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# Topological Characterization of an Asymptotic Teichmüller Space Through Measured Geodesic Laminations

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**Abstract:** Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and centered at the origin, and let $\mathcal{ML}_b(\mathbb{D})$ be the collection of Thurston bounded measured geodesic laminations on $\mathbb{D}$. We introduce an equivalence relation on $\mathcal{ML}_b(\mathbb{D})$ such that the earthquake measure map induces a bijection between the asymptotic Teichmüller space $AT(\mathbb{D})$ and the quotient space $\mathcal{AML}_b(\mathbb{D})$ of $\mathcal{ML}_b(\mathbb{D})$ under the equivalence relation. Furthermore, we introduce a topology on $\mathcal{AML}_b(\mathbb{D})$ under which the bijection is a homeomorphism between $AT(\mathbb{D})$ and $\mathcal{AML}_b(\mathbb{D})$ with respect to the Teichmüller metric on $AT(\mathbb{D})$. Corresponding results are also developed for a bijection and then a homeomorphism between the tangent space $\mathcal{AZ}(\mathbb{S}^1)$ of $AT(\mathbb{D})$ at a base point and $\mathcal{AML}_b(\mathbb{D})$ with respect to the asymptotic cross-ratio norm topology on $\mathcal{AZ}(\mathbb{S}^1)$ and the defined topology on $\mathcal{AML}_b(\mathbb{D})$.
**Keywords:** Earthquakes, Thurston bounded measured geodesic laminations, Teichmüller spaces and asymptotic Teichmüller spaces.

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and centered at the origin, and let $T(\mathbb{D})$ be the universal Teichmüller space and $\mathcal{ML}_b(\mathbb{D})$ the collection of Thurston bounded measured geodesic laminations on $\mathbb{D}$. By Thurston's earthquake theory [22], for each quasisymmetric homeomorphism $h$ of the

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unit circle $\mathbb{S}^1$, there is a unique Thurston bounded measured geodesic lamination $\lambda$ induced by any earthquake representation $E^\lambda$ of $h$. This correspondence introduces a bijection between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$, which is called *the earthquake measure map* and denoted by

$$\mathcal{EM} : T(\mathbb{D}) \to \mathcal{ML}_b(\mathbb{D}) : [h] \mapsto \lambda. \tag{1.1}$$

For background on earthquake representations of quasisymmetric maps, Thurston bounded measured geodesic laminations, and their relationships, and for further developments in Thurston's earthquake theory, we refer to [7], [10], [13], [16], [18], [20] and [21].

Furthermore, for any closed hyperbolic Riemann surface $S$, Kerckhoff [16] showed that the earthquake measure map $\mathcal{EM}$ (1.1) induces a homeomorphism between the Teichmüller space $T(S)$ of $S$ and the space $\mathcal{ML}_b(S)$ of measured geodesic laminations on $S$ with respect to the Teichmüller metric on $T(S)$ and a weak* topology on $\mathcal{ML}_b(S)$.

Lately, by introducing a uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$ (see Definition 3 in Section 2.5), Miyachi and Šarić [17] proved that the earthquake measure map $\mathcal{EM}$ is a homeomorphism between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ with respect to the Teichmüller metric on $T(\mathbb{D})$ and the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$. They also pointed out that since the invariance of a quasisymmetric map under a Fuchsian group implies the invariance of the corresponding measured lamination under the group, the same result holds for the restriction of $\mathcal{EM}$ between the Teichmüller space $T(S)$ of any geometrically infinite Riemann surface $S$ and its representation $\mathcal{ML}_b(S)$ by Thurston bounded measured geodesic laminations on $S$.

Let $\mathcal{Z}(\mathbb{S}^1)$ be the quotient space of Zygmund bounded continuous tangent vector fields on $\mathbb{S}^1$ modulo quadratic polynomials, which is the tangent space of the universal Teichmüller $T(\mathbb{D})$ space at the base point. Gardiner proved in [6] that given any element $V \in \mathcal{Z}(\mathbb{S}^1)$, there is a unique element $\lambda \in \mathcal{ML}_b(\mathbb{D})$ such that any infinitesimal earthquake map $\dot{E}^\lambda$ (see Section 7.1) satisfying $V = \dot{E}^\lambda|_{\mathbb{S}^1}$ modulo a quadratic polynomial. Therefore, there exists a bijection between $\mathcal{Z}(\mathbb{S}^1)$ and $\mathcal{ML}_b(\mathbb{D})$, which is called *the infinitesimal earthquake measure map* and denoted by

$$\dot{\mathcal{EM}} : \mathcal{Z}(\mathbb{S}^1) \to \mathcal{ML}_b(\mathbb{D}) : V \mapsto \lambda. \tag{1.2}$$

It is also proved in [17] that $\dot{\mathcal{EM}}$ is a homeomorphism with respect to the cross-ratio norm topology on $\mathcal{Z}(\mathbb{S}^1)$ (see Section 7.1) and the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$.

Let $T_0(\mathbb{D})$ be the subspace of $T(\mathbb{D})$ whose elements are represented by asymptotically conformal homeomorphisms of $\mathbb{D}$. The quotient space $T_0(\mathbb{D})\backslash T(\mathbb{D})$ is called the asymptotic Teichmüller space of quasiconformal homeomorphisms of $\mathbb{D}$, denoted by $AT(\mathbb{D})$. This space was studied by Gardiner and Sullivan in [8] and the asymptotic Teichmüller spaces of Riemann surfaces are studied in [2], [3], [5], [8] and etc.

The main work of this paper consists of the followings.

(1) Introduce an equivalence relation $\sim$ on $\mathcal{ML}_b(\mathbb{D})$ such that the earthquake measure map $\mathcal{EM}$ (1.1) induces a bijection $\widehat{\mathcal{EM}}$ between the asymptotic Teichmüller space $AT(\mathbb{D})$ and the quotient space $\mathcal{ML}_b(\mathbb{D})/\sim$.

(2) Prove that the bijection $\widehat{\mathcal{EM}}$ in (1) is a homeomorphism between $AT(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})/\sim$ with respect to the asymptotic Teichmüller metric on $AT(\mathbb{D})$ and a newly defined asymptotically uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})/\sim$.

(3) Show that the infinitesimal earthquake measure map $\dot{\mathcal{EM}}$ (1.2) induces a bijection and then a homeomorphism $\widehat{\dot{\mathcal{EM}}}$ between the tangent space $\mathcal{AZ}(\mathbb{S}^1)$ (see Section 7.3) of $AT(\mathbb{D})$ at a base point and $\mathcal{ML}_b(\mathbb{D})/\sim$ with respect to the asymptotic cross-ratio norm topology on $\mathcal{AZ}(\mathbb{S}^1)$ and the asymptotic uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})/\sim$.

Given a Thurston bounded measured geodesic lamination $\lambda$, we denote by $E^\lambda$ an earthquake map inducing $\lambda$ (see Section 2.3 for background) and by $[\lambda]$ the equivalence class of $\lambda$ (see Definition 4 in Section 3).

**Theorem 1.** *Given two points $[[h]]$ and $[[h']]$ in $AT(\mathbb{D})$, assume that $h = E^\lambda|_{\mathbb{S}^1}$ and $h' = E^{\lambda'}|_{\mathbb{S}^1}$. Then $[[h]] = [[h']]$ if and only if $[\lambda] = [\lambda']$.*

Now we let $\mathcal{AML}_b(\mathbb{D})$ be the quotient space of $\mathcal{ML}_b(\mathbb{D})$ under the equivalence relation. Theorem 1 implies that the earthquake measure map $\mathcal{EM}$ between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ induces a bijection between $AT(\mathbb{D})$ and $\mathcal{AML}_b(\mathbb{D})$, which is called *the induced earthquake measure map* and denoted by

$$\widehat{\mathcal{EM}} : AT(\mathbb{D}) \to \mathcal{AML}_b(\mathbb{D}) : [[h]] \mapsto [\lambda], \tag{1.3}$$

where $h = E^\lambda|_{\mathbb{S}^1}$.

