer the reader to the main body of this article for precise definitions. We will often view $\psi_{V,a}$ and $\phi_{V,a}$ as elements of $\mathcal{M}^{\hat{\mu}}_{\mathsf{k}}$ in this introduction.

The motivic nearby fiber is additive on the Grothendieck group $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ of varieties over $\mathbb{A}^1_{\mathsf{k}}$, as shown by F. Bittner [Bit05] and by G. Guibert, F. Loeser and M. Merle [GLM06, Thm. 3.9]. Namely, for any $a \in \mathsf{k}$, there is a map

$$K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}) \to \mathcal{M}^{\hat{\mu}}_{\mathsf{k}}$$

of $K_0(\mathrm{Var}_{\mathsf{k}})$-modules which maps the class of a proper morphism $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ with X as above to the motivic nearby fiber $\psi_{V,a}$.

The motivic Thom-Sebastiani theorem [GLM06] is a local multiplicativity result for motivic vanishing cycles. Given another morphism $W\colon Y \to \mathbb{A}^1_{\mathsf{k}}$ as above define $V \circledast W\colon X \times Y \to \mathbb{A}^1_{\mathsf{k}}$ by $(V \circledast W)(x,y) = V(x) + W(y)$. Then the motivic Thom-Sebastiani theorem states that a certain convolution of the motivic vanishing cycles $\phi_{V,a}$ and $\phi_{W,b}$ determines some part of the motivic vanishing cycles $\phi_{V \circledast W, a+b}$ (see Theorem 4.1).

Our main result states that after small adjustments - the motivic vanishing cycles $\phi_{V,a}$ we use differ by a factor $(-1)^{\dim X}$ from the usual motivic vanishing cycles (see Remark 3.2) - the motivic vanishing cycles are both additive and multiplicative.

Theorem 1.1 (see Theorem 5.9). *There is a morphism*

$$(K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \to (\mathcal{M}^{\hat{\mu}}_{\mathsf{k}}, *) \tag{1.1}$$

*of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras - called **motivic vanishing cycles measure** - which is uniquely determined by the following property: it maps the class of each proper morphism $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ from a smooth and connected k-variety X to the sum $\sum_{a \in \mathsf{k}} \phi_{V,a}$ of its motivic vanishing cycles.*

The motivic vanishing cycles measure is a motivic measure in the sense that it is a ring morphism from some Grothendieck ring of varieties to another ring. The multiplication $*$ on the target of our measure is a convolution product whose definition is due to Looijenga and involves Fermat varieties. The multiplication $\star$ on the source is given by $[X \xrightarrow{V} \mathbb{A}^1_{\mathsf{k}}] \star [Y \xrightarrow{W} \mathbb{A}^1_{\mathsf{k}}] = [X \times Y \xrightarrow{V \circledast W} \mathbb{A}^1_{\mathsf{k}}]$. Apart from the additivity and local multiplicativity results mentioned above, the main ingredient in the proof of Theorem 1.1 is a compactification construction described in [LS16a]. In fact, we prove a slightly stronger statement in Theorem 5.9: the motivic vanishing cycles measure (1.1) comes from a morphism $(K_0(\mathrm{Var}_{\mathbb{A}^1_\mathsf{k}}), \star) \to (\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_\mathsf{k}}, \star)$ of $K_0(\mathrm{Var}_\mathsf{k})$-algebras. Let us mention that our sign adjustments are already necessary for additivity (see Remark 5.5).

In the last part of this article we compare the motivic vanishing cycles measure with a motivic measure of a completely different categorical nature (in case $\mathsf{k} = \mathbb{C}$). Mapping a projective morphism $W \colon X \to \mathbb{A}^1_\mathbb{C}$ from a smooth complex variety X to its category of matrix factorizations gives rise to a "matrix factorization" motivic measure

$$\mu \colon (K_0(\mathrm{Var}_{\mathbb{A}^1_\mathbb{C}}), \star) \to K_0(\mathrm{sat}^{\mathbb{Z}_2}_\mathbb{C})$$

as we explained in [LS16b, LS16a]. The target of this ring morphism is the Grothendieck ring of saturated differential $\mathbb{Z}_2$-graded categories. Here is our comparison result.

Theorem 1.2 (see Theorem 6.3). *We have the following commutative diagram of ring homomorphisms*

$$\begin{array}{ccc} (K_0(\mathrm{Var}_{\mathbb{A}^1_\mathbb{C}}), \star) & \xrightarrow{\mu} & K_0(\mathrm{sat}^{\mathbb{Z}_2}_\mathbb{C}) \\ \downarrow & & \downarrow{\scriptstyle \chi_{\mathrm{HP}}} \\ (\mathcal{M}^{\hat{\mu}}_\mathbb{C}, *) & \xrightarrow{\chi_c} & \mathbb{Z} \end{array}$$

where the left vertical arrow is the motivic vanishing cycles measure (1.1) *from Theorem 1.1, the lower horizontal arrow is induced by forgetting the group action and taking the Euler characteristic with compact support, and the right vertical arrow is induced by taking the Euler characteristic of periodic cyclic homology.*

The main ingredients in the proof of this theorem are the comparison between the periodic cyclic homology of the dg category of matrix factorizations of a given potential V with the vanishing cohomology of V due to A. Efimov [Efi12], and the comparison between the motivic and geometric vanishing cycles due to G. Guibert, F. Loeser and M. Merle [GLM06].

1.1. Structure of the article

§2 We remind the reader of various (equivariant) Grothendieck abelian groups of varieties and multiplications (or "convolutions") on them.

We recall Looijenga's convolution product $*$ in section 2.3 and include a direct proof of associativity (see Proposition 2.12); this reproves results of [GLM06, 5.1-5.5]. We also define a variant of Looijenga's convolution product for varieties over $\mathbb{A}^1_{\mathsf{k}}$ in section 2.4.

§3 We recall the definition of the motivic nearby fiber $\psi_{V,a}$ and the motivic vanishing cycles $\phi_{V,a}$ and show that $\phi_{V,a}$ lies in $\mathcal{M}^{\hat{\mu}}_{|\mathrm{Sing}(V)\cap X_a|}$ (see Proposition 3.4). We also show an invariance property of $\phi_{V,a}$ in Corollary 3.6.

§4 We state the motivic Thom-Sebastiani theorem [GLM06, Thm. 5.18] as Theorem 4.1 and give some corollaries. In particular, we globalize the Thom-Sebastiani theorem to Corollary 4.2.

§5 A corollary of [GLM06, Thm. 3.9] is given as Theorem 5.2. We obtain additivity of the motivic vanishing cycles in Theorem 5.3. Then we deduce our main Theorem 5.9 using the previous Thom-Sebastiani results and the compactification result stated as Proposition 5.12.

§6 We remind the reader of the categorical motivic measure in [BLL04] and its relation to the matrix factorization measure. Then we prove Theorem 6.3. We finish by giving two examples and by drawing a diagram relating the motivic measures considered in this article.

1.2. Acknowledgements

We thank Daniel Bergh, Alexander Efimov, Annabelle Hartmann, François Loeser, and Anatoly Preygel for useful discussions, and the referee for careful reading.

The first author was supported by NSA grant 43.294.03. The second author was supported by a postdoctoral fellowship of the DFG and by SFB/TR 45 of the DFG.

1.3. Conventions

We fix an algebraically closed field k of characteristic zero. By a k-variety we mean a separated reduced scheme of finite type over k. A morphism of k-varieties is a morphism of k-schemes. Let $\mathrm{Var}_{\mathsf{k}}$ be the category of k-varieties. We write $\times$ instead of $\times_{\mathrm{Spec}\,\mathsf{k}}$. By our assumptions on k, the product of two k-varieties is again reduced and hence a k-variety. If X is a scheme we denote by $|X|$ the corresponding reduced closed subscheme.

2. Grothendieck rings of varieties

2.1. Grothendieck rings of varieties over a base variety

Fix a k-variety S. By an S-variety we mean a morphism $X \to S$ of k-varieties. Let Var_S be the category of S-varieties. The Grothendieck group $K_0(\mathrm{Var}_S)$ of S-varieties is the quotient of the free abelian group on isomorphism classes $\langle X \to S\rangle$ of S-varieties $X \to S$ by the subgroup generated by the scissor expressions $\langle X \to S\rangle - \langle (X - Y) \to S\rangle - \langle Y \to S\rangle$ where $Y \subset X$ is a closed reduced subscheme. Any S-variety $X \to S$ defines an element $[X \to S]$ of $K_0(\mathrm{Var}_S)$.

Given S-varieties $X \to S$ and $Y \to S$, the composition $|X \times_S Y| \to X \times_S Y \to S$ is an S-variety; this operation turns $K_0(\mathrm{Var}_S)$ into a commutative associative ring with identity element $[S \xrightarrow{\mathrm{id}} S]$ (use [GW10, Prop. 4.34] for associativity).

Let $\mathcal{M}_S := K_0(\mathrm{Var}_S)[\mathbb{L}_S^{-1}]$ be the ring obtained from $K_0(\mathrm{Var}_S)$ by inverting $\mathbb{L}_S = [\mathbb{A}^1_S \to S]$.

We usually write $K_0(\mathrm{Var}_{\mathsf{k}})$ instead of $K_0(\mathrm{Var}_{\mathrm{Spec}\,\mathsf{k}})$, $\mathbb{L} = \mathbb{L}_{\mathsf{k}}$ instead of $\mathbb{L}_{\mathrm{Spec}\,\mathsf{k}}$, and $\mathcal{M}_{\mathsf{k}}$ instead of $\mathcal{M}_{\mathrm{Spec}\,\mathsf{k}}$.

Remark 2.1. *Note that the Grothendieck ring $K_0(\mathrm{Var}_S)$ defined here is canonically isomorphic to the Grothendieck ring defined in [NS11, 3.1], by [NS11, 3.2.2].*

2.1.1. Pullback. Let $f\colon T \to S$ be a morphism of k-varieties. Then the functor $\mathrm{Var}_S \to \mathrm{Var}_T$, $(X \to S) \mapsto (|T \times_S X| \to T \times_S X \to T)$, induces a morphism

$$f^*\colon K_0(\mathrm{Var}_S) \to K_0(\mathrm{Var}_T) \tag{2.1}$$

of commutative unital rings which satisfies $f^*(\mathbb{L}_S) = \mathbb{L}_T$ and hence induces a morphism

$$f^*\colon \mathcal{M}_S \to \mathcal{M}_T \tag{2.2}$$

of rings. If $g\colon U \to T$ is another morphism of k-varieties, we have $g^* f^* = (fg)^*$, by [GW10, Prop. 4.34].

In particular, $K_0(\mathrm{Var}_S)$ (resp. $\mathcal{M}_S$) becomes a $K_0(\mathrm{Var}_{\mathsf{k}})$-algebra (resp. $\mathcal{M}_{\mathsf{k}}$-algebra), and (2.1) (resp. (2.2)) is a morphism of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras (resp of $\mathcal{M}_{\mathsf{k}}$-algebras). Note that the obvious map defines a canonical isomorphism

$$\mathcal{M}_{\mathsf{k}} \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} K_0(\mathrm{Var}_S) \xrightarrow{\sim} \mathcal{M}_S$$

of $\mathcal{M}_{\mathsf{k}}$-algebras.

2.1.2. Pushforward. Let $f\colon T \to S$ be a morphism of k-varieties. The functor $\mathrm{Var}_T \to \mathrm{Var}_S$, $(Y \xrightarrow{y} T) \mapsto (Y \xrightarrow{fy} S)$, induces a morphism

$$f_!\colon K_0(\mathrm{Var}_T) \to K_0(\mathrm{Var}_S)$$

of $K_0(\mathrm{Var}_{\mathsf{k}})$-modules. Tensoring with $\mathcal{M}_{\mathsf{k}}$ yields a morphism

$$f_!\colon \mathcal{M}_T \to \mathcal{M}_S$$

of $\mathcal{M}_{\mathsf{k}}$-modules which sends $[Y \xrightarrow{y} T] \cdot \mathbb{L}_T^{-i}$ to $[Y \xrightarrow{fy} S] \cdot \mathbb{L}_S^{-i}$.

Remark 2.2. *The canonical isomorphisms from Remark 2.1 are compatible with pullback and pushforward, by [GW10, Prop. 4.34].*

2.2. Grothendieck rings of equivariant varieties over a base variety

For $n \in \mathbb{N}_{>0}$ let $\mu_n = \mathrm{Spec}(\mathsf{k}[x]/(x^n - 1))$ be the group k-variety of n-th roots of unity. Note that actions of μ_n on a k-variety X correspond bijectively to group morphisms $\mu_n(\mathsf{k}) \to \mathrm{Aut}_{\mathrm{Var}_{\mathsf{k}}}(X)$.

Fix a k-variety S and let $n \in \mathbb{N}_{>0}$. Recall that a good μ_n-action on a k-variety is a μ_n-action such that each $\mu_n(\mathsf{k})$-orbit is contained in an affine open subset of X. An S-variety with a good μ_n-action is an S-variety $p\colon X \to S$ together with a good μ_n-action on X. So p is μ_n-equivariant if we equip S with the trivial μ_n-action. We obtain the category $\mathrm{Var}_S^{\mu_n}$ of S-varieties with good μ_n-action.

The definition of the Grothendieck ring $K_0(\mathrm{Var}_S^{\mu_n})$ of S-varieties with good μ_n-action is evident from [GLM06, 2.2-2.5]; apart from the usual scissor relations there is another family of relations, cf. [GLM06, (2.2.1)]. Any S-variety $X \to S$ with good μ_n-action gives rise to an element $[X \to S] = [X]$ of $K_0(\mathrm{Var}_S^{\mu_n})$. The product of $[X \to S]$ and $[Y \to S]$ is the element obtained from $|X \times_S Y| \to S$ with the obvious diagonal μ_n-action. Define $\mathbb{L}_S = \mathbb{L}_{S,\mu_n} = [\mathbb{A}_S^1 \to S] \in K_0(\mathrm{Var}_S^{\mu_n})$ where μ_n acts trivially on $\mathbb{A}_S^1$. Let $\mathcal{M}_S^{\mu_n} := K_0(\mathrm{Var}_S^{\mu_n})[\mathbb{L}_S^{-1}]$.

We write $K_0(\mathrm{Var}_{\mathsf{k}}^{\mu_n})$ and $\mathcal{M}_{\mathsf{k}}^{\mu_n}$ instead of $K_0(\mathrm{Var}_{\mathrm{Spec}\,\mathsf{k}}^{\mu_n})$ and $\mathcal{M}_{\mathrm{Spec}\,\mathsf{k}}^{\mu_n}$.

If $f\colon T \to S$ is a morphism of k-varieties we obtain as above a pullback morphism $f^*\colon K_0(\mathrm{Var}_S^{\mu_n}) \to K_0(\mathrm{Var}_T^{\mu_n})$ of $K_0(\mathrm{Var}_{\mathsf{k}}^{\mu_n})$-algebras satisfying $f^*(\mathbb{L}_S) = \mathbb{L}_T$ and an induced pullback morphism $f^*\colon \mathcal{M}_S^{\mu_n} \to \mathcal{M}_T^{\mu_n}$ of $\mathcal{M}_{\mathsf{k}}^{\mu_n}$-algebras. We also have a pushforward morphism $f_!\colon K_0(\mathrm{Var}_T^{\mu_n}) \to K_0(\mathrm{Var}_S^{\mu_n})$ of $K_0(\mathrm{Var}_{\mathsf{k}}^{\mu_n})$-modules, and a pushforward morphism $f_!\colon \mathcal{M}_T^{\mu_n} \to \mathcal{M}_S^{\mu_n}$ of $\mathcal{M}_{\mathsf{k}}^{\mu_n}$-modules. For $n = 1$ we recover the notions from 2.1.

Whenever n' is a multiple of n there is a morphism $\mu_{n'} \to \mu_n$, $\lambda \mapsto \lambda^{n'/n}$, of k-group varieties inducing morphisms

$$K_0(\mathrm{Var}_S^{\mu_n}) \to K_0(\mathrm{Var}_S^{\mu_{n'}}), \tag{2.3}$$

$$\mathcal{M}_S^{\mu_n} \to \mathcal{M}_S^{\mu_{n'}}, \tag{2.4}$$

of rings. These morphism are compatible with pullback and pushforward morphisms.

In particular, $K_0(\mathrm{Var}_S^{\mu_n})$ (resp. $\mathcal{M}_S^{\mu_n}$) becomes a $K_0(\mathrm{Var}_{\mathsf{k}})$-algebra (resp. $\mathcal{M}_{\mathsf{k}}$-algebra) and the morphisms (2.3) and (2.4) are morphisms of algebras. We have a canonical isomorphism

$$\mathcal{M}_{\mathsf{k}} \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} K_0(\mathrm{Var}_S^{\mu_n}) \xrightarrow{\sim} \mathcal{M}_S^{\mu_n} \tag{2.5}$$

of $\mathcal{M}_{\mathsf{k}}$-algebras given by $\frac{r}{(\mathbb{L}_{\mathsf{k}})^n} \otimes a \mapsto \frac{ra}{(\mathbb{L}_S)^n}$.

