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Calculating Colored Homflypt Invariants with Image Processing and a New Form of Gauss Code

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Abstract: We propose a new form of signed Gauss Code consisting of a list of matrices, named as intrinsic matrices (IMAT). We use IMAT to calculate colored homflypt invariants in three steps. Firstly, given a link diagram, we obtain its IMAT by image processing. Secondly, if the link is decorated by the idempotent basis, we acquire an IMAT expression for the decorated link by using the IMAT obtained in the first step. This expression is a linear combination of a few IMATs. Finally, we improve the skein template algorithm and apply it to each IMAT in the expression got in the second step, which leads to the colored homflypt invariant of the given idempotent decorated link. The procedure is illustrated with several examples. The first two steps are the main new ingredients in the scheme and it is by using them that we are able to acquire the IMAT expression for the decorated link and compute the colored homflypt invariants for arbitrary link with more intuitive inputs.
Keywords: homflypt invariant; Gauss Code; intrinsic matrices (IMAT); image processing; skein template algorithm; decorated link.

1. introduction

The features of knots and links are useful in many fields, such as theoretical physics, fluid mechanics and biology [3, 15, 22, 23]. A basic problem in the knot theory is to find out whether two link diagrams represent two topological equivalent links or not, which can be used to distinguish protein structure and periodic orbits.

The homflypt polynomial was discovered by Freyd-Yetter, Lickorish-Millet, Ocneanu, Hoste and Przytychi-Traczyk and until now, it is still one of the most useful link invariants. The colored homflypt invariant, which is more accurate when it is used to identify different link diagrams, can be obtained through the quantum group invariants of $U_q(sl_N)$ [19, 21]. The colored homflypt invariant can also be called decorated link invariant since it has an equivalent definition through the satellite invariants in homflypt skein theory [1, 6, 17].

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Let $\mathcal{L}$ be a link with L components labeled by the corresponding partitions $\mu^1, \mu^2, \ldots, \mu^L$, and from the view of homflypt skein theory, the colored homflypt invariant of $\mathcal{L}$ can be identified through the homflypt polynomial of a decorated link derived from $\mathcal{L}$ whose decoration is a list of the idempotent basis elements $Q_{\mu^1}, \ldots, Q_{\mu^L}$. Let $\vec{\mu} = (\mu^1, \ldots, \mu^L) \in \mathcal{P}^L$, and the colored homflypt invariant of the link $\mathcal{L}$ is defined by

$$W_{\vec{\mu}}(\mathcal{L}; s, v) = s^{-\sum_{\alpha=1}^{L} w(\mathcal{K}_\alpha)\kappa_{\mu^\alpha}} v^{-\sum_{\alpha=1}^{L} w(\mathcal{K}_\alpha)|\mu^\alpha|} \langle \mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha} \rangle, \tag{1}$$

where $w(\mathcal{K}_\alpha)$ is the writhe number of the α-th component $\mathcal{K}_\alpha$ of $\mathcal{L}$ and the bracket $\langle \mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha} \rangle$ denotes the framing dependent homflypt polynomial of the decorated link $\mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha}$ [14]. An idempotent basis element Q_λ is in fact an element in the homflypt skein of the annulus $\mathcal{C}$ and an explicitly skein-based version of Q_λ was developed by A. K. Aiston [1, 17]. We will introduce more details about the homflypt skein theory, decorated link and colored homflypt invariant in section 2.

Lots of efforts have been made to calculate the colored homflypt polynomials [2, 11, 13, 24]. For instance, in [13] Lin X S and Zheng H. gave the formula for the full colored homflypt invariant for the torus link T_{nL}^{mL} and in [11] Jie Gu and Hans Jockers introduced a way to calculate two-bridge hyperbolic knots.

However, in calculating the colored homflypt invariant, each study above only focused on a certain kind of knots or links. Furthermore, the inputs of the algorithm in previous studies were not quite intuitive. We aim to find a way to calculate the colored homflypt invariant with more intuitive inputs, specifically the image of the link diagram, the orientation and the partition that represents the decoration. Theoretically, our algorithm is universal for all link diagrams with easily accessible inputs, which make massive computing possible.

By analyzing equation 1, we can see that in order to calculate the invariant $W_{\vec{\mu}}(\mathcal{L}; s, v)$, we need to compute $\langle \mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha} \rangle$, which can be done in two steps: calculating $\langle \mathcal{L} \rangle$ for any given $\mathcal{L}$ and getting $\mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha}$ from $\mathcal{L}$.

Given an image of the link diagram for $\mathcal{L}$, we would like to acquire $\langle \mathcal{L} \rangle$. First, we obtain a new form of the signed Gauss Code of the link by image processing in section 3. This new form of the Gauss Code is a list of matrices named as intrinsic matrices (IMAT), denoted by $\mathcal{M}_{\mathcal{L}}$. Then, in section 5, following [5, 7, 12], we make use of the skein template algorithm (STA) and $\mathcal{M}_{\mathcal{L}}$ to calculate the framing dependence homflypt invariant $\langle \mathcal{L} \rangle$ and have $\langle \mathcal{L} \rangle = \langle \mathcal{M}_{\mathcal{L}} \rangle$. We also use IMAT to recognize RI and RII moves to improve STA in this section. Thus, with IMAT we are able to calculate the homflypt polynomials.

In addition, in section 4, we introduce a way to get $\mathcal{M}_{\mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha}}$ from $\mathcal{M}_{\mathcal{L}}$ based on two facts. The first is that the idempotent basis Q_{μ^α} can be represented by the single row idempotent h_m [1, 10, 17]. The second is that row idempotent h_m can be represented by the Turaev's basis A_λ [16]. Therefore, we can use the Turaev's basis $\{A_\lambda\}$ to represent the idempotent basis, which means $Q_\mu = \sum_{\lambda \vdash |\mu|} C_\lambda A_\lambda$, where each C_λ lies in the coefficient ring $\Gamma = \mathbb{Z}[s^{\pm 1}]$ with the elements $s^k - s^{-k}$ admitted as denominators for $k \geq 1$. Thus, we have

$$\mathcal{M}_{\mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha}} = \sum_{\vec{\lambda} \vdash |\vec{\mu}|} \left(\prod_{\alpha=1}^{L} C_{\lambda^\alpha}\right) \mathcal{M}_{\mathcal{L} \star \otimes_{\alpha=1}^{L} A_{\lambda^\alpha}}, \tag{2}$$

where $\vec{\lambda} \vdash |\vec{\mu}|$ refers to $\lambda^\alpha \vdash |\mu^\alpha|$ for all $\alpha = 1, \ldots, L$ and $|\mu^\alpha|$ refers to the weight of the partition μ^α [8]. Hence, it is sufficient to have $\mathcal{M}_{\mathcal{L} \star \otimes_{\alpha=1}^{L} A_{\lambda^\alpha}}$ if we intend to get $\mathcal{M}_{\mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\mu^\alpha}}$. To calculate $\mathcal{M}_{\mathcal{L} \star \otimes_{\alpha=1}^{L} A_{\lambda^\alpha}}$, we first propose a method called TVO (travel order method) to name the elements in the Turaev decorated link. Then we obtain the corresponding positive-negative information and up-down information for all the cross points in $\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}$. Finally, we get the IMAT $\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}}$ for the Turaev decorated link by combing the information together.

In conclusion, given an image of the link diagram for $\mathcal{L}$, its orientation and a decoration represented by $\mu^1,\dots,\mu^L$, we can first acquire the IMAT for $\mathcal{L}$, then the IMAT for $\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}$, and finally a Γ-linear combination of IMATs for $\mathcal{L}\star\otimes_{\alpha=1}^{L}Q_{\mu^\alpha}$. We use skein template algorithm to compute each IMAT $\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}}$ in equation 2. Setting $P_{\vec{\lambda}}^{\vec{\mu}}=\langle\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}}\rangle$, we have

$$\langle\mathcal{L}\star\otimes_{\alpha=1}^{L}Q_{\mu^\alpha}\rangle=\sum_{\vec{\lambda}\vdash|\vec{\mu}|}\Big(\prod_{\alpha=1}^{L}C_{\lambda^\alpha}\Big)\,P_{\vec{\lambda}}^{\vec{\mu}}. \tag{3}$$

Here $P_{\vec{\lambda}}^{\vec{\mu}}$ is an element in the ring $\Lambda=\mathbb{Z}[s^{\pm1},v^{\pm1}]$ with the elements s^k-s^{-k} admitted as denominators, where k is an integer and $k\geq 1$. If we plug equation 3 into equation 1, we obtain the colored homflypt invariant $W_{\vec{\mu}}(\mathcal{L};s,v)$ of the link $\mathcal{L}$. All the above processes can be done automatically by computer.

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2. preliminaries

2.1. Calculating the homflypt polynomial by skein relation

Define the coefficient ring $\Lambda=\mathbb{Z}[s^{\pm1},v^{\pm1}]$ with the elements s^k-s^{-k} admitted as denominators, where k is an integer and $k\geq 1$. The homflypt polynomial, which is an element in Λ, is a two-variable isotopic invariant of the oriented links. Given an oriented link diagram $D_{\mathcal{L}}$, we can use Reidemeister move and skein relation shown in figure 1 to simplify the link diagram and obtain a scalar $\langle D_{\mathcal{L}}\rangle\in\Lambda$, which is the framed homflypt invariant of the link $\mathcal{L}$. For simplicity, $\langle\mathcal{L}\rangle$ denotes $\langle D_{\mathcal{L}}\rangle$. In particular, $\mathcal{U}$ denotes the simplest unknot with no

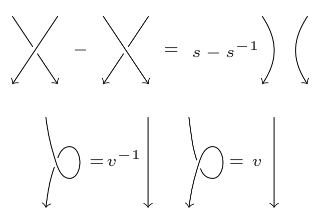


Figure 1: The skein relation.

self cross point and its framed homflypt invariant is $\langle\mathcal{U}\rangle=\frac{v-v^{-1}}{s-s^{-1}}$ shown in figure 2.

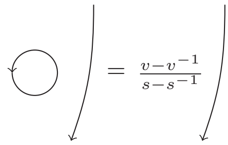


Figure 2: Removal of an unknot.

Definition 2.1. The classical (framing-independence) homflypt polynomial is defined by

$$P_{\mathcal{L}}(s,v) = \frac{v^{-w(\mathcal{L})}\langle \mathcal{L}\rangle}{\langle U\rangle}, \tag{4}$$

where $w\langle\mathcal{L}\rangle$ reffers to the writhe number of the link. Particularly, $P_{\mathcal{U}}(s,v) = 1$.

2.2. Skein $\mathcal{C}$

The homflypt skein of the annulus $\mathcal{C} = \mathcal{S}(S^1 \otimes I)$ is the Λ-linear combination of the oriented tangles in $S^1 \otimes I$ modulo local skein relation. The product of the commutative algebra $\mathcal{C}$ is induced by placing one annulus outside another.

Let $T \in H_n$ be a n-tangle and $\hat{T}$ be its closure in the annulus illustrated in figure 3 and here H_n refers to the Hecke algebra [4, 18]. The closure map is a Λ-linear map and $\mathcal{C}_n$ denotes its image. Thus, each diagram in the annulus presents an element in some $\mathcal{C}_n$. The union $\mathcal{C}_+ = \bigcup_{n\geq 0} \mathcal{C}_m$ is a submodule of $\mathcal{C}$. $\mathcal{C}_+$ is isomorphic to the algebra of the

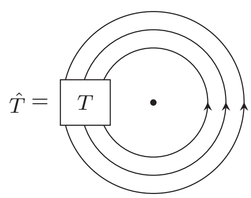


Figure 3: The closure map.

symmetric functions [16].

2.2.1. Turaev's basis for $\mathcal{C}_+$. The element $A_n \in \mathcal{C}_n$ is the closure of braid

$$\sigma_{n-1} \cdots \sigma_2\sigma_1 \in H_n$$

shown in figure 4. A partition $\lambda \vdash n$ with length l is denoted by $\lambda = (\lambda_1, \dots, \lambda_l)$ [8]. Given such a partition, we define the monomial A_λ by $A_\lambda = A_{\lambda_1} \cdots A_{\lambda_l}$. Then the monomials $\{A_\lambda\}_{\lambda\vdash n}$ constitute a basis for $\mathcal{C}_n$ and the monomials $\{A_\lambda\}_{\lambda\vdash n, n\geq 1}$ form the Turaev's geometric basis for $\mathcal{C}_+$ [20].

