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The Allcock Ball Quotient

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To Eduard Looijenga for his 69th birthday

Abstract: In this article we provide further evidence for the monstrous proposal of Daniel Allcock, by giving a plausible but still conjectural explanation for the deflation relation in the Coxeter group quotient of the orbifold fundamental group.
Keywords: ball quotients, bimonster, Allcock conjecture.

1. Introduction

A simply laced Coxeter diagram is just a graph for which any two distinct nodes are either disconnected or connected by a single bond. All Coxeter diagrams in this paper are simply laced, and therefore we shall simply write Coxeter diagram for the longer phrase simply laced Coxeter diagram. Standard examples are the Coxeter diagrams of type A_n with n nodes labeled $1, \cdots, n$ and only successive nodes are connected, or the Coxeter diagram of type $\tilde{A}_n$ with $(n+1)$ nodes labeled $0, 1, \cdots, n$ with successive vertices connected together with a connection from n to 0.

Deleting some nodes from a Coxeter diagram together with all bonds connected with at least one of them gives a Coxeter subdiagram. For example the Coxeter diagram of type $\tilde{A}_n$ has the Coxeter diagram of type A_n as subdiagram by deleting the node with label 0 and the two bonds connected to this node. A Coxeter subdiagram of type $\tilde{A}_m$ in a bigger Coxeter diagram X_n is called a free $(m+1)$-gon in X_n. For $p, q, r \in \mathbb{N}$ the Coxeter tree diagram of type Y_{pqr} has $n = (p+q+r)+1$ vertices labeled $0, 1, \cdots, (p+q+r)$, with a unique triple node 0 connected to the first nodes of three Coxeter diagrams of types A_p, A_q, A_r. Of special interest are the Y_{pqr} Coxeter diagrams of the finite type $A_n = Y_{(n-1)00}$, $D_n = Y_{(n-3)11}$ for $n \geq 4$ and $E_n = Y_{(n-4)21}$ for

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$n = 6, 7, 8$ for which

$$1/(p+1) + 1/(q+1) + 1/(r+1) > 1 ,$$

and of the affine type $\tilde{A}_n$, $\tilde{D}_n$ and $\tilde{E}_6 = Y_{222}$, $\tilde{E}_7 = Y_{331}$, $\tilde{E}_8 = Y_{521}$ for which

$$1/(p+1) + 1/(q+1) + 1/(r+1) = 1 .$$

If X_n is some Coxeter diagram with n vertices then the Artin group $A(X_n)$ is by definition the group with generators T_i for each node i of X_n and relations

$$T_iT_j = T_jT_i \ , \ T_iT_jT_i = T_jT_iT_j$$

if either i and j are disconnected or connected respectively. In the former case T_i and T_j commute and in the latter case they braid. The quotient group of $A(X_n)$ by the quadratic relations

$$T_i^2 = 1$$

is called the Coxeter group $W(X_n)$ of type X_n. For a connected Coxeter diagram X_n the group $W(X_n)$ is finite precisely for the finite type diagrams. For the affine type diagrams $\tilde{X}_n$ (with $X = A, D, E$) the Coxeter group $W(\tilde{X}_n)$ has a free Abelian normal subgroup of rank n with finite quotient group $W(X_n)$. In this case the quotient map

$$W(\tilde{X}_n) \to W(X_n)$$

is called deflation, and for the diagram $\tilde{X}_n = \tilde{A}_n$ one speaks about deflation of this free $(n+1)$-gon.

If the connected Coxeter diagram of some type X_n is neither of finite type nor of affine type then the Coxeter group $W(X_n)$ is of exponential growth. However for some special Coxeter diagrams the group $W(X_n)$ has a remarkable finite quotient with a fairly simple presentation.

Label the generators of the Artin group $A(\tilde{E}_6)$ by $a, b_1, b_2, b_3, c_1, c_2, c_3$ with a the generator corresponding to the triple node, with c_1, c_2, c_3 the three generators corresponding to the three extremal nodes and b_i the generator that braids with a and c_i for $i = 1, 2, 3$. The element

$$s = ab_1c_1ab_2c_2ab_3c_3$$

is called the spider element. The next remarkable result is due to Ivanov and Norton [27],[34].

Theorem 1.1 (Ivanov and Norton). *The group $W(Y_{555})$ modulo the spider relation $s^{10} = 1$ is equal to the wreath product $M \wr 2 = (M \times M) \rtimes S_2$ (also called the bimonster) with M the Fischer–Griess monster simple group.*

The impact of the relation $s^{10} = 1$ in $W(\tilde{E}_6)$ is to condense the six dimensional translation (root) lattice Q to the finite group $Q/3P$ of shape 3^5 with $P > Q$ the index three (weight) overlattice, as I understand from Simon Norton.

Conway and Simons showed that by increasing the number of generators this presentation takes a simpler form [14]. Let I_{26} be the incidence graph of the projective plane $\mathbb{P}^2(3)$ over a field of 3 elements. The nodes are the points and the lines of the projective plane, and two nodes are connected if they are incident. The Coxeter diagram Y_{555} is a maximal subtree of I_{26}.

Theorem 1.2 (Conway and Simons). *The bimonster $M \wr 2$ is obtained from the Coxeter group $W(I_{26})$ by deflating all free 12-gons in I_{26}.*

We shall denote $\omega = (-1 + \sqrt{-3})/2$ and $\theta = \omega - \overline{\omega} = \sqrt{-3}$. Let $\mathcal{E} = \mathbb{Z} + \mathbb{Z}\omega$ be the ring of Eisenstein integers. By an Eisenstein lattice L we shall mean a free $\mathcal{E}$-module of finite rank with a Hermitian form $\langle \cdot, \cdot \rangle$ on L such that $\langle \lambda, \mu \rangle \in \theta\mathcal{E}$ for all $\lambda, \mu \in L$. A vector $\varepsilon \in L$ with norm $\langle \varepsilon, \varepsilon \rangle = 3$ is called a root in L. The triflection

$$t_\varepsilon(\lambda) = \lambda + (\omega - 1)\frac{\langle \lambda, \varepsilon \rangle}{\langle \varepsilon, \varepsilon \rangle}\varepsilon$$

with root ε is an order three complex reflection leaving L invariant. We denote by $\mathrm{U}(L)$ the group of all unitary automorphisms of the Eisenstein lattice L. An Eisenstein lattice L is called Lorentzian if its Hermitian form $\langle \cdot, \cdot \rangle$ is nondegenerate of signature $(\mathrm{rk}(L) - 1, 1)$, and called Euclidean if $\langle \cdot, \cdot \rangle$ is positive definite. In the Lorentzian case

$$\mathbb{B}(L) = \mathbb{P}(\{z \in \mathbb{C} \otimes L; \langle z, z \rangle < 0\})$$

is the complex hyperbolic ball associated with L. The group $\Gamma(L) := \mathrm{PU}(L)$ acts properly discontinuously on $\mathbb{B}(L)$ with quotient space

$$(\mathbb{B}/\Gamma)(L) := \mathbb{B}(L)/\Gamma(L)$$

the ball quotient associated with L. For a root $\varepsilon \in L$ the hyperball

$$\mathbb{P}(\{z \in \mathbb{C} \otimes L; \langle z, z \rangle < 0, \langle z, \varepsilon \rangle = 0\})$$

is called the mirror for the root ε, and we write $\mathbb{B}^\circ(L)$ for the complement in $\mathbb{B}(L)$ of all such mirrors. The quotient of all the mirrors in $\mathbb{B}(L)$ is a divisor $\Delta(L)$ in $(\mathbb{B}/\Gamma)(L)$, called the discriminant, and so $(\mathbb{B}^\circ/\Gamma)(L)$ is called the discriminant complement.

A connected Coxeter diagram of some type X_n is called bipartite if the n nodes can be coloured black or white, such that bonds only connect black and white nodes. For a Coxeter tree diagram such a bipartition is always possible, and for the incidence diagram I_{26} one just colours points black and lines white. With a bipartite Coxeter diagram X_n we can associate an Eisenstein lattice $L(X_n)$ with basis ε_i indexed by the nodes. The Hermitian form is defined by

$$\langle\varepsilon_i,\varepsilon_i\rangle = 3\ ,\ \langle\varepsilon_i,\varepsilon_j\rangle = 0\ ,\ \langle\varepsilon_p,\varepsilon_l\rangle = \theta$$

for all i, for all disconnected $i \neq j$ and for all connected black p and white l. It is easily checked that the map

$$A(X_n) \to \mathrm{U}(L(X_n))\ ,\ T_i \mapsto t_{\varepsilon_i}$$

extends to a Hermitian representation of the Artin group $A(X_n)$ on the Eisenstein lattice $L(X_n)$. In fact, for a Coxeter tree diagram this is just the reflection representation of the Hecke algebra of type X_n (with quadratic Hecke relation $(T-1)(T+q)=0$) with parameter $q=-\omega$ as constructed by Curtis, Iwahori and Kilmoyer [18].

