Pure and Applied Mathematics Quarterly
Volume 11, Number 4, 537–557, 2015

Finiteness of Rational Curves of Degree 12 on a General Quintic Threefold

Edoardo Ballico and Claudio Fontanari

Abstract: We prove the following statement, predicted by Clemens' conjecture: *A generic quintic threefold contains only finitely many smooth rational curves of degree* 12.
Keywords: quintic threefold, rational curve, Clemens' conjecture.

1. Introduction

The present paper is entirely devoted to the proof of the following instance of Clemens' conjecture ([4]):

Theorem 1. *A generic quintic threefold contains only finitely many smooth rational curves of degree* 12.

We point out that the cases $d \leq 11$ have been previously addressed in [14] ($d \leq 7$), [17] and [13] ($d = 8, 9$), [5] ($d = 10$), [6] and [7] ($d = 11$), and we recall the general set-up.

Let M_d be the set of smooth rational curves of degree d in $\mathbb{P}^4$. It is smooth and irreducible of dimension $5d + 1$. Let $\mathbb{P}^{125}$ denote the projective space of all quintic hypersurfaces of $\mathbb{P}^4$ and consider the incidence correspondence $I_d = \{(C, W) : C \subset W\} \subset M_d \times \mathbb{P}^{125}$. Let $\pi_1 : I_d \to M_d$ and $\pi_2 : I_d \to \mathbb{P}^{125}$ denote the restrictions to I_d of the two projections.

The map π_2 turns out to be finite if for every irreducible family $\Gamma \subseteq M_d$ with general element C we have:

$$\dim \Gamma + (h^0(\mathcal{I}_C(5)) - 1) \leq 125. \tag{1}$$

From the standard exact sequence

$$0 \to H^0(\mathcal{I}_C(5)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \to H^0(\mathcal{O}_C(5)) \to H^1(\mathcal{I}_C(5)) \to 0$$

Received November 20, 2015.

it follows that

$$h^0(\mathcal{I}_C(5)) - 1 = 125 - (5d+1) + h^1(\mathcal{I}_C(5)) \tag{2}$$

and by [2] the general curve C in M_d satisfies $h^1(\mathcal{I}_C(5)) = 0$, so in order to prove Clemens' conjecture one needs to control curves C with $h^1(\mathcal{I}_C(5)) > 0$.

In the case $d = 12$, we show that if an irreducible family $\Gamma \subseteq M_{12}$ of non-degenerate curves is a potential exception to Clemens' conjecture, then its general element C satisfies $h^1(\mathcal{I}_C(2)) \geq 13$. It follows that $h^0(\mathcal{I}_C(2)) \geq 3$ and this provides a contradiction (see Lemma 1).

The key point in our reduction is to obtain $h^1(\mathcal{I}_C(2)) \geq 13$ from $h^1(\mathcal{I}_C(5)) > 0$. Indeed, Lemma 2 implies that $h^1(\mathcal{I}_C(t-1)) \geq 4 + h^1(\mathcal{I}_C(t))$ except in two special cases, which are identified by Lemma 3 and then excluded in Lemmas 7, 8, 9, 13. Finally, a careful analysis of the degenerate case is provided (see Section 3).

We remark that the strong form of Clemens' conjecture (as proved by Cotterill in [5] and [6] for $d = 10, 11$, characterizing also singular irreducible rational curves on the general quintic threefold) cannot be achieved by our methods.

We work over the complex field $\mathbb{C}$.

We thank Gilberto Bini, Gianluca Occhetta and Luis Solá Conde for helpful conversations. We are also grateful to the anonymous referee and to Carlo Madonna for useful comments.

This research was partially supported by FIRB 2012 "Moduli spaces and Applications", by GNSAGA of INdAM and by MIUR (Italy).

2. Non-degenerate case

Lemma 1. *If $C \in M_{12}$, C is non-degenerate and $h^0(\mathcal{I}_C(2)) \geq 3$, then there is no smooth quintic 3-fold containing C.*

Proof. Assume by contradiction $h^0(\mathcal{I}_C(2)) \geq 3$ and the existence of a smooth quintic 3-fold $W \subset \mathbb{P}^4$ with $W \supset C$ and let $E \subset \mathbb{P}^4$ be the intersection of 3 general element of $|\mathcal{I}_C(2)|$. Since $\deg(C) = 12 > 8$, Bezout theorem gives the existence of an integral surface F such that $C \subset F \subseteq E$. Since C is non-degenerate, F is non-degenerate and so $\deg(F) \geq 3$. Assume $E = F$, i.e. $\deg(F) = 4$. Since the complete intersection of 2 quadric hypersurfaces is contained in exactly two linearly independent quadrics and $\deg(C) > 8$, we get $h^0(\mathcal{I}_C(2)) = 2$, a contradiction. Thus $\deg(F) = 3$. The classification of minimal degree non-degenerate surfaces in $\mathbb{P}^4$ gives $h^0(\mathcal{I}_F(2)) = 3$. By assumption there is W with $C \subset W$. Since $\mathrm{Pic}(W)$ is freely generated by $\mathcal{O}_W(1)$, $F \nsubseteq W$. Hence $W \cap F$ links C to a degree 3 locally Cohen-Macaulay curve $T \subset W \cap F$. By the classification of minimal degree surfaces in $\mathbb{P}^4$, either F is a cone with vertex o over a rational normal curve $D \subset \mathbb{P}^3$ or F is isomorphic to the the Hirzebruch surface F_1 embedded by the complete linear system $|h + 2f|$, where h is a section of the ruling of F_1 and f is a fiber of the ruling of F_1.

First assume that F is a cone. Since C is smooth, it has multiplicity at most 1 at o. Hence $o \notin C$ and the linear projection from o induces a degree 4 map $\ell : C \to D$. Let $\pi : G \to F$ be the blowing up of o and C' the strict transform of C in G. G is isomorphic to the Hirzebruch surface F_3 and the map π is induced by the complete linear system $|h + 3f|$. Since $o \notin C$ and $\deg(\ell) = 4$, π induces an isomorphism $C' \to C$ and $C' \in |4h + 12f|$. We have $\omega_G \cong \mathcal{O}_G(-2h - 5f)$. The adjunction formula gives $\omega_{C'} \cong \mathcal{O}_G(2h + 7f)$. Hence $h^0(\omega_{C'}) > 0$, contradicting the rationality and smoothness of C'.

Now assume $F \cong F_1$. Take $a, b \in \mathbb{N}$ such that $C \in |ah + bf|$. Since C is irreducible and not a line, we have $b \geq a > 0$. Since $\mathcal{O}_C(1) \cong \mathcal{O}_C(h + 2f)$, $h^2 = -1$, $h \cdot f = 1$, $f^2 = 0$ and $\deg(C) = d$, we have $12 = a + b$. Since $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h - 3f)$, the adjunction formula gives $\omega_C \cong \mathcal{O}_C((a-2)h + (b-3)f)$. Since $\deg(\omega_C) = -2$, we get $(ah + bf) \cdot ((a-2)h + (b-3)f) = -2$, i.e. $-a(a-2) + a(b-3) + b(a-2) = -2$, i.e $(b-a)(a-2) + a(b-3) = -2$. Since $b \geq a > 0$ and $b = 12 - a$, we get $a = 1$ and $b = 11$. Since $\mathcal{O}_{F_1}(5) \cong \mathcal{O}_{F_1}(5h + 10f)$ and $b = 11$, we have $h^0(F_1, \mathcal{I}_{C,F_1}(5)) = 0$. Hence $W \supset F_1$, a contradiction. □

The following fact is one of key ingredients in the proof of [7, Theorem 4.1].

Lemma 2. *Fix integer $t \geq 2$, $r \geq 3$ and an integral and non-degenerate curve $T \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_T(t)) > 0$. Assume that $h^1(M, \mathcal{I}_{M \cap T, M}(t)) = 0$ for every hyperplane $M \subset \mathbb{P}^r$. Then $h^1(\mathcal{I}_T(t-1)) \geq r + h^1(\mathcal{I}_T(t))$.*

Proof. For any hyperplane $M \subset \mathbb{P}^r$ we have an exact sequence

$$0 \to \mathcal{I}_T(t-1) \to \mathcal{I}_T(t) \to \mathcal{I}_{T \cap M, M}(t) \to 0 \tag{3}$$

Since $h^1(M, \mathcal{I}_{T,M}(t)) = 0$, the map $H^1(\mathcal{I}_T(t-1)) \to H^1(\mathcal{I}_T(t))$ is surjective, hence its dual $e_M : H^1(\mathcal{I}_T(t))^\vee \to H^1(\mathcal{I}_T(t-1))^\vee$ is injective. Taking the equations of all hyperplanes we get a bilinear map map $u : H^1(\mathcal{I}_T(t))^\vee \times H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \to H^1(\mathcal{I}_T(t-1))^\vee$, which is injective with respect to the second variables, i.e. for every non-zero linear form ℓ $u_{|H^1(\mathcal{I}_T(t))^\vee \times \{\ell\}}$ is injective (it is e_M with $M := \{\ell = 0\}$). Hence if $(a, \ell) \in H^1(\mathcal{I}_T(t))^\vee \times H^0(\mathcal{O}_{\mathbb{P}^4}(1))$ with $a \neq 0$ and $\ell \neq 0$, then $u(a, \ell) = e_M(a) \neq 0$. Therefore the bilinear map u is non-degenerate in each variable. Hence $h^1(\mathcal{I}_T(t-1)) \geq h^1(\mathcal{I}_T(t)) + h^0(\mathcal{O}_{\mathbb{P}^r}(1)) - 1$ by the bilinear lemma. $\square$

The next Lemma 3 is perhaps the technical heart of this work. It relies on a particular case of a very strong result on 0-dimensional schemes in the plane, namely, [9, Corollaire 2] (see also [9, Remarque (i)]). We recall the statement in [9] for reader's convenience. Let $E \subset \mathbb{P}^2$ be a zero-dimensional scheme of degree d. Let $\tau := \max\{n : h^1(\mathcal{I}_E(n) > 0\}$. Let s be an integer such that $s \leq d/s$ and $\tau \geq s - 3 + d/s$. Then either E is the complete intersection of a curve of degree s and a curve of degree d/s and $\tau = s - 3 + d/s$, or there exists s' with $0 < s' < s$ and a subscheme $E' \subset E$ contained in a curve of degree s' such that $s'(\tau + (5 - s')/2) \geq \deg(E') \geq s'(\tau - s' + 3)$. In particular, if $\tau > d/3$, then either we have $\tau + 2$ points on a line (counted with multiplicity), or we have $2\tau + 2$ or $2\tau + 3$ points on a conic (counted with multiplicity).