Let $\hat{\mu}$ be the (inverse) limit of the $(\mu_n(\mathsf{k}))_{n \in \mathbb{N}_{>0}}$ with respect to the morphisms $\mu_{n'}(\mathsf{k}) \to \mu_n(\mathsf{k})$, $\lambda \mapsto \lambda^{n'/n}$, whenever n' is a multiple of n.

An S-variety with good $\hat{\mu}$-action is by definition an S-variety $X \to S$ together with a group morphism $\hat{\mu} \to \mathrm{Aut}_{\mathrm{Var}_{\mathsf{k}}}(X)$ that comes from a good μ_n-action on X, for some $n \in \mathbb{N}_{>0}$. As in [GLM06, 2.2] we obtain the category $\mathrm{Var}_S^{\hat{\mu}}$ of S-varieties with good $\hat{\mu}$-action. We define $K_0(\mathrm{Var}_S^{\hat{\mu}})$ and $\mathcal{M}_S^{\hat{\mu}}$ in the obvious way so that we have

$$K_0(\mathrm{Var}_S^{\hat{\mu}}) = \mathrm{colim}_n\, K_0(\mathrm{Var}_S^{\mu_n}),$$
$$\mathcal{M}_S^{\hat{\mu}} = \mathrm{colim}_n\, \mathcal{M}_S^{\mu_n}.$$

The Grothendieck ring $K_0(\mathrm{Var}_S^{\hat{\mu}})$ is an $K_0(\mathrm{Var}_{\mathsf{k}})$-algebra (even a $K_0(\mathrm{Var}_{\mathsf{k}}^{\hat{\mu}})$-algebra), and $\mathcal{M}_S^{\hat{\mu}}$ is a $\mathcal{M}_{\mathsf{k}}$-algebra (even a $\mathcal{M}_{\mathsf{k}}^{\hat{\mu}}$-algebra). We have

$$\mathcal{M}_{\mathsf{k}} \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} K_0(\mathrm{Var}_S^{\hat{\mu}}) \cong \mathcal{M}_S^{\hat{\mu}}$$

canonically as rings. If $f\colon T \to S$ is a morphism of k-varieties, we obtain a pullback morphism $f^*\colon K_0(\mathrm{Var}_S^{\hat{\mu}}) \to K_0(\mathrm{Var}_T^{\hat{\mu}})$ of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras and a pushforward morphism $f_!\colon K_0(\mathrm{Var}_T^{\hat{\mu}}) \to K_0(\mathrm{Var}_S^{\hat{\mu}})$ of $K_0(\mathrm{Var}_{\mathsf{k}})$-modules.

The base changes of these morphisms along the ring morphism $K_0(\mathrm{Var}_{\mathsf{k}}) \to \mathcal{M}_{\mathsf{k}}$ are denoted by the same symbols.

Instead of working with μ_n we could work more generally with $\mu_{n_1} \times \dots \mu_{n_r}$ (for $r \in \mathbb{N}$ and $n_1, \dots, n_r \in \mathbb{N}_{>0}$), and instead of $\hat{\mu}$ we could work with $\hat{\mu}^r$ (for $r \in \mathbb{N}$). We extend our notation accordingly.

Remark 2.3. *There is an alternative description of* $K_0(\mathrm{Var}_S^{\hat{\mu}^r})$ *and* $\mathcal{M}_S^{\hat{\mu}^r}$*, see the dictionary in [GLM06, 2.3-2.6]. When referring to results of [GLM06] we will usually translate them using this dictionary.*

Lemma 2.4. *Let S be a k-variety and $F \subset S$ a closed reduced subscheme with open complement U. Let $i \colon F \to S$ and $j \colon U \to S$ denote the inclusions. Then*

$$(j^*, i^*) \colon K_0(\mathrm{Var}_S^{\hat{\mu}}) \xrightarrow{\sim} K_0(\mathrm{Var}_U^{\hat{\mu}}) \times K_0(\mathrm{Var}_F^{\hat{\mu}}),$$
$$A \mapsto (j^*(A), i^*(A)),$$

is an isomorphism of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras, with inverse given by $(B, C) \mapsto j_!(B) + i_!(C)$. Similarly,

$$(j^*, i^*) \colon \mathcal{M}_S^{\hat{\mu}} \xrightarrow{\sim} \mathcal{M}_U^{\hat{\mu}} \times \mathcal{M}_F^{\hat{\mu}}$$

is an isomorphism of $\mathcal{M}_{\mathsf{k}}$-algebras.

Proof. This is obvious from the definitions. □

Remark 2.5. *Recall that* $K_0(\mathrm{Var}_S)$, $K_0(\mathrm{Var}_S^{\mu_n})$ *and* $K_0(\mathrm{Var}_S^{\hat{\mu}})$ *are* $K_0(\mathrm{Var}_{\mathsf{k}})$*-algebras whose multiplications are induced from the fiber product over S. In the rest of this article mainly the underlying* $K_0(\mathrm{Var}_{\mathsf{k}})$*-module structure on* $K_0(\mathrm{Var}_S)$, $K_0(\mathrm{Var}_S^{\mu_n})$ *and* $K_0(\mathrm{Var}_S^{\hat{\mu}})$ *will be important. Given* $(T \to \operatorname{Spec} \mathsf{k})$ *in* $\mathrm{Var}_{\mathsf{k}}$ *and* $(Z \to S)$ *in* Var_S *or* $\mathrm{Var}_S^{\mu_n}$ *or* $\mathrm{Var}_S^{\hat{\mu}}$ *it is given by*

$$[T \to \operatorname{Spec} \mathsf{k}].[Z \to S] = [T \times Z \to S].$$

In fact, we will introduce other multiplications on the $K_0(\mathrm{Var}_{\mathsf{k}})$*-modules* $K_0(\mathrm{Var}_S^{\mu_n})$ *and* $K_0(\mathrm{Var}_S^{\hat{\mu}})$ *turning them into* $K_0(\mathrm{Var}_{\mathsf{k}})$*-algebras.*

2.3. Convolution

After some preparations we define the convolution product $*$ on $K_0(\mathrm{Var}_S^{\mu_n})$ (Definition 2.10) and show that it turns $K_0(\mathrm{Var}_S^{\mu_n})$ into a $K_0(\mathrm{Var}_{\mathsf{k}})$-algebra

(Proposition 2.12). This is not a new result: see [GLM06, 5.1-5.5] and use the dictionary from Remark 2.3. Nevertheless we liked the exercise of showing associativity without using this dictionary.

Let S be a k-variety and $n \in \mathbb{N}_{>0}$. Let $p\colon Z \to S$ be an object of $\mathrm{Var}_S^{\mu_n \times \mu_n}$. We assume that $\mu_n \times \mu_n$ acts on Z from the right. The group $\mu_n \times \mu_n$ acts on the k-variety $Z \times \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ via $(z, x, y).(s, t) := (z.(s, t), s^{-1}x, t^{-1}y)$. The quotient with respect to this action is the balanced product $Z \times^{\mu_n \times \mu_n} \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ which is again a k-variety (use [SGA-1, Exp. V.1]). We equip it with the diagonal μ_n-action given by $[z, x, y].t = [z, tx, ty] = [z.(t, t), x, y]$. With the obvious morphism to S induced by p it is an object of $\mathrm{Var}_S^{\mu_n}$. Similarly, starting from the two closed S-subvarieties of $Z \times \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ defined by the equations $x^n + y^n = 1$ and $x^n + y^n = 0$, we obtain the two objects $(Z \times^{\mu_n \times \mu_n} \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}})|_{x^n+y^n=1}$ and $(Z \times^{\mu_n \times \mu_n} \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}})|_{x^n+y^n=0}$ of $\mathrm{Var}_S^{\mu_n}$.

Given $(Z \xrightarrow{p} S) \in \mathrm{Var}_S^{\mu_n \times \mu_n}$ as above define

(2.6)
$$\Psi(Z \xrightarrow{p} S) := -[(Z \times^{\mu_n \times \mu_n} \underset{x}{\mathbb{G}_{\mathrm{m}}} \times \underset{y}{\mathbb{G}_{\mathrm{m}}})|_{x^n+y^n=1} \xrightarrow{[z,x,y] \mapsto p(z)} S] + [(Z \times^{\mu_n \times \mu_n} \underset{x}{\mathbb{G}_{\mathrm{m}}} \times \underset{y}{\mathbb{G}_{\mathrm{m}}})|_{x^n+y^n=0} \xrightarrow{([z,x,y]) \mapsto p(z)} S] \in K_0(\mathrm{Var}_S^{\mu_n}).$$

Here the symbols x and y below $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ indicate that (x, y) forms a system of coordinates on $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$. Similar notation will be used below without further explanations.

Example 2.6. *Let $p\colon Z \to S$ be as above and assume that $\mu_n \times \mu_n$ acts trivially on Z. Then $Z \times^{\mu_n \times \mu_n} \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \xrightarrow{\sim} Z \times \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$, $[z, x, y] \mapsto (z, x^n, y^n)$, is an isomorphism which is μ_n-equivariant if we equip $Z \times \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ with the trivial μ_n-action. This implies $\Psi(Z \xrightarrow{p} S) = [Z \xrightarrow{p} S]$ in $K_0(\mathrm{Var}_S^{\mu_n})$ where Z is considered as a μ_n-variety over S with trivial action. In particular, we obtain $\Psi(S \xrightarrow{\mathrm{id}} S) = [S \xrightarrow{\mathrm{id}} S]$.*

Example 2.7. *Assume that $p = p_1 \times p_2\colon Z = Z_1 \times Z_2 \to S = S_1 \times S_2$ where S_1 and S_2 are k-varieties and $p_i\colon Z_i \to S_i$ is an object of $\mathrm{Var}_{S_i}^{\mu_n}$, for $i = 1, 2$. Moreover assume that the action of μ_n on Z_2 is trivial. Then $Z_1 \times Z_2 \times^{\mu_n \times \mu_n} \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \xrightarrow{\sim} (Z_1 \times^{\mu_n} \mathbb{G}_{\mathrm{m}}) \times (Z_2 \times \mathbb{G}_{\mathrm{m}})$,*

$[z_1, z_2, x, y] \mapsto ([z_1, x], z_2, y^n)$, *is an isomorphism over* S, *and we can simplify* (2.6) *to*

$$\begin{aligned}\Psi(Z_1 \times Z_2 \xrightarrow{p_1 \times p_2} S_1 \times S_2) = &- [(Z_1 \times^{\mu_n} \underset{x}{\mathbb{G}_m})|_{x^n \neq 1} \times Z_2 \to S_1 \times S_2] \\ &+ [(Z_1 \times^{\mu_n} \underset{x}{\mathbb{G}_m}) \times Z_2 \to S_1 \times S_2] \\ =&[(Z_1 \times^{\mu_n} \mu_n) \times Z_2 \to S_1 \times S_2] \\ =&[Z_1 \times Z_2 \to S_1 \times S_2].\end{aligned}$$

This example will be useful later on.

In fact, Ψ induces a morphism

$$\Psi\colon K_0(\mathrm{Var}_S^{\mu_n \times \mu_n}) \to K_0(\mathrm{Var}_S^{\mu_n}) \tag{2.7}$$

of $K_0(\mathrm{Var}_{\mathsf{k}})$-modules.

Our next aim is to prove Proposition 2.9 which will later on imply associativity of the convolution product.

Let $p\colon Z \to S$ be an object of $\mathrm{Var}_S^{\mu_n \times \mu_n \times \mu_n}$. Similarly as above we define

$$\begin{aligned}(2.8) \quad &\Psi_{123}(Z \xrightarrow{p} S) := \\ &- [(Z \times^{\mu_n \times \mu_n \times \mu_n} \underset{x_1}{\mathbb{G}_m} \times \underset{x_2}{\mathbb{G}_m} \times \underset{x_3}{\mathbb{G}_m})|_{x_1^n + x_2^n + x_3^n = 1} \xrightarrow{[z, x_1, x_2, x_3] \mapsto p(z)} S] \\ &+ [(Z \times^{\mu_n \times \mu_n \times \mu_n} \underset{x_1}{\mathbb{G}_m} \times \underset{x_2}{\mathbb{G}_m} \times \underset{x_3}{\mathbb{G}_m})|_{x_1^n + x_2^n + x_3^n = 0} \xrightarrow{([z, x_1, x_2, x_3]) \mapsto p(z)} S] \\ &\qquad \in K_0(\mathrm{Var}_S^{\mu_n})\end{aligned}$$

where the closed subvarieties of $Z \times^{\mu_n \times \mu_n \times \mu_n} \underset{x_1}{\mathbb{G}_m} \times \underset{x_2}{\mathbb{G}_m} \times \underset{x_3}{\mathbb{G}_m}$ are equipped with the μ_n-action $[z, x_1, x_2, x_3].t = [z, tx_1, tx_2, tx_3] = [z.(t,t,t), x_1, x_2, x_3]$. Again we obtain a morphism

$$\Psi_{123}\colon K_0(\mathrm{Var}_S^{\mu_n \times \mu_n \times \mu_n}) \to K_0(\mathrm{Var}_S^{\mu_n})$$

of $K_0(\mathrm{Var}_k)$-modules.

Similarly we associate to $(p\colon Z \to S) \in \mathrm{Var}_S^{\mu_n\times\mu_n\times\mu_n}$ the element

$$(2.9)\quad \Psi_{13}(Z) := \\ -[(Z \times^{\mu_n\times\{1\}\times\mu_n} \underset{x_1}{\mathbb{G}_\mathrm{m}} \times \{1\} \times \underset{x_3}{\mathbb{G}_\mathrm{m}})|_{x_1^n+x_3^n=1} \xrightarrow{[z,x_1,1,x_3]\mapsto p(z)} S] \\ +[(Z \times^{\mu_n\times\{1\}\times\mu_n} \underset{x_1}{\mathbb{G}_\mathrm{m}} \times \{1\} \times \underset{x_3}{\mathbb{G}_\mathrm{m}})|_{x_1^n+x_3^n=0} \xrightarrow{([z,x_1,1,x_3])\mapsto p(z)} S] \\ \in K_0(\mathrm{Var}_S^{\mu_n\times\mu_n}).$$

Here the $\mu_n \times \mu_n$-action is given by the two commuting μ_n-actions $[z, x_1, 1, x_3].s = [z, sx_1, 1, sx_3] = [z.(s, 1, s), x_1, 1, x_3]$ and $[z, x_1, 1, x_3].t = [z.(1, t, 1), x_1, 1, x_3]$, i.e. we have $[z, x_1, 1, x_3].(s, t) = [z.(s, t, s), x_1, 1, x_3]$. As above we obtain a morphism

$$\Psi_{13}\colon K_0(\mathrm{Var}_S^{\mu_n\times\mu_n\times\mu_n}) \to K_0(\mathrm{Var}_S^{\mu_n\times\mu_n})$$

of $K_0(\mathrm{Var}_k)$-modules. Similarly we define Ψ_{12} and Ψ_{23}.

Remark 2.8. *If $f\colon S \to S'$ is a morphism of k-varieties, all maps Ψ, Ψ_{123}, Ψ_{12}, Ψ_{13}, Ψ_{23} are compatible with $f_!$ and f^*, for example $\Psi(f_!(Z)) = f_!(\Psi(Z))$ for $Z \in K_0(\mathrm{Var}_S^{\mu_n\times\mu_n})$ and $\Psi(f^*(Z)) = f^*(\Psi(Z))$ for $Z \in K_0(\mathrm{Var}_{S'}^{\mu_n\times\mu_n})$. For $f_!$ this is obvious. For f^* one uses the fact that $Z \times \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \to Z \times^{\mu_n\times\mu_n} \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ is a $(\mu_n \times \mu_n)$-torsor and hence its pullback under the base change morphism f is again such a torsor.*

Proposition 2.9 ([GLM06, Prop. 5.5]). *We have*

$$\Psi_{123} = \Psi \circ \Psi_{13} = \Psi \circ \Psi_{12} = \Psi \circ \Psi_{23}$$

as morphisms $K_0(\mathrm{Var}_S^{\mu_n\times\mu_n\times\mu_n}) \to K_0(\mathrm{Var}_S^{\mu_n})$ of $K_0(\mathrm{Var}_\mathsf{k})$-modules.