Two metrics are commonly introduced on the asymptotic Teichmüller space $AT(\mathbb{D})$. As a quotient space, $AT(\mathbb{D})$ inherits a quotient metric from the Teichmüller metric on $T(\mathbb{D})$. Another metric is defined by using boundary dilatations. From [3] and [8], it is known that these two metrics are equal to each other. For this reason, they are simply called the asymptotic Teichmüller metric on $AT(\mathbb{D})$ and the topology induced by this metric is called the asymptotic Teichmüller topology on $AT(\mathbb{D})$.

Following the previous pattern in thinking, $\mathcal{AML}_b(\mathbb{D})$ first inherits a quotient topology from the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$ (see Definition 5 in Section 4). With respect to the quotient topologies on $AT(\mathbb{D})$ and $\mathcal{AML}_b(\mathbb{D})$, one can easily see that Theorem 1 and the homeomorphic property of $\mathcal{EM}$ between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ [17] imply the following corollary.

**Corollary 1.** *The induced earthquake measure map $\widehat{\mathcal{EM}}$ (1.3) is a homeomorphism with respect to the quotient topology on $AT(\mathbb{D})$ from the Teichmüller topology on $T(\mathbb{D})$ and the quotient topology on $\mathcal{AML}_b(\mathbb{D})$ from the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$.*

Corresponding to the other boundary-dilatation definition of the asymptotic Teichmüller topology on $AT(\mathbb{D})$, we introduce another topology, namely the asymptotic uniform weak* topology, on $\mathcal{AML}_b(\mathbb{D})$ (see Definition 8 in Section 6) and prove the following theorem.

**Theorem 2.** *The induced earthquake measure map $\widehat{\mathcal{EM}}$ (1.3) is a homeomorphism with respect to the asymptotic Teichmüller topology on $AT(\mathbb{D})$ and the asymptotic uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$.*

As a consequence of Theorem 2 and Corollary 1, we obtain the following.

**Corollary 2.** *The quotient topology on $\mathcal{AML}_b(\mathbb{D})$ from the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$) is equivalent to the asymptotically uniform weak* topology.*

Let $\mathcal{AZ}(\mathbb{S}^1)$ be the tangent space of the asymptotic Teichmüller space $AT(\mathbb{D})$ at a base point (see Section 7.3).

**Theorem 3.** *Given two points $[V]$ and $[V']$ in $\mathcal{AZ}(\mathbb{S}^1)$, assume that $V = \dot{E}^{\lambda}|_{\mathbb{S}^1}$ and $V' = \dot{E}^{\lambda'}|_{\mathbb{S}^1}$. Then $[V] = [V']$ if and only if $[\lambda] = [\lambda']$.*

It follows that the infinitesimal earthquake measure map $\mathcal{E}\dot{\mathcal{M}}$ (1.2) induces a bijection between $\mathcal{AZ}(\mathbb{S}^1)$ and $\mathcal{AML}_b(\mathbb{D})$, which is called *the induced infinitesimal earthquake measure map* and denoted by

$$\widehat{\mathcal{E}\dot{\mathcal{M}}} : \mathcal{AZ}(\mathbb{S}^1) \to \mathcal{AML}_b(\mathbb{D}) : [V] \mapsto [\lambda], \tag{1.4}$$

where $V = \dot{E}^{\lambda}|_{\mathbb{S}^1}$.

Under the asymptotic cross-ratio norm topology on $\mathcal{AZ}(\mathbb{S}^1)$ (see Section 7.6), we prove the following theorem.

**Theorem 4.** *The induced infinitesimal earthquake measure map* $\widehat{\dot{\mathcal{EM}}}$ *(1.4) is a homeomorphism with respect to the asymptotic cross-ratio norm topology on* $\mathcal{AZ}(\mathbb{S}^1)$ *and the asymptotic uniform weak* topology on* $\mathcal{AML}_b(\mathbb{D})$.

In the course of developing a proof of Theorem 2, we introduce and study the relationship between the asymptotic Thurston norm $||\lambda||_{\widehat{Th}}$ (see Definition 7 in Section 5) of a Thurston bounded measured geodesic lamination $\lambda$ and the strong asymptotic cross-ratio distortion norm $||h||_{\hat{\widehat{cr}}}$ (see Definition 6 in Section 5) of a quasisymmetric circle homeomorphism $h$. We obtain the following theorem.

**Theorem 5.** *Let* $h$ *be a quasisymmetric homeomorphism of* $\mathbb{S}^1$ *and* $\lambda_h$ *the measured geodesic lamination induced by an earthquake representation of* $h$. *There exists a universal constant* $C > 0$ *such that*

$$\|h\|_{\hat{\widehat{cr}}} \leq C\|\lambda_h\|_{\widehat{Th}}. \tag{1.5}$$

In Section 6, the previous theorem is used to prove the continuity of the inverse of the induced earthquake measure map $\widehat{\mathcal{EM}}$ in Theorem 2.

To prove Theorem 4, we apply the following relationship between the asymptotic cross-ratio norm $||V||_{\widehat{cr}}$ (see Definition 11 in Section 7.5) of a Zygmund bounded continuous tangent vector field $V$ on $\mathbb{S}^1$ and the asymptotic Thurston norm $||\lambda_V||_{\widehat{Th}}$ of the measured geodesic lamination $\lambda_V$ induced by any infinitesimal earthquake representation of $V$ (see Section 7.1).

**Theorem 6.** *There exists a universal constant* $C > 0$ *such that*

$$\|\lambda_V\|_{\widehat{Th}} \leq C\|V\|_{\widehat{cr}} \tag{1.6}$$

*for each* $V \in \mathcal{Z}(\mathbb{S}^1)$.

**Remark 1.** The Teichmüller space $T(S)$ of any hyperbolic Riemann surface $S$ is embedded into the universal Teichmüller $T(\mathbb{D})$ and $\mathcal{ML}_b(S)$ is embedded into $\mathcal{ML}_b(\mathbb{D})$. Therefore, one can see that the homeomorphic property of the earthquake measure map $\mathcal{EM}$ between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ continues to hold on the restriction of $\mathcal{EM}$ between $T(S)$ and $\mathcal{ML}_b(S)$. Unfortunately, the asymptotic Teichmüller space $AT(S)$ of a Riemann surface $S$ of infinite type can no longer be embedded as a subspace of the asymptotic Teichmüller space $AT(\mathbb{D})$. Therefore, one can not claim immediately that, after the work of this paper, there is a similar topological characterization of $AT(S)$ in terms of a quotient space of $\mathcal{ML}_b(S)$. To obtain such a result for a Riemann surface $S$ of infinite type, different strategies and more techniques need to be developed.

**Remark 2.** In this paper, the inequalities (1.5) and (1.6) are important since we use them to prove our main Theorems 2 and 4, respectively. They are also interesting quantitative results on their own. A natural problem is to investigate whether or not their converses hold. Based on some work in [10], it is unlikely that the converses are true. Relaxed problems are to study on what quadruples the cross-ratio distortions under $h$ control $||\lambda_h||_{\widehat{Th}}$ and on what quadruples the cross-ratio distortions under $V$ are controlled by $||\lambda_V||_{\widehat{Th}}$. We do have ideas to address these questions, which will be presented in a forthcoming paper.

**Remark 3.** The asymptotic Teichmüller topology on $AT(\mathbb{D})$ can also be characterized by a metric defined by using the shear representations of the points in $AT(\mathbb{D})$ (see [4]).

The paper is arranged as follows. Some background and definitions are given in Section 2. Then we prove Theorem 1 in Section 3, Corollary 1 in Section 4, Theorem 5 in Section 5, and finally Theorem 2 in Section 6. In the seventh and last section, we prove Theorems 3, 4 and 6.