The automorphism group $\mathrm{U}(L(A_4))$ of the Euclidean Eisenstein lattice $L(A_4)$ is generated by triflections (Theorem 5.2 of [1]), and is equal to the group ST32 in the Shephard–Todd list of finite irreducible complex reflection groups [38]. Let us denote by H the Eisenstein hyperbolic plane, with basis $\varepsilon_1,\varepsilon_2$ and Hermitian form given by $\langle\varepsilon_1,\varepsilon_1\rangle = \langle\varepsilon_2,\varepsilon_2\rangle = 0$ and $\langle\varepsilon_1,\varepsilon_2\rangle = \theta$. The following two Lorentzian Eisenstein lattices

$$\begin{aligned} L_{\mathrm{DM}} &= H \oplus L(A_4) \oplus L(A_4) \\ L_A &= H \oplus L(A_4) \oplus L(A_4) \oplus L(A_4) \end{aligned}$$

play a central role in this paper, and we shall call them the Deligne–Mostow lattice and the Allcock lattice respectively. The ball quotient $(\mathbb{B}/\Gamma)(L_{\mathrm{DM}})$ is the largest dimensional one on the list of Deligne–Mostow ball quotients associated with Lauricella hypergeometric period integrals [20],[33],[42]. The Eisenstein lattice $L(A_{11})$ has a one dimensional kernel with quotient lattice L_{DM}. Likewise the Eisenstein lattice $L(Y_{555})$ has a two dimensional kernel with quotient lattice L_{A}. Hence the triflection representations on $L(A_{11})$ and $L(Y_{555})$ induce natural homomorphisms

$$\mathrm{Br}_{12}(\mathbb{C}) = A(A_{11}) \to \mathrm{U}(L_{\mathrm{DM}})\ ,\ A(Y_{555}) \to \mathrm{U}(L_{\mathrm{A}})$$

with $\mathrm{Br}_{12}(\mathbb{C})$ the Artin braid group on 12 strands in $\mathbb{C}$. Both these homomorphisms are surjective. For the Deligne–Mostow lattice this was shown by Allcock (in Theorem 5.1 of [1]), and for the Allcock lattice this has been proven by Basak (in Theorem 1.1 of [7]).

In his monstrous proposal Allcock made a remarkable conjecture [2].

Conjecture 1.3 (Allcock). *The quotient of the orbifold fundamental group*

$$G(L_{\mathrm{A}}) = \Pi_1^{\mathrm{orb}}((\mathbb{B}^\circ/\Gamma)(L_{\mathrm{A}}))$$

by the normal subgroup N generated by the squares of the meridians is the bimonster $M \wr 2$. By a meridian is meant a small loop in $(\mathbb{B}^\circ/\Gamma)(L_{\mathrm{A}})$ that encircles the discriminant $\Delta(L_{\mathrm{A}})$ once positively at a generic point of $\Delta(L_{\mathrm{A}})$.

The original evidence for Allcock was rather modest and based on the occurrence the Y_{555} diagram both in the Ivanov–Norton theorem and in his description of the lattice L_{A}. Additional evidence for the conjecture of Allcock has been supplied by subsequent work of Basak [7],[8].

Theorem 1.4 (Basak). *The Hermitian form of the Eisenstein lattice $L(I_{26})$ has a kernel of dimension 12 and the quotient of $L(I_{26})$ by this kernel is equal to the Allcock lattice L_{A}.*

This is a remarkable observation, but the proof is straightforward. For l the index of a white node (l for line) put

$$\delta_l = -\theta\varepsilon_l + \sum_{p\sim l} \varepsilon_p$$

with $p \sim l$ meaning that the corresponding nodes are connected (p a point on l). Then an easy verification yields

$$\langle \delta_l, \varepsilon_q \rangle = 0\ ,\ \langle \delta_l, \varepsilon_m \rangle = \theta$$

for all black nodes q and white nodes m. Just distinguish q on l or not on l, and m equal l or not equal l. Hence $\delta_l - \delta_m$ is a null vector for any two white nodes l and m, and these vectors span the kernel of dimension 12.

The quotient of the triflection representation yields a homomorphism

$$A(I_{26}) \to \mathrm{U}(L_{\mathrm{A}})$$

which a fortiori is surjective. By definition the orbifold fundamental group $G(L_{\mathrm{A}})$ of $(\mathbb{B}^\circ/\Gamma)(L_{\mathrm{A}})$ gives rise to an exact sequence

$$1 \to \Pi_1(\mathbb{B}^\circ(L_{\mathrm{A}})) \to G(L_{\mathrm{A}}) \stackrel{\pi}{\to} \Gamma(L_{\mathrm{A}}) \to 1$$

of groups. The following result is due to Basak [8].

Theorem 1.5 (Basak). *There exists a natural homomorphism*

$$\psi : A(I_{26}) \to G(L_{\mathrm{A}})$$

whose composition with $\pi : G(L_{\mathrm{A}}) \to \Gamma(L_{\mathrm{A}})$ *is the triflection homomorphism* $A(I_{26}) \to \Gamma(L_{\mathrm{A}})$ *discussed above.*

Basak makes a convenient choice of base point $w_0 \in \mathbb{B}^\circ(L_{\mathrm{A}})$, which he calls the Weyl point. He shows that there are exactly 26 mirrors in $\mathbb{B}(L_{\mathrm{A}})$ at minimal distance from w_0. The loop in $G(L_{\mathrm{A}})$ starting at w_0 along the shortest geodesic to such a mirror, making a third turn near the mirror and continuing geodesically to the image $t_i w_0$ is denoted by T_i. Notably T_i becomes a meridian in $G(L_{\mathrm{A}})$. Using a computer algorithm Basak shows that these T_i satisfy the braid relations of the incidence diagram I_{26}. The following result was conjectured by Basak [8] and subsequently proved by Allcock and Basak [4].

Theorem 1.6 (Allcock and Basak). *In the notation of the previous theorem the homomorphism* $A(I_{26}) \stackrel{\psi}{\to} G(L_{\mathrm{A}})$ *is surjective.*

Let $\mathbb{B}(V_{\mathrm{A}})$ be the *real* hyperbolic ball of dimension 13 through w_0 containing these 26 geodesics departing from w_0. Each of the 26 mirrors intersects $\mathbb{B}(V_{\mathrm{A}})$ in a real hyperball. If $P \subset \mathbb{B}(V_{\mathrm{A}})$ is the hyperbolic polytope bounded by these 26 hyperballs, then P is an acute angled convex polytope of finite volume by the Vinberg criterion, as explained in more details in Section 5 and Section 8. Based on the analogy with the Deligne–Mostow ball quotient we are inclined to believe that the following conjecture holds.

Conjecture 1.7. *The interior of* P *in* $\mathbb{B}(V_{\mathrm{A}})$ *is contained in* $\mathbb{B}^\circ(L_{\mathrm{A}})$*.*

The epimorphism $\psi : A(I_{26}) \to G(L_{\mathrm{A}})$ descends to an epimorphism $\varphi : W(I_{26}) \to G(L_{\mathrm{A}})/N$ with N the normal subgroup of $G(L_{\mathrm{A}})$ generated by the squares of the meridians. Our conjecture that the interior of the polytope P does not meet any mirrors can be used to show that for each free 12-gon in I_{26} the epimorphism φ factorizes through the deflation of the corresponding subgroup $W(\tilde{A}_{11})$. Hence $\varphi : W(I_{26}) \to G(L_{\mathrm{A}})/N$ factorizes through the bimonster $M \wr 2$ by the Conway–Simons theorem. This provides a good deal of evidence for the conjecture of Allcock.