Proof. Let $p\colon Z \to S$ be an object of $\mathrm{Var}_S^{\mu_n\times\mu_n\times\mu_n}$. It is enough to show that $\Psi_{123}(Z) = \Psi(\Psi_{13}(Z)) = \Psi(\Psi_{12}(Z)) = \Psi(\Psi_{23}(Z))$ in $K_0(\mathrm{Var}_S^{\mu_n})$. We only prove $\Psi_{123}(Z) = \Psi(\Psi_{13}(Z))$ and leave the remaining cases to the reader.

From (2.6) and (2.9) we obtain

$$(2.10)\quad \Psi(\Psi_{13}(Z)) = -\Psi([(Z \times^{\mu_n\times\{1\}\times\mu_n} \underset{x_1}{\mathbb{G}_\mathrm{m}} \times \{1\} \times \underset{x_3}{\mathbb{G}_\mathrm{m}})|_{x_1^n+x_3^n=1} \to S]) \\ + \Psi([(Z \times^{\mu_n\times\{1\}\times\mu_n} \underset{x_1}{\mathbb{G}_\mathrm{m}} \times \{1\} \times \underset{x_3}{\mathbb{G}_\mathrm{m}})|_{x_1^n+x_3^n=0} \to S]) \\ = \sum_{\delta,\varepsilon\in\{0,1\}} (-1)^{\delta+\varepsilon}[D_{\delta,\varepsilon} \to S]$$

where

$$\begin{aligned} D_{\delta,\varepsilon} &:= ((Z \times^{\mu_n \times \{1\} \times \mu_n} \underset{x_1}{\mathbb{G}_\mathrm{m}} \times \{1\} \times \underset{x_3}{\mathbb{G}_\mathrm{m}})|_{x_1^n + x_3^n = \delta} \times^{\mu_n \times \mu_n} \underset{y_1}{\mathbb{G}_\mathrm{m}} \times \underset{y_2}{\mathbb{G}_\mathrm{m}})|_{y_1^n + y_2^n = \varepsilon} \\ &= (Z \times^{\mu_n \times \mu_n} \underset{x_1}{\mathbb{G}_\mathrm{m}} \times \underset{x_3}{\mathbb{G}_\mathrm{m}} \times^{\mu_n \times \mu_n} \underset{y_1}{\mathbb{G}_\mathrm{m}} \times \underset{y_2}{\mathbb{G}_\mathrm{m}})|_{\substack{x_1^n + x_3^n = \delta, \\ y_1^n + y_2^n = \varepsilon}} \\ &= \frac{(Z \times \underset{x_1}{\mathbb{G}_\mathrm{m}} \times \underset{x_3}{\mathbb{G}_\mathrm{m}} \times \underset{y_1}{\mathbb{G}_\mathrm{m}} \times \underset{y_2}{\mathbb{G}_\mathrm{m}})|_{\substack{x_1^n + x_3^n = \delta, \\ y_1^n + y_2^n = \varepsilon}}}{\mu_n \times \mu_n \times \mu_n \times \mu_n}. \end{aligned}$$

Here, by the definitions of the quotients in (2.6) and (2.9), the quotient is formed with respect to the $(\mu_n)^{\times 4}$-action

$$(z, x_1, x_3, y_1, y_2).(s, t, u, v) = (z.(su, v, tu), s^{-1}x_1, t^{-1}x_3, u^{-1}y_1, v^{-1}y_2).$$

and $D_{\delta,\varepsilon}$ is a μ_n-variety with action

$$[z, x_1, x_3, y_1, y_2].m = [z, x_1, x_3, my_1, my_2] = [z.(m, m, m), x_1, x_3, y_1, y_2].$$

The coordinate changes $a_1 = x_1 y_1$, $a_2 = x_3 y_1$, $b = y_1$, $a_3 = y_2$ in $(\mathbb{G}_\mathrm{m})^{\times 4}$ and $s' = su$, $t' = tu$, $u = u$, $v = v$ in $(\mu_n)^{\times 4}$ show that

$$D_{\delta,\varepsilon} \cong \frac{(Z \times \underset{a_1}{\mathbb{G}_\mathrm{m}} \times \underset{a_2}{\mathbb{G}_\mathrm{m}} \times \underset{b}{\mathbb{G}_\mathrm{m}} \times \underset{a_3}{\mathbb{G}_\mathrm{m}})|_{\substack{a_1^n + a_2^n = \delta b^n, \\ b^n + a_3^n = \varepsilon}}}{\mu_n \times \mu_n \times \mu_n \times \mu_n}.$$

where the quotient is formed with respect to the $(\mu_n)^{\times 4}$-action

$$(z, a_1, a_2, b, a_3).(s', t', u, v) = (z.(s', v, t'), s'^{-1}a_1, t'^{-1}a_2, u^{-1}b, v^{-1}a_3)$$

and the μ_n-action on this quotient is given by

$$[z, a_1, a_2, b, a_3].m = [z, ma_1, ma_2, mb, ma_3] = [z.(m, m, m), a_1, a_2, b, a_3].$$

The quotient of $\underset{a_1}{\mathbb{G}_\mathrm{m}} \times \underset{a_2}{\mathbb{G}_\mathrm{m}} \times \underset{b}{\mathbb{G}_\mathrm{m}} \times \underset{a_3}{\mathbb{G}_\mathrm{m}}|_{\substack{a_1^n + a_2^n = \delta b^n, \\ b^n + a_3^n = \varepsilon}}$ under the obvious action of $\{1\} \times \{1\} \times \mu_n \times \{1\}$ on the factor $\mathbb{G}_\mathrm{m}$ with coordinate b is clearly isomorphic to

$$Q_{\delta,\varepsilon} := (\underset{a_1}{\mathbb{G}_\mathrm{m}} \times \underset{a_2}{\mathbb{G}_\mathrm{m}} \times \underset{a_3}{\mathbb{G}_\mathrm{m}})|_{\substack{a_1^n + a_2^n = \delta(\varepsilon - a_3^n), \\ a_3^n \neq \varepsilon}}$$

So we obtain

$$D_{\delta,\varepsilon} \cong Z \times^{\mu_n \times \mu_n \times \mu_n} Q_{\delta,\varepsilon}$$

where the quotient is formed with respect to the $(\mu_n)^{\times 3}$-action

$$(z, a_1, a_2, a_3).(s', t', v) = (z.(s', v, t'), s'^{-1}a_1, t'^{-1}a_2, v^{-1}a_3)$$

and the μ_n-action on this quotient is given by

$$[z, a_1, a_2, a_3].m = [z.(m, m, m), a_1, a_2, a_3] = [z, ma_1, ma_2, ma_3].$$

Continuing the computation (2.10) we obtain

$$\begin{aligned}\Psi(\Psi_{13}(Z)) = &+ [Z \times^{\mu_n\times\mu_n\times\mu_n} (\underset{a_1}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_2}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_3}{\mathbb{G}_{\mathrm{m}}})|_{\substack{a_1^n+a_2^n+a_3^n=1\\ 1\neq a_3^n}} \to S]\\ &- [Z \times^{\mu_n\times\mu_n\times\mu_n} (\underset{a_1}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_2}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_3}{\mathbb{G}_{\mathrm{m}}})|_{a_1^n+a_2^n+a_3^n=0} \to S]\\ &- [Z \times^{\mu_n\times\mu_n\times\mu_n} (\underset{a_1}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_2}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_3}{\mathbb{G}_{\mathrm{m}}})|_{\substack{a_1^n+a_2^n=0\\ 1\neq a_3^n}} \to S]\\ &+ [Z \times^{\mu_n\times\mu_n\times\mu_n} (\underset{a_1}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_2}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_3}{\mathbb{G}_{\mathrm{m}}})|_{a_1^n+a_2^n=0} \to S]\end{aligned}$$

The last two summands simplify to

$$+[Z \times^{\mu_n\times\mu_n\times\mu_n} (\underset{a_1}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_2}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_3}{\mathbb{G}_{\mathrm{m}}})|_{\substack{a_1^n+a_2^n=0\\ 1=a_3^n}} \to S].$$

The two conditions $a_1^n + a_2^n = 0$ and $1 = a_3^n$ are equivalent to the two conditions $a_1^n + a_2^n + a_3^n = 1$ and $1 = a_3^n$. Hence we can further simplify and obtain

$$\begin{aligned}\Psi(\Psi_{13}(Z)) = &+ [Z \times^{\mu_n\times\mu_n\times\mu_n} (\underset{a_1}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_2}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_3}{\mathbb{G}_{\mathrm{m}}})|_{a_1^n+a_2^n+a_3^n=1} \to S]\\ &- [Z \times^{\mu_n\times\mu_n\times\mu_n} (\underset{a_1}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_2}{\mathbb{G}_{\mathrm{m}}} \times \underset{a_3}{\mathbb{G}_{\mathrm{m}}})|_{a_1^n+a_2^n+a_3^n=0} \to S]\\ =&\Psi_{123}(Z).\end{aligned}$$

where the last equality holds by definition (2.8). □

Definition 2.10 (Convolution product). *The convolution product* $*$ *on* $K_0(\mathrm{Var}_S^{\mu_n})$ *is defined as the* $K_0(\mathrm{Var}_{\mathsf{k}})$*-linear composition*
(2.11)

$$*\colon K_0(\mathrm{Var}_S^{\mu_n}) \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} K_0(\mathrm{Var}_S^{\mu_n}) \xrightarrow{\times_S} K_0(\mathrm{Var}_S^{\mu_n\times\mu_n}) \xrightarrow{\Psi} K_0(\mathrm{Var}_S^{\mu_n})$$

where the first map $\times_S$ *is the* $K_0(\mathrm{Var}_{\mathsf{k}})$*-linear map induced by mapping a pair* (A, B) *of* S*-varieties with good* μ_n*-action to the class of the* S*-variety* $|A \times_S B|$ *with good* $(\mu_n \times \mu_n)$*-action.*

More explicitly, if $A \to S$ and $B \to S$ are S-varieties with good μ_n-action, then

$$
\begin{aligned}
[A \to S] * [B \to S] &= -\,[(|A \times_S B| \times^{\mu_n \times \mu_n} \underset{x}{\mathbb{G}_\mathrm{m}} \times \underset{y}{\mathbb{G}_\mathrm{m}})|_{x^n+y^n=1} \to S] \\
&\quad + [(|A \times_S B| \times^{\mu_n \times \mu_n} \underset{x}{\mathbb{G}_\mathrm{m}} \times \underset{y}{\mathbb{G}_\mathrm{m}})|_{x^n+y^n=0} \to S] \\
&= -\,[|(A \times_S B \times^{\mu_n \times \mu_n} \underset{x}{\mathbb{G}_\mathrm{m}} \times \underset{y}{\mathbb{G}_\mathrm{m}})|_{x^n+y^n=1}| \to S] \\
&\quad + [|(A \times_S B \times^{\mu_n \times \mu_n} \underset{x}{\mathbb{G}_\mathrm{m}} \times \underset{y}{\mathbb{G}_\mathrm{m}})|_{x^n+y^n=0}| \to S].
\end{aligned} \tag{2.12}
$$

The second equality comes from the fact that taking the reduced subscheme structure commutes with fiber products ([GW10, Prop. 4.34]) and with quotients under the action of a finite group.

Remark 2.11. *Let $A \to S$ and $B \to S$ be S-varieties with good μ_n-action, and assume that the μ_n-action on B is trivial. Similar as in Example 2.7 we deduce from* (2.12) *that*

$$
\begin{aligned}
[A] * [B] &= -[|((A \times^{\mu_n} \underset{x}{\mathbb{G}_\mathrm{m}}) \times_S (B \times \underset{y'}{\mathbb{G}_\mathrm{m}}))|_{x^n+y'=1}|] \\
&\quad + [|((A \times^{\mu_n} \underset{x}{\mathbb{G}_\mathrm{m}}) \times_S (B \times \underset{y'}{\mathbb{G}_\mathrm{m}}))|_{x^n+y'=0}|] \\
&= -[|((A \times^{\mu_n} \underset{x}{\mathbb{G}_\mathrm{m}}) \times_S B)|_{x^n \neq 1}|] + [|(A \times^{\mu_n} \mathbb{G}_\mathrm{m}) \times_S B|] \\
&= [|(A \times^{\mu_n} \mu_n) \times_S B|] = [|A \times_S B|] = [A][B].
\end{aligned}
$$

Proposition 2.12 ([GLM06, Prop. 5.2]). *Let S be a k-variety and $n \geq 1$. The convolution product $*$ turns $K_0(\mathrm{Var}_S^{\mu_n})$ into an associative commutative unital $K_0(\mathrm{Var}_\mathsf{k})$-algebra. The identity element is the class of $(\mathrm{id}_S \colon S \to S)$ where μ_n acts trivially on S. We denote this ring as $(K_0(\mathrm{Var}_S^{\mu_n}), *)$.*

Proof. Clearly, the convolution product is commutative. Remark 2.11 shows that $[\mathrm{id}_S \colon S \to S]$ is the identity with respect to the convolution product. Associativity follows from Proposition 2.9:

$$
\begin{aligned}
([A] * [B]) * [C] &= \Psi(\Psi_{12}([|A \times_S B \times_S C|]) \\
&= \Psi(\Psi_{23}([|A \times_S B \times_S C|]) = [A] * ([B] * [C]).
\end{aligned}
$$

Here we again use that passing to the reduced subscheme structure commutes with fiber products and taking quotients under the action of a finite group. □

Remark 2.13. *For $n = 1$ the convolution product $*$ on $K_0(\mathrm{Var}_S^{\mu_1})$ coincides with the product on $K_0(\mathrm{Var}_S) = K_0(\mathrm{Var}_S^{\mu_1})$, so $K_0(\mathrm{Var}_S) = (K_0(\mathrm{Var}_S^{\mu_1}), *)$ as $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras. This follows immediately from Remark 2.11.*

Let $(Z \to S) \in \mathrm{Var}_S^{\mu_n \times \mu_n}$ and assume that $n' = dn$ is a multiple of n. Then the morphism $Z \times \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \to Z \times \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$, $(z, x, y) \mapsto (z, x^d, y^d)$ defines an isomorphism

$$Z \times^{\mu_{n'} \times \mu_{n'}} \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \xrightarrow{\sim} Z \times^{\mu_n \times \mu_n} \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \tag{2.13}$$

in $\mathrm{Var}_S^{\mu_{n'}}$. This implies that Ψ is compatible with the morphisms $K_0(\mathrm{Var}_S^{\mu_n \times \mu_n}) \to K_0(\mathrm{Var}_S^{\mu_{n'} \times \mu_{n'}})$ and $K_0(\mathrm{Var}_S^{\mu_n}) \to K_0(\mathrm{Var}_S^{\mu_{n'}})$, cf. (2.3), and so is the first map in (2.11). We deduce that the obvious morphism

$$(K_0(\mathrm{Var}_S^{\mu_n}), *) \to (K_0(\mathrm{Var}_S^{\mu_{n'}}), *)$$

is a map of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras. Hence convolution turns $K_0(\mathrm{Var}_S^{\hat{\mu}})$ into an associative commutative unital $K_0(\mathrm{Var}_{\mathsf{k}})$-algebra; we denote this algebra by $(K_0(\mathrm{Var}_S^{\hat{\mu}}), *)$.

If $f \colon T \to S$ is a morphism of k-varieties, the pullback maps $f^* \colon (K_0(\mathrm{Var}_S^{\mu_n}), *) \to (K_0(\mathrm{Var}_T^{\mu_n}), *)$ and $f^* \colon (K_0(\mathrm{Var}_S^{\hat{\mu}}), *) \to (K_0(\mathrm{Var}_T^{\hat{\mu}}), *)$ are maps of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras (use Remark 2.8 and that the first map in (2.11) is compatible with pullbacks).

We also want to define a convolution product on $\mathcal{M}_S^{\mu_n}$ and $\mathcal{M}_S^{\hat{\mu}}$.

Consider the localization of $(K_0(\mathrm{Var}_S^{\mu_n}), *)$ at the multiplicative set $\{1, \mathbb{L}_S, \mathbb{L}_S * \mathbb{L}_S, \dots\}$. The n-fold convolution product of $\mathbb{L}_S = [\mathbb{A}^1_S]$ with itself is $[\mathbb{A}^n_S]$ and we have $[A] * [\mathbb{A}^n_S] = [A][\mathbb{A}^n_S]$ for $[A] \in K_0(\mathrm{Var}_S^{\mu_n})$, by Remark 2.11. Hence the underlying abelian group of this localization is canonically identified with the underlying abelian group of $\mathcal{M}_S^{\mu_n}$. We can therefore denote the above localization by $(\mathcal{M}_S^{\mu_n}, *)$.

