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Point-Like Limit of the Hyperelliptic Zhang-Kawazumi Invariant

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Dedicated to Eduard Looijenga on the occasion of his 69th birthday

Abstract: The behavior near the boundary in the Deligne-Mumford compactification of many functions on $\mathcal{M}_{h,n}$ can be conveniently expressed using the notion of "point-like limit" that we adopt from the string theory literature. In this note we study a function on $\mathcal{M}_h$ that has been introduced by N. Kawazumi and S. Zhang, independently. We show that the point-like limit of the Zhang-Kawazumi invariant in a family of hyperelliptic Riemann surfaces in the direction of any hyperelliptic stable curve exists, and is given by evaluating a combinatorial analogue of the Zhang-Kawazumi invariant, also introduced by Zhang, on the dual graph of that stable curve.
Keywords: Arakelov-Green's function, hyperelliptic curve, point-like limit, stable curve, Zhang-Kawazumi invariant.

1. Introduction

The invariant from the title has been introduced around 2008 by N. Kawazumi [22] [23] and S. Zhang [32], independently, and can be defined as

$$\varphi(\Sigma) = \int_{\Sigma\times\Sigma} g_{\mathrm{Ar}}(x,y)\,\nu^2(x,y)\,. \tag{1.1}$$

Here Σ is a compact connected Riemann surface of positive genus, and g_{Ar} is the Arakelov-Green's function [2] on $\Sigma\times\Sigma$. Let Δ be the diagonal divisor on $\Sigma\times\Sigma$. Then $\nu \in A^{1,1}(\Sigma\times\Sigma)$ is the curvature form of the hermitian line

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bundle $\mathcal{O}_{\Sigma\times\Sigma}(\Delta)$ whose metric is given by the prescription

$$\log \|1\|(x,y) = g_{\mathrm{Ar}}(x,y)$$

for all $(x,y) \in \Sigma \times \Sigma$ away from Δ. Here 1 denotes the tautological global section of the line bundle $\mathcal{O}_{\Sigma\times\Sigma}(\Delta)$.

In [22] [23] the invariant φ arises in the context of a study of the first extended Johnson homomorphism [25] on the mapping class group of a pointed compact connected oriented topological surface. The results from (the unfortunately unpublished) [23] were revisited in [15]. The motivation in [32] to study φ comes from number theory, where φ appears as a local archimedean contribution in a formula that relates the height of the canonical Gross-Schoen cycle on a smooth projective and geometrically connected curve with semistable reduction over a number field with the self-intersection of its admissible relative dualizing sheaf. The connection between these two seemingly different approaches was established in [17].

The Zhang-Kawazumi invariant vanishes identically in genus $h = 1$. In genera $h \geq 2$, the invariant is strictly positive, cf. [23, Corollary 1.2] or [32, Proposition 2.5.1]. In the light of a question posed by E. Looijenga, one could speculate whether (for large enough genera h) the invariant φ would be a candidate for a $(h-2)$-pseudoconvex proper Morse function on $\mathcal{M}_h$. Several expressions for the Levi form of φ are derived in [15] [23].

The invariant in genus $h = 2$ has recently attracted attention from superstring theory [12] [13] where its integral against the volume form $\mathrm{d}\mu_2$ of the Siegel metric over $\mathcal{M}_2$ appears in the low energy expansion of the two-loop four-graviton amplitude. The detailed study of the invariant in [12] [13] has yielded the important result that φ is an eigenfunction of the Laplace-Beltrami operator with respect to $\mathrm{d}\mu_2$, with eigenvalue 5 (cf. [13, Section 4.6]). In [27], based on this result a completely explicit expression is given for the genus-two Zhang-Kawazumi invariant as a theta lift, involving a Siegel-Narain theta series and a weight $-5/2$ vector-valued modular form appearing in the theta series decomposition of the weak Jacobi form $\vartheta_1^2(\tau,z)/\eta(\tau)^6$.

One of the main starting points in deriving these results is a study of the low order asymptotics of the Zhang-Kawazumi invariant near the boundary of $\mathcal{M}_2$ in the Deligne-Mumford compactification $\overline{\mathcal{M}}_2$. As it turns out, there is a natural combinatorial description of these asymptotics in terms of the tautological stratification of $\overline{\mathcal{M}}_2$ by topological type of the dual graphs of stable curves. The following theorem summarizes the results from [12] [13]

Table 1: Point-like limit of the ZK invariant in genus two

(G,q)	$\varphi^{\mathrm{tr}}(G,q)$
$I(e_1,e_2,e_3)$	$\frac{1}{12}(x_1+x_2+x_3)-\frac{5}{12}\frac{x_1x_2x_3}{x_1x_2+x_2x_3+x_3x_1}$
$II(e_1)$	x_1
$III(e_1)$	$\frac{1}{12}x_1$
$IV(e_1,e_2)$	$x_1+\frac{1}{12}x_2$
$V(e_1,e_2)$	$\frac{1}{12}(x_1+x_2)$
$VI(e_1,e_2,e_3)$	$x_1+\frac{1}{12}(x_2+x_3)$

about the asymptotics of φ in genus $h=2$. We refer to Section 2 for an explanation of the notation and terminology.

Theorem 1.1. *Let $\bar{X}_0$ be a complex stable curve of arithmetic genus two. Let (G,q) be its dual graph and let $\{e_1,\ldots,e_r\}$ denote the edge set of G. Let $0\in U\subset\mathbb{C}^3$ be the universal deformation space of $\bar{X}_0$ as an analytic stable curve, let $\pi\colon\bar{X}\to U$ be the associated Kuranishi family where $\bar{X}_0$ is the fiber of π at 0 and suppose that $u_1\cdots u_r=0$ is an equation for the (reduced normal crossings) divisor D in U given by the points $u\in U$ such that $\bar{X}_u$ is not smooth. We assume that the choice of coordinates in $U\subset\mathbb{C}^3$ is compatible with the given ordering of the edges of G. Let $\varphi^{\mathrm{tr}}(G,q)\in\mathbb{Q}(x_1,\ldots,x_r)$ be the rational function given in Table 1.1. Then the asymptotics*

$$\varphi(\bar{X}_u)=\varphi^{\mathrm{tr}}(G,q,-\log|u_1|,\ldots,-\log|u_r|)+O(1) \tag{1.2}$$

holds as $u\to 0$ over $U\setminus D$.

