Pure and Applied Mathematics Quarterly
Volume 11, Number 4, 683–709, 2015

A 1-Dimensional Family of Enriques Surfaces in Characteristic 2 Covered by the Supersingular K3 Surface with Artin Invariant 1

Toshiyuki Katsura and Shigeyuki Kondō

Dedicated to Professor Eduard Looijenga on the occasion of his sixty-ninth birthday

Abstract: We give a 1-dimensional family of classical and supersingular Enriques surfaces in characteristic 2 covered by the supersingular $K3$ surface with Artin invariant 1. Moreover we show that there exist thirty nonsingular rational curves and ten non-effective (-2)-divisors on these Enriques surfaces whose reflection group is of finite index in the orthogonal group of the Néron-Severi lattice modulo torsion.
Keywords: Enriques surface, $K3$ surface, Classical, Supersingular.

1. Introduction

We work over an algebraically closed field k of characteristic 2. The main purpose of this paper is to give a 1-dimensional family of Enriques surfaces in characteristic 2 covered by the supersingular $K3$ surface with Artin invariant 1. In the paper [4], Bombieri and Mumford classified Enriques surfaces into three classes, namely, singular, classical and supersingular Enriques surfaces. As in the case of characteristic 0, an Enriques surface X in characteristic 2 has a canonical double cover $\pi : Y \to X$. The covering π is a separable double cover, a purely inseparable μ_2- or α_2-cover according to X being singular, classical or supersingular. The surface Y might have singularities, but it is $K3$-like in the sense that its dualizing sheaf is trivial. Bombieri

Received September 9, 2015.
Research of the first author is partially supported by Grant-in-Aid for Scientific Research (B) No. 15H03614, and the second author by (S) No. 15H05738.

and Mumford gave an explicit example of each type of Enriques surface as a quotient of the intersection of three quadrics in $\mathbf{P}^5$. In particular, they gave an α_2-covering $Y \to X$ such that Y is a supersingular $K3$ surface with 12 rational double points of type A_1. Recently Liedtke [17] showed that every Enriques surface can be realized in the form of the example by Bombieri and Mumford [4], that is, its canonical cover is a complete intersection of three quadrics in $\mathbf{P}^5$. Moreover he showed that the moduli space of Enriques surfaces with a polarization of degree 4 has two 10-dimensional irreducible components. A general member of one component (resp. the other component) consists of singular (resp. classical) Enriques surfaces. The intersection of the two components parametrizes supersingular Enriques surfaces. On the other hand, Ekedahl, Hyland and Shepherd-Barron [9] studied classical or supersingular Enriques surfaces whose canonical covers are supersingular $K3$ surfaces with 12 rational double points of type A_1. They showed that the moduli space of such Enriques surfaces is an open piece of a $\mathbf{P}^1$-bundle over the moduli space of supersingular $K3$ surfaces. Recall that the moduli space of supersingular $K3$ surfaces is 9-dimensional and is stratified by the Artin invariant σ, $1 \leq \sigma \leq 10$. Each stratum has dimension $\sigma - 1$ (Artin [1], Corollary 7.8, and Rudakov-Shafarevich [22], p.1522, Theorem 2).

In this paper, stimulated by Ekedahl, Hyland and Shepherd-Barron's work, we present a 1-dimensional family of Enriques surfaces whose canonical covers are the (unique) supersingular $K3$ surface with Artin invariant 1. These Enriques surfaces are parametrized by $a, b \in k,\ a + b = ab,\ a^3 \neq 1$. If $a = b = 0$, then the Enriques surface is supersingular, and otherwise it is classical (Theorem 4.8). We remark that this 1-dimensional family is non-isotrivial (see Remark 4.9). This family gives an explicit example of the description of the moduli space of classical and supersingular Enriques surfaces by Ekedahl, Hyland and Shepherd-Barron.

To construct these Enriques surfaces, we consider an elliptic surface defined by

$$y^2 + y + x^3 + sx(y^2 + y + 1) = 0$$

which has four singular fibers of type I_3 over $s = 1, \omega, \omega^2, \infty$ ($\omega^3 = 1, \omega \neq 1$). By taking the Frobenius base change $s = t^2$, we have an elliptic surface

$$y^2 + y + x^3 + t^2x(y^2 + y + 1) = 0.$$

which has 12 rational double points of type A_1 at the singular points of each singular fiber. By taking the minimal nonsingular model, we have an elliptic $K3$ surface $f : Y \to \mathbf{P}^1$ which is supersingular because f has four singular fibers of type I_6 and hence its Picard number should be 22. The

Artin invariant of Y is equal to 1 (cf. Dolgachev-Kondo [8], Theorem 1.1 (vi)). The Enriques surface $X = X_{a,b}$ is obtained as the quotient surface of Y by a rational vector field

$$D = \frac{1}{t-1}\left((t-1)(t-a)(t-b)\frac{\partial}{\partial t} + (1+t^2x)\frac{\partial}{\partial x}\right).$$

The construction is based on a theory of inseparable double covering due to Rudakov-Shafarevich [20], Section 2 (see also Katsura-Takeda [13], Section 2).

The supersingular $K3$ surface Y was studied by Dolgachev and the second author [8] (see also Katsura-Kondo [14]). It contains 42 nonsingular rational curves forming a $(21)_5$-configuration. These 42 curves are nothing but the 24 components of four singular fibers of type I_6 and the 18 sections of the fibration f. The automorphism group $\mathrm{Aut}(Y)$ is generated by a subgroup $\mathrm{PGL}(3, \mathbf{F}_4) \rtimes \mathbf{Z}/2\mathbf{Z}$ and the 168 involutions associated with some (-4)-divisors on Y. From this description, we see that there exist thirty nonsingular rational curves and ten non-effective (-2)-divisors on the Enriques surface X (see Sections 5, 6). The dual graph Γ of these forty divisors can be described in terms of Sylvester's duads, Sylvester's synthemes and totals related to the symmetric group $\mathfrak{S}_6$ of degree six (see Baker [2], p.220). Moreover, these forty divisors have the following remarkable property. Let $\mathrm{Num}(X) = \mathrm{NS}(X)/\{\text{torsion}\}$ be the Néron-Severi group of X modulo torsion. Then, together with the intersection pairing, it has a structure of an even unimodular lattice of signature $(1, 9)$. Let $\mathrm{O}(\mathrm{Num}(X))$ be the orthogonal group of the lattice $\mathrm{Num}(X)$ and let $W(\Gamma)$ be the subgroup of $\mathrm{O}(\mathrm{Num}(X))$ generated by reflections associated with the forty (-2)-divisors. Then $W(\Gamma)$ is of finite index in $\mathrm{O}(\mathrm{Num}(X))$ (Theorem 7.5). This property will be helpful for determining the automorphism group $\mathrm{Aut}(X)$ (Corollary 7.6).

Acknowledgement. The authors thank Shigeru Mukai for valuable conversations.

2. Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, and let S be a nonsingular complete algebraic surface defined over k. We denote by K_S a canonical divisor of S. A rational vector field D on S is said to be p-closed if there exists a rational function f on S such that $D^p = fD$. Let $\{U_i = \mathrm{Spec}A_i\}$ be an affine open covering of S. We set $A_i^D = \{D(\alpha) = 0 \mid \alpha \in A_i\}$.

The affine varieties $\{U_i^D = \mathrm{Spec} A_i^D\}$ glue together to define a normal quotient surface S^D.

Now, we assume D is p-closed. Then, the natural morphism $\pi : S \longrightarrow S^D$ is a purely inseparable morphism of degree p. If the affine open covering $\{U_i\}$ of S is fine enough, then taking local coordinates x_i, y_i on U_i, we see that there exist $g_i, h_i \in A_i$ and a rational function f_i such that the divisors defined by $g_i = 0$ and by $h_i = 0$ have no common divisor, and such that

$$D = f_i \left(g_i \frac{\partial}{\partial x_i} + h_i \frac{\partial}{\partial y_i} \right) \quad \text{on } U_i.$$

By Rudakov-Shafarevich [20], Section 1, the divisors (f_i) on U_i give a global divisor (D) on S, and zero-cycles defined by the ideal (g_i, h_i) on U_i give a global zero cycle $\langle D \rangle$ on S. A point contained in the support of $\langle D \rangle$ is called an isolated singular point of D. If D has no isolated singular point, D is said to be divisorial. Rudakov and Shafarevich showed that S^D is nonsingular if $\langle D \rangle = 0$, i.e., D is divisorial (cf. [20], Theorem 1, Corollary). When S^D is nonsingular, they also showed a canonical divisor formula

$$K_S \sim \pi^* K_{S^D} + (p-1)(D), \tag{2.1}$$

where $\pi : S \to S^D$ is the quotient map and $\sim$ means linear equivalence. As for the Euler number $c_2(S)$ of S, we have a formula

$$c_2(S) = \deg\langle D \rangle - \langle K_S, (D) \rangle - (D)^2 \tag{2.2}$$

(cf. Katsura-Takeda [13], Proposition 2.1). This is the dual version of Igusa's formula (cf. Igusa [11], p.724).