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2. Theorem of Deligne–Mostow

Let $n \geq 4$ and $0 < \mu_1, \cdots, \mu_n < 1$ be rational numbers with $\sum \mu_i = 2$. Let us write $\mu_i = m_i/m$ with $1 \leq m_1, \cdots, m_n < m$ relatively prime natural numbers. For n distinct ordered points $z_1, \cdots, z_n \in \mathbb{C}$ the curve

$$C(z) : y^m = \prod_1^n (x - z_i)^{m_i}$$

has an action of the group $\mu_m = \sqrt[m]{1}$ by multiplication on y, and the quotient map $C(z) \to \mathbb{P}$ is a ramified covering of the Riemann sphere $\mathbb{P} = \mathbb{C} \sqcup \{\infty\}$. The differential dx/y is holomorphic on $C(z)$ and the period integral

$$\int_{z_i}^{z_j} \frac{dx}{y}$$

along a path on $C(z)$ from a point above z_i to a point above z_j is a Lauricella hypergeometric function, as holomorphic function of the variable $z = (z_1, \cdots, z_n)$. They are solutions of the Lauricella hypergeometric equation, which has a solution space of dimension $n - 2$. The underlying space on which these functions and differential equation live is the moduli space $\mathcal{M}_{0,n}$, and even $\mathcal{M}_{0,n}/S_{\mathbf{m}}$ with $S_{\mathbf{m}} = \{\sigma \in S_n; m_{\sigma(i)} = m_i \ \forall\ i\}$. The dual of the local solution space (of dimension $n - 2$) around some base point has a Hermitian form of Lorentzian signature, which is invariant under the monodromy representation of the fundamental group $\Pi_1^{\mathrm{orb}}(\mathcal{M}_{0,n}/S_{\mathbf{m}})$. The projectivized evaluation map induces multivalued locally biholomorphic map from $\mathcal{M}_{0,n}/S_{\mathbf{m}}$ to the corresponding hyperbolic ball $\mathbb{B}$ (of dimension $n-3$). The image Γ of $\Pi_1^{\mathrm{orb}}(\mathcal{M}_{0,n}/S_{\mathbf{m}})$ under the projectivized monodromy representation acts on $\mathbb{B}$, and Deligne and Mostow analyzed the question for which parameters $m_1, \cdots, m_n$ and m the group Γ is a discrete cofinite volume subgroup of $\mathrm{Aut}(\mathbb{B})$.

Theorem 2.1 (Deligne and Mostow). *If for each pair $i<j$ with $\mu_i + \mu_j < 1$ we have $1-\mu_i-\mu_j = 1/m_{ij}$ with $m_{ij}\in\mathbb{N}$ (or slightly weaker $m_{ij}\in\mathbb{N}/2$ in case $\mu_i=\mu_j$) then $\Gamma < \mathrm{Aut}(\mathbb{B})$ is a discrete cofinite volume subgroup, and we have a commutative diagram with horizontal arrows period isomorphisms*

$$\begin{array}{ccc} \mathcal{M}_{0,n}/S_{\mathbf{m}} & \xrightarrow{\mathrm{Per}} & \mathbb{B}^\circ/\Gamma \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}_{0,n}/S_{\mathbf{m}}}^{\mathrm{GIT}} & \xrightarrow{\mathrm{Per}} & \overline{\mathbb{B}/\Gamma}^{\mathrm{BB}} \end{array}$$

with overline index GIT *a geometric invariant theory compactification of $\mathcal{M}_{0,n}/S_{\mathbf{m}}$, allowing stable (respectively strictly semistable) collisions of the subset of ramification points $\{z_i; i\in I\}$ in case $\sum_{i\in I}\mu_i<1$ (respectively $\sum_{i\in I}\mu_i=1$), with $\mathbb{B}^\circ/\Gamma$ a Heegner divisor complement in $\mathbb{B}/\Gamma$, and with the overline index* BB *the Baily–Borel compactification of* $\mathbb{B}/\Gamma$.

For $n=4$ this theorem goes back to the 19th century work of Schwarz and Klein on the Euler–Gauss hypergeometric equation [28]. First attempts for a multivariable extension by Picard were incomplete, and the above result was found in 1986 by Deligne and Mostow [20]. The extension $m_{ij}\in\mathbb{N}/2$ if $\mu_i=\mu_j$ was observed by Mostow [33]. The tables found by Deligne and Mostow for $n\geq 5$ had some errors, and the correct computer made table with 94 cases was given by Bill Thurston [42]. In 16 cases the group Γ is not an arithmetic group. The search for these was one of the motivations of Mostow for this work. Expositions of the work by Deligne–Mostow were given by Couwenberg in the second chapter of his PhD [15] and by Looijenga [32]. Initial steps towards a generalization of the Deligne–Mostow theory in the context of hypergeometric functions associated with root systems were taken Couwenberg in his PhD of 1994, and thanks to deep insight of Eduard Looijenga this project was finally brought in 2005 to a good end [16].

Some examples of the Deligne–Mostow list had been found before by Shimura [39]. In his PhD of 1966 (wih Shimura as advisor) Bill Casselman already found by arithmetic methods but under the restrictive assumption that m is prime number the complete list (consisting of just 3 cases for $n \geq 5$. One might wonder whether the arithmetic method of Casselman can be improved to recover the full arithmetic part of the Deligne–Mostow list (of $94 - 16 = 78$ cases for $n \geq 5$).

The largest dimensional example on the Deligne–Mostow list is in case $n = 12$ and $\mu_i = 1/6$ for all i, or equivalently $m = 6$ amd $m_i = 1$ for all i. The corresponding ball quotient comes from the Eisenstein lattice $L_{\mathrm{DM}} = L(A_{10})$. Since $L(A_{11})$ and $L(\tilde{A}_{11})$ have a kernel of dimension one and two respectively with quotient lattice L_{DM} we do get triflection homomorphisms

$$A(A_{11}), A(\tilde{A}_{11}) \to \mathrm{U}(L_{\mathrm{DM}})$$

which are in fact surjective [1]. This homomorphism is just the monodromy representation of the fundamental group $\Pi_1^{\mathrm{orb}}(\mathcal{M}_{0,12})/S_{12}$ for the related Lauricella hypergeometric system. Note that $A(A_{11})$ is just the original Artin braid group $\mathrm{Br}_{12}(\mathbb{C})$ on 12 strands in $\mathbb{C}$, and $A(\tilde{A}_{11})$ is the affine Artin braid group $\mathrm{Br}_{12}(\mathbb{C}^{\times})$ on 12 strands in $\mathbb{C}^{\times}$.

We now consider the ball quotient $(\mathbb{B}/\Gamma)(L_{\mathrm{DM}})$ of dimension 9 associated with the Deligne–Mostow lattice L_{DM}. Likewise the mirror complement is denoted $\mathbb{B}^{\circ}(L_{\mathrm{DM}})$ with quotient $(\mathbb{B}^{\circ}/\Gamma)(L_{\mathrm{DM}})$. The Deligne–Mostow period map

$$\mathrm{Per}_{\mathrm{DM}} : \mathcal{M}_{0,12}/S_{12} \to (\mathbb{B}^{\circ}/\Gamma)(L_{\mathrm{DM}})$$

is an isomorphism of orbifolds. The stable locus where no more than 5 points collide is mapped onto the full ball quotient. The minimal strictly semistable locus is a single point with the collision of the 12 points into two groups of 6 points, which corresponds to the unique cusp of the ball quotient in the Baily–Borel compactification.

3. Theorem of Couwenberg

Consider the complex vector space $\mathcal{V}_5 = \{z = (z_1, \cdots, z_5) \in \mathbb{C}^5; \sum z_i = 0\}$ with the reflection representation of the symmetric group S_5. The Coxeter group $S_5 = W(A_4)$ has standard generators s_i of order two $(i = 1, \cdots, 4)$, and together with the braid relations this is the Coxeter presentation of S_5. The elementary symmetric functions $\sigma_2, \cdots, \sigma_5$ of degrees $2, \cdots, 5$ are a basis for the ring of invariant polynomials. The discriminant polynomial

$$D(\sigma_2, \cdots, \sigma_5) = \prod_{i \neq j} (z_i - z_j)$$

is the square of the product of the 10 mirror equations, and $D = *\sigma_5^4 + \cdots$ is an explicit polynomial in $\sigma_2, \cdots, \sigma_5$.

Because the Hermitian form on the Eisenstein lattice $L(A_4)$ is positive definite the group $\mathrm{U}(L(A_4))$ is finite. Coxeter has shown that the triflection representation

$$A(A_4) \to \mathrm{U}(L(A_4))$$

is surjective, and the cubic relations $t_i^3 = 1$ together with the braid relations give a presentation of $\mathrm{U}(L(A_4))$. His proof was by computer verification [17].

By the Chevalley theorem the ring of invariant polynomials on $\mathbb{C} \otimes L(A_4)$ is a polynomial algebra on four homogeneous generators, whose degrees are computed to be $12, 18, 24, 30$. There are 40 mirrors and the discriminant is the cube of the product of the 40 mirror equations. Orlik and Solomon have shown that the generating homogeneous invariants can be chosen in such a way, that the discriminant polynomial has the exact same expression as the discriminant polynomial $D(\sigma_2, \cdots, \sigma_5)$ for the symmetric group S_5. Their proof was again by computer verification [35].