$\sigma_{n-1} \cdots \sigma_2 \sigma_1 =$

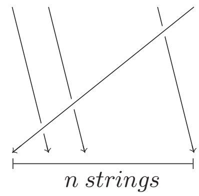


Figure 4: Braid $\sigma_{n-1} \cdots \sigma_2 \sigma_1$.

2.2.2. Idempotent basis for $\mathcal{C}_+$. H. R. Morton and A. K. Aiston established the skein-based version of the idempotent basis Q_λ with the single row idempotent elements $\{h_m\}$ for the homflypt skein of the annulus $\mathcal{C}_+$ in [1, 10].

Definition 2.2. The idempotent basis Q_λ for $\mathcal{C}_+$ is represented as follows

$$Q_\lambda = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+l-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+l-2} \\ \vdots & \vdots & & \vdots \\ h_{\lambda_l-l+1} & h_{\lambda_l-l+2} & \cdots & h_{\lambda_l} \end{pmatrix}. \tag{5}$$

Here $\{h_k\}_{k\in\mathbb{Z}}$ are the single row idempotent elements. If $k = 0$, then h_k=1 and if $k < 0$, then $h_k = 0$.

The idempotent basis$\{Q_\lambda\}$ is quite abstract. In proposition 4.2, we will prove that $\{Q_\lambda\}$ can be represented by Turaev's geometric basis which is more intuitive.

2.3. Decorated link and colored homflypt invariants

The decorated link is a link with a decoration of certain structure [24].

Definition 2.3. Given a framed link $\mathcal{L}$ with L components and diagrams $D_1, \ldots, D_L$ in the skein model of annulus in $\mathcal{C}_+$, we define the decorated link as

$$\mathcal{L} \star \otimes_{\alpha=1}^{L} D_\alpha.$$

This decorated link is derived from $\mathcal{L}$ by replacing the annulus of the components in $\mathcal{L}$ with the corresponding annulus of the diagrams $\{D_\alpha\}_{\alpha=1\ldots,L}$ so that the orientation of the cores matches.

Example 2.4. Figure 5 shows a decorated hopf link.

Definition 2.5. The framed colored homflypt invariant $\mathcal{H}(\mathcal{L}; \otimes_{\alpha=1}^{L} D_\alpha)$ of $\mathcal{L}$ is defined as the (framing-dependence) homflypt polynomial of the decorated link $\mathcal{L} \star \otimes_{\alpha=1}^{L} D_\alpha$, i.e. $\mathcal{H}(\mathcal{L}; \otimes_{\alpha=1}^{L} D_\alpha) = \langle \mathcal{L} \star \otimes_{\alpha=1}^{L} D_\alpha \rangle$.

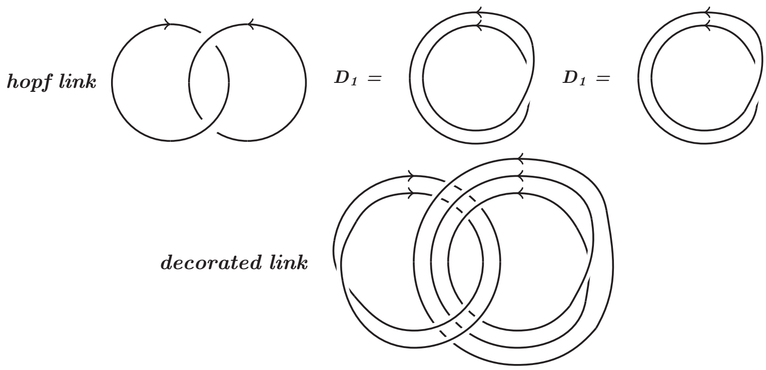


Figure 5: The hopf link decorated by D_1 and D_2.

Definition 2.6. The (framing-independence) colored homflypt invariant of $\mathcal{L}$ is defined as follows:

$$W_{\lambda_1,\ldots,\lambda_L}(\mathcal{L}) = s^{-\sum_{\alpha=1}^{L}\omega(\mathcal{K}_\alpha)\kappa_{\lambda^\alpha}}\, v^{-\sum_{\alpha=1}^{L}\omega(\mathcal{K}_\alpha)|\lambda^\alpha|}\langle \mathcal{L}\star\otimes_{\alpha=1}^{L} Q_{\lambda^\alpha}\rangle, \tag{6}$$

where Q_{λ^α} is the idempotent basis for $\mathcal{C}_+$ defined in definition 2.2.

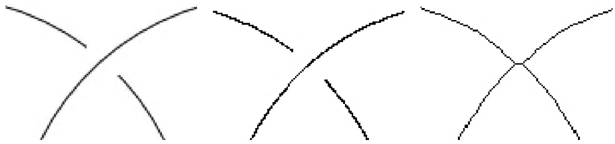
3. Extraction of the intrinsic matrices

In this section, we extract the Gauss Code for the oriented link diagram from its image. We will use some basic tricks in image processing and get a list of matrices, which is in fact a new form of signed Gauss Code named as intrinsic matrices, abbreviated to IMAT. Each matrix in the IMAT represents a component of the link.

Given an image of a link diagram, we intend to acquire the relative position, up-down information and the positive-negative information for all the cross points. In order to do this, it is sufficient to get the parameterized curves. Then, by dealing with the parameterized curves, we can extract the information we need. We fulfill our goal in three steps: preprocessing, obtaining the parameterized curve and extraction of information.

3.1. Preprocessing of the image

Firstly, we are on purpose to get two standard images: a 3D-binary image and a 2D-skeleton image. We preprocess the image and use Otsu's method [25] to obtain a binary image called the 3D-binary image. Then we apply some basic morphological methods [26, 27] to the 3D-binary image. To be specific, we use dilation to discard the up-down information and then use shrink to obtain the 2D-skeleton image. The results in a local area are shown in figure 6.



(a) input image (b) 3D-binary image (c) 2D-skeleton image

Figure 6: The results in a local area of the input image, 3D-binary image and the 2D-skeleton image.

3.2. Obtaining the parameterized curve

Secondly, we take the orientation into consideration and use 2D-skeleton image to acquire the parameterized curve for each component. Here we employ a natural coordinate system shown in figure 7. In the following description, we call the pixels belonging to the link useful pixels.

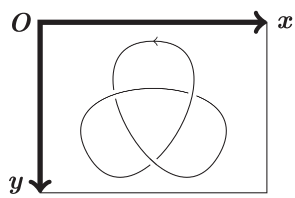


Figure 7: The natural coordinate for the image of a link diagram.

Since the cross points are of great importance in our later calculation, we need to obtain the coordinates of all the cross points before getting parameterized curve. According to the fact shown in figure 8, we introduce a time domain mask with step h_1, similar to the one in figure 9, to identify the cross points.

We attach two initial points which indicate the orientation to each component. Given the coordinates of all the cross points and the initial points, we can successively get all the linear interpolation points with step h_2. These interpolation points include all the cross points and they form the parameterized curve for the image. The main idea of this procedure is demonstrated in figure 10. An experimental result of the hopf link is shown in figure 11 and the coordinate values for all its interpolation points are shown in figure 12, where $h_2 = 60$.

3.3. Extraction of the intrinsic matrices

Finally, we extract the up-down information, positive-negative information for the cross points and store the information in a list of matrices named intrinsic matrices. Using the

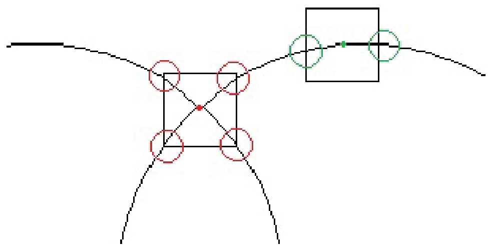


Figure 8: Using the number of local branches of useful pixels to find the cross points. Each cross point has four branches around it while any other useful pixel only has two.

1	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	0	0	0	0	1
1	0	0	0	0	0	1
1	0	0	0	0	0	1
1	0	0	0	0	0	1
1	1	1	1	1	1	1

Figure 9: The mask is used to extract useful pixels on the edge of the neighborhood of the center pixel. Here the step of the neighborhood is $h_1 = 3$.

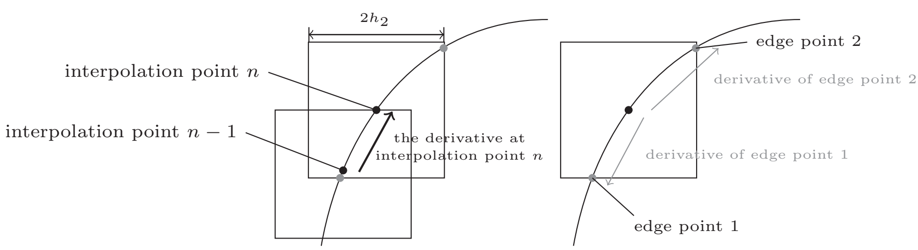


Figure 10: Since the derivative of the edge point 2 has stronger consistency with the derivative at interpolation point n than that of the edge point 1, we choose edge point 2 as the $n+1$ interpolation point.

parameterized curve, we can acquire the orientation of the link in each cross point. We use the orientation and the 3D-binary image to get the up-dwon information. And then we use the orientation and up-down information to obtain the positive-negative information. The key point is shown in figure 13. Since the coordinates of these cross points are not important in topological sense, we use names rather than coordinates to represent the

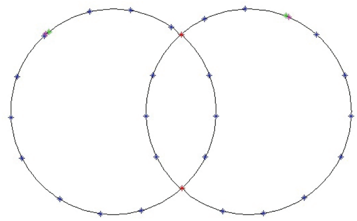


Figure 11: The interpolation points of the hopf link when $h_2 = 60$. The red points are the cross points while the pink and the green points are the given initial points.

y coordinate	x coordinate	cross mark
128	161	0
125	166	0
95	226	0
93	286	0
118	346	0
129	361	1
189	402	0
249	412	0
309	396	0
355	362	1
388	302	0
393	242	0
371	182	0
311	126	0
251	110	0
191	119	0
131	159	0
128	161	0

y coordinate	x coordinate	cross mark
103	519	0
101	515	0
91	455	0
106	395	0
129	361	1
189	319	0
249	309	0
309	324	0
355	362	1
389	422	0
392	482	0
369	542	0
309	596	0
249	611	0
189	602	0
129	560	0
103	519	0

Figure 12: The interpolation points of the hopf link when $h_2 = 60$.

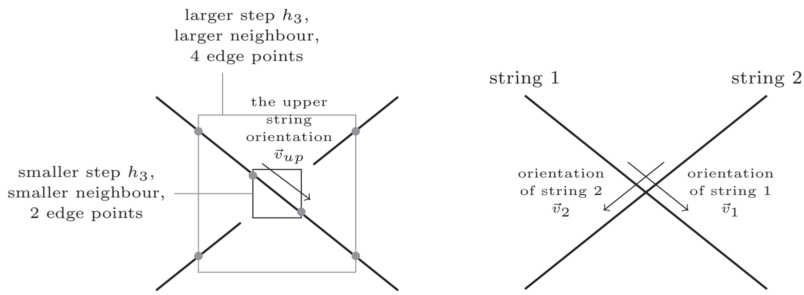
cross points. Each name is a positive integer. In the end, we gain a new form of Gauss Code which is a list of matrices named intrinsic matrices.

Definition 3.1 (Intrinsic matrices). A list of ordered matrices

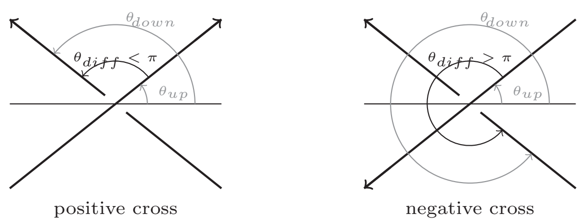
$$\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_L\}$$

is called intrinsic matrices if each matrix in it represents the Gauss Code for the corresponding component in a link diagram, i.e. if $1 \leq i \leq L$, then each matrix $\mathcal{M}_i = \mathcal{M}(\mathcal{K}_i)$ satisfies:

- The first column contains integers that represent the names of the cross points ordered by the orientation on the component, denoted by α.