For the proof of Lemma 3 we also need to introduce the notion of residual scheme. Let M be a projective scheme, A a closed subscheme and $D \subset M$ an effective Cartier divisor of M. The residual scheme $\mathrm{Res}_D(A)$ of A with respect to D is the closed subscheme of M with $\mathcal{I}_A : \mathcal{I}_D$ as its ideal sheaf. We always have $\mathrm{Res}_D(A) \subseteq A$. If A is a reduced scheme, then $\mathrm{Res}_D(A)$ is the union of the irreducible components of A not contained in D. If A is a zero-dimensional scheme, then $\deg(A) = \deg(A \cap D) + \deg(\mathrm{Res}_D(A))$. For any line bundle $\mathcal{L}$ on M we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(A)} \otimes \mathcal{L}(-D) \to \mathcal{I}_A \otimes \mathcal{L} \to \mathcal{I}_{A \cap D, D} \otimes (\mathcal{L}_{|D}) \to 0.$$

Lemma 3. *Fix an integer $t \geq 2$. Set $M := \mathbb{P}^3$ and let $Z \subset M$ a zero-dimensional scheme spanning M and with $\deg(Z) \leq 3t$. We have $h^1(M, \mathcal{I}_{Z,M}(t)) \neq 0$ if and only if either there is a line $R \subset M$ with $\deg(R \cap Z) \geq t + 2$ or there is a conic $D \subset M$ such that $\deg(D \cap Z) \geq 2t + 2$ or there is a line $L \subset M$ such that $\deg(Z \cap L) = t + 1$ and the union of the connected components of Z whose reduction is contained in L has degree $\geq 2t + 2$.*

Proof. Set $Z_0 := Z$. Let $N_1 \subset M$ be a plane such that $e_1 := \deg(Z \cap N_1)$ is maximal. Set $Z_1 := \mathrm{Res}_{N_1}(Z_0)$. For each integer $i \geq 2$ define recursively the plane N_i, the integer e_i and the zero-dimensional scheme Z_i in the following way. Let $N_i \subset M$ be any hyperplane such that $e_i := \deg(Z_{i-1} \cap N_i)$ is maximal. Set $Z_i := \mathrm{Res}_{N_i}(Z_{i-1})$. For each $i \ge 1$ we have an exact sequence

$$(4) \qquad 0 \to \mathcal{I}_{Z_i}(t-i) \to \mathcal{I}_{Z_{i-1}}(t+1-i) \to \mathcal{I}_{Z_{i-1}\cap N_i, N_i}(t+1-i) \to 0$$

We have $e_i \ge e_{i-1}$ for all i. Since any degree 3 subscheme of M is contained in a plane, if $e_i \le 2$, then $Z_{i-1} \subset N_i$ and $Z_i = \emptyset$. Since $\deg(Z) \le 3t$, there is an integer i such that $1 \le i \le t$ and $Z_i = \emptyset$. From (4) we get an integer $i \in \{1, \dots, t\}$ such that $h^1(N_i, \mathcal{I}_{Z_{i-1}\cap N_i, N_i}(t+1-i)) > 0$. Indeed, the fact that Z_i is empty for some index $1 \le i \le t$ forces the cohomology of the ideal sheaf of Z_i to be that of the ambient projective plane N_i. We call c the minimal integer i. Since $h^1(N_c, \mathcal{I}_{Z_{i-1}\cap N_c, N_c}(t+1-c)) > 0$, either $\deg(Z_{i-1} \cap N_c, N_c)) \ge 2(t+1-c)+2$ or there is a line L with $\deg(L_c \cap Z_{i-1} \cap N_c) \ge t+3-c$ ([3, Lemma 34]). In particular, since $c \le t$, we have $e_c \ge t+3-c$. Since the sequence $\{e_i\}_{i\ge 1}$ is non-increasing, we have $ce_c \ge c(t+3-c)$. Since $\sum_{i\ge 1} e_i = \deg(Z) \le 3t$, we get $c(t+3-c) \le 3t$. Set $\psi(x) = x(t+3-x)$. The function ψ is strictly increasing if $1 \le x \le (t+3)/2$ and strictly decreasing if $x > (t+3)/2$. Since $\psi(t) = 3t$ and $\psi(3) = 3t$, we get that either $1 \le c \le 3$ or $c = t$.

(a) Assume $c = 1$. Since Z spans M, we have $e_1 \le \deg(Z) - 1$. Since $e_1 \le \deg(Z) - 1$, we have $e_1 < 3t$. By [9, Corollaire 2] (see also [9, Remarque (i)]) either there is a line $R \subset N_1$ with $\deg(R \cap Z) \ge t+2$ or there is a conic $D \subset N_1$ such that $\deg(D \cap Z) \ge 2t+2$.

(b) Assume $c = 2$. Since $e_1 \ge e_2$, we have $e_2 \le \deg(Z)/2 \le 3t/2$. Since $c = 2$ and $h^1(N_2, \mathcal{I}_{Z_1\cap N_2, N_2}(t-1)) > 0$, by [3, Lemma 34] either $e_2 \ge 2t$, which is a contradiction, or there is a line $R \subset N_2$ such that $\deg(R \cap Z_1) \ge t+1$. If $\deg(R \cap Z) \ge t+2$, then we are done. Hence we may assume $\deg(Z \cap R) = t+1$. Set $W_0 := Z$. Let $M_1 \subset M$ be a plane containing R and for which $f_1 := \deg(M_1 \cap Z)$ is maximal. Since Z spans M we have $f_1 \ge t+2$. Set $W_1 := \mathrm{Res}_{M_1}(Z)$. For each integer $i \ge 2$ define recursively the plane M_i, the integer f_i and the zero-dimensional scheme W_i in the following way. Let $M_i \subset M$ be any hyperplane such that $f_i := \deg(W_{i-1} \cap M_i)$ is maximal. Set $W_i := \mathrm{Res}_{N_i}(W_{i-1})$. We have $f_i \ge f_{i+1}$ for all $i \ge 2$, but we do not claim that $f_1 \ge f_2$ (indeed, M_1 is required to contain R, while the M_i with $i \ge 2$ are not). Since any degree 3 subscheme of M is contained in a plane, if $f_i \le 2$, then $W_{i-1} \subset M_1$ and $W_i = \emptyset$. Since $\sum_{i \ge i} f_i = \deg(Z)$ and $f_1 \ge t+2$, we have $f_i = 0$ for some $i < t$. Using the residual exact sequences of the planes M_i we get the existence of a minimal integer $s \in \{1, \dots, t-1\}$ such that $h^1(M_s, \mathcal{I}_{W_{s-1}\cap M_s, M_s}(t+1-s)) > 0$. We get $f_s \ge t+3-s$. Since $f_1 \ge t+2$, we get $1 \le s \le 2$. If $s = 1$, then we use step (a) with M_1 instead of N_1. Now assume $s = 2$. Since $f_2 \le \deg(Z) - f_1 \le$

$2t-2$ and $h^1(M_2, \mathcal{I}_{Z_1 \cap M_2, M_2}(t-1)) = 0$, there is a line $L \subset M_2$ such that $\deg(L \cap Z_1) \ge t+1$. If $\deg(L \cap Z) \ge t+2$, then the lemma is true. Hence we may assume that $\deg(Z \cap L) = t+1$.

First assume $R \cap L = \emptyset$. Let $Q \subset M$ be a general quadric surface containing $L \cup R$. Call $|\mathcal{O}_Q(1,0)|$ the ruling of Q containing R and L. The residual scheme $\mathrm{Res}_Q(Z)$ of Z has degree $\deg(Z) - \deg(Z \cap Q) \le 3t - (2t+2) = t-2$ and in particular $h^1(M, \mathcal{I}_{\mathrm{Res}_Q(Z),M}(t-2)) = 0$. The residual exact sequence of Q gives $h^1(Q, \mathcal{I}_{Z \cap Q, Q}(t)) \ge h^1(M, \mathcal{I}_{Z,M}(t)) > 0$.