Because the structure morphism $K_0(\mathrm{Var}_{\mathsf{k}}) \to (K_0(\mathrm{Var}_S^{\mu_n}), *)$ sends $\mathbb{L}_{\mathsf{k}}$ to $\mathbb{L}_S$ we obtain a canonical isomorphism

$$\mathcal{M}_{\mathsf{k}} \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} (K_0(\mathrm{Var}_S^{\mu_n}), *) \xrightarrow{\sim} (\mathcal{M}_S^{\mu_n}, *) \tag{2.14}$$

of $\mathcal{M}_{\mathsf{k}}$-algebras which we will often treat as an equality in the following. Its underlying morphism of $\mathcal{M}_{\mathsf{k}}$-modules coincides with (2.5).

Similarly, we define the convolution product $*$ on $\mathcal{M}_S^{\hat{\mu}}$ and obtain the $\mathcal{M}_{\mathsf{k}}$-algebra $\mathcal{M}_{\mathsf{k}} \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} (K_0(\mathrm{Var}_S^{\hat{\mu}}), *) = (\mathcal{M}_S^{\hat{\mu}}, *)$. The map Ψ from (2.7) gives in the obvious way rise to a morphism

$$\Psi \colon \mathcal{M}_S^{\hat{\mu}\times\hat{\mu}} \to \mathcal{M}_S^{\hat{\mu}}$$

of $\mathcal{M}_{\mathsf{k}}$-modules; the convolution product $*$ on $\mathcal{M}_S^{\hat{\mu}}$ is then given as the composition

$$*\colon \mathcal{M}_S^{\hat{\mu}} \otimes_{\mathcal{M}_{\mathsf{k}}} \mathcal{M}_S^{\hat{\mu}} \xrightarrow{\times_S} \mathcal{M}_S^{\hat{\mu}\times\hat{\mu}} \xrightarrow{\Psi} \mathcal{M}_S^{\hat{\mu}}.$$

Given $f\colon T \to S$ as above, we obtain pullback maps $f^*\colon (\mathcal{M}_S^{\mu_n}, *) \to (\mathcal{M}_T^{\mu_n}, *)$ and $f^*\colon (\mathcal{M}_S^{\hat{\mu}}, *) \to (\mathcal{M}_T^{\hat{\mu}}, *)$ of $\mathcal{M}_{\mathsf{k}}$-algebras. Under the isomorphisms (2.14) they are just obtained by scalar extension along $K_0(\mathrm{Var}_{\mathsf{k}}) \to \mathcal{M}_{\mathsf{k}}$ from the previous pullback maps.

2.4. Convolution of varieties over $\mathbb{A}^1_{\mathsf{k}}$

We now use that $\mathbb{A}^1_{\mathsf{k}}$ is a commutative group k-variety. Let $\mathrm{add}\colon \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}$, $(x, y) \mapsto x + y$, be the addition morphism. Let $n \geq 1$.

Definition 2.14 (Convolution over $\mathbb{A}^1_{\mathsf{k}}$). *The convolution product $\star$ on $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n})$ is defined as the $K_0(\mathrm{Var}_{\mathsf{k}})$-linear composition*

$$(2.15)\quad \star\colon K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n}) \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n}) \xrightarrow{\times} K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}\times\mathbb{A}^1_{\mathsf{k}}}^{\mu_n\times\mu_n}) \xrightarrow{\mathrm{add}_!} K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n\times\mu_n}) \xrightarrow{\Psi} K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n})$$

where the first map $\times$ is the $K_0(\mathrm{Var}_{\mathsf{k}})$-linear map induced by mapping a pair (A, B) of $\mathbb{A}^1_{\mathsf{k}}$-varieties with good μ_n-action to the S-variety $A \times B$ with good $(\mu_n \times \mu_n)$-action.

By Remark 2.8 we have

$$A \star B = \Psi(\mathrm{add}_!(A \times B)) = \mathrm{add}_!(\Psi(A \times B))$$

for A, $B \in K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n})$.

Remark 2.15. *Let $A \xrightarrow{\alpha} \mathbb{A}^1_{\mathsf{k}}$ and $B \xrightarrow{\beta} \mathbb{A}^1_{\mathsf{k}}$ be $\mathbb{A}^1_{\mathsf{k}}$-varieties with good μ_n-action, and assume that the μ_n-action on B is trivial. Then Example 2.7* *implies that*

$$[A \xrightarrow{\alpha} \mathbb{A}^1_{\mathsf{k}}] \star [B \xrightarrow{\beta} \mathbb{A}^1_{\mathsf{k}}] = [A \times B \xrightarrow{\alpha \circledast \beta} \mathbb{A}^1_{\mathsf{k}}]$$

where $(\alpha \circledast \beta)(a,b) = \alpha(a) + \beta(b)$*; the* μ_n*-action on* $A \times B$ *is the obvious diagonal action* $(a,b).t = (a.t, b.t) = (a.t, b)$.

Remark 2.16. *In the case* $n = 1$ *the convolution product* $\star$ *on* $K_0(\mathrm{Var}^{\mu_1}_{\mathbb{A}^1_{\mathsf{k}}}) = K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ *satisfies*

$$[A \xrightarrow{\alpha} \mathbb{A}^1_{\mathsf{k}}] \star [B \xrightarrow{\beta} \mathbb{A}^1_{\mathsf{k}}] = [A \times B \xrightarrow{\alpha \circledast \beta} \mathbb{A}^1_{\mathsf{k}}]$$

where $A \xrightarrow{\alpha} \mathbb{A}^1_{\mathsf{k}}$ *and* $B \xrightarrow{\beta} \mathbb{A}^1_{\mathsf{k}}$ *are* $\mathbb{A}^1_{\mathsf{k}}$*-varieties. This is a special case of Remark 2.15.*

Proposition 2.17. *The convolution product* $\star$ *turns* $K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}})$ *into an associative commutative* $K_0(\mathrm{Var}_{\mathsf{k}})$*-algebra with identity element* $[\mathrm{Spec}\, k \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]$. *We denote this ring by* $(K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}), \star)$.

Proof. Commutativity follows from commutativity of $\mathbb{A}^1_{\mathsf{k}}$. That $[\mathrm{Spec}\, k \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]$ is the identity element with respect to $\star$ follows from Remark 2.15. Denote the morphism $(\mathbb{A}^1_{\mathsf{k}})^{\times 3} \to \mathbb{A}^1_{\mathsf{k}}$, $(x,y,z) \mapsto x+y+z$ by addd. Remark 2.8 and Proposition 2.9 show that

$$\begin{aligned}(A \star B) \star C &= \mathrm{add}_!(\Psi(\mathrm{add}_!(\Psi(A \times B)) \times C)) \\ &= \mathrm{add}_!((\mathrm{add} \times \mathrm{id})_!(\Psi(\Psi(A \times B) \times C))) \\ &= \mathrm{addd}_!(\Psi(\Psi_{12}(A \times B \times C))) \\ &= \mathrm{addd}_!(\Psi_{123}(A \times B \times C))\end{aligned}$$

A similar computation shows that the last term equals $A \star (B \star C)$. This proves associativity. □

Mapping a k-variety $(A \to \mathrm{Spec}\, \mathsf{k})$ to $(A \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}})$ induces a morphism of unital rings

$$K_0(\mathrm{Var}_{\mathsf{k}}) \to (K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \tag{2.16}$$

as follows immediately from Remark 2.15. This map is the structure map of the $K_0(\mathrm{Var}_{\mathsf{k}})$-algebra $(K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}), \star)$. Denote the image of $\mathbb{L}_{\mathsf{k}}$ under this map by

$$\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)} := [\mathbb{A}^1_{\mathsf{k}} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}] \in K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}). \tag{2.17}$$

Let us denote the localization of $(K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}), \star)$ with respect to the multiplicative set $\{1, \mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)}, \mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)} \star \mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)}, \dots\}$ by $(\tilde{\mathcal{M}}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}, \star)$. Then there is a canonical isomorphism

$$\mathcal{M}_{\mathsf{k}} \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} (K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \xrightarrow{\sim} (\tilde{\mathcal{M}}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}, \star)$$

of $\mathcal{M}_{\mathsf{k}}$-algebras given by $\frac{r}{(\mathbb{L}_{\mathsf{k}})^n} \otimes a \mapsto \frac{ra}{(\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)})^n}$. If we compare with the isomorphism (2.5) we see that $\frac{b}{(\mathbb{L}_{\mathbb{A}^1_{\mathsf{k}}})^n} \mapsto \frac{b}{(\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)})^n}$ defines an isomorphism

$$\mathcal{M}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}} \xrightarrow{\sim} \tilde{\mathcal{M}}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}} \tag{2.18}$$

of $\mathcal{M}_{\mathsf{k}}$-modules.

Similarly as above (cf. the reasoning around (2.13)), the various $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras $(K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}), \star)$ for $n \geq 1$ are compatible. Hence we obtain the $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras $(K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}), \star)$ and

$$\mathcal{M}_{\mathsf{k}} \otimes_{K_0(\mathrm{Var}_{\mathsf{k}})} (K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \xrightarrow{\sim} (\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}, \star)$$

and an isomorphism

$$\mathcal{M}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}} \xrightarrow{\sim} \tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}} \tag{2.19}$$

of $\mathcal{M}_{\mathsf{k}}$-modules.

Lemma 2.18. *Let $\varepsilon\colon \mathbb{A}^1_{\mathsf{k}} \to \operatorname{Spec} \mathsf{k}$ be the structure morphism. Then mapping an object $(A \xrightarrow{\alpha} \mathbb{A}^1_{\mathsf{k}}) \in \mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}$ to $(A \xrightarrow{\varepsilon\alpha} \operatorname{Spec} \mathsf{k}) \in \mathrm{Var}^{\mu_n}_{\mathsf{k}}$ induces a morphism*

$$\varepsilon_!\colon (\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}, \star) \to (\mathcal{M}^{\hat{\mu}}_{k}, *) \tag{2.20}$$

of $\mathcal{M}_{\mathsf{k}}$-algebras.

Proof. Certainly we have a morphism

$$\varepsilon_!\colon K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}}) \to K_0(\mathrm{Var}^{\mu_n}_{k}) \tag{2.21}$$

of $K_0(\mathrm{Var}_{\mathsf{k}})$-modules. If Z is a k-variety we denote its structure morphism $Z \to \operatorname{Spec} \mathsf{k}$ by ε^Z. Let $A, B \in K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}})$. Since $\varepsilon_!(A)$ and $\varepsilon_!(B)$ are in

$K_0(\mathrm{Var}_k^{\mu_n})$, $\varepsilon_!(A) * \varepsilon_!(B)$ is defined using the fiber product over k. Using Remark 2.8 we obtain

$$\varepsilon_!(A \star B) = \varepsilon_!(\mathrm{add}_!(\Psi(A \times B))) = \varepsilon_!^{\mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}}(\Psi(A \times B)) = \Psi(\varepsilon_!^{\mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}}(A \times B))$$
$$= \Psi(\varepsilon_!(A) \times \varepsilon_!(B)) = \varepsilon_!(A) * \varepsilon_!(B).$$

Clearly, (2.21) maps $[\mathrm{Spec}\,\mathsf{k} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]$ to $[\mathrm{Spec}\,\mathsf{k} \to \mathrm{Spec}\,\mathsf{k}]$. Therefore it is a morphism of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras $\varepsilon_! \colon (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n}), \star) \to (K_0(\mathrm{Var}_k^{\mu_n}), *)$. We can pass to $\hat{\mu}$. Then base change along $K_0(\mathrm{Var}_{\mathsf{k}}) \to \mathcal{M}_{\mathsf{k}}$ (or noting that $\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)}$ goes to $\mathbb{L}_{\mathrm{Spec}\,\mathsf{k}}$) yields a morphism $\varepsilon_! \colon (\tilde{\mathcal{M}}_{\mathbb{A}^1_{\mathsf{k}}}^{\mu_n}, \star) \to (\mathcal{M}_k^{\mu_n}, *)$ of $\mathcal{M}_{\mathsf{k}}$-algebras. The lemma follows. □

3. Motivic vanishing cycles

Let X be a smooth connected (nonempty) k-variety and let $V \colon X \to \mathbb{A}^1_{\mathsf{k}}$ be a morphism. Given $a \in \mathsf{k} = \mathbb{A}^1_{\mathsf{k}}(\mathsf{k})$ we denote by X_a the scheme theoretic fiber of V over a.

We quickly review the definition of the motivic vanishing cycles. For details we refer to [GLM06, Sect. 3]; note however that we use slightly different signs, see Remark 3.2 below. Following Denef and Loeser, the **motivic zeta function of V at a** is a certain power series

$$Z_{V,a}(T) \in \mathcal{M}^{\hat{\mu}}_{|X_a|}[[T]]$$

whose coefficients are defined using arc spaces, see [GLM06, (3.2.2)]. It is possible to evaluate $Z_{V,a}$ at $T = \infty$. This is clear if V is constant because then $Z_{V,a} = 0$. If V is not equal to a there is a formula expressing $Z_{V,a}$ in terms of an embedded resolution of $|X_a| \subset X$ which makes it evident that the evaluation at $T = \infty$ exists.

The **motivic nearby fiber $\psi_{V,a}$ of V at a** is defined to be the negative of this value at infinity, i. e.

$$\psi_{V,a} := -Z_{V,a}(\infty) \in \mathcal{M}^{\hat{\mu}}_{|X_a|}.$$

See (3.3) below for a formula for $\psi_{V,a}$ in terms of an embedded resolution. The **motivic vanishing cycles of V at a** are defined by

$$\phi_{V,a} := [|X_a| \xrightarrow{\mathrm{id}} |X_a|] - \psi_{V,a} \in \mathcal{M}^{\hat{\mu}}_{|X_a|}. \tag{3.1}$$

Here $|X_a|$ is endowed with the trivial $\hat{\mu}$-action.

Remark 3.1. *If V is constant we have $\psi_{V,a} = 0$. If V is constant $\neq a$ we have $\phi_{V,a} = 0$. If V is constant $= a$ we have $X = |X_a|$ and $\phi_{V,a} = [X \xrightarrow{\text{id}} X]$.*

Remark 3.2. *Denef and Loeser choose different signs in the definition of the motivic vanishing cycles. In [GLM06], the motivic nearby fiber (resp. motivic vanishing cycles) of V at 0 is denoted $\mathcal{S}_V$ (resp. $\mathcal{S}_V^{\phi}$). They are related to our definitions by*

$$\begin{aligned}\psi_{V,a} &= \mathcal{S}_{V-a},\\ \phi_{V,a} &= (-1)^{\dim X} \mathcal{S}^{\phi}_{V-a}.\end{aligned}$$

Our sign choice for the motivic vanishing cycles is justified in Remark 5.5.

Let $\mathrm{Sing}(V) \subset X$ be the closed subscheme defined by the vanishing of the section $dV \in \Gamma(X, \Omega^1_{X/\mathsf{k}})$ of the cotangent bundle. The closed points of $\mathrm{Sing}(V)$ are the critical points of V. Let $\mathrm{Crit}(V) = V(\mathrm{Sing}(V)(\mathsf{k})) \subset \mathbb{A}^1(\mathsf{k}) = \mathsf{k}$ be the set of critical values of W; it is finite by generic smoothness on the target. Trivially we have $\mathrm{Sing}(V) \cap X_a = \emptyset$ if a is not a critical value.

If Z is a scheme locally of finite type over k we denote its open subscheme consisting of regular points by Z^{reg}. The closed subset $Z^{\mathrm{sing}} \subset Z$ of singular points has a unique structure of a reduced closed subscheme of Z, denoted by $|Z^{\mathrm{sing}}|$.

Remark 3.3. *If $V = a$ then $\mathrm{Sing}(V) \cap X_a = X$ and $(X_a)^{\mathrm{sing}} = \emptyset$. Otherwise the singular points of X_a are precisely the elements of the scheme-theoretic intersection $\mathrm{Sing}(V) \cap X_a$, i. e. we have the equality*

$$|\mathrm{Sing}(V) \cap X_a| = |(X_a)^{\mathrm{sing}}| \tag{3.2}$$

of k-varieties. This is trivial if V is constant $\neq a$, and otherwise it follows by considering Jacobian matrices.

Let us prove that the motivic vanishing cycles $\phi_{V,a}$ live over $|\mathrm{Sing}(V) \cap X_a|$.

Proposition 3.4. *We have $\phi_{V,a} \in \mathcal{M}^{\hat{\mu}}_{|\mathrm{Sing}(V) \cap X_a|}$ canonically.*

Therefore we will often view the motivic vanishing cycles $\phi_{V,a}$ as an element of $\mathcal{M}^{\hat{\mu}}_{|(\mathrm{Sing}(V) \cap X_a|}$ in the following.