Now we consider an irreducible curve C on S and we set $C' = \pi(C)$. Take an affine open set U_i above such that $C \cap U_i$ is non-empty. The curve C is said to be integral with respect to the vector field D if $(g_i \frac{\partial}{\partial x_i} + h_i \frac{\partial}{\partial y_i})$ is tangent to C at a general point of $C \cap U_i$. Then, Rudakov-Shafarevich [20] (Proposition 1) showed the following proposition:

Proposition 2.1. (i) *If C is integral, then $C = \pi^{-1}(C')$ and $C^2 = pC'^2$.*
(ii) *If C is not integral, then $pC = \pi^{-1}(C')$ and $pC^2 = C'^2$.*

In Section 4, we will use these results to construct Enriques surfaces in characteristic 2.

A lattice is a free abelian group L of finite rank equipped with a non-degenerate symmetric integral bilinear form $\langle .,. \rangle : L \times L \to \mathbf{Z}$. For a lattice

L and an integer m, we denote by $L(m)$ the free $\mathbf{Z}$-module L with the bilinear form obtained from the bilinear form of L by multiplication by m. The signature of a lattice is the signature of the real vector space $L \otimes \mathbf{R}$ equipped with the symmetric bilinear form extended from the one on L by linearity. A lattice is called even if $\langle x, x \rangle \in 2\mathbf{Z}$ for all $x \in L$. We denote by U the even unimodular lattice of signature $(1,1)$, and by $A_m,\ D_n$ or E_k the even *negative* definite lattice defined by the Cartan matrix of type $A_m,\ D_n$ or E_k respectively. We denote by $L \oplus M$ the orthogonal direct sum of lattices L and M, and by $L^{\oplus m}$ the orthogonal direct sum of m-copies of L. Let $\mathrm{O}(L)$ be the orthogonal group of L, that is, the group of isomorphisms of L preserving the bilinear form.

3. An elliptic pencil

From here on, throughout this paper, we assume that k is an algebraically closed field of characteristic 2. On the projective plane $\mathbf{P}^2$ over k, we consider the supersingular elliptic curve E defined by

$$x_1^2 x_2 + x_1 x_2^2 = x_0^3,$$

where (x_0, x_1, x_2) is a homogeneous coordinate of $\mathbf{P}^2$. This is, up to isomorphism, the unique supersingular elliptic curve in characteristic 2. The 3-torsion points of E are given by

$$Q_0 = (0,1,0), Q_1 = (0,0,1), Q_2 = (0,1,1), Q_3 = (1,\omega,1), Q_4 = (\omega,\omega,1)$$
$$Q_5 = (\omega^2,\omega,1), Q_6 = (1,\omega^2,1), Q_7 = (\omega,\omega^2,1), Q_8 = (\omega^2,\omega^2,1).$$

The point Q_0 is the zero point of E. There exist 21 $\mathbf{F}_4$-rational points on $\mathbf{P}^2$, and among them 9 points Q_i $(i = 0, 1, \ldots, 8)$ lie on E. On the other hand, there exist 21 lines defined over $\mathbf{F}_4$ on $\mathbf{P}^2$, and among them 9 lines are triple tangents at Q_i $(i = 0, 1, \ldots, 8)$ of E. The tangent lines intersect E only at the tangent points, and the other lines intersect transversely with E at three 3-torsion points.

Now we consider the pencil of curves of degree 3 passing through the nine 3-torsion points. Then the pencil is given by the equation

$$x_1^2 x_2 + x_1 x_2^2 + x_0^3 + s x_0 (x_1^2 + x_1 x_2 + x_2^2) = 0 \qquad (3.1)$$

with a parameter s. Note that this pencil is isomorphic to the famous pencil defined by $x_0^3 + x_1^3 + x_2^3 + s x_0 x_1 x_2 = 0$ by considering the way of construction. By blowing up at the nine 3-torsion points we obtain an elliptic surface

$\psi : R \longrightarrow \mathbf{P}^1$. On the elliptic surface there exist 4 singular fibers of type I_3. Five lines defined over $\mathbf{F}_4$ pass through the point Q_i on E. They consist of one triple tangent and four lines which intersect E at Q_i transversely. Under the blowing-up, the triple tangent line goes to the purely inseparable double-section of the elliptic surface, and the 4 lines go to components of four singular fibers respectively. The 9 double sections pass through singular points of singular fibers three-by-three. The exceptional curves become nine sections of the elliptic surface which pass through the regular points of singular fibers. Each component of singular fibers intersects three sections among nine exceptional curves (see Figure 1). In Figure 1, the triangle is any singular fiber of the elliptic surface ψ of type I_3, the dotted lines are nine sections and the remaining nine lines are double sections.

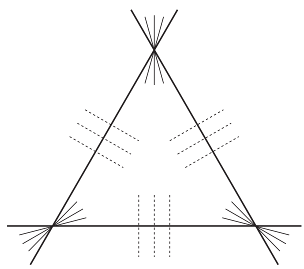

Figure 1: 9 sections and 9 double sections

4. Construction of Enriques surfaces

In characteristic 2, a minimal algebraic surface with numerically trivial canonical divisor is called an Enriques surface if the second Betti number is equal to 10. Such surfaces S are divided into three classes (for details, see Bombieri-Mumford [4], Section 3):

(i) K_S is not linearly equivalent to zero and $2K_S \sim 0$. Such an Enriques surface is called a classical Enriques surface.

(ii) $K_S \sim 0$, $\mathrm{H}^1(S, \mathcal{O}_S) \cong k$ and the Frobenius map acts on $\mathrm{H}^1(S, \mathcal{O}_S)$ bijectively. Such an Enriques surface is called a singular Enriques surface.

(iii) $K_S \sim 0$, $\mathrm{H}^1(S, \mathcal{O}_S) \cong k$ and the Frobenius map is the zero map on $\mathrm{H}^1(S, \mathcal{O}_S)$. Such an Enriques surface is called a supersingular Enriques surface.

Any elliptic fibration on a classical Enriques surface has exactly two multiple fibers. On the other hand, in case of singular or supersingular Enriques surfaces, any elliptic fibration has exactly one multiple fiber (cf. Cossec-Dolgachev [5], Theorem 5.7.2).

Lemma 4.1. *Let S be an Enriques surface. If there is a generically surjective rational map from a supersingular $K3$ surface $\tilde{S}$ to S, then S is not a singular Enriques surface.*

Proof. By Rudakov-Shafarevich [21], p.151, Corollary, $\tilde{S}$ is unirational. Therefore, S is also unirational. However, a singular Enriques surface is not unirational by Crew [6], Theorem 2.5 (see also Katsura [12], Theorem 5). □

In this section, we construct supersingular and classical Enriques surfaces, using the rational elliptic surface $\psi : R \longrightarrow \mathbf{P}^1$ constructed in Section 3 (see the equation (3.1)). We consider the base change of $\psi : R \longrightarrow \mathbf{P}^1$ by $s = t^2$. Then we get an elliptic surface with 12 rational double points of type A_1 defined by

$$x_1^2x_2 + x_1x_2^2 + x_0^3 + t^2x_0(x_1^2 + x_1x_2 + x_2^2) = 0. \tag{4.1}$$

We consider the relatively minimal model of this elliptic surface (4.1):

$$f : Y \longrightarrow \mathbf{P}^1. \tag{4.2}$$

For the readers' convenience, we give here a proof that Y is supersingular. The canonical map $\theta : Y \longrightarrow R$ is surjective and purely inseparable. We consider the relative Frobenius morphism $F : R^{(\frac{1}{2})} \longrightarrow R$. Since R is birationally isomorphic to $\mathbf{P}^2$, so is $R^{(\frac{1}{2})}$, and the Frobenius morphism factors through the canonical map θ. Therefore, we have a generically surjective rational map from $R^{(\frac{1}{2})}$ to Y and we see that Y is unirational. Hence, Y is supersingular, i.e. the Picard number $\rho(Y)$ is equal to the second Betti number $b_2(Y)$ (cf. Shioda [23], p235, Corollary 1).

As we saw in Section 3, there exist nine purely inseparable double-sections and nine sections for the elliptic surface $\psi : R \longrightarrow \mathbf{P}^1$. These make 18 sections of the elliptic surface $f : Y \longrightarrow \mathbf{P}^1$.

Now, we take an affine open set defined by $x_2 \neq 0$. Then, on the affine open set this surface is defined by

$$y^2 + y + x^3 + t^2x(y^2 + y + 1) = 0.$$

Considering the change of coordinates

$$\begin{aligned} v &= (1 + t^3)\{(1 + t^2x)y + tx^2\}/t^6 \\ u &= (1 + t^3)x/t^4 \\ T &= 1/t \end{aligned}$$

we get a surface defined by

$$v^2 + uv + T^2(T^4 + T)v + u^3 + (T^3 + 1)u^2 + T^2(T^4 + t)u = 0$$

The discriminant of this elliptic surface is given by $\Delta(T) = T^6(T^3 + 1)^6$ (cf. Tate [25], Section 1). Therefore, we have $c_2(Y) = \sum_{T\in\mathbf{P}^1} \mathrm{ord}(\Delta(T)) = 24$, and we conclude that Y is a supersingular $K3$ surface. We see there exist 4 singular fibers of type I_6. These singular fibers exist over the points given by $T = 1, \omega, \omega^2, 0$. If we denote by $m_1, m_\omega, m_{\omega^2}, m_0$ (resp. $\mu_1, \mu_\omega, \mu_{\omega^2}, \mu_0$) the number of irreducible components (resp. simple components) of these singular fibers respectively, we have

$$m_1 = m_\omega = m_{\omega^2} = m_0 = 6 \quad (\text{resp. } \mu_1 = \mu_\omega = \mu_{\omega^2} = \mu_0 = 6).$$

The Picard number $\rho(Y)$ of Y is expressed as

$$22 = \rho(Y) = r + 2 + \sum_{\nu=1}^{4}(m_\nu - 1) = r + 22,$$

where r is the rank of the group of sections of $f : Y \longrightarrow \mathbf{P}^1$. Therefore, we have $r = 0$. Then, denoting by n the order of the group of sections of $f : Y \longrightarrow \mathbf{P}^1$, we have

$$|\det \mathrm{NS}(Y)| = \prod_{\nu=1}^{4} \mu_\nu / n^2,$$

where $\det \mathrm{NS}(Y)$ is the discriminant of the Néron-Severi group of Y (Shioda [24], Corollary 1.7). Since in our case, we have $|\det \mathrm{NS}(Y)|n^2 = 6^4$ and $n \geq 18$, we conclude $|\det \mathrm{NS}(Y)| = 2^2$ and $n = 18$. Therefore, the Artin invariant of Y is equal to 1 (see also Dolgachev-Kondo [8], Theorem 1.1 (vi)).