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## 2. Preliminaries

### 2.1. Teichmüller space and asymptotic Teichmüller space

Let $\mathbb{D}$ be the open unit disk in the complex plane and centered at the origin, $\mathbb{S}^1 = \partial\mathbb{D}$, and let $QS$ be the set of all quasisymmetric homeomorphisms of $\mathbb{S}^1$. The universal Teichmüller space $T(\mathbb{D})$ is the quotient space $T(\mathbb{D}) = M\ddot{o}b(\mathbb{D})\backslash QS$, where $M\ddot{o}b(\mathbb{D})$ is the group of all Möbius transformation preserving $\mathbb{D}$ and it acts on $QS$ through post-compositions. Given any $h \in QS$, we denote by $[h]$ the corresponding point in $T(\mathbb{D})$. The Teichmüller metric on $T(\mathbb{D})$ is defined as

$$d_T([h_1],[h_2]) = \frac{1}{2}\log \inf_{f|_{\mathbb{S}^1}=h_2\circ h_1^{-1}} K(f),$$

where $f$ is a quasiconformal homeomorphism of $\mathbb{D}$ and $K(f)$ is the maximal dilatation of $f$.

An orientation-preserving homeomorphism $h$ of $\mathbb{S}^1$ is said to be *symmetric* if

$$(1+\delta(x,t))^{-1} \le \frac{|h(e^{2\pi(x+t)i}) - h(e^{2\pi xi})|}{|h(e^{2\pi xi}) - h(e^{2\pi(x-t)i})|} \le 1+\delta(x,t),$$

where $\delta(x,t) \to 0$ uniformly for all $x \in [0,1)$ as $t \to 0$. It is known that $h$ is symmetric if and only if $h = f|_{\mathbb{S}^1}$ for some asymptotic conformal homeomorphism $f$ of $\mathbb{D}$. Let $S_0$ be the collection of all symmetric homeomorphisms of $\mathbb{S}^1$. Clearly, $M\ddot{o}b(\mathbb{D}) \subset S_0$. The quotient space $S_0 \backslash QS$ is called the asymptotic Teichmüller space on $\mathbb{D}$, denoted by $AT(\mathbb{D})$. By letting $T_0(\mathbb{D})$ be the subspace of $T(\mathbb{D})$ whose elements are represented by asymptotic conformal mappings on $\mathbb{D}$, the asymptotic Teichmüller space can also be expressed as

$$AT(\mathbb{D}) = T_0(\mathbb{D}) \backslash T(\mathbb{D}).$$

Given any $h \in QS$, we denote by $[[h]]$ the corresponding point in $AT(\mathbb{D})$. As a quotient space of $T(\mathbb{D})$, the quotient metric on $AT(\mathbb{D})$ is defined as

$$\widehat{d}_{AT}([[h_1]],[[h_2]]) = \inf d_T([\tilde{h}_1],[\tilde{h}_2]), \tag{2.1}$$

where the infimum is taken over all $\tilde{h}_1 \in [[h_1]]$ and $\tilde{h}_2 \in [[h_2]]$. Using boundary dilatations, one can define another metric on $AT(\mathbb{D})$ by

$$d_{AT}([[h_1]],[[h_2]]) = \frac{1}{2}\log \inf_{f|_{\mathbb{S}^1}=h_2\circ h_1^{-1}} \inf_{E} K(f|_{\mathbb{D}\backslash E}), \tag{2.2}$$

where $f$ is a quasiconformal homeomorphism of $\mathbb{D}$ and the first acted infimum is taken over all compact subsets $E$ of $\mathbb{D}$. It is known from [3] and [8] that

$$d_{AT} = \widehat{d}_{AT},$$

which are called the *asymptotic Teichmüller metric* on $AT(\mathbb{D})$.

### 2.2. Measured laminations on $\mathbb{D}$

A complete oriented geodesic $g$ on $\mathbb{D}$ is uniquely determined by an ordered pair of two distinct endpoints, the initial and the terminal points of $g$. Thus the space of all oriented geodesics on $\mathbb{D}$ is naturally identified with $\mathbb{S}^1 \times \mathbb{S}^1 \backslash diag$, where $diag$ is the diagonal set of the product space $\mathbb{S}^1 \times \mathbb{S}^1$. Let $\mathcal{G}$ be the set of all un-oriented complete hyperbolic geodesic on $\mathbb{D}$, then

$\mathcal{G} = (\mathbb{S}^1 \times \mathbb{S}^1 \backslash diag)/\sim$, where the equivalence is defined by $(a, b) \sim (b, a)$. We denote by $\lceil a, b \rceil$ be the equivalence class of $(a, b) \in \mathbb{S}^1 \times \mathbb{S}^1 \backslash diag$. Note that the topology on $\mathcal{G}$ is the induced topology from $\mathbb{S}^1 \times \mathbb{S}^1 \backslash diag$.

A geodesic lamination $\mathcal{L}$ is a collection of disjoint un-oriented complete geodesics which foliates a closed subset of $\mathbb{D}$. Equivalently, a geodesic lamination $\mathcal{L}$ can be identified as a closed subset of $\mathcal{G}$ such that any two geodesics presented by two different elements in $\mathcal{L}$ don't intersect in $\mathbb{D}$ (they may share one common endpoint). Each complete geodesic in $\mathcal{L}$ is called a leaf of $\mathcal{L}$. A stratum of $\mathcal{L}$ is either a geodesic of $\mathcal{L}$ or a component of the complement of $\mathcal{L}$ in $\mathbb{D}$.

By a measured geodesic lamination $(\mathcal{L}, \lambda)$ we mean a nonnegative, locally finite, Borel measure on the space $\mathcal{G}$ with support equal to $\mathcal{L}$. We often briefly say that $\lambda$ is a measured lamination with support $|\lambda|$. Each measured lamination induces a transverse measure along the support $|\lambda|$. Given any hyperbolic geodesic segment $I$ of length $\leq 1$, the measure $\lambda(I)$ is equal to $\lambda(I \cap |\lambda|)$.

### 2.3. Earthquakes and earthquake measures

Earthquake maps in the hyperbolic plane $\mathbb{D}$ (and on any hyperbolic Riemann surface) were introduced by Thurston [22]. Let $\mathcal{L}$ be a geodesic lamination on $\mathbb{D}$. An earthquake $E$ along a geodesic lamination $\mathcal{L}$ is an injective and surjective map $E : \mathbb{D} \to \mathbb{D}$ satisfying

(1) the restriction of $E$ on each stratum $A$ of $\mathcal{L}$ is the restriction of a Möbius transformation, which maps $\mathbb{D}$ onto $\mathbb{D}$, on A, and,

(2) for any two strata $A$ and $B$, the comparison isometry

$$cmp(A, B) = (E|_A)^{-1} \circ E|_B : \mathbb{D} \to \mathbb{D}$$

is a hyperbolic translation whose axis weakly separates $A$ and $B$, and which translates $B$ to the left as viewed from $A$.

An earthquake $E$ on $\mathbb{D}$ continuously extends to a homeomorphism of the boundary $\mathbb{S}^1$ ([22]), and we denote by $E|_{\mathbb{S}^1}$ the restriction of the extension to $\mathbb{S}^1$. The converse statement is the so-called Thurston's theorem [22], which says that for any orientation-preserving homeomorphism $h$ of $\mathbb{S}^1$, there is an earthquake map $E^\lambda$ such that $h = E^\lambda|_{\mathbb{S}^1}$. We call $E^\lambda$ an earthquake representation induces of $h$.

Each earthquake $E$ along a lamination $\mathcal{L}$ induces a transverse measure to $\mathcal{L}$, which is called the earthquake measure $\lambda$ induced by $E$. An earthquake measure corresponds to a measured geodesic lamination. Therefore, each earthquake map $(E, \mathcal{L})$ induced a measure geodesic lamination $\lambda$ with $|\lambda| = \mathcal{L}$. It is also a fact that given a orientation-preserving homeomorphism $h$ of $\mathbb{S}^1$, although the earthquake representation of $h$ is not necessarily unique, the induced earthquake measure or measured lamination $\lambda_h$ is unique. More precisely, two homeomorphisms $h_1$ and $h_2$ determine the same measured lamination $\lambda$ if and only if $h_2 = \gamma \circ h_1$ for some $\gamma \in M\ddot{o}b(\mathbb{D})$. Therefore, Thurston's earthquake representation induces an injective map from the space of the right cosets of $M\ddot{o}b(\mathbb{D})$ in the group of orientation-preserving homeomorphisms of $\mathbb{S}^1$ into the space of measured laminations on $\mathbb{D}$ by associating each coset with the corresponding measured lamination.