Claim: We have $h^1(M, \mathcal{I}_{R \cup L}(t)) = 0$ for every $t \ge 1$.

Proof of the Claim: Take $p \in L$. Since $R \cap L = \emptyset$, $\{p\} \cup R$ spans a plane, H. We have $(L \cup R) \cap H = R \cup \{p\}$ and hence $h^1(H, \mathcal{I}_{(L \cup R) \cap H, H}(t)) = 0$ for all $t \ge 1$. The residual $\mathrm{Res}_H(L \cup R)$ of $L \cup R$ with respect to H is the line L, because $L \cup R$ is reduced and $L \nsubseteq H$. Therefore the residual sequence of H in $\mathbb{P}^3$ gives the following exact sequence:

$$0 \to \mathcal{I}_L(t-1) \to \mathcal{I}_{L \cup R}(t) \to \mathcal{I}_{(L \cup R) \cap H, H}(t) \to 0.$$

Since $h^1(\mathcal{I}_L(t-1)) = 0$ for all $t > 0$, we get the Claim.

Since $\deg(Z \cap L) = \deg(Z \cap R) = t+1$ and $R \cap L = \emptyset$, we have $h^1(R \cup L, \mathcal{I}_{(R \cup L) \cap Z}(t)) = h^1(R, \mathcal{I}_{R \cap Z}(t)) + h^1(L, \mathcal{I}_{L \cap Z}(t)) = 0$.

The Claim gives $h^1(M, \mathcal{I}_{Z \cap (R \cup L}(t)) = 0$. Hence $h^1(Q, \mathcal{I}_{Z \cap (R \cup L), Q}(t)) = 0$. The residual sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{R \cup L}(Z \cap Q)}(t-2, t)) \to \mathcal{I}_{Z \cap Q, Q}(t,t) \to \mathcal{I}_{(R \cup L) \cap Z, R \cup L}(t,t) \to 0$$

gives $h^1(Q, \mathcal{I}_{\mathrm{Res}_{R \cup L}(Z \cap Q)}(t-2,t)) > 0$. Since $\deg(\mathrm{Res}_{R \cup L}(Z \cap Q)) = \deg(Z \cap Q) - 2t - 2 \le t-2$, we get a contradiction.

Now assume $R \cap L \ne \emptyset$. If $R \ne L$, then we may take the reducible conic $R \cup L$, because $R \subset M_1$ and $\deg(L \cap \mathrm{Res}_{M_1}(Z)) = t+1$.

Now assume $R = L$. This is the last case of the statement of the lemma.

(c) Assume $c = 3$. Since $\psi(3) = 3t$, we get $e_1 = e_2 = e_3 = t$. Since $e_3 = t$ and $h^1(N_3, \mathcal{I}_{Z_2 \cap N_3, N_3}(t-2)) > 0$, there is a line $R \subset N_3$ such that $\deg(Z_2 \cap R) = t$. Since Z spans M, there is a plane N' such that $R \subset N' \subset M$ and $N' \cap Z \supsetneq Z \cap R$. Hence $e_1 \ge \deg(N' \cap Z) > t$, a contradiction.

(d) Assume $c = t$. We get $\deg(Z) = 3t$ and $e_i = 3$ for all i. In particular $e_1 = 3$, i.e. Z is in linearly general position. Since $\deg(Z) \le 3t+1$, the contradiction comes from [8, Theorem 3.2]. □

Lemma 4. *Fix an integer $t > 0$. Set $M := \mathbb{P}^3$ and let $Z \subset M$ a zero-dimensional and curvilinear scheme spanning M and with* $\deg(Z) \le 3t$.

We have $h^1(M, \mathcal{I}_{Z,M}(t)) \neq 0$ *if and only if either there is a line* $R \subset M$ *with* $\deg(R \cap Z) \geq t+2$ *or there is a conic* $D \subset M$ *such that* $\deg(D \cap Z) \geq 2t+2$.

Proof. The " if " part is trivial. To prove the other implication it is sufficient to exclude the last case of the statement of Lemma 3. By [9, Corollaire 2] (see also [9, Remarque (i)]) we may assume that $h^1(N, \mathcal{I}_{Z\cap N,N}(t)) = 0$ for every plane N.

Assume that we are in the last case of Lemma 3 and call L the associated line. We may take Z minimal with the property that $h^1(M, \mathcal{I}_{Z,M}(t)) > 0$. Let Q be a quadric surface containing L in its singular locus. Since $\deg(\mathrm{Res}_Q(Z)) \leq 3t - 2t - 2 \leq t-2$, we have $h^1(M, \mathcal{I}_{\mathrm{Res}_Q(Z)}(t-2)) = 0$. Therefore the residual exact sequence of Q gives $h^1(Q, \mathcal{I}_{Z\cap Q,Q}(t)) > 0$ and $h^1(M, \mathcal{I}_{Z\cap Q,M}(t)) > 0$. The minimality of Z gives $Z \subset Q$. Taking $Q = N_1 + N_2$ in step (b) of the proof of Lemma 3 we also get that only the connected components of Z whose reduction are contained in L arise (for a minimal Z), hence we reduce to the case $\deg(Z) = 2t+2$.

Let $W \subset Z$ be any degree $2t+1$ subscheme. Since $\deg(W \cap D) \leq \deg(Z \cap D) \leq t+1$ for each line D, Lemma 3 gives $h^1(M, \mathcal{I}_{W,M}(t)) = 0$. Hence $h^1(M, \mathcal{I}_{Z,M}(t)) = 1$. Since $h^1(N, \mathcal{I}_{Z\cap N,N}(t)) = 0$ for every plane N, as in [7] we get $h^1(M, \mathcal{I}_{Z,M}(t-1)) \geq 3 + h^1(M, \mathcal{I}_{Z,M}(t)) = 4$. Let N be any plane containing L. We have $h^1(N, \mathcal{I}_{Z\cap N}(t-1)) = 1$, because $\deg(Z \cap L) = t+1$ and $\deg(Z \cap N) \leq 2(t-1)+1$ (use the residual exact sequence of L in N). Since $\deg(\mathrm{Res}_N(Z)) \leq t+1$, we have $h^1(M, \mathcal{I}_{\mathrm{Res}_N(Z)}(t-2)) \leq 2$. Hence the residual exact sequence of N gives $h^1(M, \mathcal{I}_{\mathrm{Res}_N(Z)}(t-1)) \leq 2+1$, a contradiction. $\square$

Lemma 5. *Let* $H \subset \mathbb{P}^4$ *be a hyperplane. Let* $S \subset H$ *be a set of* 12 *points in uniform position and spanning* H.

(a) $h^1(H, \mathcal{I}_{S,H}(3)) \geq 2$ *if and only if* S *is contained in a rational normal curve of* H *and in this case we have* $h^1(H, \mathcal{I}_{S,H}(3)) = 2$;

(b) $h^1(H, \mathcal{I}_{S,H}(3)) = 1$ *if and only if* S *is contained in an integral curve* $T \subset H$, *which is the complete intersection of two quadric surfaces.*

Proof. If S is contained in a rational normal curve (resp. an integral complete intersection of two quadric surfaces), then $h^1(H, \mathcal{I}_{S,H}(3)) = 2$ (resp. $h^1(H, \mathcal{I}_{S,H}(3)) = 1$). Since S is in linearly general position, we have $h^1(H, \mathcal{I}_{S'}(3)) = 0$ for each $S' \subset S$ with $\sharp(S') = 10$. Hence $h^1(H, \mathcal{I}_{S,H}(3)) \leq 2$. If $h^0(H, \mathcal{I}_{S,H}(2)) \geq 2$, since S is in uniform position we get that S is contained in a integral curve with either degree 3 or the intersection of 2 quadric surfaces. Hence we may assume $h^0(H, \mathcal{I}_{S,H}(2)) \le 1$. There is $A \subset S$ with $\sharp(A) = 8$ and $h^0(H, \mathcal{I}_{A,H}(2)) = 2$, i.e. $h^1(H, \mathcal{I}_{A,H}(2)) = 0$. Take an ordering o_1, o_2, o_3, o_4 of $S \setminus A$. Set $A_0 := A$. For $i = 1, 2, 3, 4$ set $A_i := A \cup \{o_1, \dots, i\}$. It is sufficient to prove that $h^0(H, \mathcal{I}_{A_i,H}(3)) < h^0(H, \mathcal{I}_{A_{i-1},H}(3))$ for $i = 1, 2, 3, 4$. Let Q be a general quadric surface containing A. Since S is in uniform position, we have $Q \cap S = A$. Let N_i be any plane not containing o_i but containing o_j for all $j < i$. The cubic surface $Q \cup A_i$ gives $h^0(H, \mathcal{I}_{A_i,H}(3)) < h^0(H, \mathcal{I}_{A_{i-1},H}(3))$. □

Lemma 6. *Let $C \subset \mathbb{P}^r$, $r \ge 2$, be a smooth rational curve. Let $M(d, r)$ denote the set of all smooth rational curves of degree d in $\mathbb{P}^r$. $M(d, r)$ is smooth and irreducible of dimension $(r + 1)d + r - 3$. Set $d := \deg(C)$ and take a zero-dimensional scheme $Z \subset \mathbb{P}^n$ such that $a := \deg(Z) \le d + 1$. Then $h^1(N_C(-Z)) = 0$ and the set of all $X \in M(d, r)$ containing Z has dimension $(r + 1)d + r - 3 - (r - 1)a$.*

Proof. Fix any $Y \in M(d, r)$. Since $T\mathbb{P}^r$ is a quotient of $\mathcal{O}_{\mathbb{P}^r}(1)$ by the Euler sequence and X is smooth N_X is a quotient of $\mathcal{O}_X(1)^{(r+1)}$. Since X is a smooth rational curve, we get $h^1(N_X(-W)) = 0$ for every zero-dimensional scheme $W \subset X$ with $\deg(Z) \le d + 1$. The Hilbert scheme of all curves containing W has $H^0(N_X(-W))$ as its tangent space and $H^1(N_X(-W))$ as an obstruction space ([18, Theorem 1.5]). Taking $W = \emptyset$ we get the smoothness and dimension of $M(d, r)$. The irreducibility of $M(d, r)$ is well-known. Taking $W = Z$ we get the other statements of the lemma. □

Let $\mathcal{W}$ denote the set of all quintic hypersurfaces of Cotterill, i.e. satisfying all properties proved in [6]. In particular each $W \in \mathcal{W}$ is a smooth quintic hypersurface containing finitely many rational curves of degree ≤ 11.