From here on, we use t as a local coordinate of the base curve $\mathbf{P}^1$. For $f : Y \longrightarrow \mathbf{P}^1$, there exist 4 singular fibers over the points defined by $t = 1, \omega, \omega^2, \infty$, respectively. On each singular fiber there exist three exceptional curves derived from the resolution of the surface (4.1). We denote them by $E_{ij} (i = 1, \omega, \omega^2, \infty; j = 1, 3, 5)$. We denote by $E_{ij} (i = 1, \omega, \omega^2, \infty; j = 2, 4, 6)$ the rest of components of singular fibers of $f : Y \longrightarrow \mathbf{P}^1$. Here, $E_{i1}, E_{i2}, E_{i3}, E_{i4}, E_{i5}, E_{i6}$ are components of the singular fiber over $t = i$ $(i = 1, \omega, \omega^2, \infty)$. We have $E_{ij}^2 = -2$. Curves E_{ij} and $E_{ij'}$ intersect each other transeversely if and only if $| j - j' \pmod 6 |= 1$, and for other j, j' we have $\langle E_{ij}, E_{ij'} \rangle = 0$.

Now, we consider a rational vector field

$$D' = (t-1)(t-a)(t-b)\frac{\partial}{\partial t} + (1 + t^2 x)\frac{\partial}{\partial x} \quad \text{with } a + b = ab \text{ and } a^3 \neq 1.$$

Lemma 4.2. *Assume $a + b = ab$, $a^3 \neq 1$. Then,*

(i) *$D'^2 = t^2 D'$, namely, D' is 2-closed.*

(ii) *On the surface Y, the divisorial part of D' is given by*

$$\begin{aligned}(D') \quad &= E_{11} + E_{13} + E_{15} - E_{\omega 1} - E_{\omega 3} - E_{\omega 5} \\ &\quad - E_{\omega^2 1} - E_{\omega^2 3} - E_{\omega^2 5} - E_{\infty 2} - E_{\infty 4} - E_{\infty 6} - F_\infty,\end{aligned}$$

where F_∞ is the fiber over the point given by $t = \infty$.

(iii) *The integral curves with respect to D' in the fibers of $f : Y \longrightarrow \mathbf{P}^1$ are the following:*

the smooth fibers over $t = a, b$ (in case $a = b = 0$, the smooth fiber over $t = 0$) and

$$E_{12}, E_{14}, E_{16}, E_{\omega 1}, E_{\omega 3}, E_{\omega 5}, E_{\omega^2 1}, E_{\omega^2 3}, E_{\omega^2 5}, E_{\infty 2}, E_{\infty 4}, E_{\infty 6}.$$

Proof. These results follow from direct calculation. For example, to prove (ii) and (iii), we consider a local chart of the blowing-up at the point $(t, x, y) = (1, 1, 0)$:

$$t + 1 = TU, \ x + 1 = U, \ y = VU$$

with the new coordinates T, U, V. Then, the exceptional curve C is defined by $U = 0$ and an irreducible component C' of the fiber is given by $T = 0$ on the local chart. We can show that the surface is nonsingular along C. It is easy to see that T, U give local coordinates on a neiborhood of C in Y.

Since
$$\frac{\partial}{\partial t} = \frac{1}{U}\frac{\partial}{\partial T}, \ \frac{\partial}{\partial x} = \frac{\partial}{\partial U} + \frac{T}{U}\frac{\partial}{\partial T},$$
on the local chart we have
$$D' = U\{(T^3 + (a+b)T^2)\frac{\partial}{\partial T} + (T^2U^2 + T^2U + 1)\frac{\partial}{\partial U}\}.$$
Therefore, on the local chart we have the divisorial part $(D') = C$ and we see that C is not integral and C' is integral with respect to the vector field D'. On the other local charts for the blowing-ups, the calculation is similar. □

We set $D = \frac{1}{t-1}D'$. Then, $D^2 = abD$, that is, D is also 2-closed and D is of additive type if $a = b = 0$ and of multiplicative type otherwise. Moreover we have

$$(4.3)\qquad \begin{aligned}(D) \ &= -(E_{12} + E_{14} + E_{16} + E_{\omega 1} + E_{\omega 3} + E_{\omega 5}\\ &\quad + E_{\omega^2 1} + E_{\omega^2 3} + E_{\omega^2 5} + E_{\infty 2} + E_{\infty 4} + E_{\infty 6}).\end{aligned}$$

Lemma 4.3. *The surface Y^D is nonsingular.*

Proof. We have
$$\begin{aligned}(D)^2 \ &= E_{12}^2 + E_{14}^2 + E_{16}^2 + E_{\omega 1}^2 + E_{\omega 3}^2 + E_{\omega 5}^2\\ &\quad + E_{\omega^2 1}^2 + E_{\omega^2 3}^2 + E_{\omega^2 5}^2 + E_{\infty 2}^2 + E_{\infty 4}^2 + E_{\infty 6}^2\\ &= (-2) \times 12 = -24\end{aligned}$$
Since Y is a $K3$ surface, we have $c_2(Y) = 24$. Therefore, by the equation (2.2), we have
$$24 = c_2(Y) = \deg\langle D\rangle - \langle K_Y, (D)\rangle - (D)^2 = \deg\langle D\rangle + 24.$$
Therefore, we have $\deg\langle D\rangle = 0$ and D is divisorial. Hence, Y^D is nonsingular. □

By the result on the canonical divisor formula of Rudakov and Shafarevich (see the equation (2.1)), we have
$$K_Y = \pi^* K_{Y^D} + (D),$$
where $\pi : Y \to Y^D$ is the quotient map.

Lemma 4.4. *Let C be an irreducible curve contained in the support of the divisor (D), and set $C' = \pi(C)$. Then, C' is an exceptional curve of the first kind.*

Proof. Since C is integral with respect to D (Lemma 4.2), we have $C = \pi^{-1}(C')$ (Proposition 2.1). Since $-2 = C^2 = (\pi^{-1}(C'))^2 = 2C'^2$, we have $C'^2 = -1$. Since Y is a $K3$ surface, K_Y is linearly equivalent to zero. Therefore, we have

$$\begin{aligned} 2\langle K_{Y^D}, C'\rangle &= \langle \pi^* K_{Y^D}, \pi^*(C')\rangle \\ &= \langle K_Y - (D), C\rangle = C^2 = -2. \end{aligned}$$

Hence we have $\langle K_{Y^D}, C'\rangle = -1$. Therefore, the arithmetic genus of C' is equal to $(\langle K_{Y^D}, C'\rangle + C'^2)/2 + 1 = 0$. Hence, C' is an exceptional curve of the first kind. □

We denote the 12 exceptional curves on Y^D by E'_i $(i = 1, 2, \ldots, 12)$: These are the images of the irreducible components of $-(D)$ by π. Now we have the following commutative diagram:

$$\begin{array}{ccc} Y^D & \stackrel{\pi}{\longleftarrow} & Y \\ \varphi \downarrow & & \downarrow f \\ X & & \mathbf{P}^1 \\ g \downarrow & \swarrow_F & \\ \mathbf{P}^1 & & \end{array}$$

Here, φ is the contraction of the 12 exceptional curves and $g \circ \varphi : Y^D \longrightarrow \mathbf{P}^1$ is the fiber space constructed as the quotient of $f : Y \longrightarrow \mathbf{P}^1$ by D. F is the Frobenius morphism. Since the exceptional curves E'_i $(i = 1, 2, \ldots, 12)$ are contained in fibers of $g \circ \varphi$, we blow down them and we get the fiber space $g : X \longrightarrow \mathbf{P}^1$. Then, we have

$$K_{Y^D} = \varphi^*(K_X) + \sum_{i=1}^{12} E'_i.$$

Lemma 4.5. *The canonical divisor K_X of X is numerically equivalent to* 0.

Proof. By Lemma 4.2, all irreducible curves which appear in the divisor (D) are integral with respect to the vector field D. For an irreducible component C of (D), we set $C' = \pi(C)$. Then, we have $C = \pi^{-1}(C')$ (Proposition 2.1).