For any measured lamination $\lambda$ and $\gamma \in M\ddot{o}b(\mathbb{D})$, we denote by $\gamma^*\lambda$ a measured lamination, called the pull-back of $\lambda$ by $\gamma$, which is supported on $\gamma^{-1}(|\lambda|)$ and with the transverse measure evaluated by $\lambda \circ \gamma$. For an orientation preserving homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ and the earthquake map $E^\lambda|_{\mathbb{S}^1} = h$, we have that $h \circ \gamma = E^{\gamma^*\lambda}|_{\mathbb{S}^1}$.

### 2.4. Earthquake measure map

A measured lamination $\lambda$ is Thurston bounded if the Thurston's norm

$$\|\lambda\|_{Th} = \sup_I \lambda(I)$$

is finite, where the supremum is taken over all geodesic arcs $I$ in $\mathbb{D}$ of unit length. Let $\mathcal{ML}_b(\mathbb{D})$ be the set of bounded measured laminations on $\mathbb{D}$. The following theorem of Thurston is well known, for which we refer to [7], [10], and [18] for different proofs.

**Theorem A.** Let $h$ be an orientation preserving homeomorphism of $\mathbb{S}^1$ and let $E^\lambda$ be an earthquake on $\mathbb{D}$ such that $h = E^\lambda|_{\mathbb{S}^1}$. Then the earthquake measure $\lambda$ is Thurston bounded if and only if $h$ is quasisymmetric.

Because of Thurston's earthquake presentation of orientation-preserving homeomorphisms of $\mathbb{S}^1$ and Theorem A, a bijection between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$, called the earthquake measure map in [17], is defined by

$$\mathcal{EM} : T(\mathbb{D}) \ni [h] \to \lambda \in \mathcal{ML}_b(\mathbb{D}),$$

where $h = E^\lambda|_{\mathbb{S}^1}$.

### 2.5. Weak* topology and Uniform Weak* topology on $\mathcal{ML}_b(\mathbb{D})$

A box of geodesics $B$ in $\mathcal{G}$ is the quotient under the equivalence $\sim$ of the product $[a,b]\times[c,d]$ of two disjoint closed arcs in $\mathbb{S}^1$, where $[a,b]$ (resp. $[c,d]$) is the arc in $\mathbb{S}^1$ from $a$ (resp. $c$) to $b$ (resp. $d$) in counterclockwise order on $\mathbb{S}^1$. We will write somewhat incorrectly $B=[a,b]\times[c,d]$ instead of $B=([a,b]\times[c,d])/\sim$ . In the following of this paper, we call $B^*=[-i,1]\times[i,-1]$ the standard box.

**Definition 1.** *The Liouville measure $L$ is a non-trivial Borel measure on $\mathcal{G}$ defined by*

$$L(B)=|\log cr_L(B)|,$$

*where $B=[a,b]\times[c,d]$ and $cr_L(B)=cr_L(\{a,b,c,d\})=|\frac{(a-c)(b-d)}{(a-d)(b-c)}|$.*

One can easily see that this measure is invariant under the group $M\ddot{o}b(\mathbb{D})$.

**Remark 4.** The cross ratio $cr_L(B)=cr_L(\{a,b,c,d\})$ of a box $B=[a,b]\times[c,d]$ or a quadruple $\{a,b,c,d\}$ of four points on $\mathbb{S}^1$ arranged in counterclockwise order is used in [17] and [19]. In [7] and [10], a different cross ratio of $B$ or $\{a,b,c,d\}$ is used, that is

$$cr(B)=cr(\{a,b,c,d\})=\frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

Since we need to quote results from [17], [19], [7] and [10], we use both of the cross ratios in this paper. Their relationship is

$$cr_L(B)=cr_L(\{a,b,c,d\})=1+cr(\{a,b,c,d\})=1+cr(B).$$

**Definition 2.** *A sequence $\{\lambda_n\}_{n=1}^{\infty}$ of Borel measures on $\mathcal{G}$ converges in the weak* topology to a Borel measure $\lambda$ if*

$$\lim_{n\to\infty}\int_{\mathcal{G}} f d\lambda_n=\int_{\mathcal{G}} f d\lambda$$

*for any continuous function $f$ on $\mathcal{G}$ with compact support.*

For any box $B\in\mathcal{G}$ with $L(B)=\log 2$, in the rest of this paper, we will always use $\gamma_B$ to stand for the element of $M\ddot{o}b(\mathbb{D})$ such that $\gamma_B(B^*)=B$. Meanwhile, for any measured geodesic lamination $\lambda$ on $\mathbb{D}$, $(\gamma_B)^*\lambda$ stands for the pullback of $\lambda$ by $\gamma_B$.

**Definition 3.** *A sequence* $\{\lambda_n\}_{n=1}^{\infty} \subset \mathcal{ML}_b(\mathbb{D})$ *converges to* $\lambda \in \mathcal{ML}_b(\mathbb{D})$ *in the uniform weak* topology if for any continuous function* $f$ *on* $\mathcal{G}$ *with compact support* $supp(f) \subset B^*$,

$$\lim_{n\to\infty}\{\sup_{B}\int_{B^*} fd[(\gamma_B)^*\lambda_n - (\gamma_B)^*\lambda]\} = 0,$$

*where the supremum is taken over all boxes* $B$ *with* $L(B) = \log 2$.

**Theorem B.** [21] Let $\lambda$ and $\lambda_n$, $n = 1, 2, 3, \cdots$, be uniformly bounded earthquake measures (i.e., $\|\lambda\|_{Th}, \|\lambda_n\|_{Th} \leq M < \infty$ for all $n$) in $\mathcal{ML}_b(\mathbb{D})$. If $\lambda_n$ converges to $\lambda$ in the weak* topology, then $E^{\lambda_n}|_{\mathbb{S}^1}$ converges to $E^{\lambda}|_{\mathbb{S}^1}$ pointwise on $\mathbb{S}^1$ (i.e., for each $x \in \mathbb{S}^1$, $E^{\lambda_n}|_{\mathbb{S}^1}(x) \to E^{\lambda_n}|_{\mathbb{S}^1}(x)$ as $n \to \infty$) when the earthquakes $E^{\lambda}|_{\mathbb{S}^1}$ and $E^{\lambda_n}|_{\mathbb{S}^1}$, $n = 1, 2, 3, \cdots$, are properly normalized.

## 3. Characterization of the asymptotic Teichmüller space $AT(\mathbb{D})$ through measured geodesic laminations

In this section, we prove Theorem 1.

For any box $B = [a, b] \times [c, d]$, we define the *minimal scale* $s(B)$ of $B$ as

$$s(B) = \min\{|a - b|, |b - c|, |c - d|, |d - a|\}.$$

A sequence $\{B_n\}_{n=1}^{\infty} \subset \mathcal{G}$ of boxes is said to be *degenerating* if $L(B_n) = \log 2$ for all $n$ and

$$\lim_{n\to\infty} s(B_n) = 0.$$

**Definition 4.** *Given two bounded measured laminations* $\lambda$ *and* $\lambda'$ *in* $\mathcal{ML}_b(\mathbb{D})$, *we say that* $\lambda$ *is equivalent to* $\lambda'$ *if*

$$\sup_{\{B_i\}} \limsup_{i\to\infty} \int_{B^*} fd((\gamma_{B_i})^*\lambda - (\gamma_{B_i})^*\lambda') = 0 \tag{3.1}$$

*for any continuous function* $f$ *on* $\mathcal{G}$ *with compact support* $supp(f) \subset B^*$, *where the supremum is taken over all degenerating sequences* $\{B_i\}_{i=1}^{\infty}$ *of boxes. Denote by* $[\lambda]$ *the equivalence class of* $\lambda$.