Proof. If V is constant this follows directly from Remarks 3.1 and 3.3

So let us assume that V is not constant. As in [DL01, 3.3] let $h\colon R \to X$ be an embedded resolution of $|X_a|$ so that the ideal sheaf of $h^{-1}(|X_a|)$ is the ideal sheaf of a simple normal crossing divisor (cf. [Kol07, Thm. 3.26] or [Vil05, Thm. 2.2] for existence). Let $E = h^{-1}(X_a)$ be the divisor on R defined by $V \circ h$. Let $(E_i)_{i \in \mathrm{Irr}(|E|)}$ denote the irreducible components of $|E|$. Then $E = \sum_{i \in \mathrm{Irr}(|E|)} m_i E_i$ for unique $m_i \in \mathbb{N}_+$. Let $I \subset \mathrm{Irr}(|E|)$ be given. Define $E_I := \cap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus \bigcup_{j \in \mathrm{Irr}(|E|)\setminus I} E_j$. Let m_I be the greatest common divisor of the m_i for $i \in I$. Then Denef and Loeser define an unramified Galois cover $\widetilde{E}_I^\circ \to E_I^\circ$ with Galois group μ_{m_I}. They establish the formula

$$\psi_{V,a} = \mathcal{S}_{V-a} = \sum_{\emptyset \neq I \subset \mathrm{Irr}(|E|)} (1-\mathbb{L})^{|I|-1}[\widetilde{E}_I^\circ \to |X_a|], \tag{3.3}$$

see [DL01, Sect. 3.3 and Def. 3.5.3].

Note that h induces an isomorphism $h^{-1}(U) \xrightarrow{\sim} U$ where $U := X - |X_a|^{\mathrm{sing}}$, by part (ii) of [Vil05, Thm. 2.2]. We can also deduce this from principalization [Kol07, Thm. 3.26] as follows. Since $|X_a|$ has codimension one and h is a composition of blow-ups in smooth centers of codimension two and higher, h is an isomorphism over an open neighborhood of some regular point of $|X_a|$ if $|X_a|$ is non-empty. Since principalization is functorial with respect to étale morphisms, h is an isomorphism over all regular points of $|X_a|$.

We obviously have open embeddings $(X_a)^{\mathrm{reg}} = |(X_a)^{\mathrm{reg}}| \subset |X_a|^{\mathrm{reg}} \subset |X_a|$ and hence a closed embedding $||X_a|^{\mathrm{sing}}| \subset |(X_a)^{\mathrm{sing}}|$. Let $U' := X - (X_a)^{\mathrm{sing}} \subset U$, so $h^{-1}(U') \xrightarrow{\sim} U'$ is an isomorphism. Over $X_a \cap U' = (X_a)^{\mathrm{reg}}$ we obtain the isomorphism

$$h\colon E \cap h^{-1}(U') \xrightarrow{\sim} (X_a)^{\mathrm{reg}}, \tag{3.4}$$

so $E \cap h^{-1}(U')$ is regular.

If $|I| \geq 2$, then every element $e \in E_I$ lies in $|E|^{\mathrm{sing}} \subset E^{\mathrm{sing}}$, so $e \notin E \cap h^{-1}(U')$ and hence $h(e) \in (X_a)^{\mathrm{sing}}$. Therefore $\widetilde{E}_I^\circ \to |X_a|$ factors as $\widetilde{E}_I^\circ \to |(X_a)^{\mathrm{sing}}| \subset |X_a|$.

If $r\colon (X_a)^{\mathrm{reg}} = |(X_a)^{\mathrm{reg}}| \to |X_a|$ is the inclusion we hence obtain

$$r^*(\psi_{V,a}) = \sum_{i \in \mathrm{Irr}(|E|)} [\widetilde{E}_i^\circ|_{E_i^\circ \cap h^{-1}(U')} \to (X_a)^{\mathrm{reg}}].$$

If $E_i^\circ \cap h^{-1}(U')$ is nonempty then $m_i = 1$ because $E \cap h^{-1}(U')$ is reduced, so $\widetilde{E}_i^\circ \to E_i^\circ$ is an isomorphism. Moreover, $E \cap h^{-1}(U')$ is the disjoint union of the $E_i^\circ \cap h^{-1}(U')$, for $i \in \mathrm{Irr}(|E|)$. These facts and the isomorphism (3.4) imply that

$$r^*(\psi_{V,a}) = [(X_a)^{\mathrm{reg}} \xrightarrow{\mathrm{id}} (X_a)^{\mathrm{reg}}].$$

Hence $r^*(\phi_{V,a}) = 0$ by definition (3.1). The decomposition $(X_a)^{\mathrm{reg}} \subset X_a \supset |(X_a)^{\mathrm{sing}}|$ of X_a into an open and a closed reduced subscheme gives rise to a similar decomposition $(X_a)^{\mathrm{reg}} = |(X_a)^{\mathrm{reg}}| \subset |X_a| \supset |(X_a)^{\mathrm{sing}}|$ of $|X_a|$. Hence Lemma 2.4 shows that $\phi_{V,a} \in \mathcal{M}^{\hat{\mu}}_{|(X_a)^{\mathrm{sing}}|}$. Since V is not constant, (3.2) holds true. □

Corollary 3.5. *If V is not constant and X_a is smooth then $\phi_{V,a} = 0$.*

Proof. In this case we have $|\mathrm{Sing}(V) \cap X_a| = |(X_a)^{\mathrm{sing}}| = \emptyset$ by (3.2). More directly, we can take $h = \mathrm{id}$ as an embedded resolution in the above proof and obtain $\psi_{V,a} = [|X_a| \xrightarrow{\mathrm{id}} |X_a|]$ from (3.3) and hence $\phi_{V,a} = 0$. □

Corollary 3.6. *Assume that X is a dense open subset of a smooth k-variety X' and that $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ extends to a morphism $V'\colon X' \to \mathbb{A}^1_{\mathsf{k}}$ such that all critical points of V' are contained in X, i. e.* $\mathrm{Sing}(V') = \mathrm{Sing}(V') \cap X = \mathrm{Sing}(V)$*. Then $\phi_{V,a} = \phi_{V',a}$.*

Proof. If V is constant then V' is constant and $X = \mathrm{Sing}(V) = \mathrm{Sing}(V') = X'$, so the claim is trivial.

Assume that V is not constant. Then we can assume that the embedded resolution $h\colon R \to X$ of $|X_a|$ from the proof of Proposition 3.4 is the restriction to X of a similar embedded resolution $h'\colon R' \to X'$ of $|X'_a|$. Let $s\colon |(X_a)^{\mathrm{sing}}| \to |X_a|$ and $s'\colon |(X'_a)^{\mathrm{sing}}| \to |X'_a|$ denote the closed embeddings. Then $\phi_{V,a} = s_! s^*(\phi_{V,a})$ by (the proof of) Proposition 3.4 and

$$s^*\phi_{V,a} = [|(X_a)^{\mathrm{sing}}| \to |(X_a)^{\mathrm{sing}}|] - s^*(\psi_{V,a})$$

by definition (3.1). Similarly, we have $\phi_{V',a} = s'_! s'^*(\phi_{V',a})$ and

$$s'^*\phi_{V',a} = [|(X'_a)^{\mathrm{sing}}| \to |(X'_a)^{\mathrm{sing}}|] - s'^*(\psi_{V',a}).$$

By assumption and Remark 3.3 we have $|(X_a)^{\mathrm{sing}}| = |\mathrm{Sing}(V) \cap X_a| = |\mathrm{Sing}(V') \cap X'_a| = |(X'_a)^{\mathrm{sing}}|$. Therefore it is enough to show that $s^*(\psi_{V,a}) = s'^*(\psi_{V',a})$. But both expressions have explicit formulas obtained from equation (3.3); these expressions coincide since the Galois coverings $\widetilde{E}_I^\circ \to E_I^\circ$

and $\widetilde{E}'^{\circ}_{I'} \to E'^{\circ}_{I'}$ constructed for $h\colon R \to X$ and $h'\colon R' \to X'$ are compatible and give rise to isomorphic varieties with $\hat{\mu}$-action over $|(X_a)^{\mathrm{sing}}| = |(X'_a)^{\mathrm{sing}}|$. $\square$

4. Motivic Thom-Sebastiani theorem

Let $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ be as in the previous section 3. Let Y be a smooth connected (nonempty) k-variety with a morphism $W\colon Y \to \mathbb{A}^1_{\mathsf{k}}$. We define $V \circledast W$ to be the composition

$$V \circledast W\colon X \times Y \xrightarrow{V \times W} \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} \xrightarrow{+} \mathbb{A}^1_{\mathsf{k}}.$$

Theorem 4.1 (Motivic Thom-Sebastiani theorem, [GLM06, Thm. 5.18]). *Consider morphisms $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ and $W\colon Y \to \mathbb{A}^1_{\mathsf{k}}$ as above and let $a, b \in \mathsf{k}$. Let $i_{a,b}$ be the inclusion $|X_a| \times |Y_b| \to |(X \times Y)_{a+b}|$. Then*

$$i^*_{a,b}(\phi_{V \circledast W, a+b}) = \Psi(\phi_{V,a} \times \phi_{W,b}) \tag{4.1}$$

in $\mathcal{M}^{\hat{\mu}}_{|X_a| \times |Y_b|}$ where $\phi_{V,a} \times \phi_{W,b}$ is the obvious element of $\mathcal{M}^{\hat{\mu} \times \hat{\mu}}_{|X_a| \times |Y_b|}$.

Proof. Using Remark 3.2 this is precisely [GLM06, Thm. 5.18].

We would like to emphasize that (4.1) also holds if V or W is constant. Assume first that both V and W are constant; if $V = a$ and $W = b$ then both sides of (4.1) are equal to $[X \times Y \xrightarrow{\mathrm{id}} X \times Y]$ (use Remark 3.1 and Example 2.7); otherwise both sides are zero.

Now assume that V is not constant but W is. If $W \neq b$ then again both sides of (4.1) are zero. If $W = b$ choose an embedded resolution of $|X_a| \subset X$ as in the proof of Proposition 3.4 and obtain an explicit expression for $\phi_{V,a}$. If we take the product of this embedded resolution with $Y = Y_b$ we obtain an embedded resolution of $|(X \times Y)_{a+b}| = |X_a| \times Y \subset X \times Y$ and an explicit expression for $\phi_{V \circledast W, a+b}$. Now use again Remark 3.1 and Example 2.7. $\square$

We want to globalize this theorem. Since the set $\mathrm{Crit}(V) \subset \mathsf{k}$ of critical values of V is finite, we have

$$|\mathrm{Sing}(V)| = \coprod_{a \in \mathrm{Crit}(V)} |\mathrm{Sing}(V) \cap X_a|. \tag{4.2}$$

Proposition 3.4 shows that we can view $\phi_{V,a}$ as an element of $\mathcal{M}^{\hat{\mu}}_{|\mathrm{Sing}(V)|}$. Define

$$\widetilde{\phi}_V := \sum_{a \in \mathrm{Crit}(V)} \phi_{V,a} \in \mathcal{M}^{\hat{\mu}}_{|\mathrm{Sing}(V)|}.$$

Of course we could have equivalently taken the sum over all $a \in \mathsf{k}$, by Corollary 3.5 and Remark 3.1.

We obviously have

$$\mathrm{Sing}(V \circledast W) = \mathrm{Sing}(V) \times \mathrm{Sing}(W) \tag{4.3}$$

and hence $\mathrm{Crit}(W * V) = \mathrm{Crit}(W) + \mathrm{Crit}(V) := \{a + b \mid a \in \mathrm{Crit}(W), b \in \mathrm{Crit}(V)\}$.

Corollary 4.2 (Global motivic Thom-Sebastiani). *Let $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ and $W\colon Y \to \mathbb{A}^1_{\mathsf{k}}$ be as above. Then*

$$\widetilde{\phi}_{V \circledast W} = \Psi(\widetilde{\phi}_V \times \widetilde{\phi}_W) \tag{4.4}$$

in $\mathcal{M}^{\hat{\mu}}_{|\mathrm{Sing}(V)| \times |\mathrm{Sing}(W)|}$ where $\widetilde{\phi}_V \times \widetilde{\phi}_W$ is the obvious element of $\mathcal{M}^{\hat{\mu} \times \hat{\mu}}_{|\mathrm{Sing}(V)| \times |\mathrm{Sing}(W)|}$.

Proof. Let $s_a\colon |\mathrm{Sing}(V) \cap X_a| \to |\mathrm{Sing}(V)|$ be the closed embedding. Then obviously $s_a^*(\widetilde{\phi}) = \phi_{V,a}$. Define s'_b and s''_c similarly for W and $V \circledast W$. From (4.2) and (4.3) we see that $|\mathrm{Sing}(V \circledast W)|$ is the disjoint finite union of its closed subvarieties $|\mathrm{Sing}(V) \cap X_a| \times |\mathrm{Sing}(W) \cap Y_b|$ where $(a, b) \in \mathrm{Crit}(V) \times \mathrm{Crit}(W)$. By Lemma 2.4 it is therefore enough to show that both sides of (4.4) coincide when restricted to each of these subvarieties. Consider the following commutative diagram

$$\begin{array}{ccc}
|\mathrm{Sing}(V) \cap X_a| \times |\mathrm{Sing}(W) \cap Y_b| & \xrightarrow{\iota_{a,b}} & |\mathrm{Sing}(V \circledast W) \cap (X \times Y)_{a+b}| \\
\downarrow{\scriptstyle s_a \times s'_b} & & \downarrow{\scriptstyle s''_{a+b}} \\
|\mathrm{Sing}(V)| \times |\mathrm{Sing}(W)| & = & |\mathrm{Sing}(V \circledast W)|.
\end{array}$$

If we apply $(s_a \times s'_b)^*$ to both sides of (4.4) we obtain on the left

$$\iota^*_{a,b}((s''_{a+b})^*(\widetilde{\phi}_{V \circledast W})) = \iota^*_{a,b}(\phi_{V \circledast W, a+b})$$

and on the right

$$\Psi((s_a \times s'_b)^*(\widetilde{\phi}_V \times \widetilde{\phi}_W)) = \Psi(s_a^*(\widetilde{\phi}_V) \times (s'_b)^*(\widetilde{\phi}_W)) = \Psi(\phi_{V,a} \times \phi_{W,b})$$

where we use Remark 2.8. But $\iota^*_{a,b}(\phi_{V\circledast W,a+b}) = \Psi(\phi_{V,a} \times \phi_{W,b})$ is just a reformulation of Theorem 4.1 using Proposition 3.4 and Remark 2.8. □

Definition 4.3. *For $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ as above we define*

$$(\phi_V)_{\mathbb{A}^1_{\mathsf{k}}} := V_!(\widetilde{\phi}_V) = \sum_{a\in\mathsf{k}} V_!(\phi_{V,a}) \in \tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}} \tag{4.5}$$

where we use the isomorphism (2.19) *in order to change the target of* $V_!\colon \mathcal{M}^{\hat{\mu}}_{|\mathrm{Sing}(V)|} \to \mathcal{M}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}$ *to* $\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}$.

Recall the convolution product (2.15).

Corollary 4.4. *Let $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ and $W\colon Y \to \mathbb{A}^1_{\mathsf{k}}$ be as above. Then*

$$(\phi_{V\circledast W})_{\mathbb{A}^1_{\mathsf{k}}} = (\phi_V)_{\mathbb{A}^1_{\mathsf{k}}} \star (\phi_W)_{\mathbb{A}^1_{\mathsf{k}}}$$

in $(\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}, \star)$.

Proof. Just apply $(V \circledast W)_! = \mathrm{add}_!(V \times W)_!$ to (4.4) and note that

$$\begin{aligned}\mathrm{add}_!((V\times W)_!(\Psi(\widetilde{\phi}_V \times \widetilde{\phi}_W))) &= \mathrm{add}_!(\Psi((V\times W)_!(\widetilde{\phi}_V \times \widetilde{\phi}_W))) \\ &= \mathrm{add}_!(\Psi(V_!(\widetilde{\phi}_V) \times W_!(\widetilde{\phi}_W))) = (\phi_V)_{\mathbb{A}^1_{\mathsf{k}}} \star (\phi_W)_{\mathbb{A}^1_{\mathsf{k}}}.\end{aligned}$$

using Remark 2.8. □

Define $\underline{\phi_V}$ as the image of $(\phi_V)_{\mathbb{A}^1_{\mathsf{k}}}$ under the morphism $\varepsilon_!$ of rings from Lemma 2.18, i.e.

$$\underline{\phi_V} := \varepsilon_!((\phi_V)_{\mathbb{A}^1_{\mathsf{k}}}) = \sum_{a\in\mathsf{k}} (\varepsilon_a)_!(\phi_{V,a}) \in \mathcal{M}^{\hat{\mu}}_{\mathsf{k}} \tag{4.6}$$

where $\varepsilon_a\colon |\mathrm{Sing}(V) \cap X_a| \to \operatorname{Spec}\mathsf{k}$ denotes the structure morphism.