(a) Changing the step of the mask and comparing the orientation to get the up-down information.



(b) Comparing the shift angle with π to get the postive-negative information.

Figure 13: Acquisition of the up-down and the positive-negative information.

- The second column contains the corresponding up-down information for the cross points, $+1$ for the upper ones and -1 for the nether ones, denoted by ud.
- The third column contains the corresponding positive-negative information for the cross points, $+1$ for positive ones and -1 for negative ones, denoted by pn.

We abbreviate the intrinsic matrices to IMAT. In an IMAT, a cross point is in position (i, j) if the j-th row of the i-th matrix represents the cross point.

A cross point in a link diagram appears twice in different positions in its IMAT and we call the two positions pair positions. The cross points in pair positions are called pair cross points and they share the same name, the same positive-negative information and different up-down information. Therefore, for a cross point $(\alpha, ud_\alpha, pn_\alpha)$ in position (i, r), we can find its pair cross point in the pair position (cn_i, cn_r) by searching through the first columns in IMAT for α. It follows that the total number of the rows in IMAT is an even number. In fact the number of rows in each matrix of IMAT is an even number, which is a direct conclusion of the in-out information introduced in section 3.4.

Proposition 3.2. *Let $\mathcal{M}$ be the IMAT of a link diagram $\mathcal{L}$. The writhe number of $\mathcal{L}$ is equal to*

$$w(\mathcal{L}) = \frac{1}{2}\sum_{i=1}^{L}\sum_{r_i=1}^{n_i} \mathcal{M}_i(r_i, 3).$$

Here n_i denotes the row number of the $\mathcal{M}_i$ and $\mathcal{L}$ has L components.

The proof is evident by definition 3.1.

In the intrinsic matrices, only one column represents the name of the cross point. But in some cases, we need to use more than one column to distinguish the cross points. For instance, in section 4, we will use 3 columns to identify the cross points. We call this kind of intrinsic matrices the pseudo intrinsic matrices, defined as follows.

Definition 3.3 (Pseudo IMAT). A list of ordered matrices $\hat{\mathcal{M}} = \hat{\mathcal{M}}_1, \hat{\mathcal{M}}_2, \ldots, \hat{\mathcal{M}}_L$ is called pseudo intrinsic matrices if it is a generalized IMAT in which we use two or more integers to name each cross point. We abbreviate Pseudo IMAT to P-IMAT.

Obviously, P-IMAT can easily be transformed to IMAT by renaming the cross points.

Example 3.4. The IMAT for the hopf link:

$$\mathcal{M}_1 = \begin{pmatrix} 1 & -1 & +1 \\ 2 & +1 & +1 \end{pmatrix}, \; \mathcal{M}_2 = \begin{pmatrix} 1 & +1 & +1 \\ 2 & -1 & +1 \end{pmatrix}.$$

The integers in the first column deposit the names of the cross points while those in the second column store the up-down information and those in the third column contain the positive-negative information. The cross point in position $(1, 2)$ has Gauss Code $(2, +1, +1)$, which means the cross point is a positive upper cross point named 2. Its pair cross point also named 2 is in position $(2, 2)$. The Gauss Code for this pair cross point is $(2, -1, +1)$, indicating it is a positive nether cross point. The writhe number of the link is $w = \frac{1}{2}(\mathcal{M}_1(1, 3) + \mathcal{M}_1(2, 3) + \mathcal{M}_2(1, 3) + \mathcal{M}_2(2, 3)) = 2$.

Different ways of choosing initial points result in different IMATs, which represent the same link diagram owing to the fact that each component of a link is a closed curve. This phenomenon is called the cyclic structure of IMAT illustrated in equation 7. We will take the cyclic structure into consideration in the following description.

$$(7)\quad \begin{pmatrix} \alpha_1 & ud_1 & pn_1 \\ \alpha_2 & ud_2 & pn_2 \\ \vdots & \vdots & \vdots \\ \alpha_{n_i} & ud_{n_i} & pn_{n_i} \end{pmatrix} \sim \begin{pmatrix} \alpha_2 & ud_2 & pn_2 \\ \vdots & \vdots & \vdots \\ \alpha_{n_i} & ud_{n_i} & pn_{n_i} \\ \alpha_1 & ud_1 & pn_1 \end{pmatrix} \sim \cdots$$
$$\sim \begin{pmatrix} \alpha_{n_i-1} & ud_{n_i-1} & pn_{n_i-1} \\ \alpha_{n_i} & ud_{n_i} & pn_{n_i} \\ \vdots & \vdots & \vdots \\ \alpha_{n_i-2} & ud_{n_i-2} & pn_{n_i-2} \end{pmatrix} \sim \begin{pmatrix} \alpha_{n_i} & ud_{n_i} & pn_{n_i} \\ \alpha_1 & ud_1 & pn_1 \\ \vdots & \vdots & \vdots \\ \alpha_{n_i-1} & ud_{n_i-1} & pn_{n_i-1} \end{pmatrix}$$

Example 3.5. We can get two IMATs representing the same trefoil diagram shown in figure 14. One is obtained with initial point named 10 and the other is obtained with initial point 20.

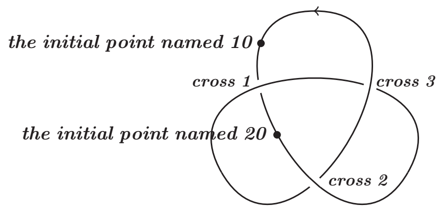


Figure 14: Different initial points for the trefoil knot.

$$\mathcal{M}_1 = \begin{pmatrix} 1 & -1 & +1 \\ 2 & +1 & +1 \\ 3 & -1 & +1 \\ 1 & +1 & +1 \\ 2 & -1 & +1 \\ 3 & +1 & +1 \end{pmatrix} \sim \mathcal{M}_1' = \begin{pmatrix} 2 & +1 & +1 \\ 3 & -1 & +1 \\ 1 & +1 & +1 \\ 2 & -1 & +1 \\ 3 & +1 & +1 \\ 1 & -1 & +1 \end{pmatrix}$$

The IMAT is in fact a new format of signed Gauss Code. The matrix format makes the IMAT more convenient and practical in calculation. We can get lots of interesting properties for the link diagram by using IMAT.

3.4. Getting the in-out information by IMAT

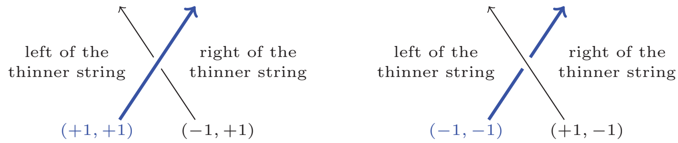
The definition for the in-out information of a component is important since it is the basis for TVO (travel order method) proposed in section 4, which will be used to get the IMAT for decorated links. We introduce the left-right information of a component before defining the in-out information.

A cross point is an intersection of two strings. Choosing one of the two strings and following its direction, we can see that when it encounters the other string, it goes through the second string either from its right to left or from left to right. We can get this information by IMAT. There are four cases shown in figure 15, which lead to the following proposition.

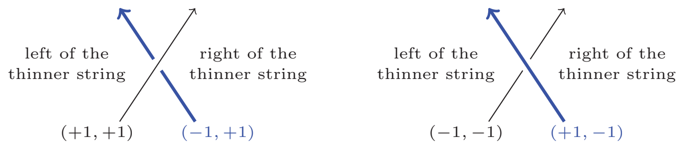
Proposition 3.6. *If string i_1 goes through string i_2:*

- *from left to right if and only if $ud_{\alpha_{i_1}} \times pn_{\alpha_{i_1}} = +1$ or $ud_{\alpha_{i_2}} \times pn_{\alpha_{i_2}} = -1$.*
- *from right to left if and only if $ud_{\alpha_{i_1}} \times pn_{\alpha_{i_1}} = -1$ or $ud_{\alpha_{i_2}} \times pn_{\alpha_{i_2}} = +1$.*

Each component of an oriented link is a closed curve. If it has no self intersection, we can define both its inside and outside by common sense illustrated in figure 16. For a clockwise component, its inside equals to its right side and its outside equals to the left side while for an anti-clockwise component, its inside is equal to the left and its outside is equal to the right. The definitions above for outside and inside do not work for components with self intersections. However, as is shown in figure 17, we can decompose this kind of components into several sub closed curves with no self intersections. For each sub closed curve of such kind, the inside and outside are defined as above. Therefore, the in-out of a component is well defined in definition 3.7.



(a) From left to right. Blue: $pn \times ud = +1$, black: $pn \times ud = -1$.



(b) From right to left. Blue: $pn \times ud = -1$, black: $pn \times ud = +1$.

Figure 15: The blue string goes through the black string from one side to another. Here $(\pm 1, \pm 1)$ denote the up-down information and positive-negative information.

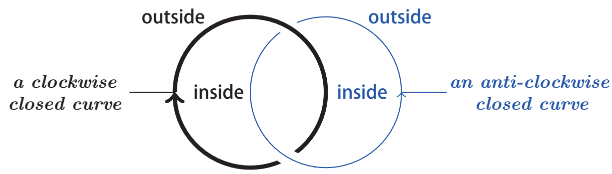


Figure 16: Each component of a link without self intersection is a closed curve with clockwise/anti-clockwise orientation. The inside and outside is well defined.

Definition 3.7 (In-out of the component). Each component of an oriented link can be decomposed into several closed curves by its own intersections. Each closed curve has either clockwise or anticlockwise orientation. The in-out of the component is defined as follows.

- For clockwise part, the right side is the inside and the left side is the outside.
- For the anti-clockwise part, the left side is the inside and the right side is the outside.

If a string encounters a closed curve, the string goes through the closed curve either from the outside to the inside or from its inside to the outside. There are eight situations shown in figure 18. Consequently, we have lemma 3.8.

Lemma 3.8. *Given an oriented link diagram, we decompose all the components into sub closed curves with no intersections.*
String i goes from the outside to the inside of closed curve j if and only if:

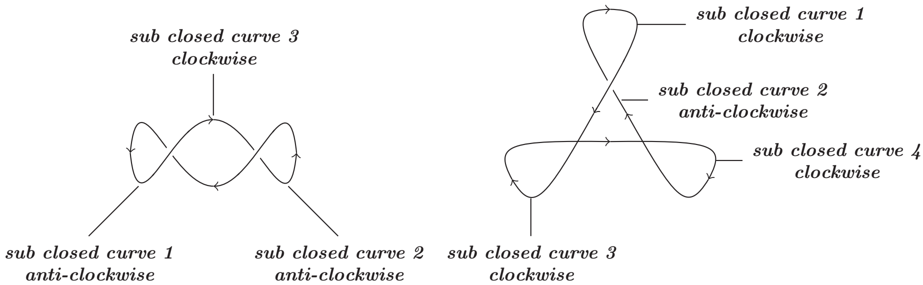


Figure 17: A component with self intersections can be decomposed into several sub closed curves.

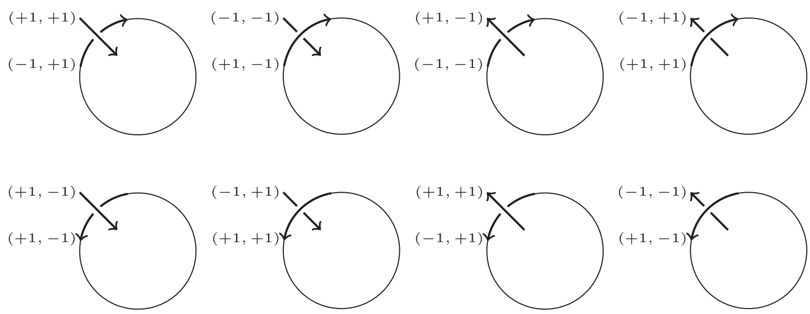


Figure 18: Eight situations when a string encountering a component. Here $(\pm 1, \pm 1)$ denote the up-down and positive-negative information for the cross point.