In particular, note that for any holomorphic arc $f\colon\Delta\to U$ with $f(0)=0$ and with image not contained in D we obtain from (1.2) that

$$\alpha'\varphi(\bar{X}_{f(t)})=\varphi^{\mathrm{tr}}(G,q,m_1,\ldots,m_r)+O(\alpha') \tag{1.3}$$

as $t\to 0$ in Δ^*, where $\alpha'=-(\log|t|)^{-1}$ and $m_i=\mathrm{ord}_0\,f^*u_i$ for $i=1,\ldots,r$. In the language of string theory, equation (1.3) says that the “point-like limit" of φ in the direction of $\bar{X}_0$ exists, and is equal to $\varphi^{\mathrm{tr}}(G,q)$. The notation $^{\mathrm{tr}}$ stands for "tropical" and is borrowed from [28] where, for a variety of string integrands on $\mathcal{M}_{h,n}$, the point-like limit is studied from the point of view of tropical modular geometry. The point-like limit of the Néron height pairing on a pair of degree zero divisors in arbitrary genus is computed, and expressed in terms of the combinatorics of the dual graph, in the paper [1].

Theorem 1.1 is proved by a case-by-case analysis. One obvious question is whether there is a general mechanism that would produce the various $\varphi^{\mathrm{tr}}(G,q)$ from Table 1.1. In particular, one would like to predict the point-like limits of φ in genera larger than two (assuming they exist).

In [32] S. Zhang introduced and studied an invariant $\varphi(G,q)$ for polarized graphs (G,q) that takes values in $\mathbb{Q}(x_1,\ldots,x_r)$ if G has r ordered edges. The invariant $\varphi(G,q)$ is homogeneous of weight one in the variables $x_1,\ldots,x_r$, and serves alongside the Zhang-Kawazumi invariant $\varphi(\Sigma)$ in a formula that relates the height $(\Delta_\xi,\Delta_\xi)\in\mathbb{R}$ of the so-called canonical Gross-Schoen cycle Δ_ξ of a smooth, projective and geometrically connected curve X with semistable reduction over a number field with the self-intersection $(\omega_a,\omega_a)\in\mathbb{R}$ of its admissible relative dualizing sheaf, cf. [32, Corollary 1.3.2]. We have calculated the graph invariant $\varphi(G,q)$ for polarized graphs of genus two in [19, Theorem 2.1]. A quick comparison between Table 1 in [19] and Table 1.1 above yields that, perhaps rather strikingly, for each of the topological types I–VI in genus two one has that $\varphi(G,q)=\varphi^{\mathrm{tr}}(G,q)$.

We expect that this is not just a coincidence. That is, apart from being a suitable non-archimedean analogue of the Zhang-Kawazumi invariant, the invariant $\varphi(G,q)$ should also be the point-like limit of the Zhang-Kawazumi invariant in the direction of each stable curve with dual graph (G,q), in the following sense.

Conjecture 1.2. *Let $\bar{X}_0$ be a complex stable curve of arithmetic genus $h\geq 2$. Let (G,q) be the polarized dual graph of $\bar{X}_0$, and write $E(G)=\{e_1,\ldots,e_r\}$ for the edge set of G. Let $0\in U\subset\mathbb{C}^{3h-3}$ be the universal deformation space of $\bar{X}_0$ as an analytic stable curve, and let $\pi\colon\bar{X}\to U$ be the associated Kuranishi family such that $\bar{X}_0$ is the fiber of π at 0. Suppose that $u_1\cdots u_r=0$ is an equation for the divisor D in U given by the points $u\in U$ such that $\bar{X}_u$ is not smooth. We assume that the choice of coordinates in $U\subset\mathbb{C}^{3h-3}$ is compatible with the given ordering of the edges of G. Let $f\colon\Delta\to U$ be a holomorphic arc with $f(0)=0$, and with image not contained in D. Let $\varphi(G,q)\in\mathbb{Q}(x_1,\ldots,x_r)$ be Zhang's graph invariant of (G,q). Then one has* *the asymptotics*

$$\alpha'\varphi(\bar{X}_{f(t)}) = \varphi(G, q, m_1, \ldots, m_r) + O(\alpha') \tag{1.4}$$

as $t \to 0$ *in* Δ^*, *where* $\alpha' = -(\log|t|)^{-1}$ *and* $m_i = \mathrm{ord}_0\, f^*u_i$ *for* $i = 1, \ldots, r$.

To obtain a feeling for the strength of this conjecture, we note that since the Zhang-Kawazumi invariant in genus $h \geq 2$ is always positive, and any polarized stable graph of genus $h \geq 2$ is the dual graph of some stable curve of genus h, we immediately obtain from (1.4) that for any polarized stable graph (G, q) of genus $h \geq 2$, and any tuple $(m_1, \ldots, m_r)$ of positive integers with $r = |E(G)|$, the rational number $\varphi(G, q, m_1, \ldots, m_r)$ should be non-negative. This statement is known to be true as follows from [8, Theorem 2.11], however the proof of the inequality in [8, Theorem 2.11] seems to be rather involved.

On the positive side, we know that in all cases where $r = 1$ Conjecture 1.2 is true. This follows from comparing [18, Corollary 1.6] and [32, Propositions 4.4.1 and 4.4.3]. The asymptotic result of [18, Corollary 1.6] has been refined in [16, Theorem A]. We note that the cases where $r = 1$ correspond to the generic points of the boundary divisor of $\mathcal{M}_h$ in $\overline{\mathcal{M}}_h$.

The aim of the present paper is to prove the following partial result, where we limit ourselves to hyperelliptic curves (of any genus). When ψ_1, ψ_2 are continuous functions on the punctured disc Δ^*, we write $\psi_1 \sim \psi_2$ if the difference $\psi_1 - \psi_2$ extends as a continuous function over Δ.