*Proof of the necessity part of Theorem 1.* Assume that $[[h]] = [[h']]$ and $\lambda$ and $\lambda'$ are the measured laminations determined by $h$ and $h'$ respectively.

We need to show $[\lambda]=[\lambda']$. Suppose that this is not true. Then the condition (3.1) does not hold. Thus there exist a degenerating sequence $\{B_i\}_{i=1}^{\infty}$ of boxes and a continuous function $f$ on $\mathcal{G}$ with compact support $supp(f)\subset B^*$ such that

$$\int_{B^*} fd((\gamma_{B_i})^*\lambda-(\gamma_{B_i})^*\lambda')>\delta \tag{3.2}$$

for some positive constant $\delta$ and all positive integers $i$. Let

$$\lambda_i=(\gamma_{B_i})^*\lambda \text{ and } \lambda_i'=(\gamma_{B_i})^*\lambda'. \tag{3.3}$$

Then $\{\|\lambda_i\|_{Th}\}_{i=1}^{\infty}$ and $\{\|\lambda_i'\|_{Th}\}_{i=1}^{\infty}$ are bounded because $\|\lambda_i\|_{Th}=\|\lambda\|_{Th}$ and $\|\lambda_i'\|_{Th}=\|\lambda'\|_{Th}$. It follows that there exist two subsequences $\{\lambda_{i_j}\}_{j=1}^{\infty}\subset\{\lambda_i\}_{i=1}^{\infty}$ and $\{\lambda_{i_j}'\}_{j=1}^{\infty}\subset\{\lambda_i'\}_{i=1}^{\infty}$ such that $\lambda_{i_j}$ weakly converges to $\widehat{\lambda}$ and $\lambda_{i_j}'$ weakly converges to $\widehat{\lambda}'$. Following (3.2), $\widehat{\lambda}\neq\widehat{\lambda}'$. For simplicity of notation, we rename the subsequences to be $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\lambda_i'\}_{i=1}^{\infty}$.

Let $A_i, A_i'\in M\ddot{o}b(\mathbb{D})$ such that $h_i=A_i\circ E^{\lambda_i}|_{\mathbb{S}^1}$ and $h_i'=A_i'\circ E^{\lambda_i'}|_{\mathbb{S}^1}$ are normalized to fix $1, i$ and $-1$. Assume also that $E^{\widehat{\lambda}}$ and $E^{\widehat{\lambda}'}$ are normalized to fix $1, i$ and $-1$. Then by Theorem B,

$$h_i=A_i\circ E^{\lambda_i}|_{\mathbb{S}^1}\to E^{\widehat{\lambda}}|_{\mathbb{S}^1}=\widehat{h} \text{ and } h_i'=A_i'\circ E^{\lambda_i'}|_{\mathbb{S}^1}\to E^{\widehat{\lambda}'}|_{\mathbb{S}^1}=\widehat{h}' \tag{3.4}$$

pointwise on $\mathbb{S}^1$. Since $\widehat{\lambda}\neq\widehat{\lambda}'$, it follows that

$$\widehat{h}\circ(\widehat{h}')^{-1}\notin M\ddot{o}b(\mathbb{D}). \tag{3.5}$$

Let $ex(\cdot)$ be the Douady-Earle extension operator. Then it follows from (3.5) that

$$ex(\widehat{h})\circ(ex(\widehat{h}'))^{-1}\notin M\ddot{o}b(\mathbb{D}). \tag{3.6}$$

Since $[[h]]=[[h']]$, using Theorem 4 in [5], $ex(h)\circ(ex(h'))^{-1}$ is asymptotically conformal. Thus given any $\epsilon>0$, there exists a compact subset $K$ of $\mathbb{D}$ such that

$$\|Belt(ex(h))|_{\mathbb{D}-K})-Belt(ex(h'))|_{\mathbb{D}-K})\|_{\infty}\leq\epsilon. \tag{3.7}$$

Since $h_i$ (resp. $h_i'$) differs from $h$ (resp. $h'$) only by precomposition and postcomposition by Möbius transformations, the conformal naturality of Douady-Earle extensions implies that $ex(h_i)$ and $ex(h_i')$ are quasiconformal mappings with maximal dilatations as the same as the extensions of $h$ and $h'$ respectively. Because $ex(h_i)$ and $ex(h_i')$ are normalized to fix three points 1, $i$ and $-1$ on $\mathbb{S}^1$, passing to subsequences we may further assume that $ex(h_i)$

and $ex(h_i')$ converge to quasiconformal homeomorphisms of $\mathbb{D}$ uniformly on the closure $\overline{\mathbb{D}}$. Since we have already known that $h_i$ and $h_i'$ converges to $\widehat{h}$ and $\widehat{h}'$ pointwise on $\mathbb{S}^1$, it follows that they converge to the limit functions uniformly on $\mathbb{S}^1$. Using the properties of Douady-Earle extensions in [1], we obtain

$$Belt(ex(h_i)) \to Belt(ex(\widehat{h})) \text{ and } Belt(ex(h_i')) \to Belt(ex(\widehat{h}')) \tag{3.8}$$

uniformly on any compact subset of $\mathbb{D}$. Again by the conformal naturality of Douady-Earle extensions,

$$ex(h_i) = A_i \circ ex(h) \circ \gamma_{B_i},\ ex(h_i') = A_i' \circ ex(h') \circ \gamma_{B_i}. \tag{3.9}$$

By taking Beltrami coefficients of the left and of the right side of (3.9), we get that

$$Belt(ex(h_i)) = Belt(ex(h)) \circ \gamma_{B_i} \frac{\overline{\partial\gamma_{B_i}/\partial z}}{\partial\gamma_{B_i}/\partial z},$$

$$Belt(ex(h_i')) = Belt(ex(h')) \circ \gamma_{B_i} \frac{\overline{\partial\gamma_{B_i}/\partial z}}{\partial\gamma_{B_i}/\partial z}. \tag{3.10}$$

Then

$$Belt(ex(h_i)) - Belt(ex(h_i'))$$

$$= [Belt(ex(h)) \circ \gamma_{B_i} - Belt(ex(h')) \circ \gamma_{B_i}] \frac{\overline{\partial\gamma_{B_i}/\partial z}}{\partial\gamma_{B_i}/\partial z}. \tag{3.11}$$

Recall that $\gamma_{B_i}$ is the Möbius transformation mapping the standard box $B^*$ to $B_i$ and $s(B_i)$ converges to 0 as $i \to \infty$. Then $\gamma_{B_i}$ maps the origin to the intersection point $O_i$ of the diagonals of $B_i$ and $O_i$ approaches $\mathbb{S}^1$ as $i \to \infty$. It follows that for any compact $K$ of $\mathbb{D}$, $\gamma_{B_i}(K)$ converges to $\mathbb{S}^1$ as $i \to \infty$. Thus $Belt(ex(h_i)) - Belt(ex(h_i'))$ converges to 0 as $i \to \infty$ uniformly on any compact subset of $\mathbb{D}$. Combining this property with (3.8), we obtain

$$Belt(ex(\widehat{h})) - Belt(ex(\widehat{h}')) = 0.$$

Hence

$$ex(\widehat{h}) \circ (ex(\widehat{h}'))^{-1} \in M\ddot{o}b(\mathbb{D}). \tag{3.12}$$

This is a contradiction to (3.6). Therefore, $[\lambda] = [\lambda']$. □

The following two lemmas are developed to prove the sufficiency of Theorem 1.