- *If the closed curve is clockwise, then* $ud_i \times pn_i = +1$ *and* $ud_j \times pn_j = -1$.
- *If the closed curve is anti-clockwise, then* $ud_i \times pn_i = -1$ *and* $ud_j \times pn_j = +1$.

String i *goes from the inside to the outside of closed curve* j *if and only if:*

- *If the closed curve is clockwise, then* $ud_i \times pn_i = -1$ *and* $ud_j \times pn_j = +1$.
- *If the closed curve is anti-clockwise, then* $ud_i \times pn_i = +1$ *and* $ud_j \times pn_j = -1$.

4. Intrinsic matrices expression of the decorated link

The framing-independent colored homflypt invariant is defined in definition 2.6 using the idempotent basis $\{Q_\lambda\}$. To compute this invariant, we need to calculate the framed homflypt invariant for idempotent decorated link $\langle \mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha} \rangle$. In this section, our purpose is to get an IMAT expression for $\mathcal{L} \star \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha}$. In section 5, we compute $\langle \mathcal{L} \rangle$ with IMAT by means of skein template algorithm.

We achieve our goal in two steps. Firstly, following [16], we use a linear combination of the Turaev's basis to present the idempotent basis, i.e. $Q_\mu = \sum_{\lambda\vdash|\mu|} C_\lambda A_\lambda$. It follows that an idempotent decorated link is a linear combination of Turaev decorated link, i.e. $\mathcal{L}\star\otimes_{\alpha=1}^{L} Q_{\mu^\alpha} = \sum_{\vec{\lambda}\vdash|\vec{\mu}|}(\prod_{\alpha=1}^{L} C_{\lambda^\alpha})\,\mathcal{L}\star\otimes_{\alpha=1}^{L} A_{\lambda^\alpha}$. Secondly, we present a method to obtain IMAT $\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L} A_{\lambda^\alpha}}$ for each Turaev decorated link in the expression got in the first step and we acquire what we want by using equation $\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L} Q_{\mu^\alpha}} = \sum_{\vec{\lambda}\vdash|\vec{\mu}|}(\prod_{\alpha=1}^{L} C_{\lambda^\alpha})\,\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L} A_{\lambda^\alpha}}$.

4.1. Intuitive expression for idempotent decorated link

In section 2.2, we have introduced two kinds of basis for skein $\mathcal{C}_+$. In fact, the idempotent basis is equal to a Γ-linear combination of Turaev's basis. Here ring Γ is a sub-ring of Λ introduced in section 2.1. Ring Γ is defined as $\Gamma = \mathbb{Z}[s^{\pm1}]$ with the elements $s^k - s^{-k}$ admitted as denominators, where k is an integer and $k \geq 1$.

In [16], H. R. Morton and P. M. G. Manchón have proved that h_m is equal to a Γ-linear combination of Turaev's basis.

Theorem 4.1. *The single row idempotent $\{h_m\}$ can be written as*

$$h_m = \sum_{\lambda\vdash m} \theta_\lambda A_\lambda, \tag{8}$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of m with length l. Let S_l be the symmetric group of order l and α be a permutation in it, then λ_α denotes the finite sequence

$$\lambda_\alpha = (\lambda_{\alpha(1)}, \ldots, \lambda_{\alpha(l)}).$$

When $\lambda = (0)$, we have $\theta_{(0)} = 1$ while when $\lambda \neq (0)$, we have

$$\theta_\lambda = \frac{1}{|Aut\lambda|} s^m \sum_{\alpha\in S_l} \prod_{i=1}^{l} \frac{1}{[\lambda_{\alpha(1)} + \cdots + \lambda_{\alpha(i)}] s^{\lambda_{\alpha(1)}+\cdots+\lambda_{\alpha(i)}}}. \tag{9}$$

Here (0) denotes the empty partition and $[k]$ denotes the quantum integer $\frac{s^k-s^{-k}}{s-s^{-1}}$.

According to theorem 4.1, definition 2.2 and definition of determinant, some tedious combinational computation leads to proposition 4.2. Then, by definition 2.3, we have proposition 4.3.

Proposition 4.2. *The idempotent basis is a Γ-linear combination of Turaev's basis, i.e. $Q_\mu = \sum_{\lambda\vdash|\mu|} C_\lambda A_\lambda$, where $C_\lambda \in \Gamma$.*

Proposition 4.3. *Let $\mathcal{L}$ be the link with L components and $\{Q_{\mu^\alpha}\}_{\alpha=1,\ldots,L}$ be the idempotent basis elements. The idempotent decorated link is a Γ-linear combination of Turaev decorated link as follows*

$$\mathcal{L}\star\otimes_{\alpha=1}^{L} Q_{\mu^\alpha} = \sum_{\vec{\lambda}\vdash|\vec{\mu}|}(\prod_{\alpha=1}^{L} C_{\lambda^\alpha})\,\mathcal{L}\star\otimes_{\alpha=1}^{L} A_{\lambda^\alpha}.$$

Here $\vec{\lambda}\vdash|\vec{\mu}|$ means $\lambda^\alpha \vdash |\mu^\alpha|$ for $\alpha = 1, 2, \ldots, L$ and A_{λ^α} refers to the Turaev's basis represented by the partition λ^α.

Each A_{λ^α} has a specific link diagram representation introduced in section 2.2.1, so $\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}$ on the right side of the equation in proposition 4.3 is a more intuitive decorated link.

4.2. IMAT expression for idempotent decorated link

We rewrite proposition 4.3 in IMAT form, illustrated in equation 10.

$$\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}Q_{\mu^\alpha}} = \sum_{\vec{\lambda}\vdash|\vec{\mu}|}(\prod_{\alpha=1}^{L}C_{\lambda^\alpha})\,\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}}. \tag{10}$$

Therefore, it is sufficient to compute IMAT for each Turaev decorated link $\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}}$ in order to get $\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}Q_{\mu^\alpha}}$. In fact, we have

$$\langle\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}Q_{\mu^\alpha}}\rangle = \sum_{\vec{\lambda}\vdash|\vec{\mu}|}(\prod_{\alpha=1}^{L}C_{\lambda^\alpha})\,\langle\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}}\rangle. \tag{11}$$

By substituting equation 11 into equation 6, we obtain

$$W_{\lambda_1,\ldots,\lambda_L}(\mathcal{L}) = s^{-\sum_{\alpha=1}^{L}\omega(\mathcal{K}_\alpha)\kappa_{\lambda^\alpha}}v^{-\sum_{\alpha=1}^{L}\omega(\mathcal{K}_\alpha)|\lambda^\alpha|}\sum_{\vec{\lambda}\vdash|\vec{\mu}|}(\prod_{\alpha=1}^{L}C_{\lambda^\alpha})\,\langle\mathcal{M}_{\mathcal{L}\star\otimes_{\alpha=1}^{L}A_{\lambda^\alpha}}\rangle. \tag{12}$$

In section 5, we will acquire the framed homflypt invariant of the decorated link with IMAT. Then we compute the (framing-independent) colored homflypt invariant using equation 12.

We call a Turaev decorated link whose decoration is $((A_1)^{k_1},(A_1)^{k_2},\cdots,(A_1)^{k_L})$ the duplicated link. Here $k_1,k_2\ldots,k_L$ are non-negative integers. To begin with, we propose a method to get the IMAT for a duplicated link in section 4.2.1. And then, for any other Turaev decorated link, we gain its IMAT in section 4.2.2. A Turaev decorated link has two kinds of points: the induced points and the structure points. The induced points are those in the duplicated link while the structure points are those that appear in Turaev structure A_i. And we get the IMAT of a Turaev decorated link by combining the induced points and the structure points.

Example 4.4. The hopf link decorated by Turaev's basis (A_2,A_2) is shown in figure 19. Cross points $\{1,2,3,4,6,7,8,9\}$ are induced points and cross points $\{5,10\}$ are structure points.

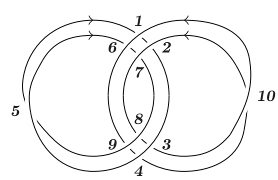


Figure 19: The hopf link decorated by Turaev's basis (A_2,A_2). Here cross points $\{1,2,3,4,6,7,8,9\}$ are induced points and cross points $\{5,10\}$ are structure points.

4.2.1. Duplicated link and induced points.

Definition 4.5 (duplicated link). A decoration of form $((A_1)^{k_1}, (A_1)^{k_2} \cdots, (A_1)^{k_L})$ is called a duplication. A duplicated link $\mathcal{L}_{(A_1)^{k_1},(A_1)^{k_2} \cdots,(A_1)^{k_L}}$ is a decorated link whose decoration is a duplication. Here $\mathcal{L}$ is a link with L components, and k_i is a non-negative integer for $i = 1, 2, \cdots, L$. Two components in the duplicated link are called duplicated components if they are induced by the same component in the original link $\mathcal{L}$.

Here are some obvious observations about duplicated links. The total number of components of the duplicated link $\mathcal{L}_{(A_1)^{k_1},\cdots,(A_1)^{k_L}}$ is $\sum_{i=1}^{L} k_i$. The i-th component of the original link $\mathcal{L}$ has k_i duplicated components in its duplicated link. We call them the i-th duplicated family. Our purpose is to develop a method to name the components in the i-th duplicated family for each $1 \le i \le L$. In section 3.4, we have already introduced that each link can be decomposed into several closed curves with clockwise/anti-clockwise orientation. We use the orientation of the closed curve to number the components in the duplicated link.

Given a duplicated link $\mathcal{L}_{(A_1)^{k_1},\ldots,(A_1)^{k_L}}$, we can decompose it into several closed curves. If a closed curve is induced by a closed curve belonging to the i-th component in the original link, we can find all its k_i duplicated closed curves (closed curves induced by the same closed curve) are of the same orientation. Then we name each of these k_i closed curves with two integers. The first one is i and the second is numbered from outside to the inside as follows:

- Clockwise: increasing order $1, 2, \ldots, k_i$,
- Anti-clockwise: decreasing order $k_i, k_{i-1}, \ldots, 1$.

We call this the travel order method.

It is illustrated in figure 20 that the travel order is compatible with the self cross point. On account of this compatibility, we can use travel order to name the components

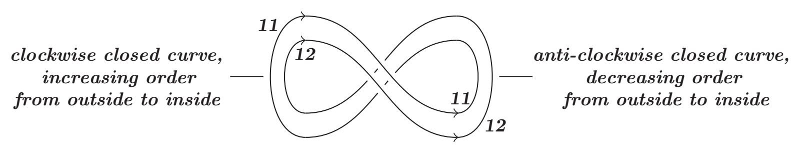


Figure 20: The travel order is compatible with the self cross point and it can be used to name the components of the duplicated link.

of the duplicated link as in proposition 4.6.

Proposition 4.6 (travel number of the component). *Let $\mathcal{L}$ be a link with L components, and the duplicated link for $\mathcal{L}$ with decoration $((A_1)^{k_1}, \cdots, (A_1)^{k_L})$ has $\sum_{i=1}^{L} k_i$ components named by the travel order*

$$\{11, \cdots, 1k_1, 21, \cdots, 2k_2, \cdots, L1, \cdots, Lk_L\}.$$

For a component named $\{ij\}$ in the duplicated link, i means that the component is induced by the i-th component in the original link while j means that the travel number for the component $\{ij\}$ is j. The i-th duplicated family consists of k_i components named

$\{i1, i2, \cdots, ik_i\}$. As a matter of fact, naming the components by travel order means that the travel numbers of the duplicated components always increase from left-to-right along the orientation of the link. We abbreviate travel order method to TVO.

Example 4.7. The hopf link decorated by Turaev's basis (A_1^2, A_1^2) is shown in figure 21. Since $k_1 = 2$, there are two components in the 1-th duplicated family. The two components are induced by the same clockwise component named 1 in the original link so we number them in increasing order from the outside to the inside $\{11, 12\}$. Similarly, we number the 2-th duplicated family.