Theorem 1.3. *Let* $\bar{X}_0$ *be a complex hyperelliptic stable curve of arithmetic genus* $h \geq 2$. *Let* (G, q) *be the polarized dual graph of* $\bar{X}_0$, *and write* $E(G) = \{e_1, \ldots, e_r\}$. *Let* $0 \in U \subset \mathbb{C}^{2h-1}$ *be the universal deformation space of* $\bar{X}_0$ *as an analytic hyperelliptic stable curve, and let* $\pi\colon \bar{X} \to U$ *be the associated Kuranishi family where* $\bar{X}_0$ *is the fiber of* π *at* 0. *Suppose that* $u_1 \cdots u_r = 0$ *is an equation for the (reduced normal crossings) divisor* D *in* U *given by the points* $u \in U$ *such that* $\bar{X}_u$ *is not smooth. Let* $f\colon \Delta \to U$ *be a holomorphic arc with* $f(0) = 0$, *and with image not contained in* D. *Let* $\varphi(G, q) \in \mathbb{Q}(x_1, \ldots, x_r)$ *be Zhang's graph invariant of* (G, q). *Then one has the asymptotics*

$$\varphi(\bar{X}_{f(t)}) \sim -\varphi(G, q, m_1, \ldots, m_r) \log|t| \tag{1.5}$$

as $t \to 0$ *in* Δ^*, *where* $m_i = \mathrm{ord}_0\, f^*u_i$ *for* $i = 1, \ldots, r$.

Note that the estimate in (1.5) is stronger than the one claimed in (1.4). We also note that Theorem 1.3 gives a uniform explanation for the six occurring point-like limits in equation (1.3). However, the asymptotics in (1.5), where the difference between the left and right hand side a priori depends on the chosen holomorphic arc, seems not strong enough to produce the $O(1)$-symbol in (1.2), which is in some sense uniform over all holomorphic arcs.

The contents of this paper are as follows. For the sake of concreteness, and also to explain the notation used in Table 1.1, we will first discuss the proof of Theorem 1.1 at some length, in Section 2 below. Then in Section 3 we review the definition of Zhang's graph invariant $\varphi(G,q)$, and recall some of its properties. In the final Section 4 we present our proof of Theorem 1.3.

2. Proof of Theorem 1.1

As noted before, Theorem 1.1 follows by gathering together various results from [12] [13] about the asymptotics of the Zhang-Kawazumi invariant in genus two. In this section, we adapt these results to our situation and provide some details about the $O(1)$-symbol in (1.2).

A polarized graph is a finite connected graph G together with a map $q\colon V(G) \to \mathbb{Z}_{\geq 0}$, called the polarization. Here $V(G)$ denotes the vertex set of G. Let (G,q) be a polarized graph, with Betti number $b_1(G)$. We call the non-negative integer $h(G,q) = b_1(G) + \sum_{x \in V(G)} q(x)$ the genus of (G,q). The polarized graph (G,q) is called stable if for each vertex x with $q(x) = 0$ the number of half-edges emanating from x is at least 3. For any given non-negative integer h, there are, up to isomorphism, only finitely many stable polarized graphs (G,q) of genus $h(G,q) = h$. If $h = 2$, there are precisely seven types. Apart from the trivial polarized graph (one vertex with $q = 2$, no edges), we list these types below as Cases I-VI.

Let $\bar{X}_0$ be a complex stable curve. The dual graph G of $\bar{X}_0$ is a finite connected graph whose vertex set $V(G)$ consists of the irreducible components of $\bar{X}_0$, and whose edge set $E(G)$ consists of the singular points of $\bar{X}_0$. The vertex assignment map is given by sending a singular point $e \in E(G)$ to the set of irreducible components of $\bar{X}_0$ that e lies on. The dual graph of $\bar{X}_0$ has a canonical polarization q given by assigning to each $x \in V(G)$ the genus of the normalization of x. The associated polarized graph (G,q) is stable. If $\bar{X}_0$ has arithmetic genus h, then the genus of (G,q) is also equal to h.

Now let $\bar{X}_0$ be a complex stable curve of arithmetic genus two, with (polarized, stable) dual graph (G,q) and with $E(G) = \{e_1, \ldots, e_r\}$. Let $0 \in$

$U \subset \mathbb{C}^3$ be the universal deformation space of $\bar{X}_0$ as an analytic stable curve, let $\pi \colon \bar{X} \to U$ be the Kuranishi family such that $\bar{X}_0$ is the fiber of π at 0, and suppose that $u_1 \cdots u_r = 0$ defines the divisor D of U given by the points $u \in U$ such that $\bar{X}_u$ is not smooth.

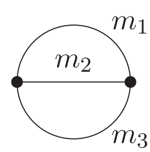
In [12] [13] one considers period matrices

$$\Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix}, \quad \operatorname{Im}\Omega > 0$$

of genus two curves near the boundary of Siegel upper half space $\mathbb{H}_2$ of degree two. In particular τ_1, τ_2 are elements of the upper half plane $\mathbb{H}_1$. In order to obtain the asymptotics as stated in Theorem 1.1, we need to translate the asymptotics from [12] [13], which are phrased in terms of Ω, in terms of the standard coordinates u_1, u_2, u_3 on the universal deformation space U. In other words, we first need to write $\Omega = \Omega(u)$ explicitly in coordinates u_1, u_2, u_3. In those cases where the period matrices associated to $\bar{X}_u$ are unbounded, one finds an expansion of Ω in terms of u_1, u_2, u_3 using a suitable several variables version of the nilpotent orbit theorem, cf. [4].

We recall that there are six topological types I-VI of (non-trivial) polarized stable genus two graphs to deal with. We discuss them case by case. For future reference we have assigned lengths $m_1, \ldots$ to the edges of each graph. In the following $\vartheta_1(z,\tau)$ denotes Riemann's theta function with modulus τ in genus one (in particular we have $z \in \mathbb{C}$, $\tau \in \mathbb{H}_1$). Finally $\eta(\tau)$ denotes Dedekind's eta function.