For any integer $b \ge 5$ let Δ_b denote the set of all non-degenerate $C \in M_{12}$ such that there is a line $L \subset \mathbb{P}^4$ with $\deg(L \cap C) = b$. Set $\Delta'_7 := \cup_{b \ge 7} \Delta_b$.

Remark 1. For any line $L \subset \mathbb{P}^4$ let $A(L, b)$ denote the set of all non-degenerate $C \in M_{12}$ such that $\deg(L \cap C) = b$. Since $\Delta_b = \emptyset$ if $b > 12$, we have $\dim(A(L, b)) = 61 - 2b$ by Lemma 6. Now, if W is any quintic threefold and $C \subset W$, then by Bezout also $L \subset W$ as soon as $b \ge 6$. Since on each $W \in \mathcal{W}$ there are finitely many lines, if W contains only finitely many $C \in A(L, b)$ for any fixed line $L \subset \mathbb{P}^4$, then W contains only finitely many $C \in \Delta_b$ as well. Hence to prove that a general $W \in \mathcal{W}$ contains only finitely many elements of Δ_b, by (1) and (2) it is sufficient to test the element $C \in \Delta_b$ with $h^1(\mathcal{I}_C(5)) \ge 2b + 1$.

Lemma 7. *A general $W \in \mathcal{W}$ contains only finitely many $C \in \Delta_7'$.*

Proof. By Remark 1 it is sufficient to test the non-degenerate curves $C \in M_{12}$ such that $h^1(\mathcal{I}_C(5)) \geq 15$. Take a general hyperplane $H \in \mathbb{P}^4$. Since $C \cap H$ is in uniform position, Lemma 4 gives $h^1(H, \mathcal{I}_{C\cap H,H}(t)) = 0$ for $t = 4, 5$. The exact sequence

$$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C\cap H,H}(t) \to 0 \tag{5}$$

gives $h^1(\mathcal{I}_C(3)) \geq h^1(\mathcal{I}_C(4)) \geq h^1(\mathcal{I}_C(5))$. By Lemma 5 we have $h^1(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3)) - 2 \geq 13$. Hence $h^0(\mathcal{I}_C(2)) \geq 3$, contradicting Lemma 1. □

Lemma 8. *A general $W \in \mathcal{W}$ contains only finitely many $C \in \Delta_6$.*

Proof. By Remark 1 it is sufficient to test the non-degenerate curves $C \in M_{12}$ such that $h^1(\mathcal{I}_C(5)) \geq 13$. By Lemma 7 we may assume that $C \notin \Delta_7'$. By Lemmas 2 and 3 we have $h^1(\mathcal{I}_C(4)) \geq 4 + h^1(\mathcal{I}_C(5)) \geq 17$. Take a general hyperplane $H \in \mathbb{P}^4$. By Lemma 4 we have $h^1(H, \mathcal{I}_{C\cap H,H}(4)) = 0$. The exact sequence (5) gives $h^1(\mathcal{I}_C(3)) \geq h^1(\mathcal{I}_C(4))$. By Lemma 5 we have $h^1(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3)) - 2 \geq 15$. Hence $h^0(\mathcal{I}_C(2)) \geq 5$, contradicting Lemma 1. □

Lemma 9. *A general $W \in \mathcal{W}$ contains only finitely many non-degenerate $C \in M_{12}$ such that there is a conic $D \subset \mathbb{P}^4$ with* $\deg(D \cap C) \geq 10$ *and if the conic is singular $C \cap D$ contains a curvilinear scheme of at least degree* 10.

Proof. A conic is either smooth or reducible or a double line. Lemmas 7 and 8 handle the case in which D is not a smooth conic and $\deg(D \cap C) \geq 11$. Assume the existence of a conic D such that $b := \deg(D \cap C) \geq 10$. Fix any $p \in C \setminus C \cap N$, where N is the plane spanned by D, and let M be the hyperplane spanned by $N \cup \{p\}$. Since $\deg(C \cap M) \geq b + 1$, we have $b \leq 11$. $\mathbb{P}^4$ contains ∞^6 planes and each plane contains ∞^5 smooth conics and ∞^4 singular conic. Fix $b \in \{10, 11\}$ and a conic D. Let $B(D, b)$ be the set of all non-degenerate $C \in M_{12}$ such that $\deg(D \cap C) = b$; if $b = 10$ and D is singular assume that $D \cap C$ is curvilinear. Since each conic contains ∞^b curvilinear subschemes of degree b, Lemma 6 gives $\dim(B(D, b)) \leq 61 - 2b$. Varying D we get that the set of all C has codimension at least 9 in M_{12}. Hence it is sufficient to test the curves C with $h^1(\mathcal{I}_C(5)) \geq 10$. Since $C \notin \Delta_7'$, we have $h^1(\mathcal{I}_C(4)) \geq 14$. Moreover, if $h^1(\mathcal{I}_C(4)) = 14$, then $\deg(D \cap C) \geq 10$ for finitely many conics $D_1, \dots, D_s$. Let N_i be the plane spanned by D_i. Fix a line $L \subset \mathbb{P}^4$ such that $L \cap N_i = \emptyset$ for all i. Set $V := H^0(\mathcal{I}_L(1))$ and take any $M \in |\mathcal{I}_L(1)|$. We have $N_i \nsubseteq M$. Since $M \cap C$ contains no line

R with $\deg(R \cap C) \geq 6$ and no conic D with $\deg(D \cap C) \geq 10$, we have $h^1(M, \mathcal{I}_{C \cap M, M}(4)) = 0$ (Lemma 4). Hence the bilinear map $H^1(\mathcal{I}_C(4))^\vee \times V \to H^1(\mathcal{I}_C(3))^\vee$ is non-degenerate in the second variable. By the bilinear lemma we have $h^1(\mathcal{I}_C(3)) \geq h^1(\mathcal{I}_C(4)) + \dim(V) - 1 = 16$. Hence in all cases we have $h^1(\mathcal{I}_C(3)) \geq 15$. By Lemma 5 we have $h^1(\mathcal{I}_C(2)) \geq 13$, contradicting Lemma 1. □

Let Δ_1 (resp. Δ_2, Δ_3) be the set of all non-degenerate $C \in M_{12}$ such that for a general hyperplane $H \subset \mathbb{P}^4$ the set $C \cap H$ is contained in a rational normal curve of H (resp., the smooth complete intersection of 2 quadric surfaces of H, resp., a singular integral curve which is the complete intersection of 2 quadric surfaces of H).

We have the following estimates:

Lemma 10. *Every irreducible component of Δ_1 has dimension ≤ 49.*

Proof. Fix a hyperplane H, a rational normal curve $D \subset H$ and $S \subset D$ such that $\sharp(S) = 12$. By Lemma 6 the set of all $C \in M_{12}$ containing S has dimension $\leq 61 - 36$. Since the set of all $S \subset D$ with $\sharp(S) = 12$ has dimension 12 and H contains ∞^{12} rational normal curves, we get the lemma. □

Lemma 11. *Every irreducible component of Δ_2 has dimension ≤ 53.*

Proof. Fix a hyperplane H. The set of all degree 4 smooth elliptic curves of H has dimension 16 and we may conclude as in the proof of Lemma 10. □

Lemma 12. *Every irreducible component of Δ_3 has dimension ≤ 52.*

Proof. Fix a hyperplane H. The set of all singular, integral and non-degenere curves $D \subset H$ with $\deg(D \cap H) = 4$, i.e. the set of all singular integral curves which are the complete intersection of 2 quadric surfaces of H, has dimension 15. Now we argue as in the proof of Lemma 10. □

Lemma 13. *A general $C \in M_{12}$ contains only finitely many elements of $\Delta_1 \cup \Delta_2 \cup \Delta_3$.*

Proof. By Lemmas 10, 11 and 12, we may assume that $h^1(\mathcal{I}_C(5)) \geq 9$. By Lemmas 7, 8 and 9 we may assume $\deg(C \cap L) \leq 5$ for all lines and $\deg(D \cap C) \leq 9$ for all conics. By Lemmas 2 and 3 for $t = 4, 5$ we get $h^1(\mathcal{I}_C(3)) \geq 4 + h^1(\mathcal{I}_C(4)) \geq 8 + h^1(\mathcal{I}_C(5)) \geq 17$. Lemma 5 gives $h^1(\mathcal{I}_C(2)) \geq 15$, i.e. $h^0(\mathcal{I}_C(2)) \geq 5$, contradicting Lemma 1. □

By Lemmas 7, 8 and 9 to prove Theorem 1 for non-degenerate $C \in M_{12}$ it is sufficient to test the ones such that $\deg(C \cap D) \le 9$ for any conic D and $\deg(L \cap C) \le 5$ for any line L. By the cases $t = 4, 5$ of Lemmas 2 and 3 we have $h^1(\mathcal{I}_C(3)) \ge 4 + h^1(\mathcal{I}_C(4)) \ge 8 + h^1(\mathcal{I}_C(5))$.