In his thesis Couwenberg has explained these results in a geometrically meaningful way [15], and for this reason we also write $L_C = L(A_4)$ and call it the Couwenberg lattice. One can think of his proof as the statement that the top horizontal arrow in the commutative period diagram

$$\begin{array}{ccc} \mathcal{V}_5^\circ/S_5 & \xrightarrow{\mathrm{Per}_C} & (\mathbb{C} \otimes L_C)^\circ/\,\mathrm{U}(L_C) \\ \downarrow & & \downarrow \\ \mathcal{V}_5/S_5 & \xrightarrow{\mathrm{Per}_C} & (\mathbb{C} \otimes L_C)/\,\mathrm{U}(L_C) \end{array}$$

is an isomorphism of manifolds, with $\mathcal{V}_5^\circ = \{z \in \mathcal{V}_5; z_i \neq z_j \ \forall\, i \neq j\}$ for the mirror complement as before. The Couwenberg period map Per_C is defined in terms of similar but algebraic Lauricella hypergeometric functions associated with configurations of 6 points on the curve $\mathbb{P} = \mathbb{C} \sqcup \{\infty\}$, with one point at ∞ of multiplicity 7 and 5 unordered points of multiplicity 1 on the affine line $\mathbb{C}$. Whereas the Deligne–Mostow period map is related to the geometric invariant theory of the semistable points for binary forms of degree 12 the Couwenberg period map is related to the unstable points in the null cone. Therefore

$$\Pi_1^{\mathrm{orb}}((\mathbb{C} \otimes L_C)^\circ/\,\mathrm{U}(L_C)) = \Pi_1(\mathcal{V}_5^\circ/S_5) = \mathrm{Br}_5(\mathbb{C})$$

is just the Artin braid group on 5 strands in $\mathbb{C}$. Note that in this case the orbifold fundamental group and the ordinary fundamental group are the same by standard finite reflection group theory.

Scholium 3.1. *The quotient of the orbifold fundamental group*

$$G(L_C) = \Pi_1^{\mathrm{orb}}((\mathbb{C} \otimes L_C)^\circ / \mathrm{U}(L_C)) = \mathrm{Br}_5(\mathbb{C})$$

by the subgroup generated by the squares of the meridians is the symmetric group S_5.

The group S_5 is just the Galois group of the ramified covering

$$\mathcal{V}_5 \to \mathcal{V}_5/S_5$$

for the natural action of S_5. Couwenberg obtained similar results for the case $S_{n+1} = W(A_n)$ acting on $\mathcal{V}_{n+1}$ and $A(A_n) \to \mathrm{U}(L(A_n))$ for $n = 1, 2, 3, 4$ [15],[16]. The finite groups $\mathrm{U}(L(A_n))$ have $1, 4, 12, 40$ mirrors and are the triflection groups STk for $k = 3, 4, 25, 32$ in the Shephard–Todd list [38],[17].

4. The orbifold fundamental group $\Pi_1^{\mathrm{orb}}(\mathcal{M}_{0,n}/S_n)$

The orbifold fundamental group of $\mathcal{M}_{0,n}/S_n$ has been described by Looijenga as a quotient of the affine Artin group $A(\tilde{A}_{n-1})$ with explicit relations [31] as follows. Let X be $\mathbb{C}^\times$, $\mathbb{C}$ or $\mathbb{P} = \mathbb{C}^\times \sqcup \{0, \infty\}$, and let us denote by $X(n)$ the configuration space of (unordered) subsets of X of cardinality n. The braid group of X with n strands $\mathrm{Br}_n(X)$ is the fundamental group of $X(n)$. The latter requires the choice of a base point and so is only defined up to conjugacy. The group Homeo(X) of homeomorphism of X acts also on $X(n)$. The image of $\Pi_1(\mathrm{Homeo}^0(X), 1)$ in $\mathrm{Br}_n(X)$ is a normal subgroup, and the quotient shall be referred to as the braid class group $\mathrm{BrCl_n}(X)$ on n strands in X.

First consider the case $X = \mathbb{C}^\times$. Take as base point $\sqrt[n]{1}$ the set of nth roots of 1. There are two special elements R and T in $\mathrm{Br}_n(\mathbb{C}^\times)$: R is given by the loop of the rotation of $\sqrt[n]{1}$ over $\exp(2\pi i t/n)$ for $t \in [0, 1]$, while T is represented by the loop that leaves all elements of $\sqrt[n]{1}$ in place except 1 and $\exp(2\pi i/n)$ which are interchanged by a counterclockwise half turn along the circle with center $[1 + \exp(2\pi i/n)]/2$ and radius $|1 - \exp(2\pi i/n)|/2$ (say $n \geq 5$). These two elements generate $\mathrm{Br}_n(\mathbb{C}^\times)$, but in order to get a more useful presentation it is better to enlarge the number of generators by putting $T_k =$

R^kTR^{-k} for $k \in \mathbb{Z}/n\mathbb{Z}$. The elements T_k satisfy the affine Artin relations

$$T_kT_{k+1}T_k = T_{k+1}T_kT_{k+1} \ , \ T_kT_l = T_lT_k$$

for all $k, l \in \mathbb{Z}/n\mathbb{Z}$ with $k - l \neq \pm 1$, and together with the obvious relations

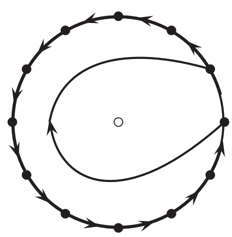
$$RT_kR^{-1} = T_{k+1}$$

this gives a presentation of $\mathrm{Br}_n(\mathbb{C}^\times)$ with generators $R, T_0, \cdots, T_{n-1}$. The element R^n comes from a loop in $\mathbb{C}^\times \subset \mathrm{Homeo}^0(\mathbb{C}^\times)$. Hence R^n dies in $\mathrm{BrCl}_n(\mathbb{C}^\times)$ and in fact $\mathrm{BrCl}_n(\mathbb{C}^\times)$ is obtained from $\mathrm{Br}_n(\mathbb{C}^\times)$ by imposing the single extra relation $R^n = 1$.

Next consider the case $X = \mathbb{C}$. It is easy to check that the elements $R, T_0, T_1, \cdots, T_{n-1}$ satisfy in $\mathrm{Br}_n(\mathbb{C})$ the additional relations

$$R = T_1T_2 \cdots T_{n-1} = T_2 \cdots T_{n-1}T_0 = \cdots = T_0T_1 \cdots T_{n-2}$$

by filling in the origin 0. For example, for $n = 12$ the picture



shows that the loop $T_1 \cdots T_{11}$ is homotopic to R if the origin is filled in. This gives the familiar presentation of $\mathrm{Br}_n(\mathbb{C})$ with generators $T_1, \cdots, T_{n-1}$ and the usual Artin relations

$$T_kT_{k+1}T_k = T_{k+1}T_kT_{k+1} \ , \ T_lT_m = T_mT_l$$

for $k, k+1, l, m \in \{1, \cdots, n-1\}$ and $l - m \neq \pm 1$. The Garside element Δ in $\mathrm{Br}_n(\mathbb{C})$ is well defined, and its square $\Delta^2 = R^n$ generates the center of $\mathrm{Br}_n(\mathbb{C})$ for $n \geq 3$ [19].

Finally consider the case that $X = \mathbb{P}$ is the projective line. It is easy to check that the elements $R, T_0, T_1, \cdots, T_{n-1}$ satisfy in $\mathrm{Br}_n(\mathbb{P})$ the additional relations

$$R = T_1 T_2 \cdots T_{n-1} \ , \ R^{-1} = T_{n-1} T_{n-2} \cdots T_1$$

by filling in 0 and ∞ respectively. Since $T_1 T_2 \cdots T_{n-1}$ and $T_{n-1} T_{n-2} \cdots T_1$ have the same nth power in $\mathrm{Br}_n(\mathbb{C})$ the above relations already imply that R^{2n} dies in $\mathrm{Br}_n(\mathbb{P})$. This gives the presentation of $\mathrm{Br}_n(\mathbb{P})$ due to Fadell and van Buskirk [22]. In the braid class group $\mathrm{BrCl}_n(\mathbb{P})$ we already have the relation $R^n = 1$ from $\mathrm{BrCl}_n(\mathbb{C}^\times)$. Since $\mathrm{BrCl}_n(\mathbb{P})$ is the same thing as the orbifold fundamental group $\Pi_1^{\mathrm{orb}}(\mathcal{M}_{0,n}/S_n)$ we arrive at the presentation with generators $T_1, \cdots, T_{n-1}$ and relations the usual Artin relations together with

$$T_1 \cdots T_{n-2} T_{n-1}^2 T_{n-2} \cdots T_1 = 1 \ , \ (T_1 T_2 \cdots T_{n-1})^n = 1$$

which was obtained by Birman [9].