Case I – This is the case of the "sunset" graph,



Both vertices have $q = 0$. The corresponding stable curve $\bar{X}_0$ consists of two projective lines, joined together at three points. If $0 \in U \subset \mathbb{C}^3$ is the universal deformation space of $\bar{X}_0$, then an analysis of the monodromy on the general fiber of $\bar{X} \to U$ together with the nilpotent orbit theorem [4] gives

$$\Omega(u_1, u_2, u_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{\log u_1}{2\pi i} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{\log u_2}{2\pi i} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{\log u_3}{2\pi i} \, ,$$

up to a bounded holomorphic function. We obtain

$$\operatorname{Im} \Omega = \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix} + O(1) \, , \quad L_i = -\frac{1}{2\pi} \log |u_i| \, , \quad i = 1, 2, 3 \, ,$$

and with this notation then following [13, Section 3.2] we have the asymptotics

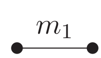
$$\varphi(\Omega) = \frac{\pi}{6} \left[L_1 + L_2 + L_3 - \frac{5 L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1} \right] + O(1/L_i^2) \tag{2.1}$$

for the Zhang-Kawazumi invariant. It follows that

$$\begin{aligned} \varphi(\bar{X}_u) = & -\frac{1}{12} (\log |u_1| + \log |u_2| + \log |u_3|) \\ & + \frac{5}{12} \frac{\log |u_1| \log |u_2| \log |u_3|}{\log |u_1| \log |u_2| + \log |u_2| \log |u_3| + \log |u_3| \log |u_1|} + O(1) \end{aligned}$$

as $t \to 0$ through $U \setminus D$. Comparing with Table 1.1 we see that in this case (1.2) is satisfied.

Case II – This is the "minimal separating degeneration limit", obtained by letting $\tau \to 0$ while keeping $\tau_1, \tau_2 \in \mathbb{H}_1$ bounded away from zero and infinity. The corresponding stable curve $\bar{X}_0$ is the join of the elliptic curves with moduli $\tau_1(0)$ and $\tau_2(0)$. The dual graph G looks like



where both vertices have $q = 1$. Following [16, Theorem A or Section 5] or [12, Section 6.1] we then have

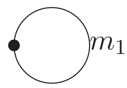
$$\varphi = -\log |2\pi\tau\eta^2(\tau_1)\eta^2(\tau_2)| + O(|\tau|^2 \log |\tau|) \tag{2.2}$$

as $\tau \to 0$. In this case we can identify $\tau = u_1$ so that we obtain the asymptotics

$$\varphi = -\log |u_1| + O(1)$$

as $t \to 0$. We thus obtain (1.2) also in this case.

Case III – This is the "minimal non-separating degeneration limit", obtained by sending $\tau_2 \to i\infty$ while keeping $\tau_1 \in \mathbb{H}$ and $\tau \in \mathbb{C}$ bounded away from zero and infinity. The corresponding stable curve $\bar{X}_0$ is the elliptic curve with modulus $\tau_1(0)$ with the points determined by $z = 0$, $z = 1$ and $z = \tau(0)$ identified. The dual graph G looks like



where the unique vertex has $q = 1$. Following [16, Theorem A or Section 5] or [12, Section 6.1] we have

$$\varphi = \frac{\pi}{6}\left(\mathrm{Im}\,\tau_2 + \frac{5(\mathrm{Im}\,\tau)^2}{\mathrm{Im}\,\tau_1}\right) - \log\left|\frac{\vartheta_1(\tau,\tau_1)}{\eta(\tau_1)}\right| + O(1/\mathrm{Im}\,\tau_2) \tag{2.3}$$

as $\tau_2 \to i\infty$. In terms of the local coordinates u_1, u_2, u_3 of U we have, by the nilpotent orbit theorem,

$$\tau_2 = \frac{\log u_1}{2\pi i}$$

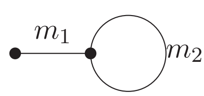
up to a bounded holomorphic function of u. We obtain the asymptotics

$$\varphi(\bar{X}_u) = -\frac{1}{12}\log|u_1| + O(1)$$

as $t \to 0$, thus verifying (1.2) again.

The three remaining cases can be obtained by successively further degenerating the elliptic curves in Cases II and III above. One then obtains (cf. [12, Section 6.2]) the required asymptotics of φ from the previous two cases by a careful study of the stated subleading terms.

Case IV – Here we take a stable curve as in Case II but further degenerate the modulus τ_2 by sending $\tau_2 \to i\infty$. The resulting stable curve is the join of an elliptic curve with modulus $\tau_1(0)$ and a rational curve with points 0 and ∞ identified. The dual graph G looks like



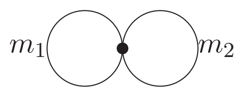
The shape of the degeneration is

$$\tau \to 0\,,\ \tau_2 \to i\infty\,, \quad \varphi(\Omega) = -\log|\tau| + \frac{\pi}{6}\mathrm{Im}\,\tau_2 + O(1)\,,$$

and this leads to the asymptotics

$$\varphi(\bar{X}_u) = -\log|u_1| - \frac{1}{12}\log|u_2| + O(1)\,.$$

Case V – We further degenerate the curve in Case III by sending $\tau_1 \to i\infty$. The resulting stable curve is a rational curve with two double points. Its dual graph looks like



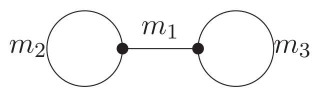
where the vertex has $q = 0$ and we have

$$\tau_1 \to i\infty\,,\ \tau_2 \to i\infty\,, \quad \varphi(\Omega) = \frac{\pi}{6}\mathrm{Im}\,\tau_1 + \frac{\pi}{6}\mathrm{Im}\,\tau_2 + O(1)\,,$$

so that

$$\varphi(\bar{X}_u) = -\frac{1}{12}\log|u_1| - \frac{1}{12}\log|u_2| + O(1)\,.$$

Case VI – In this final case we further degenerate the elliptic curve with modulus τ_1 from Case IV by sending $\tau_1 \to i\infty$. The resulting stable curve is the join of two rational curves each carrying a single double point. The dual graph of $\bar{X}_0$ in this case looks like



where both vertices have $q = 0$. We have

$$\tau \to 0\,,\ \tau_1 \to i\infty\,, \tau_2 \to i\infty\,,$$
$$\varphi(\Omega) = -\log|\tau| + \frac{\pi}{6}\mathrm{Im}\,\tau_1 + \frac{\pi}{6}\mathrm{Im}\,\tau_2 + O(1)\,,$$

so that

$$\varphi(\bar{X}_u) = -\log|u_1| - \frac{1}{12}\log|u_2| - \frac{1}{12}\log|u_3| + O(1)\,.$$

Theorem 1.1 is hereby proved. In closing this section we mention that more refined versions of the asymptotics in (2.1), (2.2) and (2.3) have been determined by B. Pioline [27].