By Lemmas 5 and 13 we may assume $h^1(H, \mathcal{I}_{C \cap H,H}(3)) = 0$. Now the case $t = 3$ of the exact sequence (5) gives

$$h^1(\mathcal{I}_C(2)) \ge h^1(\mathcal{I}_C(3)) \ge 4 + h^1(\mathcal{I}_C(4)) \ge 8 + h^1(\mathcal{I}_C(5)). \tag{6}$$

Since the stratum in M_{12} corresponding to curves with $h^1(\mathcal{I}_C(5)) > 0$ has codimension 2 (as in [7, pp. 901–902]), by (1) and (2) we may assume $h^1(\mathcal{I}_C(5)) \ge 3$, hence $h^1(\mathcal{I}_C(2)) \ge 11$. Since $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 15$ and $h^0(\mathcal{O}_C(2)) = 25$, we get $h^0(\mathcal{I}_C(2)) \ge 1$.

Now, if $h^1(\mathcal{I}_C(5)) \le 5$ (hence $h^0(\mathcal{I}_C(5)) \le 70$) we conclude by the following Lemma 14.

Lemma 14. *Let Γ be any irreducible family of non-degenerate curves of M_d, $d > 1$, contained in some quadric hypersurface. Then* $\dim \Gamma \le 14 + 3d$.

Proof. Since $\dim |\mathcal{O}_{\mathbb{P}^4}(2)| = 14$ and singular quadrics occur in codimension 1, it is sufficient to prove that for every smooth (resp., integral but singular) quadric Q the set Γ' of all $C \in M_d$ contained in Q has dimension $\le 3d$ (resp., $\le 3d + 1$).

First assume that either Q is smooth or C does not intersect the singular locus V of Q. In this case the normal sheaf $N_{C,Q}$ is a rank 2 spanned vector bundle on C, hence $h^1(N_{C,Q}) = 0$. Since $\det(N_{C,Q})$ has degree $3d - 2$ and N_C has rank 2, Riemann-Roch gives $h^0(N_{C,Q}) = 3d$, proving the lemma in this case.

Now assume $C \cap V \neq \emptyset$ and set $x := \deg(C \cap V)$. Since C is smooth, $x = 1$ if $\dim(V) = 0$. Let τ_Q denote the tangent sheaf of Q. The vector space $H^0(\tau_Q)$ is the tangent space at the identity map of the automorphism group $\mathrm{Aut}(Q)$. Since $Q \setminus V$ is homogeneous, $\tau_Q|(Q \setminus V)$ is a spanned vector bundle. Since C is not a line and $\dim V \le 1$, the set $V \cap C$ is finite. Dualizing the natural map from the conormal sheaf of C in Q to Ω^1_Q we get a map $u : \tau_Q|C \to N_{C,Q}$ which is surjective outside the finite set $C \setminus C \cap V$. Since C is smooth and rational and τ_Q is spanned at each point of $Q \setminus V$, we get $h^1(N_{C,Q}) = 0$. Since we need to prove that $\dim \Gamma' \le 3d + 1$, it is sufficient to check this inequality when C is a general element of Γ'. In particular we may assume that $\deg(C' \cap V) = x$ for a general $C' \in \Gamma'$ and use induction on the integer x, the case $x = 0$ being true by the case $C \cap V = \emptyset$ proved before. Set $\Gamma'' := \{C' \in \Gamma' : \deg(V \cap C) = x\}$. It is sufficient to prove that

$\dim \Gamma'' \le 3d+1$. Let $v : \widetilde{Q} \to Q$ be the blowing up of V, $E := v^{-1}(V)$ the exceptional divisor, and $\widetilde{C} \subset \widetilde{Q}$ the strict transform of C. Since C is smooth, v maps isomorphically $\widetilde{C}$ onto C and the numerical class of $\widetilde{C}$ with respect to $\mathrm{Pic}(\widetilde{Q})$ only depends on $\dim(V)$, d and x. Let Ψ be closure in $\mathrm{Hilb}(\widetilde{Q})$ of the strict transforms of all $C' \in \Gamma''$. It is sufficient to prove that $\dim \Psi \le 3d+1$. Take a general $D \in \Psi$. Since $\mathrm{Aut}(\widetilde{Q})$ acts transitively of $\widetilde{Q} \setminus E$, the first part of the proof gives $h^1(N_{D,\widetilde{Q}}) = 0$. Hence it is sufficient to prove that $\deg(N_{D,\widetilde{Q}}) \le 3d-1$, i.e. $\deg(\tau_{\widetilde{Q}|D}) \le 3d+1$, i.e. $\deg(\omega_{\widetilde{Q}}|D) \ge -3d-1$. The group $\mathrm{Pic}(\widetilde{Q})$ is freely generated by E and the pull-back H of $\mathcal{O}_Q(1)$. We have $D \cdot H = d$ and $D \cdot E = x$. We have $\omega_{\widetilde{Q}} \cong \mathcal{O}_{\widetilde{Q}}(-3H + cE)$ with $c = -1$ if $\dim(V) = 0$ (see for instance [12], Example 8.5 (2)) and $c = 0$ if $\dim(V) = 1$ (see for instance [12], Example 8.5 (3)). Hence $\deg(\omega_{\widetilde{Q}|D}) = -3d + cx \ge -3d-1$ and the proof is complete. $\square$

If instead $h^1(\mathcal{I}_C(5)) \ge 6$, then by (6) we have $h^1(\mathcal{I}_C(2)) \ge 14$, i.e. $h^0(\mathcal{I}_C(2)) \ge 4$, contradicting Lemma 1.

3. Degenerate case

The degenerate case occurs in codimension 10 of M_{12}. Indeed, the general curve of degree $d = 12$ in $\mathbb{P}^3$ has maximal rank ([1]), in particular it does not sit on any quintic. It follows that our codimension is $\dim(M_d) - (4d - 1 + 4) = 61 - 51 = 10$. Hence we may assume $h^1(\mathcal{I}_C(5)) \ge 11$.

We consider degenerate curves $C \in M_{12}$ with $h^1(\mathcal{I}_C(5)) \ge 11$ contained in a hyperplane M and in a general quintic W with $W' := M \cap W$.

Lemma 15. *Let $W \subset \mathbb{P}^4$ be a general quintic hypersurface. Then W contains finitely many integral curves T of degree 4 which are the complete intersection of a hyperplane and 2 quadric hypersurfaces and all of them are smooth.*

Proof. Since W contains no singular rational curves ([6]), it is sufficient to consider the smooth ones, i.e. the degree 4 elliptic curves of $\mathbb{P}^4$. Let Γ' be the set of all degree 4 elliptic curves of $\mathbb{P}^4$. Fix $T \in \Gamma'$. Since $N_T \cong \mathcal{O}_T(2)^{\oplus 2} \oplus \mathcal{O}_T(1)$, we have $h^1(N_T) = 0$, hence Γ' is smooth and of dimension $\chi(N_T) = 16$. Since T is a complete intersection, we have $h^1(\mathcal{I}_T(5)) = 0$ and $h^0(\mathcal{I}_T(5)) = \binom{9}{4} - 20$. Hence a dimension count gives the lemma. $\square$

Lemma 16. *W' contains only finitely many non-degenerate curves of degree 5 and 6.*

Proof. Fix a degree $t \in \{5,6\}$ integral and non-degenerate curve $D \subset W'$ and set $q := p_a(D)$. By [10] we have $h^1(M, \mathcal{I}_{D,M}(5)) = 0$, hence $h^0(\mathbb{P}^4, \mathcal{I}_D(5)) = 126 - 5t - 1 + q$.

First assume $t = 5$. By the genus bound for space curves we have $q \leq 2$. Since $q \leq 2$, we have $h^1(\mathcal{O}_D(1)) = 0$ and in particular $h^1(\mathcal{O}_D(5)) = 0$, i.e. $h^0(\mathcal{O}_D(5)) = 5t + 1 - q$. Since $q \leq 2$, all the irreducible components of the Hilbert scheme of M containing D have dimension 20. Since $\mathbb{P}^4$ has ∞^4 hyperplanes, it is sufficient to use that $4t + 4 \leq 5t + 1 - q$.