Combining these results with the Deligne–Mostow period map we arrive at the following conclusion, which should be thought of as a positive answer to the analogue of the conjecture of Allcock for the Deligne–Mostow lattice L_{DM} rather than the Allcock lattice L_{A}.

Scholium 4.1. *The quotient of the orbifold fundamental group*

$$G(L_{\mathrm{DM}}) = \Pi_1^{\mathrm{orb}}((\mathbb{B}^\circ/\Gamma)(L_{\mathrm{DM}}))$$

by the subgroup generated by the squares of the meridians is the symmetric group S_{12}.

The group S_{12} is just the Galois group of the covering

$$\mathcal{M}_{0,12} \to \mathcal{M}_{0,12}/S_{12}$$

for the natural action of S_{12}.

5. Acute angled polytopes in real hyperbolic space

Let V be a real vector space of finite dimension $n+1$ with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ of Lorentzian signature $(n, 1)$. The set

$$\mathbb{B}(V) = \mathbb{P}(\{v \in V; \langle v, v \rangle < 0\}) \subset \mathbb{P}(V)$$

is a model of real hyperbolic space of dimension n. Suppose we have given a spanning subset $\{e_i; i \in I\}$ of V such that its Gram matrix G with entries

$g_{ij} = \langle e_i, e_j \rangle$ satisfies $g_{ii} > 0$ and $g_{ij} \leq 0$ for all $i \neq j$. The set

$$P = \mathbb{P}(\{v \in V; \langle v, v \rangle < 0, \langle v, e_i \rangle \geq 0 \ \forall i \in I\})$$

is called an acute angled convex polytope in the hyperbolic space $\mathbb{B}(V)$. We associate with this given set $\{e_i; i \in I\}$ a Coxeter diagram with nodes labeled by I and two nodes $i, j \in I$ are connected if $g_{ij} < 0$.

For the theory of hyperbolic reflection groups such polytopes have been studied to a great extent by Vinberg [43]. A subset $J \subset I$ is called elliptic, parabolic or hyperbolic if the Gram matrix G_J of the subset $\{e_j; j \in J\}$ is positive definite, positive semidefinite, or indefinite respectively. For $J \subset I$ an elliptic subset the face

$$P^J = \mathbb{P}(\{v \in V; \langle v, v \rangle < 0, \langle v, e_i \rangle \geq 0 \ \forall i \notin J, \langle v, e_j \rangle = 0 \ \forall j \in J\})$$

of P is not empty (by the Perron–Frobenius theorem) and of codimension equal to the cardinality $|J|$ of J. It can be shown that all faces of P in $\mathbb{B}(V)$ are of this form. Moreover the orthogonal (geodesic) projection of $\mathbb{B}(V)$ onto the codimension $|J|$ hyperbolic subspace of $\mathbb{B}(V)$ containing the face P^J maps the polytope P onto its face P^J (see §3 of [43]).

The polytope P has finite hyperbolic volume if and only if

$$\mathbb{P}(\{v \in V; v \neq 0, \langle v, e_i \rangle \geq 0 \ \forall i \in I\}) \subset \mathbb{P}(\{v \in V; v \neq 0, \langle v, v \rangle \leq 0\})$$

but this can be cumbersome to check in concrete examples. A subset $J \subset I$ is called critical if J is not elliptic, but K is elliptic for all proper subsets K of J. Clearly critical subsets of I are connected subsets of the Coxeter diagram. For J a subset of I we denote by $Z(J)$ the subset of I of all nodes that are not connected to J. The next theorem is a special case of a more general result of Vinberg (see theorem 4.1 of [43]).

Theorem 5.1 (Vinberg). *Suppose P is an accute angled polytope in $\mathbb{B}(V)$ as above, such that each critical subset J of I is parabolic. Then the polytope P has finite volume in $\mathbb{B}(V)$ if and only for each critical (parabolic) subset J of I the subset $N(J) := J \sqcup Z(J)$ is still parabolic with $G_{N(J)}$ of rank $n-1$.*

Hence the subset $N(J) = J_1 \sqcup \cdots \sqcup J_r$ in the theorem is a disjoint union of parabolic subdiagrams, and corresponds to an ideal vertex $P^{N(J)}$ of P. The local structure of P near such an ideal vertex is a product of an interval $(0, \varepsilon)$ with a product of r simplices of dimensions $|J_1| - 1, \cdots, |J_r| - 1$.

6. The 12-cell of dimension 9

The Eisenstein lattice L_{DM} is equal to the quotient of $L(\tilde{A}_{11})$ by its kernel of dimension two. It has the roots ε_i for $i \in \mathbb{Z}/12\mathbb{Z}$ as a generating set. Suppose the nodes with even index are black and with odd index are white. Then the Hermitian form is given by

$$\langle \varepsilon_i, \varepsilon_i \rangle = 3 \ , \ \langle \varepsilon_i, \varepsilon_{i+1} \rangle = (-1)^i \theta \ , \ \langle \varepsilon_j, \varepsilon_k \rangle = 0$$

for all $i, j, k \in \mathbb{Z}/12\mathbb{Z}$ with $|j-k| \geq 2$. We shall extend scalars from the Eisenstein integers $\mathbb{Z}[\omega]$ to $\mathbb{Z}[\sqrt[12]{1}]$ and put

$$e_{2j} = i\varepsilon_{2j} \ , \ e_{2j+1} = \varepsilon_{2j+1}$$

for all $j \in \mathbb{Z}/12\mathbb{Z}$, and write V for their real span. The Gram matrix of $\{e_i; i \in \mathbb{Z}/12\mathbb{Z}\}$ becomes

$$\langle e_i, e_i \rangle = 3 \ , \ \langle e_i, e_{i+1} \rangle = -\sqrt{3} \ , \ \langle e_j, e_k \rangle = 0$$

for all i, j, k with $|j-k| \geq 2$. The Coxeter diagram is of type $\tilde{A}_{11}$ and the connected subdiagrams of type A_n are elliptic for $n = 1, 2, 3, 4$, parabolic for $n = 5$, and hyperbolic for $n = 6, 7, 8, 9, 10$. The critical subdiagrams are the subdiagrams of type A_5, and deleting the two adjacent nodes in the $\tilde{A}_{11}$ diagram leaves us with another subdiagram of type A_5. The rank of the Gram matrix of these two disjoint A_5 diagrams is 8, which is the rank of L^{10} minus 2. The conditions of the theorem of Vinberg are therefore satisfied and we conclude that the acute angled polytope

$$P = \mathbb{P}(\{v \in V; \langle v, v \rangle < 0, \langle v, e_i \rangle \geq 0 \ \forall i \in I\})$$

has finite volume in $\mathbb{B}(V)$. It has an isometric action by the dihedral group D_{12} of order 12, which acts transitively on the 12 codimension one faces of P. Its center is called the Weyl point w_0 which has equal distance to all 12 codimension one faces. The acute angled polytope P of dimension 9 has finite hyperbolic volume by the Vinberg criterion and will be called the 12-cell.

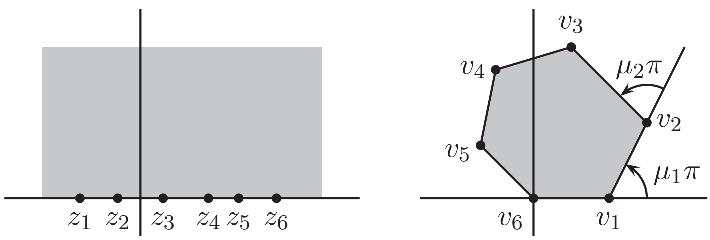
Suppose $n \geq 3$ and we are given $0 < \mu_1, \mu_2, \cdots, \mu_n < 1$ with $\sum \mu_j = 2$. If $z_1 < z_2 < \cdots < z_n$ are n successive real points and $z = (z_1, \cdots, z_n)$ then the Schwarz–Christoffel transformation

$$t \mapsto v(z;t) = \int_{z_n}^{t} (s-z_1)^{-\mu_1}(s-z_2)^{-\mu_2}\cdots(s-z_n)^{-\mu_n}ds$$

(with the integrand $s \mapsto (s-z_1)^{-\mu_1}(s-z_2)^{-\mu_2}\cdots(s-z_n)^{-\mu_n}$ single valued and holomorphic on the extended complex plane $\mathbb{C}\sqcup\{\infty\}$ minus a cut along the interval $[z_1,z_n]$ and having Laurent series expansion at ∞ of the form $s^{-2}(1+O(1/s))$, and with the integration path from z_n to t avoiding the interval $[z_1,z_n]$ except for its starting point) maps the upper half plane $\Im t>0$ conformally onto a convex polygon with vertices

$$v_1 = v(z;z_1) > 0, v_2 = v(z;z_2), \cdots, v_{n-1} = v(z;z_{n-1}), v_n = v(z;z_n) = 0$$

and interior angles $(1-\mu_j)\pi \in (0,\pi)$ at v_j summing up to $\sum(1-\mu_j)\pi = (n-2)\pi$ as should. By the reflection principle the lower half plane is mapped conformally on the mirror image of this polygon under reflection in the real axis.