3. Polarized metric graphs and Zhang's graph invariant

The aim of this section is to recall from [3] [31, Appendix] some notions related to metric graphs, and to introduce Zhang's graph invariant $\varphi(\Gamma, q)$ of a polarized metric graph (Γ, q).

A metric graph Γ is a compact connected metric space such that for each $x \in \Gamma$ there exist a natural number n and $\epsilon \in \mathbb{R}_{>0}$ such that x has a neighborhood isometric to the set

$$S(n, \epsilon) = \{z \in \mathbb{C} \, : \, z = te^{2\pi ik/n} \text{ for some } 0 \leq t < \epsilon \text{ and some } k \in \mathbb{Z}\} \, ,$$

endowed with its natural metric. If Γ is a metric graph, then for each $x \in \Gamma$ the integer n is uniquely determined, and is called the valence of x. We denote by $V_0 \subset \Gamma$ the set of points $x \in \Gamma$ with valency $\neq 2$. This is a finite set. Any finite non-empty set $V \subset \Gamma$ containing V_0 is called a vertex set of Γ.

If V is a vertex set of Γ then $\Gamma \setminus V$ is a finite union of open intervals. The closure of a connected component of $\Gamma \setminus V$ is called an edge associated to V. Let E be the set of edges associated to V. For $e \in E$ we call $e \setminus e^o$ the set of endpoints of e. This is a finite set consisting of either one (when e is a loop) or two (when e is a closed interval) elements.

Let Γ be a metric graph, and assume a vertex set V is given, with associated edge set E. For an edge $e \in E$ we let $\mathrm{d}y$ denote the usual Lebesgue measure on e, and $m(e)$ the volume of e.

Elements of $\mathbb{R}^V$ are called divisors on Γ (with respect to the given vertex set V). For a divisor $D = \sum_{x \in V} a(x)x$ we define the degree of D as $\deg D = \sum_{x \in V} a(x)$. Let $C(\Gamma)$ be the set of $\mathbb{R}$-valued continuous functions on Γ that are smooth outside V and have well-defined derivatives at each $v \in V$ along each $e \in E$ emanating from v. Denote by $C(\Gamma)^*$ the set of linear functionals on $C(\Gamma)$. The elements of $C(\Gamma)^*$ are called currents on Γ. As an important example, each $x \in \Gamma$ gives rise to a Dirac current $\delta_x \in C(\Gamma)^*$ given by sending $f \mapsto f(x)$. This prescription extends linearly to give a Dirac current δ_D associated to each divisor D on Γ. Integration of currents over Γ gives a natural linear map $C(\Gamma)^* \to \mathbb{R}$. Following [3, Section 1.2] or [31, Appendix] there exists a natural Laplace operator $\Delta \colon C(\Gamma) \to C(\Gamma)^*$.

Let K_{can} be the divisor on Γ given by $K_{\mathrm{can}}(x) = v(x) - 2$ for each $x \in V$. Note that $\deg K_{\mathrm{can}} = 2b_1(\Gamma) - 2$, where $b_1(\Gamma)$ is the Betti number of Γ. For an edge $e \in E$, we denote by $r(e)$ the effective resistance between the endpoints of e in $\Gamma \setminus e^o$, where Γ is viewed as an electric circuit with edge resistances given by the $m(e)$. We set $r(e)$ to be ∞ if $\Gamma \setminus e^0$ is disconnected.

We have a canonical current [5]

$$\mu_{\mathrm{can}} = -\frac{1}{2}\delta_{K_{\mathrm{can}}} + \sum_{e \in E} \frac{\mathrm{d}y}{m(e) + r(e)}$$

in $C(\Gamma)^*$. Here $\delta_{K_{\mathrm{can}}}$ is the Dirac current associated to the divisor K_{can} defined above. By [5, Theorem 2.11] the current μ_{can} is a probability measure, that is we have $\int_\Gamma \mu_{\mathrm{can}} = 1$.

A polarization on Γ is a map $q\colon V \to \mathbb{Z}_{\geq 0}$. Associated to a polarization q we consider the divisor

$$K_q = K_{\mathrm{can}} + 2 \sum_{x \in V} q(x) \cdot x$$

on Γ. Putting $h(\Gamma, q) = b_1(\Gamma) + \sum_{x \in V} q(x)$ we see that $\deg K_q = 2h(\Gamma, q) - 2$. We call $h(\Gamma, q)$ the genus of the polarized metric graph (Γ, q). For polarized metric graphs (Γ, q) of positive genus h, Zhang's admissible measure is defined to be the current

$$\mu = \mu(\Gamma, q) = \frac{1}{2h}\left(\delta_{K_q} + 2\mu_{\mathrm{can}}\right)$$

in $C(\Gamma)^*$. Here δ_{K_q} is the Dirac current associated to the divisor K_q defined above. We note that $\int_\Gamma \mu = 1$. Finally, Zhang's Arakelov-Green's function $g_\mu(x, y)$ on $\Gamma \times \Gamma$ is determined by the conditions

$$\Delta_y g_\mu(x, y) = \delta_x(y) - \mu(y)\,, \qquad \int_\Gamma g_\mu(x, y)\mu(y) = 0$$

for all $x \in \Gamma$. We refer to [31, Section 3 and Appendix] or [3, Section 1.5] for a proof that $g_\mu(x, y)$ exists in $C(\Gamma)$ for all $x \in \Gamma$, as well as for a discussion of its main properties.