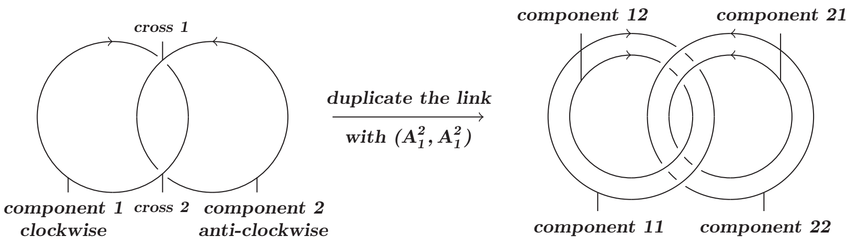


Figure 21: The component 1 (2) is clockwise (anti-clockwise) and we number the duplicated family from the outside to the inside in increasing (decreasing) order $\{11, 12\}$ ($\{22, 21\}$).

A string encounters the i-th component in the original link diagram if and only if this string encounters i-th duplicated family. We would like to figure out if the string goes through $\{i1, i2, \cdots, ik_i\}$ in increasing order or decreasing order. Recall the proposition 3.8 and we can acquire figure 22 illustrating the eight conditions when a string encounters the i-th duplicated family. This leads to proposition 4.8.

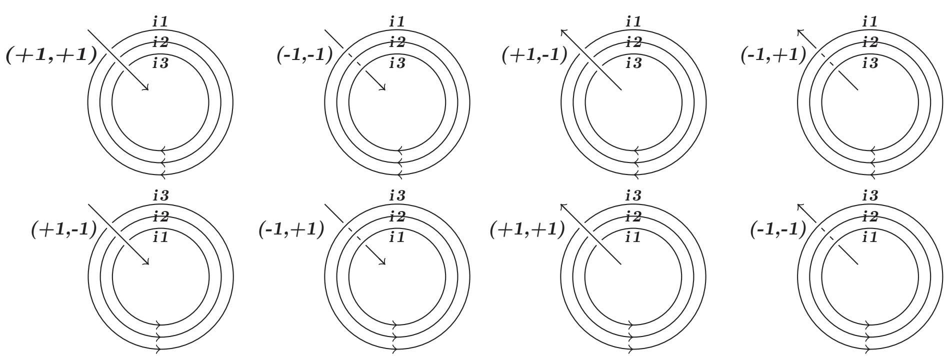


Figure 22: The eight conditions when a string encounters the i-th duplicated family. Here $k_i = 3$.

Proposition 4.8. *If string k encounters the i-th component at the cross point represented by (α_k, ud_k, pn_k) in the original link, the string will encounter the i-th duplicated family in the duplicated link in the following order:*

- *If $ud_k \times pn_k = +1$: the string goes through these k_i components $\{i1, i2, \cdots, ik_i\}$ in increasing order.*
- *If $ud_k \times pn_k = -1$: the string goes through these k_i components $\{i1, i2, \cdots, ik_i\}$ in decreasing order.*

Now, we are in the position to differentiate the cross points in the duplicated link.

Definition 4.9 (induced points and source points). Let $\mathcal{L}$ be a link with L components and $\mathcal{L}_{(A_1)^{k_1}\cdots,(A_1)^{k_L}}$ be its duplicated link. If a cross point α is formed by component i_1 and component i_2 in $\mathcal{L}$, then a cross point formed by component $i_1 j_1$ and component $i_2 j_2$ in the duplicated link is an induced point of α and α is its source point. Here $0 \le j_1 \le k_{i_1}$, $0 \le j_2 \le k_{i_2}$.

All the points in the duplicated link are induced points. The total number of induced points of a source point α formed by component i_1 and component i_2 is $k_{i_1} \times k_{i_2}$. We intend to distinguish these induced points. If an induced cross point is formed by upper string belonging to component $i_1 j_{i_1}$ and nether string belonging to component $i_2 j_{i_2}$, we name it with three integers $(\alpha, j_{i_1}, j_{i_2})$. The first part α is the name of the source point. The second part j_{i_1} is the travel number of the upper string. The third part j_{i_2} is the travel number of the nether string. Here $1 \le j_{i_1} \le k_{i_1}$, $1 \le j_{i_2} \le k_{i_2}$. We call this naming method travel order method of induced points, also abbreviated to TVO.

Example 4.10. Given a duplicated hopf link $\mathcal{L}_{A_1^2, A_1^3}$, we use different names to distinguish the cross points in duplicated hopf link shown in figure 23. Here $L = 2$, $k_1 = 2$, $k_2 = 3$. Taking the cross point named 121 as an example, the first part 1 means that the source point for 121 is 1, the second part 2 represents the travel number for the upper string in the duplicated link and the third part 1 represents the travel number for the nether string in the duplicated link.

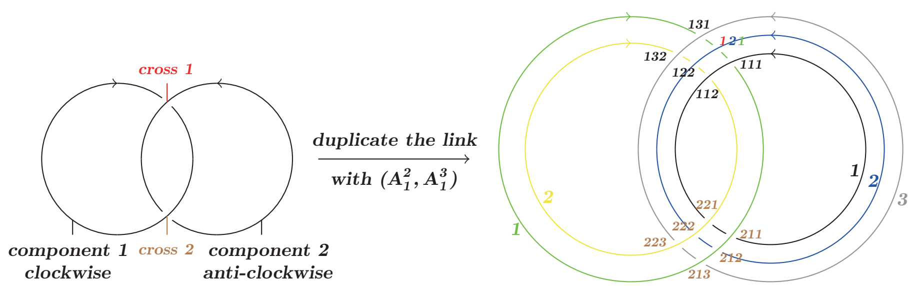


Figure 23: Naming the induced cross points in the duplicated hopf link by travel order.

Since an induced cross point obviously share the same up-down and positive-negative information with its corresponding source point, we have the unique name and the corresponding information for each cross point in the duplicated link. Therefore, it remains to get the relative position of all the induced points in order to acquire the P-IMAT of the duplicated link if we are given the IMAT of the original link and a duplication.

As we have mentioned before, each component $\mathcal{L}_i$ in the original link $\mathcal{L}$ has k_i duplicated components named $i1, i2, \ldots, ik_i$ in the duplicated link and the duplicated link has $\sum_{i=1}^{L} k_i$ components. We would like to get P-IMAT of ij_i component for each i and j_i. It is obvious that the cross points in the duplicated component ij_i are all induced points whose source points belong to the i-th component in the original link. Besides, the induced points with the same source point are successive in certain order on account of proposition 4.8. This leads to a way to get the P-IMAT of the duplicated link.

Let $\mathcal{M}_{\mathcal{L}}$ be the IMAT of a link and $((A_1)^{k_1} \cdots, (A_1)^{k_L})$ be its duplicated decoration. The matrix shown in equation 13 represents the i-th component.

$$\begin{pmatrix} \alpha_1 & ud_1 & pn_1 \\ \alpha_2 & ud_2 & pn_2 \\ \vdots & \vdots & \vdots \\ \alpha_r & ud_r & pn_r \\ \vdots & \vdots & \vdots \\ \alpha_{n_i} & ud_{n_i} & pn_{n_i} \end{pmatrix} \tag{13}$$

As we have mentioned in section 3.3, for each cross point in position (i, r), we can get the pair position (cn_r, cr_r) by searching the names in IMAT. Component ij_i in the duplicated link has $\sum_{s=1}^{n_i} k_{cn_r}$ induced cross points. All these induced cross points belong to n_i kinds, and each of the induced cross points has the same source point in its kind. The r-th kind has k_{cn_r} induced cross points whose source point is α_r. These k_{cn_r} induced points are successive in the ij_i-th component following the order illustrated in proposition 4.8. We use TVO to name all the cross points. Therefore, we have proposition 4.11.

Proposition 4.11. *Let $\mathcal{L}$ be a link with L components, $\mathcal{M}_{\mathcal{L}}$ be the IMAT of $\mathcal{L}$, and $((A_1)^{k_1}, \cdots, (A_1)^{k_L})$ be its duplication. A cross point (α_r, ud_r, pn_r) in position (i, r) has k_{cn_r} induced cross points. The sub-matrix is as follows, representing these k_{cn_r} induced cross points in the ij_i-th P-IMAT of the duplicated link.*

- *if $ud > 0$ and $pn > 0$*

$$\mathcal{MI}(i, j_i, r) = \begin{pmatrix} \alpha_r & j_i & 1 & ud_r & pn_r \\ \alpha_r & j_i & 2 & ud_r & pn_r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_r & j_i & k_{cn_r} & ud_r & pn_r \end{pmatrix}.$$

- *if $ud > 0$ and $pn < 0$*

$$\mathcal{MI}(i, j_i, s_i) = \begin{pmatrix} \alpha_r & j_i & k_{cn_r} & ud_r & pn_r \\ \alpha_r & j_i & k_{cn_r} - 1 & ud_r & pn_r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_r & j_i & 1 & ud_r & pn_r \end{pmatrix}.$$

- *if* $ud < 0$ *and* $pn > 0$

$$\mathcal{MI}(i, j_i, s_i) = \begin{pmatrix} \alpha_r & k_{cn_r} & j_i & ud_r & pn_r \\ \alpha_r & k_{cn_r} - 1 & j_i & ud_r & pn_r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_r & 1 & j_i & ud_r & pn_r \end{pmatrix}.$$

- *if* $ud < 0$ *and* $pn < 0$

$$\mathcal{MI}(i, j_i, s_i) = \begin{pmatrix} \alpha_r & 1 & j_i & ud_r & pn_r \\ \alpha_r & 2 & j_i & ud_r & pn_r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_r & k_{cn_r} & j_i & ud_r & pn_r \end{pmatrix}.$$

We apply proposition 4.11 to each cross point (α_r, ud_r, pn_r) in the i-th component, where $1 \le r \le n_i$. This brings about the P-IMAT of the ij_i-th component in the duplicated link. Hence, we have proposition 4.12.

Proposition 4.12. *The P-IMAT (pseudo intrinsic matrices) of the* ij_i*-th component in the duplicated link* $\mathcal{MD}$ *is as follows.*

$$\mathcal{PMI}(i, j_i) = \begin{pmatrix} \mathcal{MI}(i, j_i, 1) \\ \mathcal{MI}(i, j_i, 2) \\ \vdots \\ \mathcal{MI}(i, j_i, n_i) \end{pmatrix}$$

We calculate the P-IMAT of the ij_i-th component for all $1 \le i \le L$, $1 \le j_i \le k_i$. Then we get the P-IMAT for the duplicated link. Renaming all the cross points one by one, we finally acquire the IMAT for the duplicated link.

Example 4.13. We are to calculate the P-IMAT of the duplicated link illustrated in example 4.10. The IMAT of the original link is:

$$\mathcal{M}_1 = \begin{pmatrix} 1 & -1 & +1 \\ 2 & +1 & +1 \end{pmatrix} \mathcal{M}_2 = \begin{pmatrix} 1 & +1 & +1 \\ 2 & -1 & +1 \end{pmatrix}$$

For $i = 1, 2$, $j_1 = 1, 2$, $j_2 = 1, 2, 3$, $r_1 = 1, 2$, $r_2 = 1, 2$ we compute $M(i, j_i, r_i)$:

$$\mathcal{MI}(1,1,1) = \begin{pmatrix} 1 & 3 & 1 & -1 & +1 \\ 1 & 2 & 1 & -1 & +1 \\ 1 & 1 & 1 & -1 & +1 \end{pmatrix} \mathcal{MI}(1,1,2) = \begin{pmatrix} 2 & 1 & 1 & +1 & +1 \\ 2 & 1 & 2 & +1 & +1 \\ 2 & 1 & 3 & +1 & +1 \end{pmatrix}$$

$$\mathcal{MI}(1,2,1) = \begin{pmatrix} 1 & 3 & 2 & -1 & +1 \\ 1 & 2 & 2 & -1 & +1 \\ 1 & 1 & 2 & -1 & +1 \end{pmatrix} \mathcal{MI}(1,2,2) = \begin{pmatrix} 2 & 2 & 1 & +1 & +1 \\ 2 & 2 & 2 & +1 & +1 \\ 2 & 2 & 3 & +1 & +1 \end{pmatrix}$$

$$\mathcal{MI}(2,1,1) = \begin{pmatrix} 1 & 1 & 1 & +1 & +1 \\ 1 & 1 & 2 & +1 & +1 \end{pmatrix} \mathcal{MI}(2,1,2) = \begin{pmatrix} 2 & 2 & 1 & -1 & +1 \\ 2 & 1 & 1 & -1 & +1 \end{pmatrix}$$

$$\mathcal{MI}(2,2,1) = \begin{pmatrix} 1 & 2 & 1 & +1 & +1 \\ 1 & 2 & 2 & +1 & +1 \end{pmatrix} \mathcal{MI}(2,2,2) = \begin{pmatrix} 2 & 2 & 2 & -1 & +1 \\ 2 & 1 & 2 & -1 & +1 \end{pmatrix}$$

$$\mathcal{MI}(2,3,1) = \begin{pmatrix} 1 & 3 & 1 & +1 & +1 \\ 1 & 3 & 2 & +1 & +1 \end{pmatrix} \mathcal{MI}(2,3,2) = \begin{pmatrix} 2 & 2 & 3 & -1 & +1 \\ 2 & 1 & 3 & -1 & +1 \end{pmatrix}$$