The directed edge functions

$$w_j = w_j(z) = \int_{z_j}^{z_{j+1}} (s-z_1)^{-\mu_1}(s-z_2)^{-\mu_2}\cdots(s-z_n)^{-\mu_n}ds$$

satisfy $w_j = v_{j+1} - v_j$ and are called Lauricella F_D hypergeometric functions of the variable z. If we put $\omega_j = \exp\pi i(\mu_1+\cdots+\mu_j)$ then the edge lengths $l_j = \overline{\omega}_j w_j$ are positive real numbers (or functions of z) and satisfy the two linear relations

$$\sum \omega_j l_j(z) = \sum \overline{\omega}_j l_j(z) = 0$$

making the span V of the vectors $l=(l_1,\cdots,l_n)$ a real vector space of dimension $(n-2)$.

The cone $V_+ = \{l \in V; l_j > 0\ \forall j\}$ gets identified with the space of all such polygons with vertices $v_1 > 0, v_2, \cdots, v_n = 0$ and edge lengths $l_j > 0$ from v_j to v_{j+1}, and is called the polygon space of type $\mu = (\mu_1, \cdots, \mu_n)$.

The spanning vector space V carries a natural Lorentzian inner product for which the norm $\langle l, l\rangle$ of $l \in V_+$ is equal to minus (due to our signature convention) the area of the corresponding polygon. The Hermitian extension to the complexification $\mathbb{C} \otimes_{\mathbb{R}} V$ is a monodromy invariant Lorentzian Hermitian form on the space of Lauricella functions with parameter μ. For proofs and further details we refer to the discussion of the Lauricella F_D function by Couwenberg in his thesis [15].

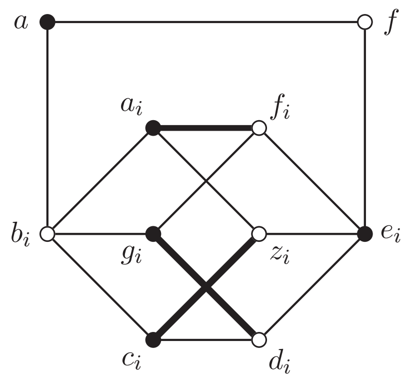
If the parameter $\mathbf{m} = (m_1, \cdots, m_n)$ occurs on the Deligne–Mostow list then it follows from the Deligne–Mostow theorem that a point in $V_+/\mathbb{R}_+$ uniquely determines the configuration $z = (z_1, \cdots, z_n) \in \mathcal{M}_{0,n}/S_{\mathbf{m}}$, which by the Schwarz–Christoffel theory a fortiori should be real. Hence the real hyperbolic polytope $V_+/\mathbb{R}_+$ as subset of the complex ball quotient $\mathbb{B}/\Gamma$ lies in fact in $\mathbb{B}^\circ/\Gamma$.

The parameter $\mu = (1/6, \cdots, 1/6)$ is the relevant example. The set V_+ is identified with the space of 12-gons with vertices $v_1 > 0, v_2, \cdots, v_{12} = 0$ and all interior angles equal to $5\pi/6$. The 12-cell $P = V_+/\mathbb{R}_+$ is just the space of such 12-gons up to a positive scale factor. The central Weyl point w_0 in P at equal distance to all 12 codimension one faces corresponds to the regular 12-gon with the lengths of all edges equal.

Scholium 6.1. *The interior of the* 12*-cell P of dimension* 9 *is contained in the mirror complement $\mathbb{B}^\circ(L_{\mathrm{DM}})$ of the Deligne–Mostow ball. The Weyl point w_0 in P lies at equal distance to all* 12 *codimension one faces. The cyclic group C_{12} of order* 12 *acts on P by isometries leaving w_0 fixed.*

7. The Coxeter diagram I_{26}

The projective plane $\mathbb{P}^2(3)$ over a field of 3 elements has 13 points and 13 lines. The incidence diagram I_{26} has 26 nodes of which 13 are marked bold (the points) and 13 hollow (the lines) with index i taking values $1, 2, 3$. A thin bond in the figure below indicates that the two end nodes are incident if their indices coincide, while a thick bond indicates that the end nodes are incident if their indices differ. So a thick bond represents altogether 6 different bonds, and a thin bond just 3. The diagram I_{26} has valency 4. The group of diagram automorphisms of I_{26} preserving the marking of the nodes is the group $\mathrm{L}_3(3) = \mathrm{PGL}_3(3)$ of order $5616 = 2^4 \cdot 3^3 \cdot 13$. The group $\mathrm{L}_3(3){:}2$ of order 11232, obtained by adjoining an outer automorphism of projective duality between points and lines, acts as group of automorphisms of the diagram I_{26} with all nodes unmarked.



Note that the subdiagram with nodes $ab_ic_id_ie_if_i$ (all i) by deleting the remaining nodes $fa_ig_iz_i$ (all i) and all bonds connected to these remaining nodes is the Y_{555} diagram, which is just a maximal subtree of I_{26}. Deleting the triple node a of this Y_{555} diagram shows that the I_{26} diagram has a subdiagram of type $3A_5$. Adjoining a_3 and deleting b_3 shows that I_{26} also has a subdiagram of type $A_4 + \tilde{A}_{11}$ with the 4 nodes $c_3d_3e_3f_3$ making A_4 and the 12 nodes $ab_1c_1d_1e_1f_1a_3f_2e_2d_2c_2b_2$ making $\tilde{A}_{11}$. The $\tilde{A}_{11}$ subdiagram is also called a free 12-gon. The remaining 10 nodes $a_1a_2b_3fg_iz_i$ (all i) are each connected to both this A_4 subdiagram and this $\tilde{A}_{11}$ subdiagram. Hence A_4 and $\tilde{A}_{11}$ determine each other uniquely as the maximal disjoint complementary subdiagram in I_{26}. In our previous notation $Z(A_4) = \tilde{A}_{11}$ and $Z(\tilde{A}_{11}) = A_4$. Observe also that ab_i and $d_ie_if_iz_i$ (all i) yields a subdiagram of I_{26} of type $4D_4$.

8. The 26-cell of dimension 13

The set $I = \mathcal{P} \sqcup \mathcal{L}$ of 26 vertices of the Coxeter diagram I_{26} splits as a disjoint union of the 13 points and the 13 lines of $\mathbb{P}^2(3)$. If ε_i is the generating set of the Allcock lattice L_{A} with Gram matrix

$$\langle \varepsilon_i, \varepsilon_i \rangle = 3 \ , \ \langle \varepsilon_j, \varepsilon_k \rangle = 0 \ , \ \langle \varepsilon_p, \varepsilon_l \rangle = \theta$$

for all i, for all disconnected $j \neq k$ and for all connected $p \in \mathcal{P}$ and $l \in \mathcal{L}$ then we introduce a new set $\{e_i\}$ simply by

$$e_p = i\varepsilon_p, e_l = \varepsilon_l$$

for $p \in \mathcal{P}$ and $l \in \mathcal{L}$. The Gram matrix of e_i becomes the real symmetric matrix

$$\langle e_i, e_i \rangle = 3 \ , \ \langle e_i, e_j \rangle = 0 \ , \ \langle e_j, e_k \rangle = -\sqrt{3}$$

for all i, j, k with $i \neq j$ disconnected and $j \neq k$ connected. If for each line $l \in \mathcal{L}$ we put $d_l = \sqrt{3} e_l + \sum_{p \sim l} e_p$ then it is easy to check that

$$\langle d_l, e_q \rangle = 0 \ , \ \langle d_l, e_m \rangle = -\sqrt{3}$$

for all $q \in \mathcal{P}$ and $m \in \mathcal{L}$. Hence $d_l - d_m$ is a null vector for all $l, m \in \mathcal{L}$ and we conclude that the real subspace V spanned by the vectors $\{e_i; i \in I\}$ is a real Lorentzian vector space of dimension 14.

The acute angled hyperbolic polytope of dimension 13

$$P = \mathbb{P}(\{v \in V; \langle v, v \rangle < 0, \langle v, e_i \rangle \geq 0 \ \forall i \in I\})$$

in $\mathbb{B}(V)$ will be called the 26-cell. It is easy to check that the critical subdiagrams are the connected parabolic diagrams of type A_5 or D_4. Since $N(A_5) = 3A_5$ and $N(D_4) = 4D_4$ are both parabolic and have both Gram matrices of rank 12 it follows from the Vinberg criterion that P is a finite volume convex hyperbolic polytope.