Let (Γ, q) be a polarized metric graph of genus h. We denote by $\delta(\Gamma)$ the total volume of Γ. Zhang's graph invariant $\varphi(\Gamma, q)$ is defined via the formula [32, Section 1.3]

$$\varphi(\Gamma, q) = -\frac{1}{4}\delta(\Gamma) + \frac{1}{4}\int_\Gamma g_\mu(x, x)((10h + 2)\mu(x) - \delta_{K_q}(x))\,. \tag{3.1}$$

For later reference we define the similar epsilon-invariant [31, Theorem 4.4]

$$\varepsilon(\Gamma, q) = \int_\Gamma g_\mu(x, x)((2h - 2)\mu(x) + \delta_{K_q}(x))\,. \tag{3.2}$$

Let G be a finite connected graph with set of edges $E(G)$ and set of vertices $V(G)$. Then to G we have naturally associated a metric graph by glueing $|E(G)|$ closed intervals $[0,1]$ according to the vertex assignment map. More generally, if a weight $m \in \mathbb{R}_{>0}^{E(G)}$ of $E(G)$ is given, then one has an associated metric graph $\Gamma = (G, m)$ by glueing the closed intervals $[0, m(e)]$, where e runs through $E(G)$, according to the vertex assignment map. Usually we assume that the edge set $E(G)$ is ordered, so that we may write $\Gamma = (G, m_1, \ldots, m_r)$ where $r = |E(G)|$. Note that Γ comes with a natural vertex set $V(G)$. A polarization $q\colon V(G) \to \mathbb{Z}_{\geq 0}$ of G naturally gives rise to a polarized metric graph $(\Gamma, q) = (G, q, m_1, \ldots, m_r)$.

We see that if (G, q) is a polarized graph, then for each $m \in \mathbb{R}_{>0}^{E(G)}$ we have an element $\varphi(G, q, m) \in \mathbb{R}$. We thus obtain a natural map $\varphi(G, q)\colon \mathbb{R}_{>0}^{E(G)} \to \mathbb{R}$. By [11, Proposition 4.6] we have the following important property of $\varphi(G, q)$. Let $b_1(G)$ be the first Betti number of G. Then there exist homogeneous integral polynomials P, Q of degree $2b_1(G) + 1$ resp. $2b_1(G)$ such that

$$\varphi(G, q, m_1, \ldots, m_r) = P(m_1, \ldots, m_r)/Q(m_1, \ldots, m_r)$$

for all $(m_1, \ldots, m_r) \in \mathbb{R}_{>0}^r$. In particular, the map $\varphi(G, q)\colon \mathbb{R}_{>0}^{E(G)} \to \mathbb{R}$ is continuous, and homogeneous of weight one. Moreover, we may view $\varphi(G, q)$ as an element of the rational function field $\mathbb{Q}(x_1, \ldots, x_r)$.

In [7] [10] one finds algorithms to calculate $\varphi(G, q) \in \mathbb{Q}(x_1, \ldots, x_r)$, together with many examples in low genera. In [19] a list is given of the $\varphi(G, q)$ for all stable polarized graphs of genus two. In [6] a complete study is made of $\varphi(G, q)$ for all stable polarized graphs of genus three. In [29] [30] the invariant $\varphi(G, q)$ is studied for so-called hyperelliptic polarized graphs. In our proof of Theorem 1.3 below we will make essential use of a result from [30] that relates $\varepsilon(G, q)$ and $\varphi(G, q)$ for hyperelliptic polarized graphs. In [8] an effective lower bound is proven for the invariant $\varphi(\Gamma, q)$ for general polarized metric graphs, establishing Conjecture 1.4.2 from [32].

4. Proof of Theorem 1.3

As in Theorem 1.3, let $\bar{X}_0$ be a complex hyperelliptic stable curve of arithmetic genus $h \geq 2$, with canonically polarized dual graph (G, q). Let $0 \in U \subset \mathbb{C}^{2h-1}$ be the universal deformation space of $\bar{X}_0$ as an analytic hyperelliptic stable curve, and let $\pi\colon \bar{X} \to U$ be the associated Kuranishi family where $\bar{X}_0$ is the fiber of π at 0. Assume that the locus D of points $u \in U$

such that $\bar{X}_u$ is not smooth is given by the equation $u_1 \cdots u_r = 0$, in particular $|E(G)| = r$. We consider a holomorphic arc $f \colon \Delta \to U$, with $f(0) = 0$, and with image not contained in D. We denote by $\varphi(G,q) \in \mathbb{Q}(x_1,\ldots,x_r)$ Zhang's graph invariant of (G,q).

Let $m_i = \mathrm{ord}_0 f^* u_i$ for $i = 1,\ldots,r$. Then as in Section 3 we let $(G,q,m_1,\ldots,m_r)$ denote the polarized metric graph associated to (G,q) and the weight $(m_1,\ldots,m_r) \in \mathbb{R}^r_{>0}$. Let $\bar{Y} \to \Delta$ be the stable curve over Δ obtained by pulling back, in the category of analytic spaces, the stable curve $\bar{X} \to U$ along the holomorphic arc $f \colon \Delta \to U$. Then in a neighborhood of a node e of $\bar{X}_0$, the surface $\bar{Y}$ is given by an equation $uv - t^{m(e)}$ and hence a local minimal resolution of singularities of $\bar{Y}$ at e is obtained by replacing e by a chain of $m(e) - 1$ projective lines. Let $\bar{Y}' \to \bar{Y}$ be the minimal resolution of singularities of $\bar{Y}$. Then $\bar{Y}' \to \Delta$ is a semistable curve with smooth total space $\bar{Y}'$, and by construction of $\bar{Y}'$ the polarized metric graph obtained by taking the dual graph (G',q') of the fiber of $\bar{Y}' \to \Delta$ over 0 and giving each edge of G' unit length is canonically isometric to the polarized metric graph $(G,q,m_1,\ldots,m_r)$.