So the P-IMAT of duplicated hopf link shown in figure 23.

$$\mathcal{PMI}(1,1) = \begin{pmatrix} 1 & 3 & 1 & -1 & +1 \\ 1 & 2 & 1 & -1 & +1 \\ 1 & 1 & 1 & -1 & +1 \\ 2 & 1 & 1 & +1 & +1 \\ 2 & 1 & 2 & +1 & +1 \\ 2 & 1 & 3 & +1 & +1 \end{pmatrix} \mathcal{PMI}(1,2) = \begin{pmatrix} 1 & 3 & 2 & -1 & +1 \\ 1 & 2 & 2 & -1 & +1 \\ 1 & 1 & 2 & -1 & +1 \\ 2 & 2 & 1 & +1 & +1 \\ 2 & 2 & 2 & +1 & +1 \\ 2 & 2 & 3 & +1 & +1 \end{pmatrix}$$

$$\mathcal{PMI}(2,1) = \begin{pmatrix} 1 & 1 & 1 & +1 & +1 \\ 1 & 1 & 2 & +1 & +1 \\ 2 & 2 & 1 & -1 & +1 \\ 2 & 1 & 1 & -1 & +1 \end{pmatrix} \mathcal{PMI}(2,2) = \begin{pmatrix} 1 & 2 & 1 & +1 & +1 \\ 1 & 2 & 2 & +1 & +1 \\ 2 & 2 & 2 & -1 & +1 \\ 2 & 1 & 2 & -1 & +1 \end{pmatrix}$$

$$\mathcal{PMI}(2,3) = \begin{pmatrix} 1 & 3 & 1 & +1 & +1 \\ 1 & 3 & 2 & +1 & +1 \\ 2 & 2 & 3 & -1 & +1 \\ 2 & 1 & 3 & -1 & +1 \end{pmatrix}$$

Renaming the components and the cross points, we get the IMAT of the duplicated link shown in figure 24.

$$\begin{pmatrix} 1 & -1 & +1 \\ 2 & -1 & +1 \\ 3 & -1 & +1 \\ 4 & +1 & +1 \\ 5 & +1 & +1 \\ 6 & +1 & +1 \end{pmatrix} \begin{pmatrix} 7 & -1 & +1 \\ 8 & -1 & +1 \\ 9 & -1 & +1 \\ 10 & +1 & +1 \\ 11 & +1 & +1 \\ 12 & +1 & +1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & +1 & +1 \\ 9 & +1 & +1 \\ 10 & -1 & +1 \\ 4 & -1 & +1 \end{pmatrix} \begin{pmatrix} 2 & +1 & +1 \\ 8 & +1 & +1 \\ 11 & -1 & +1 \\ 5 & -1 & +1 \end{pmatrix} \begin{pmatrix} 1 & +1 & +1 \\ 7 & +1 & +1 \\ 12 & -1 & +1 \\ 6 & -1 & +1 \end{pmatrix}$$

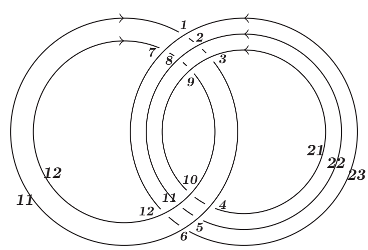


Figure 24: Renaming the induced cross points to get the IMAT for the duplicated link.

4.2.2. Structure points and Turaev decorated link.

Definition 4.14 (structure points). Let $\mathcal{L}$ be a link with L components and let $(\lambda^1, \cdots, \lambda^L)$ be the Turaev decoration. The Turaev decorated link $\mathcal{L}_{A_{\lambda^1}, A_{\lambda^2}, \cdots, A_{\lambda^L}}$ has a corresponding duplicated link $\mathcal{L}_{(A_1)^{k_1},(A_1)^{k_2}\cdots,(A_1)^{k_L}}$, where $|\lambda^i|= k_i$ for $1 \leq i \leq L$. A cross point in a Turaev decorated link is a structure point if it has no source point. Structure points belong to the decoration of the link.

A simple example of structure points is given in example 4.4.

Here we propose a way to name the structure points. Let $(\lambda^1, \cdots, \lambda^L)$ be a Turaev decoration. A structure point in structure A_{λ^i} is named by three integers $(-i, p_i, q_{p_i})$. Here "$-$" means this is a structure point rather than an induced point. Integer i means the cross point belongs to the structure A_{λ^i} which is the decoration of the i-th component of the original link. Integer p_i means that the cross point belongs to structure $A_{\lambda^i_p}$. The total number of the structure points belonging to structure $A_{\lambda^i_p}$ is $\lambda^i_p - 1$. We number them $1, 2, \ldots, \lambda^i_p - 1$ and employ $1 \leq q_{p_i} \leq \lambda^i_p - 1$ to denote these numbers. Here λ^i_p is the p-th integer in partition λ^i. We call this procedure the travel order method to name the structure points, still abbreviated to TVO.

Example 4.15. If the i-th component is anti-clockwise and the i-th decoration partition is $\lambda^i = (3, 2)$, $\lambda^i_1 = 3$, $\lambda^i_2 = 2$, the i-th component in the original link is decorated by A_3A_2. There are two kinds of structure points. The 1-th and 2-th structure points belong to structure A_3, and the 3-th structure point belongs to structure A_2, shown in figure 25. For a cross point named $(-i, 1, 2)$, "$-$" means that the cross point is a structure point, "i" indicates that the structure point belongs to the decoration of the i-th component, "1" means that the cross point belongs to A_3 (the 1-th part of decoration, $\lambda^i_1 = 3$), and 2 means that the cross point is the 2-th cross point in structure A_3.

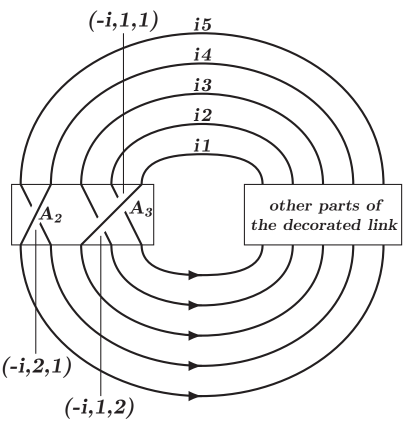


Figure 25: Naming the structure points by TVO.

Let $\mathcal{L}$ be a link with L component and $(\lambda^1, \cdots, \lambda^L)$ be its Turaev decoration. The corresponding duplication is $(|\lambda^1|, |\lambda^2|, \ldots, |\lambda^L|)$. By simple observation, we see that the Turaev decorated link consists of $\sum_{i=1}^{L} \ell(\lambda^i)$ components. There are L kinds of components in the decorated link, each of which has $\ell(\lambda^i)$ components, where $1 \leq i \leq L$. The

p-th P-IMAT of the component of the i-th kind can be obtained by combining the structure points in $A_{\lambda_p^i}$ and the P-IMAT of the $|\lambda_p^i|$ corresponding duplicated components $\{(i, \sum_{x=1}^{p-1})|\lambda_x^i|+1), \cdots, (i, \sum_{x=1}^{p}|\lambda_x^i|)\}$, where $1 \le p \le \ell(\lambda^i)$. Here $\ell(\lambda)$ refers to the length of the partition λ and $|\lambda|$ refers to the weight of the partition [8].

Proposition 4.16. *Given the IMAT for the link diagram $\mathcal{L}$ with L components and its Turaev's decoration $(\lambda^1, \cdots, \lambda^L)$, we obtain the P-IMAT of the p_i-th component of the i-th kind:*

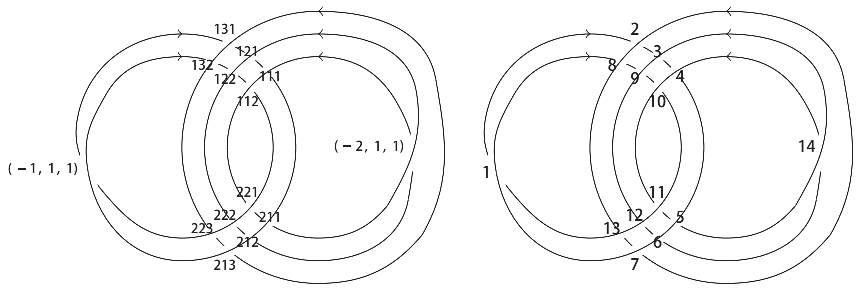
$$\mathcal{MP}(i, p_i) = \begin{pmatrix} -i & p_i & 1 & +1 & +1 \\ -i & p_i & 2 & +1 & +1 \\ & & \vdots & & \\ -i & p_i & |\lambda_{p_i}^i|-1 & +1 & +1 \\ \multicolumn{5}{c}{\mathcal{PMI}(i, \sum_{x=1}^{p_i-1}|\lambda_x^i|+|\lambda_{p_i}^i|)} \\ -i & p_i & |\lambda_{p_i}^i|-1 & -1 & +1 \\ \multicolumn{5}{c}{\mathcal{PMI}(i, \sum_{x=1}^{p_i-1}|\lambda_x^i|+|\lambda_{p_i}^i|-1)} \\ -i & p_i & |\lambda_{p_i}^i|-2 & -1 & +1 \\ & & \vdots & & \\ \multicolumn{5}{c}{\mathcal{PMI}(i, \sum_{x=1}^{p_i-1}|\lambda_x^i|+2)} \\ -i & p_i & 1 & -1 & +1 \\ \multicolumn{5}{c}{\mathcal{PMI}(i, \sum_{x=1}^{p_i-1}|\lambda_x^i|+1)} \end{pmatrix}.$$

Here $1 \le p_i \le \ell(\lambda^i)$ for each $1 \le i \le L$.

So far, we have the P-IMAT for the Turaev decorated link. By renaming the cross points, we easily get its IMAT.

Example 4.17. We employ the same example as the one shown in figure 5 to calculate its IMAT. The decorated hopf link $\mathcal{L}_{A_2, A_2 A_1}$ whose Turaev's decoration is represented by partitions $\lambda^1 = (2)$ and $\lambda^2 = (2, 1)$ has a corresponding duplicated link $\mathcal{L}_{(A_1)^2, (A_1)^3}$. The P-IMATs for the duplicated components $\mathcal{PMI}(i, j_i)$ are shown in example 4.13.

As is shown in figure 26(a), the P-IMAT for the Turaev decorated link is constituted of two kinds of components, the first of which contains one component and the second contains two components.



(a) P-IMAT for the decorated link (b) IMAT for the decorated link

Figure 26: The P-IMAT/IMAT for the Turaev decorated link is in fact a recombination of the structure points and the induced points.