**Lemma 1.** *Suppose that* $\{\gamma_{B_i}\}_{i=1}^{\infty} \subset M\ddot{o}b(\mathbb{D})$ *and* $s(\gamma_{B_i}(B^*)) \to 0$ *as* $i \to \infty$*. Then given any box* $B = [a, b] \times [c, d]$*,* $s(\gamma_{B_i}(B)) \to 0$ *as* $i \to \infty$*.*

*Proof.* Suppose on the contrary that there exists a box $B = [a, b] \times [c, d]$ such that

$$s(\gamma_{B_i}(B)) \nrightarrow 0$$

as $i \to \infty$. Then there is a subsequence $\{\gamma_{B_{i_j}}\}_{j=1}^{\infty}$ of $\{\gamma_{B_i}\}_{i=1}^{\infty}$ such that $\gamma_{B_{i_j}}(B) \to [a', b'] \times [c', d']$ as $j \to \infty$, where $a', b', c'$ and $d'$ are four distinct points on $\mathbb{S}^1$. For simplicity of notation, we rename the subsequence $\{\gamma_{B_{i_j}}\}_{j=1}^{\infty}$ to be $\{\gamma_{B_i}\}_{i=1}^{\infty}$. Let $\gamma_{B_i}(B^*) = [a_i, b_i] \times [c_i, d_i]$ and $\gamma_{B_i}(B) = [a_i', b_i'] \times [c_i', d_i']$. Since $s(\gamma_{B_i}(B^*)) \to 0$ as $i \to \infty$, passing to a subsequence we may assume that two of the four points $\{a_i, b_i, c_i, d_i\}$ converge to a point $x$ on $\mathbb{S}^1$. Since $L(B_i) = \log 2$ for each $i$, it follows that there are at least three of $\{a_i, b_i, c_i, d_i\}$ converging to $x$. Passing to a subsequence one more time we may assume that $a_i$, $b_i$ and $c_i$ converge to $x$ as $i \to \infty$. The fourth point $d_i$ may converge to another point $y$ on $\mathbb{S}^1$. There are two points of $\{a', b', c', d'\}$ different from $x$ and $y$, namely, $a'$ and $b'$. Then the hyperbolic distance $\delta_i$ between the geodesic connecting $a_i'$ and $b_i'$ and the geodesic connecting $a_i$ and $b_i$ converges to $\infty$ as $i \to \infty$ since $a_i'$ and $b_i'$ converge to two different points $a'$ and $b'$ and $a_i$ and $b_i$ converge to the same point $x$. On the other hand, since the hyperbolic distance is preserved under Möbius transformation, it follows that $\delta_i$ is equal to the hyperbolic distance between the geodesic connecting $a$ and $b$ and the geodesic connecting $-i$ and 1, which is a constant. This is a contradiction. Therefore, the conclusion of the lemma follows. □

**Lemma 2.** *Let* $h_1$ *and* $h_2$ *be two quasisymmetric homeomorphisms of* $\mathbb{S}^1$*. Then* $h_1 \circ (h_2)^{-1}$ *is symmetric provided that*

$$\sup_{\{B_n\}} \limsup_{n\to\infty} |L(h_1(B_n)) - L(h_2(B_n))| = 0, \tag{3.13}$$

*where the supremum is taken over all degenerating sequences* $\{B_n\}_{n=1}^{\infty}$ *of boxes.*

*Proof.* Suppose that (3.13) is satisfied but $h_1 \circ (h_2)^{-1}$ is not symmetric. Let $ex(h_1)$ and $ex(h_2)$ be the Douady-Earle extensions of $h_1$ and $h_2$ respectively. Then $ex(h_1) \circ (ex(h_2))^{-1}$ is not asymptotic conformal on $\mathbb{D}$, which means that there exist a constant $\epsilon > 0$ and a sequence $\{D_n\}_{n=1}^{\infty}$ of hyperbolic disks in $\mathbb{D}$ of diameter 1 with the Euclidean distance from $D_n$ to $\mathbb{S}^1$ approaching 0 as $n \to \infty$ such that

$$\|Belt(ex(h_1)|_{D_n}) - Belt(ex(h_2)|_{D_n})\|_{L^\infty} \geq \epsilon \tag{3.14}$$

for all $n$. Let $D_0$ be the hyperbolic disk on $\mathbb{D}$ of diameter 1 and centered in 0, and assume that $\gamma_n \in M\ddot{o}b(\mathbb{D})$ and $\gamma_n(D_0) = D_n$. Let $A_{1,n}$ and $A_{2,n}$ $\in M\ddot{o}b(\mathbb{D})$ such that $A_{1,n} \circ h_1 \circ \gamma_n$ and $A_{2,n} \circ h_2 \circ \gamma_n$ fix $1, -1, i$ for all $n$. Using the assumption (3.13) and applying Lemma 1 to $\gamma_n$, we obtain

$$\lim_{n\to\infty} |L(A_{1,n} \circ h_1 \circ \gamma_n(B)) - L(A_{2,n} \circ h_2 \circ \gamma_n(B))| = 0 \tag{3.15}$$

for any box $B$ with $L(B) = \log 2$. Let $ex(A_{1,n} \circ h_1 \circ \gamma_n)$ and $ex(A_{2,n} \circ h_2 \circ \gamma_n)$ be the Douady-Earle extensions of $A_{1,n} \circ h_1 \circ \gamma_n$ and $A_{2,n} \circ h_2 \circ \gamma_n$ respectively. Since these quasiconformal mappings fix three common points and have constant maximal dilatations, passing to subsequences we may assume that $A_{1,n} \circ h_1 \circ \gamma_n$ and $A_{2,n} \circ h_2 \circ \gamma_n$ converge uniformly to quasisymmetric mappings $\widehat{h}_1$ and $\widehat{h}_2$ respectively. Then it follows from (3.15) that

$$L(\widehat{h}_1(B)) = L(\widehat{h}_2(B))$$

for any given box $B$ with $L(B) = \log 2$. By the normalized condition at three points, we conclude that $\widehat{h}_1 = \widehat{h}_2$. By the convergence properties of Douady-Earle extensions, $Belt(ex(A_{1,n} \circ h_1 \circ \gamma_n))$ and $Belt(ex(A_{2,n} \circ h_2 \circ \gamma_n))$ converge to $Belt(ex(\widehat{h}_1))$ and $Belt(ex(\widehat{h}_2))$ uniformly on $D_0$; that is,

$$||Belt(ex(A_{1,n} \circ h_1 \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2,n} \circ h_2 \circ \gamma_n)|_{D_0})||_{L^\infty} \to 0 \tag{3.16}$$

as $n \to \infty$. On the other hand, by the conformal naturality of Douady-Earle extensions and (3.14),

$$\begin{aligned} & ||Belt(ex(A_{1,n} \circ h_1 \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2,n} \circ h_2 \circ \gamma_n)|_{D_0})||_{L^\infty} \\ = \; & ||Belt(ex(h_1)|_{D_n}) - Belt(ex(h_2)|_{D_n})||_{L^\infty} \geq \epsilon > 0. \end{aligned}$$

This is a contradiction to (3.16), so $h_1 \circ (h_2)^{-1}$ is symmetric. □

*Proof of the sufficiency part of Theorem 1.* We prove $[[h]] = [[h']]$ if $[\lambda] = [\lambda']$. Suppose on the contrary that $[[h]] \neq [[h']]$, which means $h' \circ h^{-1}$ is not symmetric. Then the condition (3.13) in Lemma 2 does not hold. Thus there exists a degenerating sequence $\{B_i\}_{i=1}^{\infty}$ of boxes such that

$$|L(h(B_i)) - L(h'(B_i))| \geq \delta \tag{3.17}$$

for some positive constant $\delta$ and all positive integers $i$.