Therefore, we have

$$(D) = -\pi^*(\sum_{i=1}^{12} E'_i).$$

Since Y is a $K3$ surface,

$$\begin{aligned} 0 \sim K_Y &= \pi^* K_{Y^D} + (D) \\ &= \pi^*(\varphi^*(K_X) + \textstyle\sum_{i=1}^{12} E'_i) + (D) = \pi^*(\varphi^*(K_X)) \end{aligned}$$

Therefore, K_X is numerically equivalent to zero. □

Lemma 4.6. *The morphism $g : X \longrightarrow \mathbf{P}^1$ makes X into an elliptic surface.*

Proof. Take a general fiber G of $g : X \longrightarrow \mathbf{P}^1$. Then, there exists a purely inseparable morphism from a fiber G' of $f : Y \longrightarrow \mathbf{P}^1$ to G. Since G' is an elliptic curve, the nonsingular model of G is also an elliptic curve. On the other hand, by $G^2 = 0$ and Lemma 4.5 the arithmetic genus of G is equal to $(G^2 + \langle G, K_X\rangle)/2 + 1 = 1$. Therefore, we conclude that G is a nonsingular elliptic curve. □

Lemma 4.7. *The surface X has $b_2(X) = 10$ and $c_2(X) = 12$.*

Proof. Since $\pi : Y \longrightarrow Y^D$ is finite and purely inseparable, the étale cohomology of Y is isomorphic to the étale cohomology of Y^D. Therefore, we have $b_1(Y^D) = b_1(Y) = 0$, $b_3(Y^D) = b_3(Y) = 0$ and $b_2(Y^D) = b_2(Y) = 22$. Since φ is blowing-downs of 12 exceptional curves of the first kind, we see $b_0(X) = b_4(X) = 1$, $b_1(X) = b_3(X) = 0$ and $b_2(X) = 10$. Therefore, we have

$$c_2(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) + b_4(X) = 12.$$

□

Theorem 4.8. *Under the notation above, the following statements hold.*

(i) *X is a supersingular Enriques surface if $a = b = 0$.*

(ii) *X is a classical Enriques surface if $a + b = ab$ and $a \notin \mathbf{F}_4$.*

Proof. We give here a proof without using the classification theory of algebraic surfaces.

(i) Assume $a = b = 0$. Then, the only fiber of the elliptic fibration $f : Y \longrightarrow \mathbf{P}^1$ that is integral for D is the one over the point P_0 defined by $t = 0$ (Lemma 4.2). Since $f^{-1}(P_0)$ is a supersingular elliptic curve, the reduced part of the fiber $g^{-1}(F(P_0))$ is also a supersingular elliptic curve, and we have only one multiple fiber on the elliptic surface $g : X \longrightarrow \mathbf{P}^1$. Let $g^{-1}(F(P_0)) = 2E_0$ be the multiple fiber. Then, since E_0 is a supersingular elliptic curve, it has no 2-torsion points. Therefore, $\mathrm{Pic}^0(E_0)$ has also no 2-torsion points. Since the normal bundle $\mathcal{O}(E_0)|_{E_0} \in \mathrm{Pic}^0(E_0)$ and $(\mathcal{O}(E_0)|_{E_0})^{\otimes 2}$ is a trivial invertible sheaf, $\mathcal{O}(E_0)|_{E_0}$ itself is trivial. Therefore, $2E_0$ is a wild fiber (See Bombieri-Mumford [3], and Katsura-Ueno [15]). The canonical divisor formula is given by

$$\begin{aligned} &K_X = g^*(K_{\mathbf{P}^1} - L) + mE_0 \quad \text{with an integer } m \ (0 \leq m \leq 1), \\ &-\deg\, L = \chi(X, \mathcal{O}_X) + t. \end{aligned}$$

Here, $L = R^1 g_* \mathcal{O}_X / \mathcal{T}$ with $\mathcal{T}$ the torsion of $R^1 g_* \mathcal{O}_X$, and t is the rank of the torsion part of $\mathrm{R}^1 g_* \mathcal{O}_X$. There exist wild fibers if and only if $t \geq 1$ (cf. Bombieri-Mumford [3]). Since $2E_0$ is wild, we see $t \geq 1$. Since K_X is numerically trivial and $\deg K_{\mathbf{P}^1} = -2$, considering the intersection of K_X with a hyperplane section, we have

$$0 = (-2 + 1 + t) + \frac{m}{2}.$$

Since $t \geq 1$ and $m \geq 0$, we conclude that $t = 1$ and $m = 0$. Therefore, we have $K_X \sim 0$. Since the second Betti number $b_2(X) = 10$, X is either singular Enriques surface or supersingular Enriques surface. On the other hand, since Y is a supersingular $K3$ surface, X is not a singular Enriques surface by Lemma 4.1. Hence, we conclude that X is a supersingular Enriques surface.

(ii) We assume $a + b = ab$ and $a \notin \mathbf{F}_4$. Then, the only fibers of the elliptic fibration $f : Y \longrightarrow \mathbf{P}^1$ that are integral for D are the ones over the point P_a defined by $t = a$ and over the point P_b defined by $t = b$ (Lemma 4.2). Let $g^{-1}(F(P_b)) = 2E_b$ and $g^{-1}(F(P_a)) = 2E_a$ be two multiple fibers. Then, the canonical divisor formula is given by

$$\begin{aligned} &K_X = g^*(K_{\mathbf{P}^1} - L) + m_a E_a + m_b E_b \\ &\qquad \text{with integers } m_a \text{ and } m_b \ (0 \leq m_a, m_b \leq 1) \\ &-\deg\, L = \chi(X, \mathcal{O}_X) + t. \end{aligned}$$

Here, $L = R^1 g_* \mathcal{O}_X / \mathcal{T}$ with $\mathcal{T}$ the torsion of $R^1 g_* \mathcal{O}_X$, and t is the rank of the torsion part of $\mathrm{R}^1 g_* \mathcal{O}_X$. Suppose both E_a and E_b are wild. Then we have $t \geq 2$. Therefore, we have $\deg(K_{\mathbf{P}^1} - L) \geq -2 + 1 + 2 = 1$. Hence, K_X is not numerically equivalent to zero, a contradiction.

Now, suppose only one of E_a and E_b, say E_b, is wild. Then, $K_X = g^*(K_{\mathbf{P}^1} - L) + E_a + m_b E_b$ with an integer m_b $(0 \leq m_b \leq 1)$ and $t \geq 1$. Then, we have $\deg(K_{\mathbf{P}^1} - L) \geq -2 + 1 + 1 = 0$. Therefore, we have $K_X \succ E_a$ and K_X is not numerically equivalent to zero, a contradiction.

Therefore, both E_a and E_b are tame, and the canonical divisor is given by

$$K_X = g^*(K_{\mathbf{P}^1} - L) + E_a + E_b \quad \text{with } \chi(X, \mathcal{O}_X) = 1, t = 0.$$

Therefore, K_X is not linearly equivalent to zero and $2K_X \sim 0$. Since $b_2(X) = 10$, we conclude that X is a classical Enriques surface. □

Remark 4.9. According to Namikawa [18], Theorem (6.7), there are only a finite number of ellitptic fibrations on an Enriques surface up to automorphism in characteristic 0. We don't know whether the number is finite or not in positive characteristic. However, by a similar result to Namikawa [18], Proposition (6.6), we see that the number is at most countably infinite up to automorphism in positive charcacteristic. The j-invariant of the elliptic curve which appears as the fiber E_a defined by $t = a$ of $f : Y \longrightarrow \mathbf{P}^1$ is equal to $a^{24}/(1 + a^3)^6$. Consider the multiple fiber $2E'_a$ on the Enriques surface X which is the image of E_a. Since we have a purely inseparable morphism of degree 2 from E_a to E'_a, we see that the j-invariant of E'_a is equal to $a^{48}/(1 + a^3)^{12}$. We take an algebraically closed field K such that $K \supset k$ and such that the cardinality of K is uncountably infinite. We consider $f : Y \longrightarrow \mathbf{P}^1$ with parameter a as a family defined over K. Then, the number of different elliptic curves which appear as multiple fibers of Enriques surfaces in our family is uncountably infinite. Therefore, by taking account of Namikawa's result, in our family of Enriques surfaces there are infinitely many non-isomorphic ones over K and so the family is not isotrivial. Therefore, we conclude that our family of Enriques surfaces defined over k is not isotrivial, either.

5. Thirty nodal curves

We use the same notation in the previous sections. We call a nonsingular rational curve on a $K3$ or an Enriques surface a nodal curve. In this section and the next we will show that there exist thirty nodal curves and 10 non-effective (-2)-divisors on X.

First we recall some results for the supersingular $K3$ surface Y with Artin invariant 1 in Dolgachev-Kondo [8]. The Néron-Severi lattice $\mathrm{NS}(Y)$ is an even lattice of signature $(1, 21)$ isomorphic to $U \oplus D_{20}$. The $K3$ surface

Y is obtained as the minimal resolution of a purely inseparable double cover of the projective plane $\mathbf{P}^2$. We denote by

$$p : Y \to \mathbf{P}^2$$

the morphism obtained by the composition of the purely inseparable double cover of $\mathbf{P}^2$ and the minimal resolution. The purely inseparable double cover of $\mathbf{P}^2$ has 21 ordinary nodes over 21 $\mathbf{F}_4$-rational points $\mathbf{P}^2(\mathbf{F}_4)$. Thus we have a set $\mathcal{A}$ consisting of 21 disjoint nodal curves on Y as exceptional divisors. On the other hand the pullbacks of 21 lines in $\mathbf{P}^2(\mathbf{F}_4)$ form a set $\mathcal{B}$ consisting of 21 disjoint nodal curves on Y. Therefore Y contains 42 nodal curves. These curves form a $(21)_5$-configuration, that is, they are divided into two families $\mathcal{A}$ and $\mathcal{B}$ each of which consists of 21 disjoint curves, and each curve in one family meets exactly 5 curves in another family at one point transversely. Recall that Y has a structure of an elliptic fibration

$$f : Y \to \mathbf{P}^1$$

with four singular fibers of type I_6 and 18 sections (see (4.2)). The above 42 nodal curves coincide with the set of 24 irreducible components of singular fibers and 18 sections of the fibration f.