Corollary 4.5. *We have $\underline{\phi_{V\circledast W}} = \underline{\phi_V} * \underline{\phi_W}$ in $\mathcal{M}^{\hat{\mu}}_{\mathsf{k}}$.*

Proof. This is obvious from Lemma 2.18 and Corollary 4.4. □

5. Motivic vanishing cycles measure

5.1. Some reminders

We recall some facts from [GLM06, 3.7-3.9]. Let X be a smooth connected k-variety and $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ a morphism. Let $U \subset X$ be a dense open subvariety. Then Guibert, Loeser and Merle define in [GLM06, Prop. 3.8] an element

$$\mathcal{S}_{V,U,X} \in \mathcal{M}^{\hat{\mu}}_{|X_0|}$$

(they denote this element just by $\mathcal{S}_{V,U}$).

Remark 5.1. *The remarks at the end of [GLM06, 3.8] (and Remarks 3.2 and 3.1) imply: if $U = X$ we have*

$$\mathcal{S}_{V,X,X} = \mathcal{S}_V = \psi_{V,0}; \tag{5.1}$$

if $V = 0$ we have

$$\mathcal{S}_{V,U,X} = 0 = \psi_{V,0} = \mathcal{S}_V$$

Theorem 5.2 ([GLM06, Thm. 3.9]). *Let $\alpha\colon A \to \mathbb{A}^1_{\mathsf{k}}$ be an $\mathbb{A}^1_{\mathsf{k}}$-variety. Then there exists a unique $\mathcal{M}_{\mathsf{k}}$-linear map*

$$\mathcal{S}^{\mathcal{M}_A}_\alpha\colon \mathcal{M}_A \to \mathcal{M}^{\hat{\mu}}_{|A_0|}$$

such that for every proper morphism $V'\colon Z \to A$ where Z is a smooth and connected k-variety, and every dense open subvariety U of Z we have

$$\mathcal{S}^{\mathcal{M}_A}_\alpha([U \to A]) = V'_!(\mathcal{S}_{\alpha\circ V',U,X}).$$

Note that given any morphism $V\colon U \to A$ where U is a smooth connected k-variety, there is a smooth connected k-variety Z containing U as a dense open subscheme and a proper morphism $V'\colon Z \to A$ extending V (use Nagata compactification and resolve the singularities).

In particular, if V is a proper morphism, then by definition of $\mathcal{S}^{\mathcal{M}_A}_\alpha$ and using (5.1) we have

$$\mathcal{S}^{\mathcal{M}_A}_\alpha([U \xrightarrow{V} A]) = V_!(\mathcal{S}_{\alpha\circ V,U,U}) = V_!(\mathcal{S}_{\alpha\circ V}) = V_!(\psi_{\alpha\circ V,0}). \tag{5.2}$$

In particular, if A is smooth and connected we obtain $\mathcal{S}^{\mathcal{M}_A}_\alpha([A \xrightarrow{\mathrm{id}} A]) = \mathcal{S}_\alpha = \psi_{\alpha,0}$ which justifies the notation $\mathcal{S}^{\mathcal{M}_A}_\alpha$.

We will apply Theorem 5.2 only in the case that α is a translation $\mathbb{A}^1_{\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}$, $x \mapsto x - a$, for some $a \in \mathsf{k}$.

5.2. Additivity of the motivic vanishing cycles

Theorem 5.2 has the following consequence.

Theorem 5.3. *There exists a unique $\mathcal{M}_{\mathsf{k}}$-linear map*

$$\Phi' \colon \mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \to \tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}$$

such that

$$\Phi'([X \xrightarrow{V} \mathbb{A}^1_{\mathsf{k}}]) = (\phi_V)_{\mathbb{A}^1_{\mathsf{k}}}$$

for all proper morphisms $V \colon X \to \mathbb{A}^1_{\mathsf{k}}$ of k-varieties where X is smooth over k and connected (for the definition of $(\phi_V)_{\mathbb{A}^1_{\mathsf{k}}}$ see Definition 4.3).

Proof. Uniqueness is clear since $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ is generated by the classes of proper morphisms $V \colon X \to \mathbb{A}^1_{\mathsf{k}}$ of k-varieties with X connected and smooth over k (and relations given by the blowing-up relations), see [Bit04, Thm. 5.1].

Let $a \in \mathsf{k}$ and let $\gamma_a \colon |X_a| \to \{a\} = \operatorname{Spec} \mathsf{k}$ be the obvious morphism. Apply Theorem 5.2 to the morphism $\alpha \colon \mathbb{A}^1_{\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}$, $x \mapsto x - a$. We obtain an $\mathcal{M}_{\mathsf{k}}$-linear map $\mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \to \mathcal{M}^{\hat{\mu}}_{\{a\}}$ that maps $[V \colon X \to \mathbb{A}^1_{\mathsf{k}}]$ to $-(\gamma_a)_!(\psi_{V,a})$ (use (5.2); we add a global minus sign) whenever $V \colon X \to \mathbb{A}^1_{\mathsf{k}}$ is proper with X connected and smooth over k.

Obviously there is a unique $\mathcal{M}_{\mathsf{k}}$-linear map $\mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \to \mathcal{M}^{\hat{\mu}}_{\{a\}}$ mapping $[V \colon X \to \mathbb{A}^1_{\mathsf{k}}]$ to $[|X_a| \to \{a\}] = (\gamma_a)_!([|X_a| \to |X_a|])$ for any morphism $V \colon X \to \mathbb{A}^1_{\mathsf{k}}$ of k-varieties.

Let $\Phi'_a \colon \mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \to \mathcal{M}^{\hat{\mu}}_{\{a\}}$ be the sum of these two maps. If $V \colon X \to \mathbb{A}^1_{\mathsf{k}}$ is proper with X connected and smooth over k we have $\Phi'_a([V \colon X \to \mathbb{A}^1_{\mathsf{k}}]) = (\gamma_a)_!(\phi_{V,a})$ by the definition of the motivic vanishing cycles (3.1).

For any $a \in \mathsf{k}$, let $i_a \colon \{a\} \to \mathbb{A}^1_{\mathsf{k}}$ be the inclusion. Observe that

$$\sum_{a \in \mathsf{k}} (i_a)_!(\Phi'_a) \colon \mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \to \mathcal{M}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}$$

is well defined since for any given $m \in \mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}}$ only finitely many $\Phi'_a(m)$ are nonzero. The composition of this morphism with the isomorphism (2.19) has the required properties. This proves existence. □

Remark 5.4. *If Z is a smooth k-variety we have $\Phi'([Z \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]) = [Z \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]$. If Z is proper over k this follows from Remark 3.1. Otherwise we can compactify Z to a smooth proper k-variety $\overline{Z}$ such that $\overline{Z} - Z$ is a simple normal crossing divisor and then express the class of Z in terms of $\overline{Z}$ and the various smooth intersections of the involved smooth prime divisors.*

In particular we have $\Phi'([\operatorname{Spec} \mathsf{k} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]) = [\operatorname{Spec} \mathsf{k} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]$ and $\Phi'([\mathbb{A}^1_{\mathsf{k}} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]) = [\mathbb{A}^1_{\mathsf{k}} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]$

Remark 5.5. *We keep our promise from Remark 3.2 to justify our sign choice. We do this by showing that Theorem 5.3 does not hold if the right hand side of (4.5) is replaced by $\sum_{a\in\mathsf{k}} V_!(\mathcal{S}^{\phi}_{V-a})$. Assume that there is morphism $\Xi\colon \mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \to \mathcal{M}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}} \cong \tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}$ of abelian groups such that $\Xi([X \xrightarrow{V} \mathbb{A}^1_{\mathsf{k}}]) = \sum_{a\in\mathsf{k}} V_!(\mathcal{S}^{\phi}_{V-a})$ for all proper morphisms $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ of k-varieties where X is smooth over k and connected. Remark 3.1 implies that $\Xi([Z \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]) = (-1)^{\dim Z}[Z \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}]$ for all smooth proper connected k-varieties Z.*

Let X be a smooth proper connected 2-dimensional k-variety and $\widetilde{X}$ its blowup in a closed point $Y = \{x\} \subset X$. Let E be the exceptional divisor. We view $X, \widetilde{X}, Y, E$ as $\mathbb{A}^1_{\mathsf{k}}$-varieties via the zero morphism to $\mathbb{A}^1_{\mathsf{k}}$. In $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ we obviously have $[X] - [Y] = [\widetilde{X}] - [E]$. So if we apply Ξ we obtain $[X] - [Y] = [\widetilde{X}] + [E]$ since E has odd dimension. We obtain $2[E] = 0$ in $\mathcal{M}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}$. Let us explain why this is a contradiction. Note that $E \cong \mathbb{P}^1_{\mathsf{k}}$. Pulling back via the inclusion $\operatorname{Spec} \mathsf{k} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}$ and forgetting the group action shows that $2[\mathbb{P}^1_{\mathsf{k}}] = 0$ in $\mathcal{M}_{\mathsf{k}}$. Taking the topological Euler characteristic with compact support (see [NS11, Example 4.3]) yields the contradiction $4 = 0$ in $\mathbb{Z}$.

Remark 5.6. *If $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ is a smooth and proper morphism then $\Phi'([X \xrightarrow{V} \mathbb{A}^1_{\mathsf{k}}]) = 0$. This follows from Corollary 3.5; note that X is smooth over k and V is not constant (if X is nonempty).*

Remark 5.7. *We claim that $\Phi'(\mathbb{L}_{\mathbb{A}^1_{\mathsf{k}}}) = 0$. Indeed, we have*

$$\mathbb{L}_{\mathbb{A}^1_{\mathsf{k}}} = [\mathbb{A}^1_{\mathbb{A}^1_{\mathsf{k}}}] = [\mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}] = [\mathbb{A}^1_{\mathsf{k}} \times \mathbb{P}^1_{\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}] - [\mathbb{A}^1_{\mathsf{k}} \times \operatorname{Spec} \mathsf{k} \to \mathbb{A}^1_{\mathsf{k}}]$$

in $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$. Now apply Remark 5.6. In fact, this argument together with the compactification argument from Remark 5.4 shows: if Z is any smooth k-variety, then Φ' maps the class of the projection $\mathbb{A}^1_{\mathsf{k}} \times Z \to \mathbb{A}^1_{\mathsf{k}}$ to zero.

Remark 5.8. *Remark 5.7 shows that the morphism Φ' from Theorem 5.3 is not a morphism of rings if we consider the usual multiplication on $\mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}}$:*

it maps the invertible element $\mathbb{L}_{\mathbb{A}^1_{\mathsf{k}}}$ to zero and hence would be the zero morphism (which it is not, by Remark 5.4). Therefore it seems presently more appropriate to restrict Φ' to $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$. See however Remark 5.11 below.

5.3. The motivic vanishing cycles measure

We define Φ to be the $K_0(\mathrm{Var}_{\mathsf{k}})$-linear composition

$$\Phi\colon K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}) \to \mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \xrightarrow{\Phi'} \tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}} \tag{5.3}$$

where the second map is the morphism Φ' from Theorem 5.3.

Now we can state our main theorem which says that Φ is a ring morphism if we equip source $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}) = K_0(\mathrm{Var}^{\mu_1}_{\mathbb{A}^1_{\mathsf{k}}})$ and target $\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}$ with the convolution product $\star$ from section 2.4, see in particular Remark 2.16 and Proposition 2.17.

Theorem 5.9. *The map* (5.3) *from Theorem 5.3 is a morphism*

$$\Phi\colon (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \to (\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}, \star)$$

of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras. By composing with (2.20) *we obtain a morphism*

$$\varepsilon_! \circ \Phi\colon (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \to (\mathcal{M}^{\hat{\mu}}_{\mathsf{k}}, *) \tag{5.4}$$

*of $K_0(\mathrm{Var}_{\mathsf{k}})$-algebras. We call these two morphisms **motivic vanishing cycles measures**.*

Proof. The second claim is obvious from Lemma 2.18, so let us prove the first claim. Remark 5.4 shows that Φ maps the identity element to the identity element. Remark 5.4 shows that Φ is compatible with the algebra structure maps, cf. (2.16).

We use that $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ is generated by the classes of projective morphisms $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ with X a connected quasi-projective k-variety that is smooth over k (and relations given by the blowing-up relations), see [Bit04, Thm. 5.1].

So let X and Y be connected quasi-projective k-varieties that are smooth over k and let $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ and $W\colon Y \to \mathbb{A}^1_{\mathsf{k}}$ be projective morphisms. Then we know by Theorem 5.3 and Corollary 4.4 that

$$\Phi([X \xrightarrow{V} \mathbb{A}^1_{\mathsf{k}}]) \star \Phi([Y \xrightarrow{W} \mathbb{A}^1_{\mathsf{k}}]) = (\phi_V)_{\mathbb{A}^1_{\mathsf{k}}} \star (\phi_W)_{\mathbb{A}^1_{\mathsf{k}}} = (\phi_{V \circledast W})_{\mathbb{A}^1_{\mathsf{k}}}.$$

Our aim is to show that

$$(\phi_{V \circledast W})_{\mathbb{A}^1_{\mathsf{k}}} = \Phi([X \times Y \xrightarrow{V \circledast W} \mathbb{A}^1_{\mathsf{k}}]).$$

This is not obvious since $V \circledast W \colon X \times Y \to \mathbb{A}^1_{\mathsf{k}}$ is not proper in general.

We apply Proposition 5.12 below and use notation from there. We obtain the equality

$$\begin{aligned}[X \times Y \xrightarrow{V \circledast W} \mathbb{A}^1_{\mathsf{k}}] = [Z \xrightarrow{h} \mathbb{A}^1_{\mathsf{k}}] - \sum_i [D_i \xrightarrow{h_i} \mathbb{A}^1_{\mathsf{k}}] + \sum_{i<j} [D_{ij} \xrightarrow{h_{ij}} \mathbb{A}^1_{\mathsf{k}}] \\ - \dots + (-1)^s [D_{12\dots s} \xrightarrow{h_{12\dots s}} \mathbb{A}^1_{\mathsf{k}}]\end{aligned}$$

in $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$. On the right-hand side, Z and all $D_{i_1 \dots i_p}$ are smooth quasi-projective k-varieties, h is a projective morphism, and all $h_{i_1 \dots i_p}$ are projective and smooth morphisms, by part (iv) of Proposition 5.12. Hence we can compute $\Phi([X \times Y \xrightarrow{V \circledast W} \mathbb{A}^1_{\mathsf{k}}]$ now. Remark 5.6 shows that Φ vanishes on all $[D_{i_1 \dots i_p} \xrightarrow{h_{i_1 \dots i_p}} \mathbb{A}^1_{\mathsf{k}}]$. We obtain

$$\Phi([X \times Y \xrightarrow{V \circledast W} \mathbb{A}^1_{\mathsf{k}}]) = \Phi([Z \xrightarrow{h} \mathbb{A}^1_{\mathsf{k}}]) = (\phi_h)_{\mathbb{A}^1_{\mathsf{k}}}$$

and are left to show that

$$(\phi_h)_{\mathbb{A}^1_{\mathsf{k}}} = (\phi_{V \circledast W})_{\mathbb{A}^1_{\mathsf{k}}}.$$

But this holds true by Corollary 3.6 which we can apply by parts (i), (ii), (iii) of Proposition 5.12. □

Remark 5.10. *If X and Y are smooth connected k-varieties and $V \colon X \to \mathbb{A}^1_{\mathsf{k}}$ and $W \colon Y \to \mathbb{A}^1_{\mathsf{k}}$ are proper morphisms then Theorems 5.9 and 4.1 show that*

$$\begin{aligned}\Phi([X \times Y \xrightarrow{V \circledast W} \mathbb{A}^1_{\mathsf{k}}]) = \Phi([X \xrightarrow{V} \mathbb{A}^1_{\mathsf{k}}]) * \Phi([Y \xrightarrow{W} \mathbb{A}^1_{\mathsf{k}}]) \\ = (\phi_V)_{\mathbb{A}^1_{\mathsf{k}}} * (\phi_V)_{\mathbb{A}^1_{\mathsf{k}}} = (\phi_{V \circledast W})_{\mathbb{A}^1_{\mathsf{k}}}.\end{aligned}$$

So even though $V \circledast W$ might not be proper, the motivic vanishing cycles measure Φ sends it to $(\phi_{V \circledast W})_{\mathbb{A}^1_{\mathsf{k}}}$.