For $i = 1, 2$, $p_1 = 1$, $p_2 = 1, 2$ we have:

$$\mathcal{MP}(1,1) = \begin{pmatrix} -1 & 1 & 1 & +1 & +1 \\ & & \mathcal{PMI}(1,2) & & \\ -1 & 1 & 1 & -1 & +1 \\ & & \mathcal{PMI}(1,1) & & \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & +1 & +1 \\ 1 & 3 & 2 & -1 & +1 \\ 1 & 2 & 2 & -1 & +1 \\ 1 & 1 & 2 & -1 & +1 \\ 2 & 2 & 1 & +1 & +1 \\ 2 & 2 & 2 & +1 & +1 \\ 2 & 2 & 3 & +1 & +1 \\ -1 & 1 & 1 & -1 & +1 \\ 1 & 3 & 1 & -1 & +1 \\ 1 & 2 & 1 & -1 & +1 \\ 1 & 1 & 1 & -1 & +1 \\ 2 & 1 & 1 & +1 & +1 \\ 2 & 1 & 2 & +1 & +1 \\ 2 & 1 & 3 & +1 & +1 \end{pmatrix},$$

$$\mathcal{MP}(2,1) = \begin{pmatrix} -2 & 1 & 1 & +1 & +1 \\ & & \mathcal{PMI}(2,2) & & \\ -2 & 1 & 1 & -1 & +1 \\ & & \mathcal{PMI}(2,1) & & \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 & +1 & +1 \\ 1 & 2 & 1 & +1 & +1 \\ 1 & 2 & 2 & +1 & +1 \\ 2 & 2 & 2 & -1 & +1 \\ 2 & 1 & 2 & -1 & +1 \\ -2 & 1 & 1 & -1 & +1 \\ 1 & 1 & 1 & +1 & +1 \\ 1 & 1 & 2 & +1 & +1 \\ 2 & 2 & 1 & -1 & +1 \\ 2 & 1 & 1 & -1 & +1 \end{pmatrix},$$

$$\mathcal{MP}(2,2) = \begin{pmatrix} \mathcal{PMI}(2,3) \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & +1 & +1 \\ 1 & 3 & 2 & +1 & +1 \\ 2 & 2 & 3 & -1 & +1 \\ 2 & 1 & 3 & -1 & +1 \end{pmatrix}.$$

Renaming the component and the cross points, we finally get the IMAT for the Turaev decorated hopf link $\mathcal{L}_{A_2,A_2A_1}$:

$$\mathcal{M}_1 = \begin{pmatrix} 1 & +1 & +1 \\ 2 & -1 & +1 \\ 3 & -1 & +1 \\ 4 & -1 & +1 \\ 5 & +1 & +1 \\ 6 & +1 & +1 \\ 7 & +1 & +1 \\ 1 & -1 & +1 \\ 8 & -1 & +1 \\ 9 & -1 & +1 \\ 10 & -1 & +1 \\ 11 & +1 & +1 \\ 12 & +1 & +1 \\ 13 & +1 & +1 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 14 & +1 & +1 \\ 9 & +1 & +1 \\ 3 & +1 & +1 \\ 6 & -1 & +1 \\ 12 & -1 & +1 \\ 14 & -1 & +1 \\ 10 & +1 & +1 \\ 4 & +1 & +1 \\ 5 & -1 & +1 \\ 11 & -1 & +1 \end{pmatrix}, \mathcal{M}_3 = \begin{pmatrix} 8 & +1 & +1 \\ 2 & +1 & +1 \\ 7 & -1 & +1 \\ 13 & -1 & +1 \end{pmatrix}.$$

The final result is shown in figure 26(b).

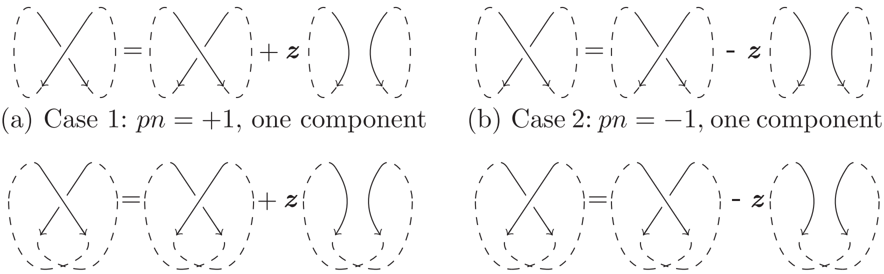
5. Calculating homflypt polynomials by Skein-Template Algorithm with IMAT

Given the intrinsic matrices for a link diagram $\mathcal{L}$, we propose an auto computational calculation for $\langle\mathcal{L}\rangle$ by using IMAT. Following [5, 7, 12], we introduce a way to apply the skein-template algorithm (STA) to IMAT. First, we show how to use skein relation by IMAT in section 2.1. Then we employ STA on IMAT in section 5.2. In the end, we use Reidemeister move to improve the algorithm so that the algorithm can be more efficient.

5.1. Employing skein relation on IMAT

We employ skein relation on IMAT by cutting and recombining the IMAT. Each time, employing skein relation on an IMAT ends up with two IMATs.

Given an IMAT and a chosen position, we would like to apply skein relation shown in figure 1 to a cross point in order to change the up-down information of the cross point. If the cross point in the chosen position is formed by string 1 and string 2, in view of the positive-negative information and the belongings of the two strings, there are four cases when we employ the skein relation shown in figure 27. The first is that string 1 and 2 belong to the same component and the chosen cross point is positive. The second is that string 1 and 2 belong to the same component and the chosen cross point is negative. The third is that string 1 and 2 belong to different components and the chosen cross point is positive. The fourth is that string 1 and 2 belong to different components and the chosen cross point is negative. According to the four cases, we have the following equation, where α is the name of the chosen cross point.



(a) Case 1: $pn = +1$, one component (b) Case 2: $pn = -1$, one component

(c) Case 3: $pn = +1$, two components (d) Case 4: $pn = -1$, two component

Figure 27: The four cases when we apply skein relation to a cross point.

- case 1:

$$\left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & +1 & +1 \\ & B & \\ \alpha & -1 & +1 \\ & C & \end{pmatrix}, \cdots \right\rangle = \left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & -1 & -1 \\ & B & \\ \alpha & +1 & -1 \\ & C & \end{pmatrix}, \cdots \right\rangle$$

$$+ z \left\langle \cdots, \begin{pmatrix} A \\ C \end{pmatrix}, \cdots, \begin{pmatrix} B \end{pmatrix} \right\rangle.$$

- case 2:

$$\left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & -1 & -1 \\ & B & \\ \alpha & +1 & -1 \\ & C & \end{pmatrix}, \cdots \right\rangle = \left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & +1 & +1 \\ & B & \\ \alpha & -1 & +1 \\ & C & \end{pmatrix}, \cdots \right\rangle$$

$$- z \left\langle \cdots, \begin{pmatrix} A \\ C \end{pmatrix}, \cdots, \begin{pmatrix} B \end{pmatrix} \right\rangle.$$

- case 3:

$$\left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & +1 & +1 \\ & B & \end{pmatrix}, \cdots, \begin{pmatrix} & C & \\ \alpha & -1 & +1 \\ & D & \end{pmatrix}, \cdots \right\rangle$$

$$= \left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & -1 & -1 \\ & B & \end{pmatrix}, \cdots, \begin{pmatrix} & C & \\ \alpha & +1 & -1 \\ & D & \end{pmatrix}, \cdots \right\rangle + z \left\langle \cdots, \begin{pmatrix} A \\ D \\ C \\ B \end{pmatrix} \right\rangle.$$

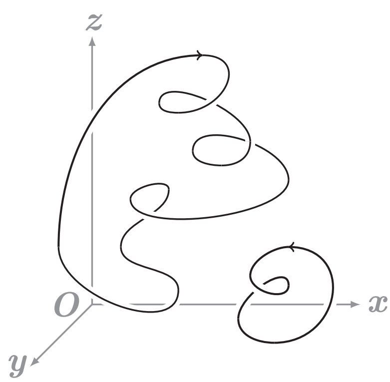
- case 4:

$$\left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & +1 & -1 \\ & B & \end{pmatrix}, \cdots, \begin{pmatrix} & C & \\ \alpha & -1 & -1 \\ & D & \end{pmatrix}, \cdots \right\rangle$$

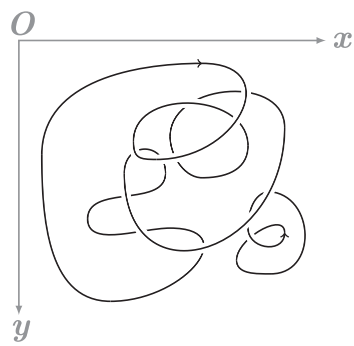
$$= \left\langle \cdots, \begin{pmatrix} & A & \\ \alpha & -1 & +1 \\ & B & \end{pmatrix}, \cdots, \begin{pmatrix} & C & \\ \alpha & +1 & +1 \\ & D & \end{pmatrix}, \cdots \right\rangle - z \left\langle \cdots, \begin{pmatrix} A \\ D \\ C \\ B \end{pmatrix} \right\rangle.$$

5.2. Employing skein Template Algorithm on IMAT

The key point of skein-template algorithm (STA) is that you will always draw unknots if you follow the rule: along the orientation of the link, for each pair of cross points, you always reach the upper one and then its nether pair. This is because that the strings with different vertical coordinate value cannot be indeed tangled with each other, which is illustrated in figure 28. The first skein model for STA was given by Francois Jaeger [5] and L. H. Kauffman showed how to interpret and generalize this model as a direct consequence of skein calculation [12]. In [7], G. Gouesbet, S. Meunier-Guttin-Cluzel and C.Letellier realized SAT with microcomputer programming. We make use of the main idea in the STA and calculate the homflypt polynomial for the link with IMAT. We also show that IMAT can be used to recognize certain structure in the link diagram. Specifically, we use Reidemeister move to make the STA more efficient.



(a) The link embedded in $\mathbb{R}^3$



(b) The link diagram in $\mathbb{R}^2$

Figure 28: Along the orientation of the link, for each pair of cross points, you always reach the upper one and then its nether pair. This is because that the strings with different z coordinate values cannot be indeed tangled with each other.

Our goal is to change the up-down information for the unhandled nether cross points by using skein relation so that the link diagram follow the rules we mentioned before. We attach a value mk to each row in IMAT to mark the handled cross points. Here, we set $mk = 1$ for the unhandled cross points and $mk = 0$ for the handled cross point. Then, we deal with the cross points one by one until all the cross points are handled, which means that all the components in the link diagrams are unknots. In fact, if all the components of a link are unknots, we have equation 14.

$$\langle \mathcal{L}^u \rangle = v^w (\frac{v - v^{-1}}{s - s^{-1}})^L. \tag{14}$$

Here w is the writhe number. We omit the proof of this equation. Therefore, it is possible for us to use STA to compute homflypt invariants. The following is an example of calculating homflypt invariant by applying STA to IMAT.

Example 5.1. Given the IMAT for the hopf link, we add a column to the IMAT and get the marked IMAT which is still denoted by $\mathcal{M}_i$.

$$\mathcal{M}_1 = \begin{pmatrix} 1 & -1 & +1 & 1 \\ 2 & +1 & +1 & 1 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 1 & +1 & +1 & 1 \\ 2 & -1 & +1 & 1 \end{pmatrix}.$$

We choose the cross point in position $(1,1)$ to start the iteration. The name of the cross point is 1. Here $\mathcal{M}_1(1,2) = -1$ means the cross point is a nether one. So we apply skein relation to this cross point to change it to a upper one. The pair cross point for $(1,-1,+1,1)$ is in position $(2,1)$. Therefore, the two strings that form the chosen cross point belong to different components. $\mathcal{M}_1(1,3) = +1$ means the chosen cross point is positive. Accordingly, the cross point $(1,-1,+1,1)$ belongs to case 3 in section 5.1. By using skein relation and marking the cross point as handled $mk = 0$, we have

$$\begin{aligned}\langle \mathcal{M} \rangle &= \left\langle \begin{pmatrix} 1 & -1 & +1 & 1 \\ 2 & +1 & +1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & +1 & +1 & 1 \\ 2 & -1 & +1 & 1 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 1 & +1 & -1 & 0 \\ 2 & +1 & +1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 0 \\ 2 & -1 & +1 & 1 \end{pmatrix} \right\rangle \\ &\quad + (s - s^{-1}) \left\langle \begin{pmatrix} 2 & -1 & +1 & 1 \\ 2 & +1 & +1 & 1 \end{pmatrix} \right\rangle \\ &= \langle \mathcal{M}^{\mathcal{L}^+} \rangle + (s - s^{-1}) \langle \mathcal{M}^{\mathcal{L}^{||}} \rangle.\end{aligned}$$

Then we deal with $\mathcal{M}^{\mathcal{L}^+}$ and $\mathcal{M}^{\mathcal{L}^{||}}$.