The action of the projective transformation group $\mathrm{PGL}(3, \mathbf{F}_4)$ on the plane can be lifted to an action by automorphisms of Y. Also there exists an involution σ of Y, called a switch, changing two families $\mathcal{A}$ and $\mathcal{B}$. The semi-direct product $\mathrm{PGL}(3, \mathbf{F}_4) \rtimes \mathbf{Z}/2\mathbf{Z}$ preserves the 42 nodal curves. Here $\mathbf{Z}/2\mathbf{Z}$ is generated by σ. Moreover there exist 168 involutions of Y as follows. A set of six points in $\mathbf{P}^2(\mathbf{F}_4)$ is called general if any three points in the set are not collinear. There are 168 general sets of six points. For each general set of six points, we associate the Cremona transformation of the plane which can be lifted to an involution of Y. We call this involution the Cremona transformation associated with a general set I of six points and denote it by Cr_I. The action of Cr_I on $\mathrm{NS}(Y)$ is the reflection associated with a (-4)-vector

$$2\ell - (C_1 + \cdots + C_6) \tag{5.1}$$

in $\mathrm{NS}(Y)$. Here ℓ is the class of the pullback of a line in the projective plane by p and $C_1, \ldots, C_6$ are exceptional curves over the six points in I. It is known that the group $\mathrm{Aut}(Y)$ is generated by $\mathrm{PGL}(3, \mathbf{F}_4)$, σ and 168 Cremona transformations (Dolgachev-Kondo [8], Main theorem 1.1).

Let X be the Enriques surface given in Theorem 4.8. It is known that the Néron-Severi lattice modulo torsions, denoted by $\mathrm{Num}(X)$, is isomorphic to $U \oplus E_8$ which is an even unimodular lattice of signature $(1, 9)$ (see Cossec-Dolgachev [5], Theorem 2.5.1). Consider the map

$$\tilde{\pi} = \varphi \circ \pi : Y \to X$$

where $\pi : Y \to Y^D$ and $\varphi : Y^D \to X$ are given in Section 4. Then $\tilde{\pi}^*(\mathrm{Num}(X))$ is a primitive sublattice in $\mathrm{NS}(Y)$ isomorphic to $U(2) \oplus E_8(2)$ because $\langle \tilde{\pi}^* D, \tilde{\pi}^* D' \rangle = 2\langle D, D' \rangle$. Denote by $E_1, \ldots, E_{12}$ the 12 disjoint integral nodal curves on Y which are contracted under the map $\tilde{\pi}$ (In the equation (4.3) in Section 4, we denote them by E_{12}, E_{14}, E_{16}, $E_{\omega 1}$, $E_{\omega 3}$, $E_{\omega 5}$, $E_{\omega^2 1}$, $E_{\omega^2 3}$, $E_{\omega^2 5}$, $E_{\infty 2}$, $E_{\infty 4}$, $E_{\infty 6}$). Note that these 12 curves consist of 6 curves in $\mathcal{A}$ and 6 curves in $\mathcal{B}$. Let $A_1^{\oplus 12}$ be the sublattice in $\mathrm{NS}(Y)$ generated by $E_1, \ldots, E_{12}$. Obviously $A_1^{\oplus 12}$ is orthogonal to $\tilde{\pi}^*(\mathrm{Num}(X))$.

As mentioned above, there are 42 nodal curves on Y. Among them, 12 curves $E_1, \ldots, E_{12}$ are integral and contracted by $\tilde{\pi}$. In the following we discuss the remaining thirty non-integral curves. Let F be a remaining non integral nodal curve. Note that F meets exactly two curves among $E_1, \ldots, E_{12}$ and the image $\pi(F)$ has the self-intersection number -4 by Proposition 2.1. Therefore, the image $\tilde{\pi}(F)$ is a nodal curve. Let F' be another remaining curve. If $\langle F, F' \rangle = 1$, then $\tilde{\pi}(F)$ meets $\tilde{\pi}(F')$ at one point with multiplicity 2. Assume that F belongs to the family $\mathcal{A}$. Recall that F meets 5 curves in $\mathcal{B}$. Denote by E, E', F_1, F_2, F_3 the curves meeting with F where E, E' are integral, that is, they belong to $\{E_1, \ldots, E_{12}\}$. Assume that E meets $F, G_1, \ldots, G_4$ and E' meets $F, G'_1, \ldots, G'_4$. Obviously $G_1, \ldots, G_4, G'_1, \ldots, G'_4$ belong to $\mathcal{A}$. Then the image $\tilde{\pi}(F)$ meets three curves $\tilde{\pi}(F_i)$ $(i = 1, 2, 3)$ with multiplicity 2 and meets 4 curves $\tilde{\pi}(G_i), 1 \leq i \leq 4$, (resp. $\tilde{\pi}(G'_i), 1 \leq i \leq 4$) at the point $\tilde{\pi}(E)$ (resp. $\tilde{\pi}(E')$). We now get the following lemma (see also Figure 2).

Lemma 5.1. *There exist thirty nodal curves on X which are the images of the* 30 *nodal curves not belonging to $\{E_1, \ldots, E_{12}\}$. Let $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ be the families of nodal curves which are the images of curves in $\mathcal{A}$ and $\mathcal{B}$ respectively. Each nodal curve in one family is tangent to three nodal curves in another family. Each nodal curve C in one family meets eight nodal curves in the same family transversely. Moreover, four of these eight nodal curves meet at one point on C, and the remaining four meet at a different point on C.*

Remark 5.2. Recall that there exists an elliptic fibration $g : X \to \mathbf{P}^1$ which is induced from the elliptic fibration $f : Y \to \mathbf{P}^1$. The fibration f has four singular fibers of type I_6 and 18 sections. Since 12 disjoint nodal curves in the singular fibers of f are contracted to points in X, g has four singular fibers of type I_3 and eighteen 2-sections. The thirty nodal curves in Lemma 5.1 are the twelve components of singular fibers of g and the eighteen 2-sections. Let C be a nodal curve in $\bar{\mathcal{A}} \cup \bar{\mathcal{B}}$. If C is a component of a singular fiber F of g, then C meets two componets of F and six 2-sections transversely at singular points of F, and is tangent to three 2-sections. If C is a 2-section of g, then C passes through a singular point of two singular fibers and is tangent to a component of the remaining two singular fibers. And C is tangent to one more nodal curve C' which is a 2-section. Two nodal curves C and C' meet with multiplicity 2 at a point on the unique fiber of g which is the supersingular elliptic curve.

In the following we show that the incidence relation between nodal curves in $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ is the same as that of Sylvester's duads and synthemes. First we recall Sylvester's duads and synthemes (see Baker [2], p.220). We denote by ij the transposition of i and j $(1 \leq i \neq j \leq 6)$ which is classically called Sylvester's duad. Six letters $1, 2, 3, 4, 5, 6$ can be arranged in three pairs of duads, for example, (12, 34, 56), called Sylvester's syntheme. (We understand that (12, 34, 56) is the same as (12, 56, 34) or (34, 12, 56)). Duads and Synthemes are in $(3, 3)$ correspondence, that is, each syntheme consists of three duads and each duad belongs to three synthemes. It is possible to choose a set of five synthemes which together contain all the fifteen duads. Such a family is called a total. The number of possible totals is six. And each pair of totals has exactly one syntheme in common. The following table gives the six totals $A, B, \ldots, F$ in its rows, and also in its columns (see Baker [Ba], p.221) :

	A	B	C	D	E	F
A		14,25,36	16,24,35	13,26,45	12,34,56	15,23,46
B	14,25,36		15,26,34	12,35,46	16,23,45	13,24,56
C	16,24,35	15,26,34		14,23,56	13,25,46	12,36,45
D	13,26,45	12,35,46	14,23,56		15,24,36	16,25,34
E	12,34,56	16,23,45	13,25,46	15,24,36		14,26,35
F	15,23,46	13,24,56	12,36,45	16,25,34	14,26,35	

Now we consider the six numbers $1, \ldots, 6$ as the six points on X which are the images of curves in $\mathcal{A}$ contracted by $\tilde{\pi}$, and the six totals $A, \ldots, F$ as the six points on X which are the images of curves in $\mathcal{B}$ contracted by $\tilde{\pi}$. Also consider fifteen duads as fifteen nodal curves in $\bar{\mathcal{A}}$. The transposition ij corresponds to the nodal curve through the two points i and j. On the other hand, consider fifteen synthemes as fifteen nodal curves in $\bar{\mathcal{B}}$. A syntheme corresponds to the nodal curve through the two points corresponding to two totals containg the syntheme. Then two curves in $\bar{\mathcal{A}}$ meet if the corresponding two duads have a common letter, and two curves in $\bar{\mathcal{B}}$ meet if the corresponding two synthemes have no common duads. And the $(3,3)$ correspondence between duads and synthemes describes the intersection relation between fifteen curves in $\bar{\mathcal{A}}$ and fifteen curves in $\bar{\mathcal{B}}$. For example, the nodal curve $(12, 34, 56)$ is tangent to nodal curves $12, 34, 56$ and meets eight nodal curves in $\bar{\mathcal{B}}$ belonging to the totals A or E at the points A and E (see Figure 2).