Remark 5.11. *Recall the element* $\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)} = [\mathbb{A}^1_{\mathsf{k}} \xrightarrow{0} \mathbb{A}^1_{\mathsf{k}}] \in K_0(\mathrm{Var}^{\mu_n}_{\mathbb{A}^1_{\mathsf{k}}})$ *defined in* (2.17)*. For* $n = 1$ *it is an element of* $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ *and we have* $(K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}), \star)[(\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)})^{-1}] = (\tilde{\mathcal{M}}_{\mathbb{A}^1_{\mathsf{k}}}, \star)$*. Remark 5.4 shows that* $\Phi(\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)}) = \mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)}$ *where we view* $\mathbb{L}_{(\mathbb{A}^1_{\mathsf{k}},0)}$ *in the obvious way as an element of* $K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}})$*. Therefore,* Φ *factors as the composition*

$$\Phi\colon (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \to (\tilde{\mathcal{M}}_{\mathbb{A}^1_{\mathsf{k}}}, \star) \to (\tilde{\mathcal{M}}^{\hat{\mu}}_{\mathbb{A}^1_{\mathsf{k}}}, \star).$$

The second map is a morphism of $\mathcal{M}_{\mathsf{k}}$*-algebras. It is, up to the isomorphism* $\mathcal{M}_{\mathbb{A}^1_{\mathsf{k}}} \xrightarrow{\sim} \tilde{\mathcal{M}}_{\mathbb{A}^1_{\mathsf{k}}}$ *from* (2.18) *(for* $n = 1$*), the morphism* Φ' *from Theorem 5.3. This makes up for Remark 5.8. As observed in Remarks 5.6 and 5.7,* Φ *vanishes on many other elements, for example on* $[\mathbb{A}^1_{\mathsf{k}} \xrightarrow{\mathrm{id}} \mathbb{A}^1_{\mathsf{k}}]$ *or on* $\mathbb{L}_{\mathbb{A}^1_{\mathsf{k}}}$*.*

5.4. Compactification

For the convenience of the reader we recall our compactification result from [LS16a].

Proposition 5.12 ([LS16a, Prop. 6.1]). *Let* k *be an algebraically closed field of characteristic zero. Let* X *and* Y *be smooth* k*-varieties and let* $V\colon X \to \mathbb{A}^1_{\mathsf{k}}$ *and* $W\colon Y \to \mathbb{A}^1_{\mathsf{k}}$ *be projective morphisms (hence* X *and* Y *are quasi-projective* k*-varieties). Consider the convolution*

$$V \circledast W\colon X \times Y \xrightarrow{V \times W} \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} \xrightarrow{+} \mathbb{A}^1_{\mathsf{k}}.$$

Then there exists a smooth quasi-projective k*-variety* Z *with an open embedding* $X \times Y \hookrightarrow Z$ *and a projective morphism* $h\colon Z \to \mathbb{A}^1_{\mathsf{k}}$ *such that the following conditions are satisfied.*

(i) The diagram

$$\begin{array}{ccc} X \times Y & \hookrightarrow & Z \\ \downarrow{\scriptstyle V \circledast W} & & \downarrow{\scriptstyle h} \\ \mathbb{A}^1_{\mathsf{k}} & = & \mathbb{A}^1_{\mathsf{k}} \end{array}$$

commutes.

(ii) All critical points of h *are contained in* $X \times Y$*, i. e.* $\mathrm{Sing}(V \circledast W) = \mathrm{Sing}(h)$*.*

(iii) *We have* $Z \setminus X \times Y = \bigcup_{i=1}^{s} D_i$ *where the* D_i *are pairwise distinct smooth prime divisors. More precisely,* $Z \setminus X \times Y$ *is the support of a simple normal crossing divisor.*

(iv) *For every* p*-tuple* $(i_1, \dots, i_p)$ *of indices (with* $p \geq 1$*) the morphism*

$$h_{i_1 \dots i_p} \colon D_{i_1 \dots i_p} := D_{i_1} \cap \dots \cap D_{i_p} \to \mathbb{A}^1_{\mathsf{k}}$$

induced by h *is projective and smooth. In particular, all* $D_{i_1 \dots i_p}$ *are smooth quasi-projective* k*-varieties.*

6. Comparison with the matrix factorization motivic measure

We would like to place Theorem 5.9 in a certain context and compare the motivic measure Φ or rather $\varepsilon_! \circ \Phi$ with another motivic measure of a different nature.

6.1. Categorical motivic measures

First let us recall the *categorical* measure

$$\nu \colon K_0(\mathrm{Var}_{\mathsf{k}}) \to K_0(\mathrm{sat}^{\mathbb{Z}}_{\mathsf{k}})$$

constructed in [BLL04]. Here $K_0(\mathrm{sat}^{\mathbb{Z}}_{\mathsf{k}})$ is the free abelian group generated by quasi-equivalence classes of saturated differential $\mathbb{Z}$-graded k-categories with relations coming from semiorthogonal decompositions into admissible subcategories on the level of homotopy categories. The map ν sends the class $[X]$ of a smooth projective k-variety X to the class $[\mathrm{D^b}(\mathrm{Coh}(X))]$ of (a suitable differential $\mathbb{Z}$-graded k-enhancement of) its derived category $\mathrm{D^b}(\mathrm{Coh}(X))$. The tensor product of differential $\mathbb{Z}$-graded k-categories induces a ring structure on $K_0(\mathrm{sat}^{\mathbb{Z}}_{\mathsf{k}})$ and ν is a ring homomorphism. In recent papers [LS16b, LS16a] we have constructed a motivic measure

$$\mu \colon (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}), \star) \to K_0(\mathrm{sat}^{\mathbb{Z}_2}_{\mathsf{k}})$$

which is a relative analogue of the measure ν. Here $K_0(\mathrm{sat}^{\mathbb{Z}_2}_{\mathsf{k}})$ is defined in exactly the same way as $K_0(\mathrm{sat}^{\mathbb{Z}}_{\mathsf{k}})$ except that this time we consider saturated differential $\mathbb{Z}_2$-graded k-categories. If $[X \xrightarrow{W} \mathbb{A}^1_{\mathsf{k}}] \in K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ where X is smooth over k and W is proper, then $\mu([X \xrightarrow{W} \mathbb{A}^1_{\mathsf{k}}])$ is defined as the class $[\mathbf{MF}(W)]$ of (a suitable differential $\mathbb{Z}_2$-graded k-enhancement of) the category

$$\mathbf{MF}(W) := \prod_{a \in k} \mathbf{MF}(X, W-a)^{\natural}$$

Here $\mathbf{MF}(X, W-a)^{\natural}$ is the Karoubi envelope of the category $\mathbf{MF}(X, W-a)$ of matrix factorizations of the potential $W-a$.

The measures ν and μ are related by the commutative diagram of ring homomorphisms (as announced in the introduction of [LS16b])

$$\begin{array}{ccc} K_0(\mathrm{Var}_{\mathsf{k}}) & \xrightarrow{\nu} & K_0(\mathrm{sat}^{\mathbb{Z}}_{\mathsf{k}}) \\ \downarrow & & \downarrow \\ (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}}), \star) & \xrightarrow{\mu} & K_0(\mathrm{sat}^{\mathbb{Z}_2}_{\mathsf{k}}) \end{array} \tag{6.1}$$

where $K_0(\mathrm{Var}_{\mathsf{k}}) \to K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathsf{k}}})$ is the ring homomorphism (2.16) (for $n=1$) and $K_0(\mathrm{sat}^{\mathbb{Z}}_{\mathsf{k}}) \to K_0(\mathrm{sat}^{\mathbb{Z}_2}_{\mathsf{k}})$ is induced from the *folding* (see [Sch]) which assigns to a differential $\mathbb{Z}$-graded k-category the corresponding differential $\mathbb{Z}_2$-graded k-category.

6.2. Comparing vanishing cycles and matrix factorization measures

To each saturated differential $\mathbb{Z}_2$-graded k-category A one assigns the finite dimensional $\mathbb{Z}_2$-graded vector space

$$\mathrm{HP}(A) = \mathrm{HP}_0(A) \oplus \mathrm{HP}_1(A)$$

over the Laurent power series field $\mathsf{k}((u))$ - the periodic cyclic homology of A (see [Kel98]).

Put $\chi_{\mathrm{HP}}(A) := \dim_{\mathsf{k}((u))} \mathrm{HP}_0(A) - \dim_{\mathsf{k}((u))} \mathrm{HP}_1(A)$. Since HP is additive on semiorthogonal decompositions of triangulated categories (see [Kel99]) this assignment descends to a group homomorphism

$$\chi_{\mathrm{HP}} \colon K_0(\mathrm{sat}^{\mathbb{Z}_2}_{k}) \to \mathbb{Z}$$

Because of the Künneth property for HP (see [Shk12] and references therein) the map χ_{HP} is in fact a ring homomorphism.

On the other hand, if $\mathsf{k} = \mathbb{C}$ we have the usual ring homomorphism (see [Loo02])

$$\chi_{\mathrm{c}} := \sum (-1)^i \dim \mathrm{H}^i_{\mathrm{c}} \,:\, \mathcal{M}_{\mathbb{C}} \to \mathbb{Z} \tag{6.2}$$

Notice that $\chi_{\mathrm{c}}(\mathbb{L}) = 1$, hence χ_{c} is indeed well-defined as a homomorphism from $\mathcal{M}_{\mathbb{C}}$.

Forgetting the action of $\hat{\mu}$ obviously defines a map

$$\mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \to \mathcal{M}_{\mathbb{C}} \tag{6.3}$$

of $\mathcal{M}_{\mathbb{C}}$-modules. Clearly, this map is a ring homomorphism if we equip its source with the usual multiplication. However, this is not true if we equip its source with the convolution product $*$ as we will explain in Lemma 6.2 below. Nevertheless we have the following result.

Proposition 6.1. *The composition of* χ_{c} *(see* (6.2)*) with the map "forget the* $\hat{\mu}$*-action"* (6.3) *defines a ring homomorphism*

$$\chi_{\mathrm{c}} \colon (\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}, *) \to \mathbb{Z} \tag{6.4}$$

which we denote again by χ_{c}.

Proof. Let A and B be complex varieties with a good μ_n-action for some $n \geq 1$. We need to show that $A \times B$ and

$$\begin{aligned}[A] * [B] = [((A \times^{\mu_n} \underset{x}{\mathbb{G}_{\mathrm{m}}}) \times (B \times^{\mu_n} \underset{y}{\mathbb{G}_{\mathrm{m}}}))|_{x^n+y^n=0}] \\ - [((A \times^{\mu_n} \underset{x}{\mathbb{G}_{\mathrm{m}}}) \times (B \times^{\mu_n} \underset{y}{\mathbb{G}_{\mathrm{m}}}))|_{x^n+y^n=1}]\end{aligned}$$

(see (2.12)) have the same Euler characteristic with compact support. Since $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, $x \mapsto x^n$, is a μ_n-torsor (in the étale topology) we have a pullback square

$$\begin{array}{ccc} A \times \mathbb{G}_{\mathrm{m}} & \xrightarrow{(a,x)\mapsto x} & \mathbb{G}_{\mathrm{m}} \\ \downarrow & & \downarrow{\scriptstyle x\mapsto x^n} \\ A \times^{\mu_n} \mathbb{G}_{\mathrm{m}} & \xrightarrow{[a,x]\mapsto x^n} & \mathbb{G}_{\mathrm{m}}. \end{array}$$

Its lower horizontal morphism is an étale-locally trivial fibration with fiber A. Therefore it is a locally trivial fibration if we pass to the analytic topologies.

In this way we obtain a locally trivial fibration

$$f\colon (A^{\mathrm{an}} \times^{\mu_n(\mathbb{C})} \mathbb{C}^\times) \times (B^{\mathrm{an}} \times^{\mu_n(\mathbb{C})} \mathbb{C}^\times) \xrightarrow{([a,x],[b,y])\mapsto(x^n,y^n)} \mathbb{C}^\times \times \mathbb{C}^\times$$

with fiber $A^{\mathrm{an}} \times B^{\mathrm{an}}$. Consider the subsets $N := \{x' + y' = 0\} \cong \mathbb{C}^\times$ and $E := \{x' + y' = 1\} \cong \mathbb{C}^\times - \{1\}$ of the base of this fibration where x' and y' are the obvious coordinates. Then

$$\begin{aligned}\chi_{\mathrm{c}}([A] * [B]) &= \chi_{\mathrm{c}}(f^{-1}(N)) - \chi_{\mathrm{c}}(f^{-1}(E)) \\ &= \chi_{\mathrm{c}}(A^{\mathrm{an}} \times B^{\mathrm{an}})(\chi_{\mathrm{c}}(N) - \chi_{\mathrm{c}}(E)) = \chi_{\mathrm{c}}(A^{\mathrm{an}} \times B^{\mathrm{an}}).\end{aligned}$$

This proves what we need. □

Although not strictly needed for our purposes we would like to include the following result (which is also true for k instead of $\mathbb{C}$).

Lemma 6.2. *The map "forget the $\hat{\mu}$-action" $f\colon \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \to \mathcal{M}_{\mathbb{C}}$ (see* (6.3)*) does not define a ring homomorphisms $(\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}, *) \to \mathcal{M}_{\mathbb{C}}$.*

Proof. Let $M = \mu_2 \in \mathrm{Var}_{\mathbb{C}}^{\mu_2}$ with obvious action of μ_2. We claim that $f([M] * [M]) \neq f([M])f([M])$.

We clearly have $f([M])f([M]) = 4[\mathrm{Spec}\,\mathbb{C}] = 4$. On the other hand multiplication defines an isomorphism $M \times^{\mu_2} \mathbb{G}_{\mathrm{m}} \xrightarrow{\sim} \mathbb{G}_{\mathrm{m}}$ and therefore (2.12) yields

$$[M] * [M] = [(\underset{x}{\mathbb{G}_{\mathrm{m}}} \times \underset{y}{\mathbb{G}_{\mathrm{m}}})|_{x^2+y^2=0}] - [(\underset{x}{\mathbb{G}_{\mathrm{m}}} \times \underset{y}{\mathbb{G}_{\mathrm{m}}})|_{x^2+y^2=1}]$$

The μ_2-action on $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ is the diagonal action. Instead of using the coordinates (x, y) on $\mathbb{A}^2_{\mathbb{C}}$ let us use the coordinates (a, b) where $a = x + iy$ and $b = x - iy$. Then $x^2 + y^2 = ab$ and the conditions $x \neq 0$ and $y \neq 0$ are equivalent to $a + b \neq 0$ and $a - b \neq 0$. Hence

$$[M] * [M] = [\underset{(a,b)}{\mathbb{A}^2_{\mathbb{C}}}|_{ab=0,\ a\neq\pm b}] - [\underset{(a,b)}{\mathbb{A}^2_{\mathbb{C}}}|_{ab=1,\ a\neq\pm b}]$$

The μ_2-action on $\mathbb{A}^2_{\mathbb{C}}$ is again the diagonal action. The first summand is the coordinate cross without the origin and equal to $2[\mathbb{G}_{\mathrm{m}}]$ with obvious μ_2-action. To treat the second summand note that the map $(\mathbb{G}_{\mathrm{m}} - \mu_4) \to \mathbb{A}^2_{\mathbb{C}}|_{ab=1,a\neq\pm b}$, $a \mapsto (a, a^{-1})$ defines a μ_2-equivariant isomorphism. Hence

$$[M] * [M] = 2[\mathbb{G}_{\mathrm{m}}] - [\mathbb{G}_{\mathrm{m}}] + [\mu_4] = [\mathbb{G}_{\mathrm{m}}] + 2[\mu_2] \tag{6.5}$$

and $f([M] * [M]) = [\mathbb{G}_{\mathrm{m}}] + 4$.

But the element $[\mathbb{G}_{\mathrm{m}}] = f([M] * [M]) - f([M])f([M])$ is certainly not zero in $\mathcal{M}_{\mathbb{C}}$: taking the Hodge-Deligne polynomial defines a ring homomorphism $\mathcal{M}_{\mathbb{C}} \to \mathbb{Z}[u, v, u^{-1}, v^{-1}]$ which sends $[\mathbb{G}_{\mathrm{m}}]$ to $uv - 1$, cf. [NS11, Example 4.6]. □

Theorem 6.3. *We have the following commutative diagram of ring homomorphisms*

$$\begin{array}{ccc} (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathbb{C}}}), \star) & \xrightarrow{\mu} & K_0(\mathrm{sat}^{\mathbb{Z}_2}_{\mathbb{C}}) \\ \Big\downarrow{\scriptstyle \varepsilon_! \circ \Phi} & & \Big\downarrow{\scriptstyle \chi_{\mathrm{HP}}} \\ (\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}, *) & \xrightarrow{\chi_{\mathrm{c}}} & \mathbb{Z} \end{array}$$

where the left vertical arrow is the map (5.4) *from Theorem 5.9 and the lower horizontal map is the ring homomorphism* (6.4)*.*

Proof. The abelian group $K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathbb{C}}})$ is generated by classes $[X \xrightarrow{W} \mathbb{A}^1_{\mathbb{C}}]$ where X is smooth over $\mathbb{C}$ and the map W is projective (see [Bit04]). So it suffices to prove commutativity on such generators.