For $\mathcal{M}^{\mathcal{L}^+}$, the cross point next to $(1,+1,-1,0)$ along its orientation is in position $(1,2)$. We now handle this cross point and the one in its pair position $(2,2)$. Since $\mathcal{M}_1^{\mathcal{L}^+}(2,2) = +1$, the cross point is an upper cross point. So we do nothing about the cross point, and mark it as the handled point. Then we have:

$$\begin{aligned}\langle \mathcal{M}^{\mathcal{L}^+} \rangle &= \left\langle \begin{pmatrix} 1 & +1 & -1 & 0 \\ 2 & +1 & +1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 0 \\ 2 & -1 & +1 & 1 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 1 & +1 & -1 & 0 \\ 2 & +1 & +1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 0 \\ 2 & -1 & +1 & 0 \end{pmatrix} \right\rangle \\ &= \langle \mathcal{M}^{end,1} \rangle.\end{aligned}$$

All the cross points in $\langle \mathcal{M}^{end,1} \rangle$ are handled.

For $\mathcal{M}_{\mathcal{L}^{||}}$, the cross point next to the previous chosen point is in position $(1,2)$ in $\mathcal{M}_{\mathcal{L}^{||}}$. So we now handle the cross point in position $(1,2)$ of $\mathcal{M}_{\mathcal{L}^{||}}$. This is a nether cross point, so we apply skein relation to the cross point and then we have:

$$\langle \mathcal{M}^{\mathcal{L}^{||}} \rangle = \left\langle \begin{pmatrix} 2 & -1 & +1 & 1 \\ 2 & +1 & +1 & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 & +1 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{pmatrix} \right\rangle + (s - s^{-1}) \left\langle \Theta, \Theta \right\rangle.$$

Here Θ refers to the empty matrix. All the cross points in $\mathcal{M}_{\mathcal{L}^{||}}$ have been handled. Thus,

$$\langle \mathcal{M}^{end,2} \rangle = \left\langle \begin{pmatrix} 2 & +1 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{pmatrix} \right\rangle, \langle \mathcal{M}^{end,3} \rangle = \left\langle \Theta, \Theta \right\rangle.$$

We have:

$$\langle \mathcal{M} \rangle = \langle \mathcal{M}^{end,1} \rangle + (s - s^{-1})(\langle \mathcal{M}^{end,2} \rangle + (s - s^{-1}) \langle \mathcal{M}^{end,3} \rangle).$$

Using equation 14, we compute $\mathcal{M}^{end,i}$, $i = 1, 2, 3$. For instance, we calculate the writhe number

$$w(\mathcal{L}^{end,1}) = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\mathcal{M}_i^{end,1}(j,3) = 0,$$

and the number of component in $\mathcal{L}^{end,1}$ which is obvious 2. Hence,

$$\langle\mathcal{M}^{end,1}\rangle = v^0(\frac{v - v^{-1}}{s - s^{-1}})^2.$$

As a result, we obtain:

$$\langle\mathcal{M}\rangle = v^0(\frac{v - v^{-1}}{s - s^{-1}})^2 + (s - s^{-1})v^{-1}\frac{v - v^{-1}}{s - s^{-1}} + (s - s^{-1})^2 v^0(\frac{v - v^{-1}}{s - s^{-1}})^2.$$

This is the framing-dependent invariant for the hopf link. Then we calculate the writhe number

$$w(\mathcal{L}) = \frac{1}{2}\sum_{i=1}^{2}\sum_{s=1}^{2}\mathcal{M}_i(s,3) = 2,$$

and by

$$P_{\mathcal{L}}(s,v) = \frac{v^{-w(\mathcal{L})}\langle\mathcal{L}\rangle}{\langle U\rangle},$$

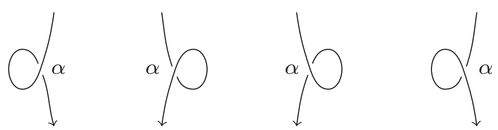
We have:

$$P_{\{\mathcal{M}_1,\mathcal{M}_2\}}(s,v) = \frac{s^2 - (v + v^{-1})v^{-1} + s^{-2}}{v(s - s^{-1})}.$$

5.3. Improved STA: recognize RI and RII structures in IMAT

In this section, we introduce a few ways to recognize RI and RII structures in an IMAT to make the STA more efficient. In our experiment, by recognizing RI and RII moves, it takes only two-thirds of the previous time to get the final result.

5.3.1. Reidemeister move I. We search the IMAT for structures shown in figure 29. In an IMAT, these structures are sub matrices illustrated in equation 15 and 16. Then,



(a) $\mathcal{R}_{I1}$ (b) $\mathcal{R}_{I2}$ (c) $\mathcal{R}_{I3}$ (d) $\mathcal{R}_{I4}$

Figure 29: The four structure for RI cross point.

we use the positive-negative information to get the coefficient for the simplification.

$$\mathcal{R}_{I1} = \begin{pmatrix} \alpha & +1 & +1 \\ \alpha & -1 & +1 \end{pmatrix} \mathcal{R}_{I2} = \begin{pmatrix} \alpha & -1 & +1 \\ \alpha & +1 & +1 \end{pmatrix} \tag{15}$$

$$\mathcal{R}_{I3} = \begin{pmatrix} \alpha & +1 & -1 \\ \alpha & -1 & -1 \end{pmatrix} \mathcal{R}_{I4} = \begin{pmatrix} \alpha & -1 & -1 \\ \alpha & +1 & -1 \end{pmatrix} \tag{16}$$

Owing to skein relation, we have equation 17.

$$\left\langle \cdots, \begin{pmatrix} A \\ \mathcal{R}_{Ii} \\ B \end{pmatrix}, \cdots \right\rangle = v^{\mathcal{R}_{Ii}(1,3)} \left\langle \cdots, \begin{pmatrix} A \\ B \end{pmatrix}, \cdots \right\rangle \tag{17}$$

5.3.2. Reidemeister move II. Our goal is to recognize structures shown in figure 30. First, we search the IMAT for sub matrices shown in equation 18.

(a) $ud = +1$ (b) $ud = -1$

Figure 30: The RII structure $\mathcal{R}_{II}$.

$$\overline{\mathcal{R}_{II}} = \begin{pmatrix} \alpha_1 & ud & pn \\ \alpha_2 & ud & -pn \end{pmatrix} \tag{18}$$

Then we get the pair position (cn_1, cr_1) for the cross (α_1, ud, pn) and pair position (cn_2, cr_2) for the cross $(\alpha_2, ud, -pn)$. If $cn_1 = cn_2$ and $|cr_1 - cr_2| = 1$, then the RII structure is a sub-matrix $\mathcal{R}_{II} = \overline{\mathcal{R}_{II}}$. Accordingly, we have equation 19

$$\left\langle \cdots, \begin{pmatrix} A \\ \mathcal{R}_{II} \\ B \end{pmatrix}, \cdots, \begin{pmatrix} C \\ \mathcal{R}_{II} \\ D \end{pmatrix}, \cdots \right\rangle = \left\langle \cdots, \begin{pmatrix} A \\ B \end{pmatrix}, \cdots, \begin{pmatrix} C \\ D \end{pmatrix}, \cdots \right\rangle. \tag{19}$$

6. Computer programming, problems and discussion

All the processes in the article have been implemented in computer programs by using MATLAB language. The programs can run on PC computer smoothly. Some results of the programs have already been shown in the previous sections in our description.

We end this paper with some future directions and challenges.

It should be noted that our algorithm introduced in section 5 still has some limitations. It is not quite efficient. For some complex link diagrams with more than 30 cross points, the program cannot get the result even in a whole day. This problem could be improved if we change the language of the programs into C language. But unfortunately, even in C language, the algorithm still cannot deal with link diagrams with too many cross points because the time it needs to get the result grows exponentially with the increase of the number of the cross points. We will continue working on this problem in the future. Specifically, we may optimize the algorithm by recognizing some known structures.

Despite the drawbacks, the framework for the algorithm is flexible and extendable with the available and simple inputs. In fact, one may even change the algorithm in section 5 into a more efficient algorithm to compute invariants for links of a certain type, maintaining the simplicity in inputs and accelerating the computing at the same time. Also, the framework for algorithm can be extended to calculate full colored homflypt invariants, Kauffman polynomial, and composite invariants. These are interesting problems that can be discussed further in the future.

References

[1] A. K. Aiston. *Skein theoretic idempotents of Hecke algebras and quantum group invariants*. PhD dissertation, University of Liverpool (1996).

[2] Mironov, Andrei, Alexei Morozov, Andrey Morozov. *On colored HOMFLY polynomials for twist knots*. Modern Physics Letters A, 29(34): 1450183 (2014).

[3] Khesin B, Arnold V I., *Topological fluid dynamics*. Notices AMS, 52: 9-19(2005).

[4] Qingtao Chen, Shengmao Zhu, *Full colored homflypt invariants, composite invariants and congruent skein relation*, arXiv:1410.2211 [math.QA] (2014).

[5] Jaeger F., *A combinatorial model for the Homfly polynomial*. European Journal of Combinatorics, 11(6): 549-557(1990).

[6] Lukac, S. G., *Homfly skeins and the Hopf link*. PhD. thesis, University of Liverpool(2001).

[7] Gouesbet G., Meunier-Guttin-Cluzel S, Letellier C. *Computer evaluation of Homfly polynomials by using Gauss Codes, with a skein-template algorithm*. Applied mathematics and computation, 105(2): 271-289(1999).

[8] Macdonald Ian Grant., *Symmetric functions and Hall polynomials*. Oxford university press(1998).

[9] Richatd J. Hadji, *Homfly skein theory of the reversed string satellites*, PhD thesis, University of Liverpool (2003).

[10] Richatd J. Hadji, H. R. Morton, *A basis for the full Homfly skein of the annulus*, Math. Proc. Camb. Philos. Soc. 141, 81-100 (2006). arXiv:math/0408078.

[11] Gu J, Jockers H., *A note on colored HOMFLY polynomials for hyperbolic knots from WZW models*. Communications in Mathematical Physics, 338(1): 393-456(2015).

[12] L H. Kauffman, *State models for link polynomials*. (1988).

[13] X S Lin, H. Zheng, *On the Hecke algebra and the colored HOMFLY polynomial*. Transactions of the American Mathematical Society, 362(1): 1-18 (2010).

[14] Kefeng Liu, Pan Peng, *New structures of knot invariants*. Commun. Number Theory Phys. 5 (2011), no. 3, 601?615.

[15] Mallam A L., Jackson S E., *Folding studies on a knotted protein*, Journal of molecular biology, 346(5): 1409-1421(2005).

[16] H. R. Morton, P. M. G. Manchón, *Geometrical relations and the plethysms in the Homfly skein of the annulus*, J. Lond. Math. Soc. 78, 305-328 (2008).

[17] H. R. Morton and A. K. Aiston, *Idempotents of Hecke algebras of type A*. Journal of Knot Theory and Its Ramifications, 7(04): 463-487(1998).

[18] Kosuda M, Murakami J, *Centralizer algebras of the mixed tensor representations of quantum group* $U_q(gl(n,\mathbb{C}))$, Osaka J. Math. 30 (1993), 475-507.

[19] N.Yu. Reshetikhin, V.G Turaev, *Ribbon graphs and their invaraints derived from quantum groups.* Communications in Mathematical Physics 127.1 (1990): 1-26.

[20] V. G. Turaev, *The Conway and Kauffman modules of solid torus*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167(1988), Issled. Topol. 6, 79-89.

[21] V. G. Turaev, *The Yang-Baxter equation and invariants of links.* Inventiones mathematicae, 92(3): 527-553(1988).

[22] Schmid F X., *How Proteins Knot Their Ties*, journal of molecular biology, 427(2): 225-227(2015).

[23] Akutsu Y, Wadati M.,*Knot invariants and the critical statistical systems.* Journal of the Physical Society of Japan, 56(3): 839-842(1987).

[24] Shengmao Zhu, *Colored HOMFLY polynomials via skein theory*, J. High Energy. Phys. 10(2013), 229.

[25] `https://en.wikipedia.org/wiki/Otsu%27s_method`

[26] `https://www.cs.auckland.ac.nz/courses/compsci773s1c/lectures/ImageProcessing-html/topic4.htm`

[27] `https://en.wikipedia.org/wiki/Mathematical_morphology`

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