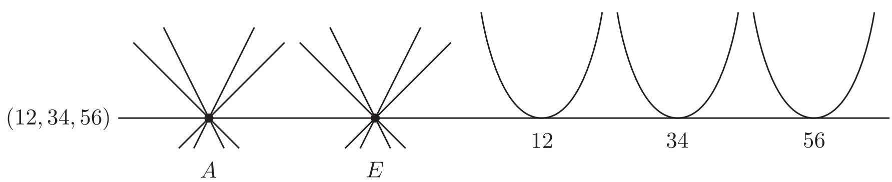

Figure 2: Eleven nodal curves meeting a nodal curve

The nodal curve 12 is tangent to the nodal curves $(12, 34, 56), (12, 35, 46)$ and $(12, 36, 45)$. It also meets eight nodal curves in $\bar{\mathcal{A}}$ containing the letter 1 or 2 at the points 1 and 2. Thus fifteen duads, fifteen synthemes, six letters and six totals are realized on the Enriques surface X geometrically.

6. Ten (-2)-divisors

We keep the same notation as in the previous section. Recall that the $K3$ surface Y has the 168 divisors given in (5.1). In this section, we will see that exactly ten divisors among these 168 divisors descend to (-2)-divisors on the Enriques surface X, and will study the dual graph of thirty nodal curves given in Lemma 5.1 and these ten divisors.

Recall that $E_1, ..., E_{12}$ are the 12 disjoint integral curves on Y which are contracted to points on X under the map $\tilde{\pi}$. For simplicity, we assume that $E_1, \ldots, E_6$ are the pullbacks of six lines $\ell_1, \ldots, \ell_6$ in $\mathbf{P}^2(\mathbf{F}_4)$. For each partition of $\{\ell_1, \ldots, \ell_6\}$ into two sets $\{\ell_i, \ell_j, \ell_k\}$, $\{\ell_l, \ell_m, \ell_n\}$ of three lines, we have six points $q_1, \dots, q_6$ which are the intersection points of three lines ℓ_i, ℓ_j, ℓ_k and those of ℓ_l, ℓ_m, ℓ_n. Let $C_1, \dots, C_6$ be the exceptional curves on Y over $q_1, \dots, q_6$. Thus we have ten divisors

$$2\ell - (C_1 + \cdots + C_6) \tag{6.1}$$

according to the ten partitions of $\{\ell_1, \dots, \ell_6\}$.

Lemma 6.1. *The ten divisors given in* (6.1) *are the divisors among* 168 *divisors which are orthogonal to the root lattice* $A_1^{\oplus 12}$ *generated by* $E_1, \dots, E_{12}$.

Proof. As above, we assume that $E_1, \dots, E_6$ are the pullbacks of six lines $\ell_1, \dots, \ell_6$ in $\mathbf{P}^2(\mathbf{F}_4)$ and $E_7, \dots, E_{12}$ are exceptional curves over $\mathbf{F}_4$-rational points $p_1, \dots, p_6$ of $\mathbf{P}^2$. Obviously $p_1, \dots, p_6$ do not lie on ℓ_i, $1 \le i \le 6$. Moreover the set $\{p_1, \dots, p_6\}$ of six points is general by construction. Let $\tilde{r} = 2\ell - (C_1 + \cdots + C_6)$ be a divisor such that $C_1, \dots, C_6$ are exceptional curves over general six points $q_1, \dots, q_6$ on $\mathbf{P}^2(\mathbf{F}_4)$. Assume that $\langle \tilde{r}, E_j \rangle = 0$ for $j = 1, \dots, 12$. Since $\langle \ell, C_i \rangle = 0$, we see $\langle \tilde{r}, C_i \rangle = 2$. Hence we have $E_j \ne C_i$ $(i = 1, \dots, 6; j = 7, \dots, 12)$. The condition $\langle \tilde{r}, E_j \rangle = 0$ implies that each E_j $(j = 1, \dots, 6)$ meets exactly two curves in $\{C_1, \dots, C_6\}$. This means that the six points $q_1, \dots, q_6$ are intersection points of six lines $\ell_1, \dots, \ell_6$. Thus the divisors $\tilde{r}$ satisfying $\langle \tilde{r}, E_j \rangle = 0$ $(j = 1, \dots, 12)$ correspond to the set of general six points $q_1, \dots, q_6$ which are intersections between $\ell_1, \dots, \ell_6$. We will show that six lines $\ell_1, \dots, \ell_6$ are divided into two sets $\{\ell_i, \ell_j, \ell_k\}$ and $\{\ell_l, \ell_m, \ell_n\}$ such that six points $q_1, \dots, q_6$ coincide with the intersection points of three lines ℓ_i, ℓ_j, ℓ_k and those of ℓ_l, ℓ_m, ℓ_n. Denote by ij the intersection point of ℓ_i and ℓ_j. If six points are given by ij, jk, ki, mn, nl, lm, then we have the desired one. Otherwise six points are given by ij, jk, kl, lm, mn, ni because each letter appears twice. In this case, the line ℓ through ij and kl does not appear in $\{\ell_1, \dots, \ell_6\}$. Since the set $\{p_1, \dots, p_6\}$ of six points is general, ℓ passes exactly two points in $\{p_1, \dots, p_6\}$. Since ℓ contains five $\mathbf{F}_4$-rational points, it should pass one more point not lying on $\ell_i \cup \ell_j \cup \ell_k \cup \ell_l$ because $\ell \cap \{\ell_i \cup \ell_j \cup \ell_k \cup \ell_l\} = \{ij, kl\}$. This implies that ℓ passes the remaining point mn. This contradicts the generality of the six points ij, jk, kl, lm, mn, ni. Thus we have the assertion. □

Let $\tilde{r}_a, \tilde{r}_b, \dots, \tilde{r}_j$ be the ten divisors in $\mathrm{NS}(Y)$ indexed by ten letters $a, b, \dots, j$ which are given in Lemma 6.1. Let $r_a, r_b, \dots, r_j \in \mathrm{Num}(X)$ be the images of $\tilde{r}_a, \tilde{r}_b, \dots, \tilde{r}_j$. Since $\tilde{r}_a^2 = \cdots = \tilde{r}_j^2 = -4$, we have $r_a^2 = \cdots = r_j^2 = -2$. Consider two distinct divisors $\tilde{r}$ and $\tilde{r}'$. Assume that $\tilde{r}$ (resp.

$\tilde{r}'$) corresponds to six points $q_1, ..., q_6$ (resp. $q'_1, ..., q'_6$) which are the union of intersection points of ℓ_i, ℓ_j, ℓ_k and those of ℓ_l, ℓ_m, ℓ_n (resp. the union of intersections of $\ell_{i'}, \ell_{j'}, \ell_{k'}$ and those of $\ell_{l'}, \ell_{m'}, \ell_{n'}$). Note that either $|\{i,j,k\} \cap \{i',j',k'\}| = 2$ or $|\{i,j,k\} \cap \{l',m',n'\}| = 2$. This implies that

$$|\{q_1, \ldots, q_6\} \cap \{q'_1, \ldots, q'_6\}| = 2.$$

Therefore we have $\langle \tilde{r}, \tilde{r}' \rangle = 4$, and hence $\langle r, r' \rangle = 2$. Thus we have the following Lemma.

Lemma 6.2. *The dual graph of* $\{r_a, r_b, \ldots, r_j\}$ *is a complete graph whose edges are double lines.*

Now, we discuss the incidence relation between the ten (-2)-vectors $r_a, \ldots, r_j$ and the fifteen duads, the fifteen synthemes.

Lemma 6.3. *Each vector in* $\{r_a, \ldots, r_j\}$ *meets exactly six duads and six synthemes with intersection multiplicity two.*

Proof. We use the same notation as in the proof of Lemma 6.1. Let C be the nodal curve on Y corresponding to a duad. Then C meets exactly two nodal curves E, E' in $\{E_1, \ldots . E_6\}$. Then $2C + E + E'$ is perpendicular to $A_1^{\oplus 12}$, that is, $2C + E + E' \in \tilde{\pi}^*(\mathrm{Num}(X)) = U(2) \oplus E_8(2)$. Let $\tilde{r} = 2\ell - (C_1 + \cdots + C_6)$ be a divisor in $\{\tilde{r}_a, \ldots, \tilde{r}_j\}$. Then $\langle E, C_1 + \cdots + C_6 \rangle = \langle E', C_1 + \cdots + C_6 \rangle = 2$. If C appears in $\{C_1, \ldots, C_6\}$, then

$$\langle \tilde{r}, 2C + E + E' \rangle = 4,$$

and if C does not appear in $\{C_1, \ldots, C_6\}$, then

$$\langle \tilde{r}, 2C + E + E' \rangle = 0.$$

The proof for which C corresponds to a syntheme is similar. Thus we have the assertion. □

We can identify the ten divisors $r_a, \ldots, r_j$ with the ten symbols

$$(123, 456), (124, 356), (125, 346), (126, 345), (134, 256),$$

$$(135, 246), (136, 245), (145, 236), (146, 235), (156, 234).$$

For example, $(123, 456)$ meets six duads $12, 13, 23, 45, 46, 56$ and six synthemes

$$(14, 25, 36), (14, 26, 35), (15, 24, 36), (15, 26, 34), (16, 24, 35), (16, 25, 34).$$

We denote by Γ the dual graph of the thirty nodal curves and the ten (-2)-divisors.