Now assume $t = 6$. By the genus bound for space curves ([11, Theorems 3.7 and 3.13]), $q \leq 3$ and $q = 3$ if and only if D is contained in a quadric surface Q. Assume $q = 3$. In this case D is the complete intersection of Q and a cubic surface ([11, Corollary 3.14]) and so D is a locally complete intersection, $\omega_D \cong \mathcal{O}_D(1)$ and $N_{D,M} \cong \mathcal{O}_D(2) \oplus \mathcal{O}_D(3)$. Since $h^1(N_D) = 0$ the Hilbert scheme of M at D is smooth and of dimension $4t$. We conclude as in the case $t = 5$. The case $q \leq 2$ is done as in the case $t = 5$. □

A theorem of Zak (see for instance [22]) states that the Gauss map of any smooth projective variety is finite, hence W' has only finitely many singular points, all of them being hypersurface singularities. By [15, p. 733] W' has only rational double points of type A_i, $i \leq 4$, and D_4 as singularities.

We may improve the lower bound $h^1(\mathcal{I}_C(5)) \geq 11$ if we restrict the set of hyperplanes or rather if we restrict the pairs $(W, M) \in |\mathcal{O}_{\mathbb{P}^4}(5)| \times |\mathcal{O}_{\mathbb{P}^4}(1)|$.

Remark 2. If M is tangent to W, i.e. if W' is singular, then we may assume $h^1(\mathcal{I}_C(5)) \geq 12$. Since the Gauss map is birational, if W' has at least two singular points, then we may assume $h^1(\mathcal{I}_C(5)) \geq 13$.

Remark 3. For any line $L \subset \mathbb{P}^4$ we have $h^0(\mathcal{I}_L(1)) = 3$. A general W contains only finitely many lines ([6]). Hence if W' contains a line, then we may assume $h^1(\mathcal{I}_C(5)) \geq 13$. Since any two lines of W are disjoint ([6]), any two lines of W span a hyperplane. Hence if W' contains two different lines, then we may assume $h^1(\mathcal{I}_C(5)) \geq 15$. Fix a line $L \subset W$. For any $p \in L$, the hyperplane T_pW is the only hyperplane singular at p. Since $\dim(L) = 1$, we get that if W' is singular at one point of L, then we may assume $h^1(\mathcal{I}_C(5)) \geq 14$.

Remark 4. For any smooth conic $D \subset \mathbb{P}^4$ we have $h^0(\mathcal{I}_D(1)) = 2$. A general W contains only finitely many conics ([6]). Hence if W' contains a smooth conic, then we may assume $h^1(\mathcal{I}_C(5)) \geq 14$.

Remark 5. For any integer x with $3 \leq x \leq 11$, W contains only finitely many smooth rational curve of degree x, none of them contained in a plane.

Hence if W' contains a smooth rational curve of degree x, then we may assume $h^1(\mathcal{I}_C(5)) \geq 15$. The same is true if W' contains a line and a conic or 2 conics.

For any hyperplane U let $M_{12}(U)$ denote the set of all $C \in M_{12}$ contained in U. The locus $M_{12}(U)$ is smooth and irreducible and $\dim(M_{12}(U)) = 48$.

Remark 6. Fix an integer $e > 0$ and assume the existence of a line $L \subset W'$ such that $\deg(L \cap C) = e$. Let $\mathcal{J}(e)$ be the set of all quadruples (W, H, L, C) with $W \in \mathcal{W}$, H a hyperplane, $L \subset W \cap U$ a line, $C \in M_{12}(U)$ and $\deg(L \cap C) = e$. Fix any $(W, H, L, C) \in \mathcal{J}(e)$. We have $\mathcal{J}(e) = \emptyset$ if $e \geq 12$. Now assume $e \leq 11$. Fix a line $L \subset M$ and a degree e zero-dimensional scheme $Z \subset U$ with $\deg(Z) = e$ and take any $C \in M_{12}(U)$ such that $Z \subset C$. As in Lemma 6 we see that $h^1(N_{C,M}(-Z)) = 0$, hence the set of all $C \in M_{12}(U)$ with $Z \subset C$ has dimension $48 - 2e$. Varying Z in L we see that the set of all $C \in M_{12}(U)$ the set of all $C \in M_{12}(U)$ such that $\deg(C \cap L) = e$ has dimension $\leq 48 - e$. Since each $W \in \mathcal{W}$ contains only finitely many lines, to show that for all $(W, M, L, C) \in \mathcal{J}(e)$ we have $C \nsubseteq W$ it is sufficient to exclude the ones with $h^1(\mathcal{I}_C(5)) \geq 13 + e$.

Since the Gauss map of the smooth projective variety W is finite, W' has only finitely many singular points. Since W' is locally a complete intersection, W' is normal. By [6] W has only finitely many lines and only finitely many conics and no singular rational curve of degree ≤ 11. By Lemma 15 W has only finitely many smooth elliptic curves of degree 4.

Remark 7. Let $W \subset \mathbb{P}^4$ be a general quintic. Let $\mathcal{D}_i$, $i \geq 1$, be the set of all irreducible plane curves of degree i contained in W. Since W contains no plane, we have $\mathcal{D}_i = \emptyset$ for all $i \geq 6$, and the set $\mathcal{D}_5$ is formed by the irreducible degree 5 curves of the form $W \cap N$ with $N \subset \mathbb{P}^4$ a plane. Hence $\mathcal{D}_5$ is irreducible and of dimension 8. By [6], $\mathcal{D}_1 \cup \mathcal{D}_2$ is finite and any two elements of it are disjoint. Fix $D \in \mathcal{D}_3$ and let $N \subset \mathbb{P}^4$ be the plane spanned by D. The plane curve $W \cap N$ is the union of some $D \in \mathcal{D}_3$ together with a smooth conic, a reducible conic, or a double line. Since $\mathcal{D}_2$ is finite, the first case may occur only for finitely many planes and these are exactly the planes N such that $W \cap N = T_2 \cup T_3$ with $T_2 \in \mathcal{D}_2$ and $T_3 \in \mathcal{D}_3$. The second case does not occur, because the lines of W are disjoint. Now assume that $W \cap N = D \cup 2L$ with L a line. By Zak's tangency theorem the restriction to L of the Gauss map of W is finite. Therefore the third case occurs only for at most one plane $N \supset L$ Now take $T \in \mathcal{D}_4$ and let N be the plane spanned by T. We have $N \cap W = T \cup R$ with $R \in \mathcal{D}_1$, hence all elements of $\mathcal{D}_4$ are obtained in the following way. Since the set of all planes of $\mathbb{P}^4$ containing a line is a 2-dimensional projective space, each irreducible component of $\mathcal{D}_4$ has dimension 2. Fix any $L \in \mathcal{D}_1$ and take the intersection with W of the element of the net of all planes of $\mathbb{P}^4$ containing L. For a fixed hyperplane $M \subset L$ the set of all planes containing L and contained in M has dimension 1.

Let α be the minimal degree of a surface of M containing C. Since C is irreducible, every degree α surface containing C is irreducible.

Lemma 17. *We have $\alpha > 3$.*

Proof. Since C spans M, we have $\alpha > 1$. Assume $\alpha = 2$ and take $Q \in |\mathcal{I}_{C,M}(2)|$. Since W' is irreducible, $W' \cap Q$ is a degree 10 curve containing C, a contradiction. Now assume $\alpha = 3$. Since $\deg(C) > 9$, C is contained in a unique cubic surface S. Let $J \subset S \cap W'$ be the locally Cohen-Macaulay curve linked to C by the complete intersection $S \cap W'$. We have $\deg(J) = 3$ and $p_a(J) = -18$ ([19, Proposition 3.1]). Since $\deg(J) < -p_a(J)$, J has a multiple component. Since $\deg(J) = 3$, the multiple component is a line, L. Since $|\mathcal{I}_{C,M}(5)|$ contains all quintic surfaces $S \cup Q$ with $Q \in |\mathcal{O}_M(2)|$ and W' is irreducible, we have $h^0(M, \mathcal{I}_{C,M}(5)) \geq 11$, i.e. $h^1(M, \mathcal{I}_{C,M}(5)) \geq 16$. Assume for the moment the non-existence of a line $R \subset M$ with $\deg(R \cap C) \geq 7$. By Lemmas 2 and 3 we get $h^1(M, \mathcal{I}_{C,M}(4)) \geq 19$. Fix a general plane $N \subset M$. We have an exact sequence

$$0 \to \mathcal{I}_{C,M}(t-1) \to \mathcal{I}_{C,M}(t) \to \mathcal{I}_{C \cap N,N}(t) \to 0 \tag{7}$$

Since N is general, the plane cubic $C \cap N$ is irreducible and $C \cap N$ is formed by 12 points of the smooth locus of $C \cap N$. Hence $h^1(N, \mathcal{I}_{C \cap N,N}(4)) \leq 1$ with equality if and only if $C \cap N$ is the complete intersection of $S \cap N$ with a plane quartic. Since $h^0(M, \mathcal{I}_{C,M}(2)) = 0$ and (by the genus formula) C is not a complete intersection of two surfaces, [21, Theorem 6] gives $h^1(N, \mathcal{I}_{C \cap N,N}(4)) = 0$. The case $t = 4$ of (7) gives $h^1(M, \mathcal{I}_{C,M}(3)) \geq 19$, i.e. $h^0(M, \mathcal{I}_{C,M}(3)) \geq 2$, a contradiction. Now assume the existence of a line $R \subset M$ such that $e := \deg(R \cap C) \geq 7$. There are at most finitely many such R, because they cannot be all the lines of a ruling of S. Take a line $L \subset M$ disjoint from all R. Set $V := H^0(M, \mathcal{I}_{L,M}(1))$. Take any plane $U \subset M$ containing L. Since $\deg(K \cap C) \leq 6$ for each line $K \subset U$, we have $h^1(U, \mathcal{I}_{C \cap U,U}(5)) = 0$. Hence the bilinear map $H^1(M, \mathcal{I}_{C,M}(5))^\vee \times V \to H^1(\mathcal{I}_{C,M}(4))^\vee$ is non-degenerate. Since $\dim(V) = 2$, the bilinear lemma gives $h^1(M, \mathcal{I}_{C,M}(4)) \geq h^1(M, \mathcal{I}_{C,M}(5)) + 2 - 1$. Since $e > 5$, Bezout gives

$R \subset W'$. By Remark 6 we may assume $h^1(M, \mathcal{I}_{C,M}(5)) \geq 20$. Since we just proved that $h^1(N, \mathcal{I}_{C\cap N,N}(4)) = 0$, we get $h^1(M, \mathcal{I}_{C,M}(3)) \geq 21$, hence the contradiction $h^0(M, \mathcal{I}_{C,M}(3)) \geq 3$. □

Lemma 18. *W' contains no $C \in M_{12}(M)$ such that $h^0(M, \mathcal{I}_{C,M}(4)) \geq 3$ and no $C \in M_{12}(M)$ with a line $L \subset M$ with* $\deg(L \cap C) \geq 7$.