With this in mind, Theorem 1.3 reduces to the following statement.

Theorem 4.1. *Let $\bar{X}$ be a smooth complex surface and let $\pi \colon \bar{X} \to \Delta$ be a hyperelliptic semistable curve, smooth over Δ^*. Let (G,q) be the polarized dual graph of the special fiber $\bar{X}_0$, and let (Γ,q) be the associated polarized metric graph where each edge has unit length. Then the Zhang-Kawazumi invariant of the fibers $\bar{X}_t$ satisfies the asymptotics*

$$\varphi(\bar{X}_t) \sim -\varphi(\Gamma,q) \log|t|$$

as $t \to 0$.

The remainder of this paper is devoted to a proof of Theorem 4.1. The proof consists of a number of steps. First we write the hyperelliptic Zhang-Kawazumi invariant in terms of the Faltings delta-invariant [11] and the Petersson norm of the modular discriminant [24]. This step was already accomplished in [18]. We then focus on the asymptotics of the Faltings delta-invariant and the Petersson norm of the modular discriminant. In [14] we have found the asymptotics of the Faltings delta-invariant in an arbitrary semistable family of curves over the unit disc. Next, a result of I. Kausz [21] can be interpreted as giving the asymptotics of the Petersson norm of the modular discriminant. In our final step we relate the resulting asymptotics for φ to Zhang's graph invariant via a result from [30].

When $\pi\colon \bar{X} \to \Delta$ is a semistable curve over the unit disc, we denote by ω the relative dualizing sheaf of π, and by λ_1 the determinant of the Hodge bundle $\lambda_1 = \det R\pi_*\omega$ on Δ.

We start by reviewing from [24, Section 3] the construction of a canonical discriminant modular form Δ_h on the moduli space $\mathcal{I}_h$ of complex hyperelliptic curves of genus h. The form Δ_h generalizes the usual discriminant Δ of weight 12 occurring in the theory of moduli of elliptic curves. It is well known that for a semistable family of elliptic curves $\bar{X} \to \Delta$, with $\bar{X}$ smooth, the section

$$\Lambda = (2\pi)^{12}\Delta(\tau)(\mathrm{d}z)^{\otimes 12}$$

of the line bundle $\lambda_1^{\otimes 12}$ is trivializing over Δ^*, and acquires a zero of multiplicity δ at the origin, where δ is the number of singularities of the special fiber $\bar{X}_0$. These facts generalize to semistable families of hyperelliptic curves of genus h over the unit disc as follows.

Let n be the binomial coefficient $\binom{2h}{h+1}$. Let $\mathbb{H}_h$ be Siegel's upper half-space consisting of symmetric complex $h \times h$-matrices with positive definite imaginary part. Then for z in $\mathbb{C}^h$ (viewed as a column vector), matrices $\Omega \in \mathbb{H}_h$ and η', η'' in $\frac{1}{2}\mathbb{Z}^h$ one has the classical theta function with characteristic $\eta = \begin{bmatrix} \eta' \\ \eta'' \end{bmatrix}$ given by the Fourier series

$$\theta[\eta](z,\Omega) = \sum_{n \in \mathbb{Z}^h} \exp(\pi i {}^t(n+\eta')\Omega(n+\eta') + 2\pi i {}^t(n+\eta')(z+\eta'')) \,.$$

For a given set $\mathcal{T}$ of even theta characteristics in $\frac{1}{2}\mathbb{Z}^h$ we let $S(\mathcal{T})$ be the set of matrices $\Omega \in \mathbb{H}_h$ such that the equivalence

$$\vartheta[\eta](0,\Omega) \neq 0 \quad \Longleftrightarrow \quad \eta \in \mathcal{T}$$

holds for Ω. Following [26, Theorem III.9.1] there exists a canonical set $\mathcal{T}_0$ of even theta characteristics such that $S(\mathcal{T}_0)$ is precisely the set of hyperelliptic period matrices. Here and in what follows, by a hyperelliptic period matrix we mean a normalized period matrix of a hyperelliptic Riemann surface Σ, formed on a canonical symplectic basis of $H_1(\Sigma,\mathbb{Z})$. We refer to [26, Section III.5] for the notion of a canonical symplectic basis, determined by an ordering of the hyperelliptic branch points of Σ.

The discriminant modular form Δ_h is defined to be the function

$$\Delta_h(\Omega) = 2^{-(4h+4)n} \prod_{\eta \in \mathcal{T}_0} \theta[\eta](0,\Omega)^8$$

on $\mathbb{H}_h$. A verification shows that Δ_h is a modular form on the congruence normal subgroup $\Gamma_h(2) \subset \mathrm{Sp}(2h,\mathbb{Z})$ of weight $(8h+4)n/h$. If $\Omega \in \mathbb{H}_h$ is a hyperelliptic period matrix, we put

$$\|\Delta_h\|(\Omega) = (\det \operatorname{Im} \Omega)^{(4h+2)n/h} |\Delta_h(\Omega)| \,. \tag{4.1}$$

Then for a given hyperelliptic curve Σ of genus h the value of $\|\Delta_h\|$ on a period matrix Ω of Σ on a canonical basis of homology does not depend on the choice of Ω. We conclude that $\|\Delta_h\|$ induces a well-defined real-valued function on $\mathcal{I}_h$. In [18] we found that the restriction of φ to $\mathcal{I}_h$ can be expressed in terms of $\|\Delta_h\|$ and the Faltings delta-invariant [11].

Theorem 4.2. *The Zhang-Kawazumi invariant of a hyperelliptic Riemann surface Σ with hyperelliptic period matrix Ω satisfies the following formula:*

$$(2h-2)\varphi(\Sigma) = -8(2h+1)h\log(2\pi) - 3(h/n)\log\|\Delta_h\|(\Sigma) - (2h+1)\delta_F(\Sigma) \,.$$

Here $\delta_F(\Sigma)$ is the Faltings delta-invariant of Σ.