Remark 6.4. The graph Γ appears in other places. For example, consider the moduli space of principally polarized abelian surfaces with level 2-structure over the field $\mathbf{C}$ of complex numbers. It has fifteen 0-dimensional and fifteen 1-dimensional boundary components and contains ten divisors parametrizing abelian surfaces of product type (e.g. see [10], Proposition 1.1). The incidence relation between these boundary components and the ten divisors is given by the graph Γ. On the other hand, S. Mukai found the existence of the above configuration of 30 nodal curves and ten (-2)-vectors on an Enriques surface defined over $\mathbf{C}$ (unpublished).

Proposition 6.5. *The automorphism group of the graph Γ is isomorphic to the automorphism group $Aut(\mathfrak{S}_6)$ of the symmetric group $\mathfrak{S}_6$ of degree 6.*

Proof. Recall that $\mathrm{Aut}(\mathfrak{S}_6)$ is generated by $\mathfrak{S}_6$ and an outer automorphism. An outer automophism interchanges duads with synthemes, and six letters $1, \ldots, 6$ with six totals $A, \ldots, F$ respectively. Obviously $\mathrm{Aut}(\mathfrak{S}_6)$ preserves the graph Γ. Let g be an automorphism of Γ. If necessary, by composing with an outer automorphism, we assume g preserves six letters. If g fixes each of six letters, then g acts on Γ identically. Thus g is contained in $\mathfrak{S}_6$. □

Remark 6.6. The Néron-Severi lattice $\mathrm{NS}(Y)$ is isomorphic to the orthogonal complement of the root lattice D_4 in the even unimodular lattice $\mathrm{II}_{1,25}$ of signature $(1, 25)$. If we embed $\mathrm{NS}(Y)$ into $\mathrm{II}_{1,25}$ as the orthogonal complement, then 42 nodal curves and 168 (-4)-divisors on Y are the projections of Leech roots into $\mathrm{NS}(Y)$ (see [8], §3.3). The lattice $\tilde{\pi}^*(\mathrm{Num}(X))$ $(\cong U(2) \oplus E_8(2))$ is the orthogonal complement of $D_4 \oplus A_1^{\oplus 12}$ in $\mathrm{II}_{1,25}$, and the above thirty nodal curves and 10 (-2)-divisors on X correspond to the projections of some Leech roots.

7. Automorphisms

Let S be an Enriques surface. Let $\mathrm{Num}(S)$ be the Néron-Severi lattice modulo torsions. Then $\mathrm{Num}(S)$ is an even unimodular lattice of signature $(1, 9)$

(Cossec-Dolgachev [5]). We denote by $\mathrm{O}(\mathrm{Num}(S))$ the orthogonal group of $\mathrm{Num}(S)$. The set

$$\{x \in \mathrm{Num}(S) \otimes \mathbf{R} \ : \ \langle x, x \rangle > 0\}$$

has two connected components. Denote by $P(S)$ the connected component containing an ample class of S. For $\delta \in \mathrm{Num}(S)$ with $\delta^2 = -2$, we define an isometry s_δ of $\mathrm{Num}(S)$ by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta, \quad x \in \mathrm{Num}(S).$$

The s_δ is called the reflection associated with δ.

Let $W(S)$ be the subgroup of $\mathrm{O}(\mathrm{Num}(S))$ generated by reflections associated with all nodal curves on S. Then $P(S)$ is divided into chambers each of which is a fundamental domain with respect to the action of $W(S)$ on $P(S)$. There exists a unique chamber containing an ample class which is nothing but the closure of the ample cone $D(S)$ of S. It is known that the natural map Consider the natural map

$$(7.1) \qquad \rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Num}(S)).$$

has a finite kernel (Dolgachev [7], Theorem 4). Denote by $\mathrm{Aut}(S)^*$ the image of the map (7.1). Define

$$\mathrm{Aut}(D(S)) = \{\varphi \in \mathrm{O}(\mathrm{Num}(S)) : \varphi(D(S)) = D(S)\}.$$

Since $\mathrm{Aut}(S)$ preserves $D(S)$, we see $\mathrm{Aut}(S)^* \subset \mathrm{Aut}(D(S))$ and $\mathrm{Aut}(S)^* \cap W(S) = \{1\}$. On the other hand, $\mathrm{Aut}(S)^*$ and $W(S)$ generate a subgroup of $\mathrm{O}(\mathrm{Num}(S))$. In particular $\mathrm{Aut}(S)$ is finite if $W(S)$ is of finite index in $\mathrm{O}(\mathrm{Num}(S))$. Over the field of complex numbers, Enriques surfaces with finite group of automorphisms were classified by Nikulin [19] and the second author [16]. In general it is difficult to describe the group $\mathrm{Aut}(D(S))$.

Now, we recall Vinberg's criterion which guarantees that a group generated by finite number of reflections is of finite index in $\mathrm{O}(\mathrm{Num}(S))$.

Let Δ be a finite set of (-2)-vectors in $\mathrm{Num}(S)$. Let Γ be the graph of Δ, that is, Δ is the set of vertices of Γ and two vertices δ and δ' are joined by m-tuple lines if $\langle \delta, \delta' \rangle = m$. We assume that the cone

$$K(\Gamma) = \{x \in \mathrm{Num}(S) \otimes \mathbf{R} \ : \ \langle x, \delta_i \rangle \geq 0, \ \delta_i \in \Delta\}$$

is a strictly convex cone. Such Γ is called non-degenerate. A connected parabolic subdiagram Γ' in Γ is a Dynkin diagram of type $\tilde{A}_m$, $\tilde{D}_n$ or $\tilde{E}_k$

(see [26], p. 345, Table 2). If the number of vertices of Γ' is $r+1$, then r is called the rank of Γ'. A disjoint union of connected parabolic subdiagrams is called a parabolic subdiagram of Γ. We denote by $\tilde{K_1} \oplus \tilde{K_2}$ a parabolic subdiagram which is a disjoint union of two connected parabolic subdiagrams of type $\tilde{K_1}$ and $\tilde{K_2}$, where K_i is A_m, D_n or E_k. The rank of a parabolic subdiagram is the sum of the rank of its connected components. Note that the dual graph of singular fibers of an elliptic fibration on S gives a parabolic subdiagram. For example, a singular fiber of type III, IV or I_{n+1} defines a parabolic subdiagram of type $\tilde{A}_1$, $\tilde{A}_2$ or $\tilde{A}_n$ respectively. We denote by $W(\Gamma)$ the subgroup of $\mathrm{O}(\mathrm{Num}(S))$ generated by reflections associated with $\delta \in \Gamma$.

Proposition 7.1. (Vinberg [26], Theorem 2.3) *Let Δ be a set of (-2)-vectors in Num(S) and let Γ be the graph of Δ. Assume that Δ is a finite set, Γ is non-degenerate and Γ contains no m-tuple lines with $m \geq 3$. Then $W(\Gamma)$ is of finite index in $\mathrm{O}(Num(S))$ if and only if every connected parabolic subdiagram of Γ is a connected component of some parabolic subdiagram in Γ of rank 8 (= the maximal one).*

For the proof of Proposition 7.1, see Vinberg [26] (also see [16], Theorem 1.9).

Let X be the Enriques surface given in Theorem 4.8. In the following, as Δ we take forty (-2)-vectors in $\mathrm{Num}(X)$ corresponding to the fifteen duads, the fifteen synthemes and the ten (-2)-vectors given in the previous section. Let Γ be the graph of these forty vectors. We directly see the following Lemma.

Lemma 7.2. *The maximal parabolic subdiagrams of Γ are*

$$\tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2,\ \ \tilde{A}_4 \oplus \tilde{A}_4,\ \ \tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1,\ \ \tilde{A}_3 \oplus \tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$$

each of which has the maximal rank 8.

In the following we give an example of each maximal parabolic subdiagrams.

(i) The diagram $\tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2$ corresponds to an elliptic fibration on X with four singular fibers of type I_3. For example, four sets

$$\{12, 23, 13\}, \{45, 46, 56\}, \{(14, 25, 36), (15, 26, 34), (16, 24, 35)\},$$

$$\{(14, 26, 35), (15, 24, 36), (16, 25, 34)\}$$

are components of singular fibers of an elliptic fibration of this type. The syntheme $(12, 35, 46)$ is a 2-section of this fibration.

(ii) The diagram $\tilde{A}_4 \oplus \tilde{A}_4$ corresponds to an elliptic fibration on X with two singular fibers of type I_5. For example, two sets $\{12, 23, 34, 45, 15\}$ and $\{(13, 25, 46), (14, 26, 35), (13, 24, 56), (14, 25, 36), (16, 24, 35)\}$ are components of singular fibers of an elliptic fibration and the duad 46 is a 2-section of this fibration.