Proof. The statement is made of two parts.

(a) Take general $S_1, S_2 \in |\mathcal{I}_{C,M}(4)|$ and take a general $(S_1, S_2) \in |\mathcal{I}_{C,M}(4)|^2$. Since $\alpha > 3$, S_i is irreducible. The complete intersection $S_1 \cap S_2$ links C to a degree 4 curve J with $p_a(J) = -16$ ([19, Proposition 3.1]), hence J has at least one multiple component, say B with multiplicity $c \geq 2$. Since $h^0(M, \mathcal{I}_{C,M}(4)) \geq 3$, J has also a movable component A. Hence B is a line and either A is a line or it is a smooth conic.

First assume that A is a smooth conic, $c = 2$, and J has no other component. We have $p_a(C \cup B) = \deg(C \cap B) - 1 \leq 10$ and $p_a(A \cup B) = \deg(A \cap B) - 1 \geq -1$. Since $A \cup B$ is linked to $C \cup A$ by the complete intersection $S_1 \cap S_2$, we have $p_a(A \cup B) = p_a(C \cup B) - 20 \leq -10$, a contradiction.

Now assume $\deg(A) = 1$. Moving S_2 we get that S_1 is ruled by lines. Since $\deg(S_1) > 2$, S_1 has a unique ruling. This case cannot occur if $h^0(M, \mathcal{I}_{C,M}(4)) \geq 4$, because the plane is the only surface with ∞^2 lines.

First assume that $c = 2$. In this case J contains a line $R \notin \{B, A\}$. We have $p_a(C \cup B) \leq 10$, $p_a(B \cup A \cup R) \geq -2$, while [19, Proposition 3.1] gives $p_a(B \cup A \cup R) = p_a(C \cup B) - 20$, a contradiction. Now assume $c = 3$. C, A and B are the unique components of $S_1 \cap S_2$. S_i and S_2 do not contain B in their singular locus, because $S_1 \cap S_2$ would contain B with multiplicity 2. Since the line B is not a line of the ruling of S_1, S_1 is not a cone, it is rational and it is a linear projection from a minimal degree surface $S \subset \mathbb{P}^5$ (neither the Veronese surface not a cone). S is a Hirzebruch surface, either $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ embedded by the complete linear system $|\mathcal{O}_{F_0}(h + 2f)|$ or F_2 embedded by the complete linear system $|h + 3f|$. S_1 is not a linear projection of F_0, because it has a line, B, not in the ruling and not in the singular locus (i.e. the image of a conic of F_0). Hence S_1 is a linear projection of F_0. Any smooth rational curve $C_1 \subset F_0$ with C_1 not a line is an element of $|h + xf|$ for some $x \geq 3$. We have $\deg(C_1) = (h + xf) \cdot (h + 3f) = x$, hence if C_1 has C as its projection, then $x = 12$. B is the image of h. We have $\deg(h \cap C_1) = 10$ and so $\deg(C \cap B) \geq 7$. Hence to prove the lemma it is sufficient to prove the second assertion.

(b) Take $C \in M_{12}(M)$ with a line $L \subset M$ with $\deg(L \cap C) \geq 7$. By part (a) to get a contradiction it is sufficient to prove that $h^0(M, \mathcal{I}_{C,M}(4)) \geq 4$. Bezout gives $R \subset W'$. By Remark 6 we may assume $h^1(M, \mathcal{I}_{C,M}(5)) \geq 19$. As in the proof of Lemma 17 we get $h^1(M, \mathcal{I}_{C,M}(4)) \geq 20$, i.e. $h^0(M, \mathcal{I}_{C,M}(4)) \geq 6$. □

Remark 8. Take $C \in M_{12}(M)$ without lines L with $\deg(L \cap C) \geq 7$. By Lemma 2 to prove that $h^0(M, \mathcal{I}_{C,M}(4)) \geq 3$ (hence to prove that $C \nsubseteq W'$ by Lemma 18) it is sufficient to prove that $h^1(\mathcal{I}_C(5)) \geq 14$. By Remarks 3, 4 and 5 this is always the case if W' contains a smooth rational curve of degree ≥ 2 or if it contains two lines. So from now on we assume that W' has no such curves, hence no smooth elliptic curve of degree 3 by Remark 7. We also assume that W' has no smooth elliptic curve of degree 4 by Lemma 15.

Now we are going to apply all of the dimension-counting remarks and lemmas above and to use liaison in order to show that degenerate rational curves which are sufficiently generic (with respect to the properties described in the remarks and lemmas) must in fact have $h^1(\mathcal{I}_C(5)) < 11$, contradiction. Our argument hinges on a careful case-by-case analysis involving the types of divisors that that arise as components of certain residuals C_T to C inside of complete intersections of type $(5, 5)$.

Since $h^1(\mathcal{I}_C(5)) \geq 11$ we have $h^0(M, \mathcal{I}_C(5)) \geq 6$. Hence $h^0(W', \mathcal{I}_{C,W'}(5)) \geq 5$. For any $T \in |\mathcal{I}_{C,M}(5))|$ with $T \neq W'$ let $C_T \subset T \cap W'$ denote the curve linked to C by the complete intersection $T \cap W'$.

We have $\deg(C_T) = 13$, $\chi(\mathcal{O}_{C_T}) = -2$ ([19, Proposition 3.1]) and $h^1(\mathcal{I}_{C_T}(1)) = h^1(\mathcal{I}_C(5))$ ([16, Theorem 1.1 (a)], [20]). Since $\deg(C_T) = 13$, $\chi(\mathcal{O}_{C_T}) = -2$, C_T is not a plane curve (i.e. $h^0(M, \mathcal{I}_{C_T}(1)) = 0$), hence $h^0(\mathcal{O}_{C_T}(1)) = h^1(\mathcal{I}_{C_T}(1)) + 4 = h^1(\mathcal{I}_C(5)) + 4 \geq 15$. Since $\deg(C_T) > 2p_a(C_T) - 2$, we see that C_T is not integral.

Varying T we find inside W' a positive dimensional family of effective divisors C_T, all of them linked to C and with the same arithmetic genus, hence a flat family of effective divisors of W'. Therefore some of the effective divisors whose sum gives C_T moves in W'.

Let $D_1, \dots, D_k$ be all all movable divisors of C_T and let $R_1, \dots, R_u$ the fixed divisors with multiplicities $e_1, \dots, e_u$. Hence for a general T we have $C_T = D_1 + \cdots + d_k + e_1R_1 + \cdots e_uR_u$ as effective Weil divisors of W'.

Let $m(D_i)$, $1 \leq i \leq k$, be the dimension of D_i in the family $|\mathcal{I}_{C,W'}(5))|$. We have $m(D_1) + \cdots + m(D_k) \geq 4$. We also proved that $m(D_1) + \cdots + m(D_k) \geq h^1(\mathcal{I}_C(5)) - 7$. We saw that if $\deg(D_i) = 4$, then W' contains a line L and $m(D_i) = 1$, because the moving family is induced by the family of planes of M containing L. We saw that if $\deg(D_i) = 5$, then D_i is a plane section of W', hence $m(D_i) = 3$.

Let $R_1, \dots, R_u$, $u \geq 0$, be the fixed divisors of $|\mathcal{I}_{W',C}(5)|$ and call b_i the multiplicity of R_i in C_T (for a general T).