Let $\lambda_i = (\gamma_{B_i})^*\lambda$ and $\lambda_i' = (\gamma_{B_i})^*\lambda'$. We show that $\lambda_i - \lambda_i'$ converges to 0 in the weak* topology as $i \to \infty$. It suffices to show that for any box $B$ with $L(B) = \log 2$ and any continuous function $f$ on $\mathcal{G}$ with a compact support $supp(f) \subset B$,

$$\lim_{i\to\infty} \int_B f d(\lambda_i - \lambda_i') = 0. \tag{3.18}$$

Let $\gamma \in M\ddot{o}b(\mathbb{D})$ such that $B = \gamma(B^*)$. Then

$$\begin{aligned} \int_B f d(\lambda_i - \lambda_i') &= \int_{B^*} f \circ \gamma d(\gamma^*(\lambda_i) - \gamma^*(\lambda_i')) \\ &= \int_{B^*} f \circ \gamma d((\gamma_{B_i} \circ \gamma)^*\lambda - (\gamma_{B_i} \circ \gamma)^*\lambda'). \end{aligned} \tag{3.19}$$

By Lemma 1, we know $s(\gamma_{B_i} \circ \gamma(B^*)) \to 0$ as $i \to \infty$. Then the definition of $[\lambda] = [\lambda']$ implies that the last integral in the previous expression (3.19) converges to 0 as $i \to \infty$, which means (3.18) holds.

Since $\{\|\lambda_i\|_{Th}\}_{i=1}^{\infty}$ and $\{\|\lambda_i'\|_{Th}\}_{i=1}^{\infty}$ are uniformly bounded, passing to subsequences we may assume that they converge in the weak* topology. Then the two weak* limits are equal to each other. Now by applying Theorem B, there exist $\{A_i\}_{i=1}^{\infty}$ and $\{A_i'\}_{i=1}^{\infty}$ in $M\ddot{o}b(\mathbb{D})$ such that the two sequences $\{E^{\lambda_i}|_{S^1} = A_i \circ h \circ \gamma_{B_i}\}_{i=1}^{\infty}$ and $\{E^{\lambda_i'}|_{S^1} = A_i' \circ h' \circ \gamma_{B_i}\}_{i=1}^{\infty}$ converge to the same quasisymmetric map pointwise on $\mathbb{S}^1$. Thus

$$\lim_{i\to\infty} |L(A_i \circ h \circ \gamma_{B_i}(B^*)) - L(A_i' \circ h' \circ \gamma_{B_i}(B^*)| = 0. \tag{3.20}$$

On the other hand,

$$|L(A_i \circ h \circ \gamma_{B_i}(B^*)) - L(A_i' \circ h' \circ \gamma_{B_i}(B^*))| = |L(h(B_i)) - L(h'(B_i))|.$$

It follows that (3.20) is a contradiction to (3.17). Thus our assumption $[[h]] \neq [[h']]$ is false. Therefore, $[[h]] = [[h']]$ if $[\lambda] = [\lambda']$. $\square$

## 4. Quotient uniform weak* topology

In this section, we first give the definition of the quotient uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$; then we prove Corollary 1.

**Definition 5.** *A sequence* $\{[\lambda_n]\}_{n=1}^{\infty} \subset \mathcal{AML}_b(\mathbb{D})$ *converges to* $[\lambda] \in \mathcal{AML}_b(\mathbb{D})$ *in the quotient uniform weak* topology if for any continuous function* $f$ *on* $\mathcal{G}$ *with compact support* $supp(f) \subset B^*$*,*

$$\inf_{\widehat{\lambda}_n \in [\lambda_n], \widehat{\lambda} \in [\lambda]} \sup_B \left| \int_{B^*} f d((\gamma_B)^* \widehat{\lambda}_n - (\gamma_B)^* \widehat{\lambda}) \right| \to 0 \text{ as } n \to \infty,$$

*where the supremum is taken over all boxes* $B \in \mathcal{G}$ *with* $L(B) = \log 2$.

*Proof of Corollary 1.* We first show that the induced earthquake measure map $\widehat{\mathcal{EM}}$ is continuous. Let $[\lambda_n] = \widehat{\mathcal{EM}}([[h_n]])$ and $[\lambda] = \widehat{\mathcal{EM}}([[h]])$. If $d_{AT}([[h_n]], [[h]]) \to 0$ as $n \to \infty$, then there exist $h'_n \in [[h_n]]$ and $h' \in [[h]]$ such that $d_T([h'_n], [h']) \to 0$ as $n \to \infty$. Let $\lambda'_n = \mathcal{EM}([h'_n])$ and $\lambda' = \mathcal{EM}([h'])$. Theorem 1 implies that $\lambda' \in [\lambda]$ and $\lambda'_n \in [\lambda_n]$. Since $\mathcal{EM}$ is continuous (Theorem 1 in [17]), it follows that $\lambda'_n \to \lambda'$ in the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$. Using Definition 5, we conclude that $[\lambda_n] \to [\lambda]$ in the quotient uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$.

Now we show that $\widehat{\mathcal{EM}}^{-1}$ is continuous. Suppose $[\lambda_n] = \widehat{\mathcal{EM}}([[h_n]]) \to [\lambda] = \widehat{\mathcal{EM}}([[h]])$ in the quotient uniform weak* topology as $n \to \infty$. By definition, there exist $\lambda'_n \in [\lambda_n]$ and $\lambda' \in [\lambda]$ such that $\lambda'_n \to \lambda'$ in the uniform weak* topology as $n \to \infty$. Let $[h'_n] = \mathcal{EM}^{-1}(\lambda'_n)$ and $[h'] = \mathcal{EM}^{-1}(\lambda')$. Theorem 1 implies that $h'_n \in [[h_n]]$ and $h' \in [[h]]$. Since $\mathcal{EM}^{-1}$ is continuous (Theorem 1 in [17]), it follows that $d_T([h'_n], [h']) \to 0$ as $n \to \infty$. Thus $d_{AT}([[h_n]], [[h]]) \to 0$ as $n \to \infty$. □

## 5. Asymptotic Thurston's norm and strong asymptotic cross-ratio distortion norm

Let $Q$ be a quadruple consisting of four points $a, b, c, d$ on the unit circle arranged in the counter-clockwise direction, denoted by $Q = \{a, b, c, d\}$. We use $cr(Q)$ to denote the following cross ratio:

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

For a quasisymmetric homeomorphism $h$ of $\mathbb{S}^1$, $cr(h(Q))$ denotes

$$cr(h(Q)) = \frac{(h(b)-h(a))(h(d)-h(c))}{(h(c)-h(b))(h(d)-h(a))}.$$

In [7] and [10], the cross-ratio distortion norm $||h||_{cr}$ is defined to be

$$||h||_{cr} = \sup_{cr(Q)=1} |\log cr(h(Q))|,$$

where the supremum is taken over all quadruples $Q$ with $cr(Q) = 1$.

Let $\lambda_h$ be the measured lamination induced by an earthquake representation of $h$. It is shown in [7] that there is a universal positive constant $C$ such that

$$||\lambda_h||_{Th} \leq C||h||_{cr}$$

for any quasisymmetric homeomorphism $h$ of $\mathbb{S}^1$. Then the converse is proved in [10]; that is

$$||h||_{cr} \leq C||\lambda_h||_{Th} \tag{5.1}$$

for a universal positive constant $C$. Therefore, the cross-ratio distortion norm and the Thurston norm are comparable. In this section, we also briefly denote $\lambda_h$ by $\lambda$.

**Definition 6.** *The strong asymptotic cross-ratio distortion norm of a quasisymmetric homeomorphism $h$ of $\mathbb{S}^1$ is defined as*

$$\|h\|_{\widehat{\widehat{cr}}} = \sup_{\{Q_i\}} \limsup_{i\to\infty} |\log cr(h(Q_i))|,$$

*where the supremum is taken over all sequences $\{Q_i\}_{i=1}^{\infty}$ of quadruples such that $cr(Q_i) = 1$ for all $i$ and $S_{max}(Q_i) \to 0$ as $i \to \infty$, and where $S_{max}(Q)$ is the maximum scale of $Q$; that is,*

$$S_{max}(Q) = \max\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

**Remark 5.** Note first that in the previous definition of $||h||_{\widehat{\widehat{cr}}}$, we require the maximal scale $S_{max}(Q)$ of $Q$ to approach 0. Note secondly that $\|h\|_{\widehat{\widehat{cr}}} \leq M$ if and only if for any arbitrary small positive $\epsilon$, there exists $\delta > 0$ such that for any quadruple $Q$ with $S_{max}(Q) < \delta$,

$$|\log cr(h(Q))| < M + \epsilon.$$

The proof for the "if" part is trivial; the "only if" part can be easily shown by proof by contradiction.