(iii) The diagram $\tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1$ corresponds to an elliptic fibration on X with singular fibers of type I_6, IV and I_2. For example, six synthemes

$$(14, 25, 36), (15, 26, 34), (14, 23, 56), (15, 24, 36), (14, 26, 35), (15, 23, 46)$$

are components of a singular fiber of type I_6, three duads $12, 13, 16$ are components of a singular fiber of type IV. The pair of the duad 45 and (-2)-vector $(145, 236)$ forms the subdiagram of type $\tilde{A}_1$. The duad 56 is a 2-section of this fibration.

Remark 7.3. Note that there exists a nodal curve C such that C and the duad 45 form the singular fiber of type I_2. If we denote by $2f$ the class of a multiple fiber of this fibration, then

$$(145, 236) = C - f.$$

The 2-section 56 meets C, but not $(145, 236)$. Note that C does not appear in forty (-2)-vectors.

(iv) The diagram $\tilde{A}_3 \oplus \tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$ corresponds to an elliptic fibration on X with two singular fibers of type I_4 and one singular fiber of type III. For example, four duads $24, 25, 34, 35$ and four synthemes

$$(12, 36, 45), (14, 23, 56), (13, 26, 45), (15, 23, 46)$$

define two singular fibers of type I_4 respectively, and the pair of the duad 16 and the syntheme $(16, 23, 45)$ defines a singular fiber of type III. The remaining subdiagram of type $\tilde{A}_1$ consists of two (-2)-vectors $(123, 456)$ and $(145, 236)$. The duad 13 is a 2-section of this fibration.

Remark 7.4. The pullback of an elliptic fibration given in (i), (ii), (iii) or (iv) to the covering $K3$ surface Y gives an elliptic fibration on Y with reducible singular fibers of type $(\mathrm{I}_6, \mathrm{I}_6, \mathrm{I}_6, \mathrm{I}_6)$, of type $(\mathrm{I}_{10}, \mathrm{I}_{10}, \mathrm{I}_2, \mathrm{I}_2)$, of type $(\mathrm{I}_{12}, \mathrm{IV}^*, \mathrm{I}_4)$, or of type $(\mathrm{I}_8, \mathrm{I}_8, \mathrm{I}_1^*)$, respectively.

Denote by $D(\Gamma)$ the finite polyhedron defined by forty (-2)-vectors in Γ. Combining Proposition 7.1 and Lemma 7.2, we have the following theorem.

Theorem 7.5. *The group $W(\Gamma)$ is of finite index in $\mathrm{O}(Num(X))$. The symmetry group of the graph Γ coincides with $Aut(D(\Gamma))$ which is isomorphic to the semi-direct product $\mathfrak{S}_6 \rtimes \mathbf{Z}/2\mathbf{Z}$ where $\mathfrak{S}_6$ is the symmetric group of the six letters $\{1,\ldots,6\}$ and $\mathbf{Z}/2\mathbf{Z}$ is generated by an outer automorphism of $\mathfrak{S}_6$.*

Recall that $\mathrm{Aut}(Y)$ is generated by $\mathrm{PGL}(3,\mathbf{F}_4)$, a switch and 168 Cremona transformations, where Y is the covering $K3$ surface of X. Among these automorphisms, the subgroup $\mathfrak{S}_6 \rtimes \mathbf{Z}/2\mathbf{Z}$ and the ten Cremona transformations associated with the ten divisors given in (6.1) preserve the 12 nodal curves $E_1,\ldots,E_{12}$ contracted under the map $\tilde{\pi}$.

Conjecture. *The subgroup $\mathfrak{S}_6 \rtimes \mathbf{Z}/2\mathbf{Z}$ and the ten Cremona transformations descend to automorphisms of X.*

Let G be the subgroup of $\mathrm{O}(\mathrm{Num}(X))$ generated by the reflections associated with the ten non-effective (-2)-divisors in Γ. If the conjecture is true, then the ten Cremona transformations descend to the ten generators of G. Next we show that ρ is injective (see (7.1)). Let $\varphi \in \mathrm{Ker}(\rho)$. Then φ preserves each nodal curve in Δ. Since each nodal curve in Δ meets eleven nodal curves at five points, φ fixes all 30 nodal curves in Δ pointwisely. Consider the elliptic fibration $g : X \to \mathbf{P}^1$ discussed in Remark 5.2 which has eighteen 2-sections contained in Δ. Then φ fixes a general fiber of g, and hence φ is identity.

Now, by an argument in Vinberg [27], 1.6, we have the following Corollary.

Corollary 7.6. *Assume the conjecture holds. Then $Aut(X)$ is generated by $Aut(D(\Gamma))(\cong \mathfrak{S}_6 \rtimes \mathbf{Z}/2\mathbf{Z})$ and G.*

References

[1] M. Artin, Supersingular $K3$ surfaces, Ann. Sci. École Norm. Sup., 4 (1974), 543–567.

[2] H. F. Baker, Principles of geometry, II, Cambridge University Press 1922.

[3] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. p, II, in "Complex Analysis and Algebraic Geometry" (W. L.

Bailly and T. Shioda, eds.), Iwanami Shoten, Tokyo, and Princeton Univ. Press, Princeton, NJ, 1977, 22–42.

[4] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. p, III, Inventiones Math., 35 (1976), 197–232.

[5] F. Cossec and I. Dolgachev, Enriques surfaces I, Progress in Math., vol. 76, 1989, Birkhäuser.

[6] R. M. Crew, Etale p-covers in characteristic p, Compositio Math., 52 (1984), 31–45.

[7] I. Dolgachev, Numerically trivial automorphisms of Enriques surfaces in arbitrary characteristic, in "Arithmetic and Geometry of $K3$ surfaces and Calabi-Yau threefolds", Fields Institute Communications 67 (2013), 267–283.

[8] I. Dolgachev and S. Kondō, A supersingular $K3$ surface in characteristic 2 and the Leech lattice, IMRN 2003 (2003), 1–23.

[9] T. Ekedahl, J. M. E. Hyland and N. I. Shepherd-Barron, Moduli and periods of simply connected Enriques surfaces, arXiv:1210.0342.

[10] G. van der Geer, On the geometry of a Siegel modular threefold, Math. Ann., 260 (1982), 317–350.

[11] J. Igusa, Betti and Picard numbers of abstract algebraic surfaces, Proc. Nat. Acad. Sci. U.S.A., 46 (1960), 724–726

[12] T. Katsura, Surfaces unirationnelles en caractéristique p, C. R. Acad. Sc. Paris, t. 288 (1979), 45–47.

[13] T. Katsura and Y. Takeda, Quotients of abelian and hyperelliptic surfaces by rational vector fields, J. Algebra, 124 (1989), 472–492.

[14] T. Katsura and S. Kondō, A note on a supersingular $K3$ surface in chrateristic 2, EMS Series of Congress Reports, in "Geometry and Arithmetic" (C. Faber, G. Farkas, R. de Jong, eds.), 2012, 243–255.

[15] T. Katsura and K. Ueno, On elliptic surfaces in characteristic p, Math. Ann., 272 (1985), 291–330.

[16] S. Kondō, Enriques surfaces with finite automorphism groups, Japanese J. Math., 12 (1986), 191–282.

[17] C. Liedtke, Arithmetic moduli and liftings of Enriques surfaces, J. Reine Angew. Math., **706** (2015), 35–65.

[18] Y. Namikawa, Periods of Enriques surfaces, Math. Ann. 270 (1985), 201–222.

[19] V. Nikulin, On a description of the automorphism groups of Enriques surfaces, Soviet Math. Dokl., 30 (1984), 282–285.

[20] A. N. Rudakov and I. R. Shafarevich, Inseparable morphisms of algebraic surfaces, Izv. Akad. Nauk SSSR Ser. Mat., 40 (1976), 1269–1307.

[21] A. N. Rudakov and I. R. Shafarevich, Supersingular surfaces of type $K3$ over fields of characteristic 2, Izv. Akad. Nauk SSSR Ser. Mat., 42 (1978), 848–869.

[22] A. N. Rudakov and I. R. Shafarevich, Surfaces of type $K3$ over fields of finite characteristic, J. Soviet Math. 22 (1983), 1476–1533.

[23] T. Shioda, An example of unirational surfaces in characteristic p, Math. Ann., 211 (1974), 233–236.

[24] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan, 24 (1972), 20–59.

[25] J. Tate, Algorithm for determining the type of a singular fiber in an elliptic pencil, in "Modular Functions of One Variable IV", Lecture Notes in Math. 476, Springer-Verlag, Berlin·Heidelberg·New York, 1970, 33–52.

[26] E. B. Vinberg, Some arithmetic discrete groups in Lobachevskii spaces, in "Discrete subgroups of Lie groups and applications to Moduli", Tata-Oxford (1975), 323–348.

[27] E. B. Vinberg, The two most algebraic $K3$ surfaces, Math. Ann., 265 (1983), 1–21.

Toshiyuki Katsura
Faculty of Science and Engineering, Hosei University,
Koganei-shi, Tokyo 184-8584, Japan
E-mail: toshiyuki.katsura.tk@hosei.ac.jp

Shigeyuki Kondō
Graduate School of Mathematics, Nagoya University,
Nagoya, 464-8602, Japan
E-mail: kondo@math.nagoya-u.ac.jp