The 26-cell P has two natural vertices $w_{\mathcal{P}}$ perpendicular to all e_p with $p \in \mathcal{P}$ and $w_{\mathcal{L}}$ perpendicular to all e_l with $l \in \mathcal{L}$. The midpoint w_0 on the geodesic from $w_{\mathcal{P}}$ to $w_{\mathcal{L}}$ is called the Weyl point. The group $L_3(3){:}2$ of diagram automorphisms of the unmarked Coxeter diagram I_{26} acts a group of isometries of P leaving the Weyl point w_0 fixed. Under this symmetry group the 26-cell P has two inequivalent ideal vertices of the above types $3A_5$ and $4D_4$. The next conjecture is the analogue of Scholium 6.1 for the 26-cell P.

Conjecture 8.1. *The interior of the* 26*-cell* P *is the connected component of* $\mathbb{B}(V) \cap \mathbb{B}^\circ(L_{\mathrm{A}})$ *containing* w_0*. In other words, the interior of* P *does not meet any mirror of the complex Allcock ball* $\mathbb{B}(L_{\mathrm{A}})$*.*

Partial results towards this conjecture are due to Basak [8]. He shows that the 26 mirrors supported by the codimension one faces of the 26-cell P are exactly those mirrors in the Allcock ball $\mathbb{B}(L_{\mathrm{A}})$ that are nearest to the Weyl point w_0. The real subball $\mathbb{B}(V) \subset \mathbb{B}(L_{\mathrm{A}})$ supported by P contains all 26 shortest geodesics from w_0 to these nearest mirrors, and this characterizes $\mathbb{B}(V)$. In particular for each vertex i of I_{26} the geodesic from w_0 to the orthogonal projection w_i of w_0 on the codimension one face P^i of P does not meet any mirror in $\mathbb{B}(L)$ before it reaches w_i.

Basak defines a curve γ_i in $\mathbb{B}^\circ$ with begin point the Weyl point w_0 and end point $t_i(w_0)$. Here t_i is the triflection with eigenvalue ω leaving the codimension one face P^i fixed. The curve γ_i is almost the geodesic from w_0 to w_i and then continues geodesically to $t_i(w_0)$. However this curve hits the mirror supported by P^i at w_i and so instead shortly before arriving at w_i it makes a one third turn in the complex line through $w_0, w_i, t_i(w_0)$. The curve γ_i defines the meridian element T_i of $\Pi_1^{\mathrm{orb}}((\mathbb{B}^\circ/\Gamma)(L_{\mathrm{A}}), w_0)$.

For i, j two different nodes of I_{26} Basak proves the Artin braid relations

$$T_iT_jT_i = T_jT_iT_j \ , \ T_iT_j = T_jT_i$$

in case i, j are connected or disconnected respectively along the following lines. Let w_{ij} be the orthogonal projection of w_0 on the codimension two face P^{ij} of P. Basak shows that the interior of the convex hull of the 4 points w_0, w_i, w_j, w_{ij} does not meet any mirror of $\mathbb{B}(L_{\mathrm{A}})$. The curve γ_i can be continuously deformed in $\mathbb{B}^\circ(L_{\mathrm{A}})$ to a curve γ_{ij} going geodesically from w_0 to w_{ij} and shortly before arriving at w_{ij} making a one third turn around the mirror supported by P^i. Likewise γ_j can be deformed to γ_{ji}. The braid relation for the two corresponding meridians is a local relation of the mirror arrangement near w_{ij} and follows from the work of Couwenberg as described in Section 3, or by giving the explicit homotopy as Basak did. If i, j are connected then four mirrors pass through P^{ij} while in case i, j are disconnected only two orthogonal mirrors pass trough P^{ij}.

The group $\mathrm{L}_3(3){:}2$ of diagram automorphisms of the unmarked diagram I_{26} acts by isometries on the 26-cell P. The Weyl point w_0 is a fixed point for this action. The infinitesimal action of $\mathrm{L}_3(3){:}2$ on the tangent space of $\mathbb{B}(V)$ at w_0 decomposes as a direct sum of a one dimensional representation (coming from the geodesic through $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$) and an irreducible representation of dimension 12 on the orthogonal complement. This is the smallest dimensional irreducible representation of $\mathrm{L}_3(3){:}2$ that is nontrivial on $\mathrm{L}_3(3)$.

Let J be a subdiagram of I_{26} of type A_4. Any two such subdiagrams are conjugated under $\mathrm{L}_3(3){:}2$ and so we can assume that J consists of the nodes $c_3d_3e_3f_3$ in the notation of Section 7. The complementary subdiagram $Z(J)$ obtained by deleting all nodes of J and those connected to J contains the 12 nodes $ab_1c_1d_1e_1f_1a_3f_2e_2d_2c_2b_2$ and is of type $\tilde{A}_{11}$. The face P^J of P of codimension 4 is just the 12-cell of dimension 9 in the Deligne–Mostow ball as discussed in Section 6. We denote by $\mathbb{B}(U)$ the real hyperbolic space supported by P^J, viewed as subspace of the real hyperbolic space $\mathbb{B}(V)$

supported by P. The subgroup of $\mathrm{L}_3(3){:}2$ preserving the face P^J is the dihedral group D_{12} of order 24 permuting the nodes $ab_1c_1d_1e_1f_1a_3f_2e_2d_2c_2b_2$ in cyclic way or inverting them.

Lemma 8.2. *Any positive definite Eisenstein lattice of rank* 5 *containing* $L(\mathrm{A}_4)$ *as a primitive sublattice and spanned by* $L(\mathrm{A}_4)$ *and a complementary root is of the form* $L(\mathrm{A}_4) \oplus L(\mathrm{A}_1)$.

Proof. By assumption the lattice has a root basis $\varepsilon_1, \cdots, \varepsilon_4, \varepsilon_5$ with the first four vectors the standard basis of $L(\mathrm{A}_4)$. If we assume that

$$\langle \varepsilon_1, \varepsilon_5 \rangle = x\theta \ , \ \langle \varepsilon_2, \varepsilon_5 \rangle = y\theta \ , \ \langle \varepsilon_3, \varepsilon_5 \rangle = z\theta \ , \ \langle \varepsilon_4, \varepsilon_5 \rangle = w\theta$$

then the determinant of the Gram matrix (divided by 9) is easily found to be

$$3 - x\overline{x} - w\overline{w} - 2(y\theta - x)(\overline{y\theta - x}) - 2(z\theta + w)(\overline{z\theta + w}) + \\ -\theta(y\theta - x)(\overline{z\theta + w}) + \theta(z\theta + w)(\overline{y\theta - x})$$

with $x, y, z, w \in \mathcal{E}$. Since

$$2a\overline{a} + 2b\overline{b} + \theta a\overline{b} - \theta b\overline{a} = (a - b\omega)(\overline{a - b\omega}) + (a + b\omega)(\overline{a + b\omega})$$

the above expression becomes

$$3 - x\overline{x} - w\overline{w} - (a - b\omega)(\overline{a - b\omega}) - (a + b\omega)(\overline{a + b\omega})$$

with $a = y\theta - x, b = z\theta + w$. This expression should be positive, and so

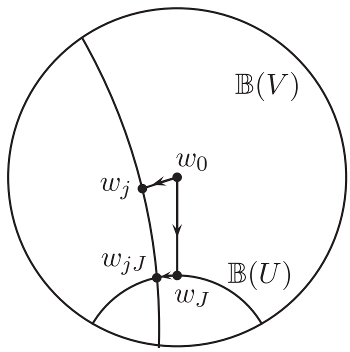
$$x\overline{x} \le 1 \ , \ w\overline{w} \le 1 \ , \ (a - b\omega)(\overline{a - b\omega}) \le 1 \ , \ (a + b\omega)(\overline{a + b\omega}) \le 1$$

and their sum is at most 2, so at least two terms are 0.

If $x = w = 0$ then $a = y\theta, b = z\theta$ which implies $y = z = 0$. Similarly if $x = 0, a = b\omega$ then $a = b = 0$ which in turn implies $y = z = w = 0$. Finally if $a = b\omega = -b\omega$ then $a = b = 0$ and so $x = y = z = w = 0$. $\square$

Hence the complexification $\mathbb{B}(L_{\mathrm{DM}})$ of $\mathbb{B}(U)$ in the Allcock ball $\mathbb{B}(L_{\mathrm{A}})$ is the intersection of 40 mirrors in $\mathbb{B}(L_{\mathrm{A}})$, and $\mathbb{B}(U) = \mathbb{B}(V) \cap \mathbb{B}(L_{\mathrm{DM}})$. By the above lemma all other mirrors in $\mathbb{B}(L_{\mathrm{A}})$ intersecting $\mathbb{B}(L_{\mathrm{DM}})$ do so in a perpendicular way. The local structure of the 26-cell P near its face P^J is a product of P^J with a real simplicial chamber P_J of dimension 4 of the group $\mathrm{U}(L(\mathrm{A}_4))$ corresponding to 5 ordered points on $\mathbb{R}$ with zero sum, as discussed in Section 3.