Fix a projective map $W \colon X \to \mathbb{A}^1_{\mathbb{C}}$ of a smooth $\mathbb{C}$-variety X. Then by definition

$$\mu(W) = \sum_{a \in \mathbb{C}} [\mathbf{MF}(X, W - a)^{\natural}] \in K_0(\mathrm{sat}^{\mathbb{Z}_2}_{\mathbb{C}})$$

and

$$\varepsilon_! \circ \Phi(W) = \sum_{a \in \mathbb{C}} (\varepsilon_a)_! \phi_{W,a} \in \mathcal{M}_{\mathbb{C}}$$

with notation as in (4.6). So it suffices to prove that

$$\chi_{\mathrm{HP}}(\mathbf{MF}(X, W - a)^{\natural}) = \chi_{\mathrm{c}}((\varepsilon_a)_! \phi_{W,a})$$

for any given $a \in \mathbb{C}$. We may and will assume that $a = 0$.

Let X^{an} denote the space X with the analytic topology. Recall the classical functors of nearby and vanishing cycles

$$\psi^{\mathrm{geom}}_W, \phi^{\mathrm{geom}}_W \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathrm{an}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathrm{an}}_0)$$

between the corresponding derived categories of constructible sheaves with complex coefficients. For $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathrm{an}})$ we have a distinguished triangle

$$F|_{X^{\mathrm{an}}_0} \to \psi^{\mathrm{geom}}_W F \to \phi^{\mathrm{geom}}_W F \to F|_{X^{\mathrm{an}}_0}[1] \tag{6.6}$$

in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathrm{an}}_0)$ (see [SGA-7II, Exp. XIII]).

In particular for the constant sheaf $\mathbb{C}_{X^{\mathrm{an}}}$ we have the complex $\phi_W^{\mathrm{geom}}\mathbb{C}_{X^{\mathrm{an}}}$ of sheaves on X_0^{an}. Consider its hypercohomology with compact supports $\mathrm{H}_\mathrm{c}^\bullet(X_0^{\mathrm{an}}, \phi_W^{\mathrm{geom}}\mathbb{C}_{X^{\mathrm{an}}})$ and its Euler characteristic $\sum_i(-1)^i \dim \mathrm{H}_\mathrm{c}^i(X_0^{\mathrm{an}}, \phi_W^{\mathrm{geom}}\mathbb{C}_{X^{\mathrm{an}}})$. (Note that in our case we may as well consider the hypercohomology $\mathrm{H}^\bullet$ instead of $\mathrm{H}_\mathrm{c}^\bullet$, since X_0^{an} is compact.) It follows from [Efi12, Thm. 1.1] that

$$\chi_{\mathrm{HP}}(\mathbf{MF}(X,W)) = -\sum_i (-1)^i \dim \mathrm{H}_\mathrm{c}^i(X_0^{\mathrm{an}}, \phi_W^{\mathrm{geom}}\mathbb{C}_{X^{\mathrm{an}}}).$$

By the localization theorem in cyclic homology it follows that the Karoubi closure $\mathbf{MF}(X,W)^\natural$ has the same cyclic homology as $\mathbf{MF}(X,W)$, i. e. $\chi_{\mathrm{HP}}(\mathbf{MF}(X,W)) = \chi_{\mathrm{HP}}(\mathbf{MF}(X,W)^\natural)$. Hence it remains to prove the equality

$$\chi_\mathrm{c}((\varepsilon_0)_!\phi_{W,0}) = -\sum_i (-1)^i \dim \mathrm{H}_\mathrm{c}^i(X_0^{\mathrm{an}}, \phi_W^{\mathrm{geom}}\mathbb{C}_{X^{\mathrm{an}}}). \tag{6.7}$$

Lemma 6.4. *(a) For every variety Y there exists a unique group homomorphism*

$$\mathrm{SH}_Y\colon K_0(\mathrm{Var}_Y) \to K_0(\mathrm{D}_\mathrm{c}^\mathrm{b}(Y^{\mathrm{an}}))$$

such that $\mathrm{SH}_Y([Z \xrightarrow{f} Y]) = [\mathbf{R}f_!\mathbb{C}_{Z^{\mathrm{an}}}]$.

(b) Given a morphism of varieties $g\colon Y \to T$ the diagram

$$\begin{array}{ccc} K_0(\mathrm{Var}_Y) & \xrightarrow{\mathrm{SH}_Y} & K_0(\mathrm{D}_\mathrm{c}^\mathrm{b}(Y^{\mathrm{an}})) \\ \downarrow{\scriptstyle g_!} & & \downarrow{\scriptstyle K_0(\mathbf{R}g_!)} \\ K_0(\mathrm{Var}_T) & \xrightarrow{\mathrm{SH}_T} & K_0(\mathrm{D}_\mathrm{c}^\mathrm{b}(T^{\mathrm{an}})) \end{array}$$

commutes.

(c) If $Y = \operatorname{Spec}\mathbb{C}$, then $K_0(\mathrm{D}_\mathrm{c}^\mathrm{b}((\operatorname{Spec}\mathbb{C})^{\mathrm{an}})) = \mathbb{Z}$ (by taking the alternating sum of the cohomologies) and $\mathrm{SH}_{\operatorname{Spec}\mathbb{C}}([Z \to \operatorname{Spec}\mathbb{C}]) = \chi_\mathrm{c}([Z])$.

Proof. (a) For a variety S and an open embedding $j\colon U \hookrightarrow S$ with complementary closed embedding $i\colon Z = S - U \hookrightarrow S$ recall the short exact sequence of sheaves

$$0 \to j_!\mathbb{C}_{U^{\mathrm{an}}} \to \mathbb{C}_{S^{\mathrm{an}}} \to i_!\mathbb{C}_{Z^{\mathrm{an}}} \to 0.$$

This implies that the map $\mathrm{SH}_Y([Z \xrightarrow{f} Y]) = [\mathbf{R}f_!\mathbb{C}_{Z^{\mathrm{an}}}]$ indeed descends to a homomorphism $\mathrm{SH}_Y\colon K_0(\mathrm{Var}_Y) \to K_0(\mathrm{D}_\mathrm{c}^\mathrm{b}(Y^{\mathrm{an}}))$. Uniqueness is obvious.

(b) Given a morphism $f\colon Z \to Y$ we have by definition

$$K_0(\mathbf{R}g_!) \cdot \mathrm{SH}_Y([Z \xrightarrow{f} Y]) = K_0(\mathbf{R}g_!)[\mathbf{R}f_!\mathbb{C}_{Z^{\mathrm{an}}}] = [\mathbf{R}(gf)_!\mathbb{C}_{Z^{\mathrm{an}}}]$$

and

$$\mathrm{SH}_T \cdot g_!([Z \xrightarrow{f} Y]) = \mathrm{SH}_T([Z \xrightarrow{gf} T] = [\mathbf{R}(gf)_!\mathbb{C}_{Z^{\mathrm{an}}}]$$

(c) This is clear. □

Now [GLM06, Prop. 3.17] implies the following equality in $K_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_0^{\mathrm{an}}))$:

$$\mathrm{SH}_{X_0}(\psi_{W,0}) = [\psi_W^{\mathrm{geom}}(\mathbb{C}_{X^{\mathrm{an}}})].$$

Applying part (b) of Lemma 6.4 to the map $\varepsilon_0\colon X_0 \to \operatorname{Spec}\mathbb{C}$ and using part (c) we conclude that

$$\chi_{\mathrm{c}}((\varepsilon_0)_!\psi_{W,0}) = \sum_i (-1)^i \dim \mathrm{H}^i_{\mathrm{c}}(X_0^{\mathrm{an}}, \psi_W^{\mathrm{geom}}(\mathbb{C}_{X^{\mathrm{an}}})).$$

Notice that on one hand by definition of $\phi_{W,0}$ we have

$$\chi_{\mathrm{c}}((\varepsilon_0)_!\phi_{W,0}) = \chi_{\mathrm{c}}((\varepsilon_0)_![X_0 \xrightarrow{\mathrm{id}} X_0]) - \chi_{\mathrm{c}}((\varepsilon_0)_!\psi_{W,0})$$

and on the other hand by the distinguished triangle (6.6) we have

$$\sum_i (-1)^i \dim \mathrm{H}^i_{\mathrm{c}}(X_0^{\mathrm{an}}, \phi_W^{\mathrm{geom}}(\mathbb{C}_{X^{\mathrm{an}}})) = \sum_i (-1)^i \dim \mathrm{H}^i_{\mathrm{c}}(X_0^{\mathrm{an}}, \psi_W^{\mathrm{geom}}(\mathbb{C}_{X^{\mathrm{an}}})) - \sum_i (-1)^i \dim \mathrm{H}^i_{\mathrm{c}}(X_0^{\mathrm{an}}, \mathbb{C}_{X_0^{\mathrm{an}}})$$

It remains to notice that

$$\chi_{\mathrm{c}}((\varepsilon_0)_![X_0 \xrightarrow{\mathrm{id}} X_0]) = \sum_i (-1)^i \dim \mathrm{H}^i_{\mathrm{c}}(X_0^{\mathrm{an}}, \mathbb{C}_{X_0^{\mathrm{an}}})$$

This proves equality (6.7) and finishes the proof of the theorem. □

We give two simple examples in which the equality (6.7) can be verified directly.

Example 6.5. *Let $X = \mathbb{A}^1_{\mathbb{C}}$ and $W(a) = a^n$ for some $n \geq 1$. Then $\phi_W^{\mathrm{geom}}\mathbb{C}_{X^{\mathrm{an}}} = \mathbb{C}_{(0)}^{\oplus n-1}$. Hence the right-hand side of equation* (6.7) *is equal to $-(n-1)$.*

On the other hand, in the notation of the proof of Proposition 3.4 (with the identity as embedded resolution) the divisor E is $n \cdot (0)$ and hence its μ_n-Galois covering $\tilde{E}$ is isomorphic to μ_n. From (3.3) *we obtain*

$$\phi_{W,0} = [|X_0|] - \psi_{W,0} = [(0)] - \mu_n.$$

Thus $\chi_c((\varepsilon_0)_! \phi_{W,0})$ is also equal to $-(n-1)$.

Example 6.6. *Let $X = \mathbb{A}^2_{\mathbb{C}}$ and $W\colon X \to \mathbb{A}^1_{\mathbb{C}}$, $W(a,b) = ab$. (This is not proper, but should make no difference since the complex $\phi_W^{\mathrm{geom}} \mathbb{C}_{X^{\mathrm{an}}}$ has compact support.) Then $\phi_W^{\mathrm{geom}} \mathbb{C}_{X^{\mathrm{an}}} = \mathbb{C}_{(0,0)}[-1]$. Hence the right-hand side of* (6.7) *is equal to 1.*

On the other hand, in the notation of the proof of Proposition 3.4 the divisor E is the coordinate cross (with components of multiplicity one) and so (3.3) *yields*

$$\phi_{W,0} = [X_0] - \psi_{W,0} = (\mathbb{G}_{\mathrm{m}} + \mathbb{G}_{\mathrm{m}} + pt) - (\mathbb{G}_{\mathrm{m}} + \mathbb{G}_{\mathrm{m}} - \mathbb{G}_{\mathrm{m}}) = \mathbb{L}$$

Hence $\chi_c((\varepsilon_0)_! \phi_{W,0}) = 1$.

Here is another way to compute this example. Using coordinates (s,t) on $\mathbb{A}^2_{\mathbb{C}}$ so that $a = s + it$ and $b = s - it$ we have $W(a,b) = ab = s^2 + t^2 = s^2 \circledast t^2$. Example 6.5 shows that $\phi_{s^2,0} = [(0)] - \mu_2$ and $\chi_c((\varepsilon_0)_! \phi_{s^2,0}) = -1$. We have $\varepsilon_! \Phi(s^2) = (\varepsilon_0)_! \phi_{s^2,0}$ and

$$\chi_c((\varepsilon_0)_! \phi_{W,0}) = \chi_c(\varepsilon_! \Phi(ab)) = \chi_c(\varepsilon_! \Phi(s^2)) \chi_c(\varepsilon_! \Phi(t^2)) = (-1)^2 = 1$$

using multiplicativity of our motivic measures. We can also use the motivic Thom-Sebastiani Theorem 4.1 and compute (use Remark 2.11 and (the computation leading to) equation (6.5)*)*

$$\begin{aligned} \phi_{W,0} &= \Psi(\phi_{s^2,0} \times \phi_{t^2,0}) \\ &= \Psi([(0)] \times [(0)]) - 2\Psi([(0)] \times [\mu_2]) + \Psi([\mu_2] \times [\mu_2]) \\ &= [(0)] - 2[\mu_2] + ([\mathbb{G}_{\mathrm{m}}] + 2[\mu_2])) = \mathbb{L}. \end{aligned}$$

Here the μ_2-action on $\mathbb{G}_m$ is a priori the obvious one but can then also be assumed to be trivial by the defining relations of the equivariant Grothendieck group.

6.3. Summarizing diagram

We collect the motivic measures considered in this paper in the following commutative diagram (in case $\mathsf{k} = \mathbb{C}$; see (6.1) and Theorem 6.3).

$$\begin{array}{ccc} K_0(\mathrm{Var}_{\mathbb{C}}) & \xrightarrow{\nu} & K_0(\mathrm{sat}^{\mathbb{Z}}_{\mathbb{C}}) \\ \downarrow & & \downarrow \\ (K_0(\mathrm{Var}_{\mathbb{A}^1_{\mathbb{C}}}), \star) & \xrightarrow{\mu} & K_0(\mathrm{sat}^{\mathbb{Z}_2}_{\mathbb{C}}) \\ {\scriptstyle \varepsilon_! \circ \Phi}\downarrow & & \downarrow{\scriptstyle \chi_{\mathrm{HP}}} \\ (\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}, *) & \xrightarrow{\chi_c} & \mathbb{Z} \end{array}$$

The upper left vertical arrow and the vertical composition on the left are the algebra structure maps. The composition from the top left corner to the bottom right corner is induced by mapping a complex variety to its Euler characteristic with compact support.

References

[Bit04] Franziska Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), no. 4, 1011–1032.

[Bit05] ______, *On motivic zeta functions and the motivic nearby fiber*, Math. Z. **249** (2005), no. 1, 63–83.

[BLL04] Alexey I. Bondal, Michael Larsen, and Valery A. Lunts, *Grothendieck ring of pretriangulated categories*, Int. Math. Res. Not. (2004), no. 29, 1461–1495.

[DL98] Jan Denef and François Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537.

[DL99] ______, *Motivic exponential integrals and a motivic Thom-Sebastiani theorem*, Duke Math. J. **99** (1999), no. 2, 285–309.

[DL01] ______, *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 327–348.

[Efi12] Alexander I. Efimov, *Cyclic homology of categories of matrix factorizations*, Preprint (2012), arXiv:1212.2859.

[GLM06] Gil Guibert, François Loeser, and Michel Merle, *Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink*, Duke Math. J. **132** (2006), no. 3, 409–457.

[GW10] Ulrich Görtz and Torsten Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010.

[Kel98] Bernhard Keller, *Invariance and localization for cyclic homology of DG algebras*, J. Pure Appl. Algebra **123** (1998), no. 1-3, 223–273.

[Kel99] ______, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), no. 1, 1–56.

[Kol07] János Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007.

[Loo02] Eduard Looijenga, *Motivic measures*, Astérisque (2002), no. 276, 267–297, Séminaire Bourbaki, Vol. 1999/2000.

[LS16a] Valery A. Lunts and Olaf M. Schnürer, *Matrix factorizations and motivic measures*, J. Noncommut. Geom. **10** (2016), no. 3, 981–1042.

[LS16b] ______, *Matrix factorizations and semi-orthogonal decompositions for blowing-ups*, J. Noncommut. Geom. **10** (2016), no. 3, 907–979.

[NS11] Johannes Nicaise and Julien Sebag, *The Grothendieck ring of varieties*, Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume I, LMS Lecture Note Ser., vol. 383, Cambridge Univ. Press, Cambridge, 2011, pp. 145–188.

[Sch] Olaf M. Schnürer, in preparation.

[SGA-1] *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris), 3, SMF, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck. Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin].

[SGA-7II] Pierre Deligne and Nicholas Katz, *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol 340, Springer-Verlag, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7, II).

[Shk12] Dmytro Shklyarov, *On a Hodge theoretic property of the Kuenneth map in periodic cyclic homology*, Preprint (2012), arXiv:1207.5533.

[Vil05] Orlando Villamayor U., *On constructive desingularization*, J. Symbolic Comput. **39** (2005), no. 3-4, 465–491.

Valery A. Lunts
Department of Mathematics
Indiana University
Rawles Hall
831 East 3rd Street
Bloomington, IN 47405
USA
E-mail: vlunts@indiana.edu

Olaf M. Schnürer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany
E-mail: olaf.schnuerer@math.uni-bonn.de