Proof. This is [18, Corollary 1.8]. □

Let $\pi \colon \bar{X} \to \Delta$ be a hyperelliptic semistable curve, with $\bar{X}$ smooth over $\mathbb{C}$, and with π smooth over Δ^*. Following the paper [21] by I. Kausz, the algebraic discriminant of a generic hyperelliptic equation for $\bar{X} \to \Delta$ gives rise to a canonical global section Λ of λ_1^{8h+4} over Δ which is trivializing over Δ^*. In holomorphic coordinates on the associated jacobian family we may write (cf. the proof of [20, Theorem 8.2])

$$\Lambda^{\otimes n} = (2\pi)^{(8h+4)hn} \Delta_h(\Omega)^{\otimes h} (\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_h)^{\otimes (8h+4)n} \,. \tag{4.2}$$

We say that a double point x of the special fiber $\bar{X}_0$ is of type 0 if the local normalization of $\bar{X}_0$ at x is connected. We say that x is of type i, where $i = 1, \ldots, [h/2]$, if the local normalization of $\bar{X}_0$ at x is the disjoint union of a curve of genus i and a curve of genus $h-i$. Let ι be the involution on $\bar{X}_0$ induced by the hyperelliptic involution on $\bar{X}$. Let x be a double point of type 0 on $\bar{X}_0$. If x is not fixed by ι, the local normalization of $\bar{X}_0$ at $\{x, \iota(x)\}$ consists of two connected components, of genus j and $h-j-1$, say, where $0 \leq j \leq [(h-1)/2]$. In this case we say that the pair $\{x, \iota(x)\}$ is of subtype j. Let ξ_0' be the number of double points of type 0 fixed by ι, let ξ_j for $j = 0, \ldots, [(h-1)/2]$ be the number of pairs $\{x, \iota(x)\}$ of double points of subtype j, and let δ_i for $i = 1, \ldots, [h/2]$ be the number of double points of type i. Let the integer d be given by

$$d = h\xi_0' + \sum_{j=0}^{[(h-1)/2]} 2(j+1)(h-j)\xi_j + \sum_{i=1}^{[h/2]} 4i(h-i)\delta_i \,. \tag{4.3}$$

Theorem 4.3. *Let $\Omega(t)$ be a family of hyperelliptic period matrices associated to $\bar{X} \to \Delta$. Then the asymptotic formula*

$$-(h/n)\log\|\Delta_h\|(\bar{X}_t) \sim -d\log|t| - (4h+2)\log\det\operatorname{Im}\Omega(t) \tag{4.4}$$

holds as $t \to 0$.

Proof. By [21, Theorem 3.1] we have, since $\bar{X}$ is smooth, the identity $\operatorname{ord}_0(\Lambda) = d$. Combining with (4.2) we obtain the identity $\operatorname{ord}_0(\Delta_h) = dn/h$. Formula (4.4) then follows from (4.1). □

Remark 4.4. The identity $\operatorname{ord}_0(\Lambda) = d$ is a refined version of an identity in $\operatorname{Pic}_{\mathbb{Q}}(\bar{\mathcal{I}}_h)$ due to M. Cornalba and J. Harris [9, Proposition 4.7]. Here $\bar{\mathcal{I}}_h$ denotes the stack theoretic closure of $\mathcal{I}_h$ inside $\overline{\mathcal{M}}_h$.

Let (G, q) be the dual graph of the special fiber $\bar{X}_0$, and let (Γ, q) be the associated polarized metric graph where each edge has unit length.

Theorem 4.5. *Let δ be the total volume of (Γ, q), and let ε be Zhang's epsilon-invariant (3.2) of (Γ, q). Let $\Omega(t)$ be any family of normalized period matrices associated to $\bar{X} \to \Delta$. Then the Faltings delta-invariant satisfies the asymptotics*

$$\delta_F(\bar{X}_t) \sim -(\delta+\varepsilon)\log|t| - 6\log\det\operatorname{Im}\Omega(t) \tag{4.5}$$

as $t \to 0$.

Proof. This is a special case of [14, Theorem 1.1]. □

Proof of Theorem 4.1. Upon combining Theorems 4.2, 4.3 and 4.5 we obtain the asymptotic estimate

$$(2h-2)\varphi(\bar{X}_t) \sim -(3d - (2h+1)(\delta+\varepsilon))\log|t|$$

as $t \to 0$. Hence we are done once we show that the equality

$$(2h-2)\varphi = 3d - (2h+1)(\delta+\varepsilon) \tag{4.6}$$

holds, where $\varphi = \varphi(\Gamma, q)$ is Zhang's graph invariant of (Γ, q). In [30, Section 1.7] the invariant

$$\psi = \varepsilon + \frac{2h-2}{2h+1}\varphi$$

is considered. With this notation formula (4.6) is equivalent to the formula

$$(2h+1)\psi = 3d - (2h+1)\delta\,. \tag{4.7}$$

A combination of [30, Section 1.9] and [30, Theorem 3.5] yields that for the hyperelliptic polarized graph (Γ, q) the equality

$$\begin{aligned}(2h+1)\psi =&(h-1)\delta_0 + \sum_{j=1}^{[(h-1)/2]} 6j(h-1-j)\xi_j \\ &+ \sum_{i=1}^{[h/2]} (12i(h-i) - (2h+1))\delta_i\end{aligned} \tag{4.8}$$

holds. On the other hand, from (4.3) and the identities

$$\delta = \delta_0 + \sum_{i=1}^{[h/2]} \delta_i\,, \quad \delta_0 = \xi_0' + 2\sum_{j=0}^{[(h-1)/2]} \xi_j$$

we obtain

$$\begin{aligned}3d - (2h+1)\delta =&(h-1)\delta_0 + \sum_{j=0}^{[(h-1)/2]} (6(j+1)(h-j) - 6h)\xi_j \\ &+ \sum_{i=1}^{[h/2]} (12i(h-i) - (2h+1))\delta_i\,.\end{aligned} \tag{4.9}$$

One readily checks that the right hand sides of (4.8) and (4.9) are equal, and (4.7) follows. This finishes the proof of Theorem 4.1. □

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