Let J be the given subset of I_{26} of type A_4 with complement $Z(J)$ of type $\tilde{A}_{11}$. Let w_J be the orthogonal projection on the face P^J of the Weyl point w_0 of P. The point w_J is the central point of P^J corresponding to the regular 12-gon in the Deligne–Mostow picture. For $j \in Z(J)$ let w_{jJ} be the projection on w_0 on the face P^{jJ} (with jJ standing for $\{j\} \sqcup J$), which is the same as the orthogonal projection of w_J on the the codimension one face P^{jJ} of P^J. Now Conjecture 8.1 implies that the curve γ_j can be continuously deformed to a curve γ_{jJ}, which is a curve $\tilde{\gamma}_j$ in the tubular neighborhood of $\mathbb{B}_{\mathrm{DM}}$ in $\mathbb{B}_{\mathrm{A}}$ with base point a nearby point $\tilde{w}_J$ of w_J conjugated by a geodesic from this nearby point to w_0.



Indeed the desired homotopy is obtained using the orthogonal projection of P onto its face P^J. Under the identification of $Z(J)$ with $\mathbb{Z}/12\mathbb{Z}$ the meridian elements $T_i \in \Pi_1^{\mathrm{orb}}(\mathbb{B}^\circ/\Gamma, w_0)$ for $i \in \mathbb{Z}/12\mathbb{Z}$ satisfy the Artin braid relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}\ ,\ T_iT_j = T_jT_i$$

for $i - j \neq \pm 1$ of the affine Artin group of type $\tilde{A}_{11}$.

The inclusion map of the face P^J of P gives rise to a holomorphic map from the Deligne–Mostow ball quotient $(\mathbb{B}/\Gamma)(L_{\mathrm{DM}})$ to the Allcock ball quotient $(\mathbb{B}/\Gamma)(L_{\mathrm{A}})$. This map is an immersion, but not an injection, since the image of $(\mathbb{B}/\Gamma)(L_{\mathrm{DM}})$ in $(\mathbb{B}/\Gamma)(L_{\mathrm{A}})$ has triple self intersection along a one dimensional ball quotient, which is isomorphic to the modular curve $\mathbb{H}_+/\mathrm{PSL}(2,\mathbb{Z})$. This one dimensional ball quotient is associated with the Eistenstein hyperbolic plane H inside $L_{\mathrm{DM}} = H \oplus L(\mathrm{A}_4) \oplus L(\mathrm{A}_4)$ and the order three symmetry comes from a permutation of the three factors $L(\mathrm{A}_4)$ inside the Allcock lattice $L_{\mathrm{A}} = H \oplus L(\mathrm{A}_4) \oplus L(\mathrm{A}_4) \oplus L(\mathrm{A}_4)$.

Let $\mathbb{N}^\circ$ be the pull back of a small tubular neighbourhood of $(\mathbb{B}/\Gamma)(L_{\mathrm{DM}})$ inside the mirror complement $(\mathbb{B}^\circ/\Gamma)(L_{\mathrm{A}})$ under the natural immersion $(\mathbb{B}/\Gamma)(L_{\mathrm{DM}}) \to (\mathbb{B}/\Gamma)(L_{\mathrm{A}})$. Then we have a fiber bundle

$$\mathbb{N}^\circ \to \mathbb{B}^\circ/\Gamma(L_{\mathrm{DM}})$$

with fiber a small ball around the origin in $(\mathbb{C} \otimes L(\mathrm{A}_4))^\circ$ modulo $\mathrm{U}(L(\mathrm{A}_4))$. This gives rise to an exact homotopy sequence

$$1 \to \Pi_1^{\mathrm{orb}}((\mathbb{C} \otimes L(\mathrm{A}_4))^\circ/\mathrm{U}(L(\mathrm{A}_4))) \to \Pi_1^{\mathrm{orb}}(\mathbb{N}^\circ) \to \Pi_1^{\mathrm{orb}}(\mathbb{B}^\circ/\Gamma(L_{\mathrm{DM}})) \to 1$$

and taking the quotient by squares of meridians we conclude by Scholium 3.1 and Scholium 4.1 that the group $\Pi_1^{\mathrm{orb}}(\mathbb{N}^\circ)$ modulo squares of meridians is isomorphic to $S_5 \times S_{12}$. Indeed, the only action of S_{12} by automorphisms on S_5 is the trivial action. Hence the image of the subgroup generated by the T_i for $i \in \mathbb{Z}/12\mathbb{Z}$ under the homomorphism $\varphi : W(I_{26}) \to G/N$ is a factor group of S_{12}. In other words, the free 12-gons are deflated in G/N.

As a consequence of Conjecture 8.1 in combination with Theorem 1.6 and Theorem 1.2 we find that the orbifold fundamental group $G(L_{\mathrm{A}})$ of the Allcock ball quotient$(\mathbb{B}^\circ/\Gamma)(L_{\mathrm{A}})$ modulo the squares of the meridians is a factor group of the bimonster $M \wr 2$. The factor groups of $M \wr 2$ are either $M \wr 2$ or have order 2 or 1. This provides additional evidence for the Allcock conjecture. The following remark I learned from Eduard Looijenga.

Remark 8.3. *One can show that the orbifold fundamental group of the image of* $\mathbb{N}^\circ$ *in* $\mathbb{B}^\circ/\Gamma(L_{\mathrm{A}})$ *is obtained from that of* $\mathbb{N}^\circ$ *by means of an* HNN *extension (after Higman, Neumann and Neumann [37]). To be precise, the fiber orbifold fundamental group* $\Pi_1^{\mathrm{orb}}(\mathrm{Fiber}) \subset \Pi_1^{\mathrm{orb}}(\mathbb{N}^\circ)$ *also appears as the image of an embedding* $h : \Pi_1^{\mathrm{orb}}(\mathrm{Fiber}) \to \Pi_1^{\mathrm{orb}}(\mathrm{Base})$ *and the* HNN *extension in question simply adds an extra generator* t *to* $\Pi_1^{\mathrm{orb}}(\mathbb{N}^\circ)$ *subject to the relation that conjucagy with* t *restricted to* $\Pi_1^{\mathrm{orb}}(\mathrm{Fiber})$ *is a lift of* h*. So if we subsequently divide out by the (normal) subgroup generated by the squares of the meridians, then we get an* HNN *extension of* $S_5 \times S_{12}$ *relative to the standard inclusion of* S_5 *in the second factor. Note that Conway and Pritchard [13] characterize the bimonster as the smallest quotient of this* HNN *extension, which still contains* $S_5 \times S_{12}$ *and is not isomorphic to* S_{17}.

Remark 8.4. *In our joint preprint with Sander Rieken [25] Conjecture 1.3 was proved, but as pointed out to us by Daniel Allcock the proof is incomplete. What we did check correctly is that the interior of the 26-cell* P *does not have a real codimension one intersection with the norm* 3 *mirrors. This* *proves that the interior of P minus the complex mirrors is connected. But we overlooked the possibility of real codimension two intersections. Hence it is still not proven that the interior of P minus the mirrors is contractible, and that is what is needed in the above argument. We intend to return to this problem in the future.*

Remark 8.5. *The analogue of Conjecture 8.1 for similar ball quotients has now been checked for the Allcock–Carlson–Toledo ball quotient corresponding to cubic surfaces [5], [24] and for the Kondo ball quotient corresponding to quartic curves [29], [36]. In both cases the interior of the analogous real cell P does not meet any complex mirrors. In turn, we have given a geometric explanation of the corresponding odd presentations for the Weyl group $W(\mathrm{E}_6)$ as factor group of the Coxeter group $W(\mathrm{P}_{10})$ of the Petersen graph P_{10} modulo deflation of the free hexagons [24], a presentation found by Christopher Simons [41]. Likewise the Weyl group $W(\mathrm{E}_7)$ is the factor group of the Coxeter group $W(\mathrm{T}_{10})$ of the graph T_{10} modulo deflation of the free octagons. Here T_{10} is the tetrahedral graph, which has 10 nodes at the 4 vertices and the 6 midpoints of the edges of the tetrahedron, and has 12 simply laced branches along the half edges of the tetrahedron [36].*

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