(a) Assume the existence of $i \in \{1, \dots, k\}$ such that D_i is a plane curve of degree 5. With no loss of generality we may assume $i = 1$. Let N be the plane containing D_1. Since $m(D_1) = 3$, we have $k \geq 2$ and $h^0(W', \mathcal{I}_{C\cup D_1,W'}(5)) \geq 2$. Since $h^0(W', \mathcal{I}_{C\cup D_1,W'}(5)) = h^0(W', \mathcal{I}_{C,W'}(4))$. Since $h^1(M, \mathcal{I}_{W',M}(4)) = h^1(\mathcal{O}_M(-1)) = 0$, we get $h^0(M, \mathcal{I}_C(4)) \geq 2$. Since $\alpha > 3$ (Lemma 17), there are integral quartic surfaces $T_i \in |\mathcal{I}_{C,M}(4)|$, $i = 1, 2$, with $T_1 \neq T_2$. The complete intersection $T_1 \cap T_2$ links C to a locally Cohen-Macaulay curve G such that $\deg(G) = 4$. and $p_a(G) = -16$ ([19, Proposition 3.1]). Since $p_a(G) \leq -4$ and $\deg(G) = 4$, G has at least one multiple component, J, with multiplicity $e \geq 2$. Our proof of the existence of T_1 and T_2 shows that we may take T_1, T_2 such that D_2 is a subcurve of G. Since $\deg(D_2) \geq 4 = \deg(G)$ and G has a multiple component, we get a contradiction.

(b) Assume the existence of $i \in \{1, \dots, k\}$ such that D_i is a plane curve of degree 4. Just to fix the notation we assume $i = 1$. Let N be the plane spanned by D_1. We have $W' \cap N = D_1 \cup L$ with L a line. Remark 3 gives $h^0(\mathcal{I}_C(5)) \geq 7$, hence $m(D_1) + \cdots + m(D_k) \geq 6$. We saw that $m(D_1) = 1$. Since $m(D_2) + \cdots + m(D_k) \geq 5$, we have $h^0(W', \mathcal{I}_{C\cup D_1,W'}(5)) \geq 6$. Since $\deg(L \cap D_1) = 4$, we get $h^0(W', \mathcal{I}_{C\cup D_1 \cup L}(5)) \geq 4$. Hence we may find a movable divisor E in $|\mathcal{I}_{C\cup L\cup D_1}(5)|$. We saw that $\deg(E) \geq 4$. As in step (a) we get $h^0(M, \mathcal{I}_{C,M}(4)) \geq 4$, contradicting Lemma 18.

(c) From now on we assume that each D_i is non-degenerate. By Lemma 16 we may assume $\deg(D_i) \geq 7$ for all i. By Remark 8 we cannot have $2 \leq \deg(R_i) \leq 4$ and we have $\deg(R_i) = 1$ at most one index i.

Recall that $13 = \sum_{i=1}^{k} \deg(D_i) + \sum_{i=1}^{u} b_i R_u$ and we proved that $k + u > 1$. Since $\deg(D_i) \geq 7$ for all i, we have $k = 1$.

Assume that C_T has no multiple component. We have $h^0(\mathcal{O}_{D_1}(1)) + h^0(\mathcal{O}_{R_1}(1)) + \cdots + h^0(\mathcal{O}_{R_u}(1)) \geq 2 + h^1(\mathcal{I}_C(5))$. Since D_1 moves, we have $p_a(D_1) > 0$ ([6]), hence $h^0(\mathcal{O}_{D_1}(1)) \leq \deg(D_1)$. Since $h^0(\mathcal{O}_{R_i}(1)) = \deg(R_i) + 1$ for at most one index i, we get a contradiction.

Hence C_T has at least one multiple component, say R_1. Since $\deg(D_1) \geq 6$, we get $b_1 \deg(R_1) \leq 13 - \deg(D_1) \leq 7$ and in particular $\deg(R_1) \leq 3$. Since W' has no curve of degree 2 or 3 (Remark 8), R_1 is a line, hence we may assume $h^1(\mathcal{I}_C(5)) \geq 13$. Set $b := b_1$, $R := R_1$ and $e := \deg(C \cap R)$.

We have $\deg(D_1) = 13 - b$. By Remarks 6 and 8 we may assume $e = 0$, i.e. $R \cap C = \emptyset$ and that R is contained in the smooth locus of W'.

(d) Recall that $h^1(M, \mathcal{I}_{C,M}(5)) \geq 13$. By Lemma 18 we may assume that L has no line T with $\deg(T \cap C) \geq 7$. By Lemma 2 we have $h^1(M, \mathcal{I}_{C,M}(4)) \geq 16$, i.e. $h^0(M, \mathcal{I}_{C,M}(4)) \geq 2$. Since $\alpha > 3$ (Lemma 17), each $S \in |\mathcal{I}_{C,M}(4)|$ is irreducible. Let $\mathcal{B}$ denote the linear system on W' induced by $|\mathcal{I}_{C,M}(4)|$ and fix a general $S \in |\mathcal{I}_{C,M}(4)|$. Write $S \cap W' = C + C' \in \mathcal{B}$. Since C' is linked to C by the complete intersection $S \cap W'$, we have $\deg(C') = 8$ and $p_a(C') = -10$ ([19, Proposition 3.1]). Hence C' has a multiple component. Since W' contains no curve of degree $x \in \{2, 3\}$, the multiple component is a line. Since W' has a unique line, R, R is the multiple component. We saw that $R \cap C = \emptyset$ and $C \subset W_{\text{reg}}$. Since $\dim(\mathcal{B}) > 0$, $\mathcal{B}$ has at least one movable component, A. By Lemma 16 A is a plane curve of degree $x \in \{4, 5\}$. We have $C \cup R \cup A \subset S$. First assume $x = 5$. Since $A \in |\mathcal{O}_S(1)|$, $C \cup R$ is contained in an element of $|\mathcal{O}_S(3)|$. Since the restriction map $H^0(M, \mathcal{O}_M(3)) \to H^0(S, \mathcal{O}_S(3))$ we get $\alpha \leq 3$, contradicting Lemma 17. Now assume $x = 4$. Since R is the only line of W' we get $A \cup R \in |\mathcal{O}_S(1)|$. As above we get $\alpha \leq 3$, a contradiction.

References

[1] E. Ballico and Ph. Ellia, The maximal rank conjecture for non-special curves in $\mathbf{P}^3$, Invent. Math. 79 (1985), 541–555.

[2] E. Ballico and Ph. Ellia, On postulation of curves in $\mathbb{P}^4$, Math. Z. 188 (1985), 215–223.

[3] A. Bernardi, A. Gimigliano and M. Idà, Computing symmetric rank for symmetric tensors, J. Symbolic. Comput., 46 (2011), 34–53.

[4] H. Clemens, Some results about Abel-Jacobi mappings. Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), 289–304, Ann. of Math. Stud., 106, Princeton Univ. Press, Princeton, NJ, 1984.

[5] E. Cotterill, Rational curves of degree 10 on a general quintic threefold, Comm. Algebra 33 (2005), 1833–1872.

[6] E. Cotterill, Rational curves of degree 11 on a general quintic 3-fold, Quart. J. Math. 63 (2012), 539–568.

[7] D'Almeida, Courbes rationnelles de degré 11 sur une hypersurface quintique générale de $\mathbb{P}^4$, Bull. Sci. math. 136 (2012), 899–903.

[8] D. Eisenbud and J. Harris, Finite projective schemes in linearly general position, J. Algebraic Geom. 1 (1992), 15–30.

[9] Ph. Ellia and Ch. Peskine, Groupes de points de $\mathbf{P}^2$: caractère et position uniforme, in: Algebraic geometry (L' Aquila, 1988), 111–116, Lecture Notes in Math., 1417, Springer, Berlin, 1990.

[10] L. Gruson, R. Lazarsfeld and Ch. Peskine, On a theorem of Castelnuovo, and the equations defining space curves, Invent. Math. 72 (1983), 491–506.

[11] J. Harris, with the collaboration of D. Eisenbud, Curves in Projective Space, Les Presses de l'Université de Montréal, Montréal, 1982.

[12] P. Jahnke, T. Peternell, and I. Radloff, Some recent developments in the classification theory of higher dimensional manifolds, Global aspects of complex geometry, 311–357, Springer, Berlin, 2006.

[13] T. Johnsen and S. Kleiman, Rational curves of degree at most 9 on a general quintic threefold, Comm. Algebra 24 (1996), 2721–2753.

[14] S. Katz, On the finiteness of rational curves on quintic threefolds, Compositio Math. 60 (1986), 151–162.

[15] C. McCrory, T. Shifrin and R. Varley, The Gauss map of a generic hypersurface in $\mathbf{P}^4$, J. Differential Geom. 30 (1989), 689–759.

[16] J. Migliore, Geometric invariants for liaison of space curves, J. Algebra 99 (1986), 548–572.

[17] P. G. J. Nijsse, Clemens' conjecture for octic and nonic curves. Indag. Math. 6 (1995), 213–221.

[18] D. Perrin, Courbes passant par m points généraux de $\mathbb{P}^3$, Bull. Soc. Math. France, Mémoire 28/29 (1987).

[19] C. Peskine and L. Szpiro, Liaison des variétés algébriques. I, Invent. Math. 26 (1974), 271–302.

[20] P. Rao, Liaison among curves in $\mathbb{P}^3$, Invent. Math. 50 (1979), 205–217.

[21] R. Strano, A characterization of complete intersection curves in $\mathbf{P}^3$, Proc. Amer. Math. Soc. 104 (1998), 711–715.

[22] F. L. Zak, Tangents and secants of algebraic varieties. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993.

Edoardo Ballico
Dipartimento di Matematica
Università di Trento
38123 Povo (TN), Italy
E-mail: edoardo.ballico@unitn.it

Claudio Fontanari
Dipartimento di Matematica
Università di Trento
38123 Povo (TN), Italy
E-mail: claudio.fontanari@unitn.it