**Definition 7.** *The asymptotic Thurston norm* $\|\lambda\|_{\widehat{Th}}$ *of a bounded measured geodesic lamination* $\lambda$ *is defined as*

$$\|\lambda\|_{\widehat{Th}} = \sup_{\{I_n\}} \limsup_{n\to\infty} \lambda(I_n),$$

*where the supremum is taken over all sequences* $\{I_n\}_{n=1}^{\infty}$ *of closed geodesic segments in* $\mathbb{D}$ *of hyperbolic length* 1 *such that the Euclidean distance from* $I_n$ *to* $\mathbb{S}^1$ *goes to* 0 *as* $n \to \infty$.

Similarly, one can see that $\|\lambda\|_{\widehat{Th}} \leq M$ if and only if for any arbitrary small positive $\epsilon$, there exists $\delta > 0$ such that for any geodesic segment $I$ of hyperbolic length 1, if the Euclidean distance from $I$ to $\mathbb{S}^1$ is less than $\delta$, then $\lambda(I) < M + \epsilon$.

In this section, we prove Theorem 5. The strategy of the proof is as the same as the one used in [10] to prove the inequality $||h||_{cr} \leq C||\lambda_h||_{Th}$, but all considerations have to be arranged near the boundary of the open unit disk. For the completeness of the paper, we sketch the proof. In order to do so, we first recall a few technical results developed in [7] and [10]. The following two lemmas are enumerated as Corollaries 1 and 2 in [7] with proofs.

**Lemma 3 ([7]).** *Let* $Q = \{a, b, c, d\}$ *be a quadruple on the real line with* $-\infty \leq a < b < c < d$, $c \leq s \leq d$ *and* $d < t$. *Suppose that* $A_{(s,t)}$ *is the hyperbolic Möbius transformation with repelling fixed point at* $s$ *and attracting fixed point at* $t$ *and derivative at the repelling fixed point equal to* $\lambda > 1$. *Suppose* $f_{(s,t)} : \mathbb{R} \to \mathbb{R}$ *is defined to be equal to* $A_{(s,t)}$ *on the interval* $[s, t]$ *and equal to the identity on the complement of* $[s, t]$. *Then the cross-ratio of the image quadruple* $f_{(s,t)}(Q)$ *considered as a function of two variables* $s \in [c, d]$ *and* $t \in (d, +\infty)$ *decreases in* $s$ *for each fixed* $t$ *and increases in* $t$ *for each fixed* $s$.

**Lemma 4 ([7]).** *With the same notation as in the previous lemma, suppose* $b \leq s \leq c$ *and* $d \leq t$. *Then the cross-ratio of the image quadruple* $f_{(s,t)}(Q)$ *is increasing in* $s$ *for each fixed* $t$ *and also increasing in* $t$ *for each fixed* $s$.

By using Lemma 3 and 4, the proof of the inequality (5.1) given in [10] is reduced to deriving similar inequalities in three cases, which are summarized into Propositions 3, 4 and 5 there. In order to sketch a proof for our Theorem 5, we recall them too.

Let $h$ denote an orientation-preserving circle homeomorphism, $(E, \mathcal{L})$ an earthquake representation of $h$, and $\lambda$ the induced earthquake measure by

$(E, \mathcal{L})$. There are three universal positive constants $C_0$, $C_1$ and $C_2$, independent of $f$, $\lambda$ and $Q$, such that the following three propositions hold.

**Proposition 1 ([10]).** *If $cr(Q) = 1$ and $a, b, c$ belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of $h$, then*

$$0 \leq \log cr(h(Q)) \leq C_1 C_2 ||\lambda||_{Th}.$$

**Proposition 2 ([10]).** *If $cr(Q) = 1$ and $a, c$ belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of $h$, then*

$$0 \leq \log cr(h(Q)) \leq 2C_1 C_2 ||\lambda||_{Th}.$$

**Proposition 3 ([10]).** *If $cr(Q) = 1$ and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ which separates $a, b$ from $c, d$, then*

$$|\log(cr(Q))| \leq (C_0 + 2C_1 C_2) ||\lambda||_{Th}.$$

For a quadruple $Q = \{a, b, c, d\}$ of four points $a, b, c, d$ on $\mathbb{S}^1$ arranged in the counter-clockwise direction, we notice that $cr(Q) = 1$ if and only if the geodesic $\overline{ac}$ between $a$ and $c$ is perpendicular to the one $\overline{bd}$ between $b$ and $d$. Denote by $e$ the intersection point between $\overline{ac}$ and $\overline{bd}$, and by $\overline{ea}$ (resp. $\overline{eb}$, $\overline{ec}$, $\overline{ed}$) the geodesic ray from $e$ to $a$ (resp. $b$, $c$, $d$).

Given two points $x$ and $y$ on the unit circle, we use $[x, y]$ (resp. $(x, y)$, $[x, y)$, $(x, y]$) to denote the closed (resp. open, half open and half closed) arc on $\mathbb{S}^1$ from $x$ to $y$ in the counter-clockwise direction. Careful examinations of the proofs of the previous propositions in [10] enable us to state them in more elaborated ways as follows.

**Corollary 3.** *If $cr(Q) = 1$ and $a, b, c$ belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of $h$, then*

$$0 \leq \log cr(h(Q)) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ed}} \lambda|_{\overline{ed}}(I),$$

*where $\lambda|_{\overline{ed}}$ is the restriction of $\lambda$ on the collection of leaves of $\mathcal{L}$ connecting points in $[c, d)$ to points in $(d, a]$.*

**Corollary 4.** *If $cr(Q) = 1$ and $a, c$ belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of $h$, then*

$$0 \leq \log cr(h(Q)) \leq 2C_1C_2 \sup_{l(I)=1, I \subset \overline{bd}} \lambda|_{\overline{bd}}(I),$$

*where $\lambda|_{\overline{bd}}$ is the restriction of $\lambda$ on the collection of the leaves of $\mathcal{L}$ connecting points in $[a, c)$ to points in $[c, a)$.*

**Corollary 5.** *If $cr(Q) = 1$ and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ which separates $a, b$ from $c, d$, then*

$$-C \max\{\sup_{l(I)=1, I \subset \beta} \lambda|_{L_I}(I), \sup_{l(I)=1, I \subset \overline{ea}} \lambda|_{L_{II}}(I), \sup_{l(I)=1, I \subset \overline{ec}} \lambda|_{L_{IV}}(I)\}$$

$$\leq \log(cr(Q)) \leq$$

$$C \max\{\sup_{l(I)=1, I \subset \beta} \lambda|_{L_I}(I), \sup_{l(I)=1, I \subset \overline{eb}} \lambda|_{L_{III}}(I), \sup_{l(I)=1, I \subset \overline{ed}} \lambda|_{L_V}(I)\},$$

*where $C = C_0 + 2C_1C_2$, $\beta$ is the common perpendicular segment between the geodesics $\overline{ab}$ and $\overline{cd}$, $L_I$ is the collection of the geodesic lines in $\mathcal{L}$ that connect points of the arc $[d, a]$ to points of the arc $[b, c]$, $L_{II}$ is the collection of the lines in $\mathcal{L}$ that connect points of the arc $(d, a)$ to points of the arc $(a, b)$, $L_{III}$ is the collection of the lines in $\mathcal{L}$ that connect points of the arc $(a, b)$ to points of the arc $(b, c)$, $L_{IV}$ is the collection of the lines in $\mathcal{L}$ that connect points of the arc $(b, c)$ to points of the arc $(c, d)$, and finally $L_V$ is the collection of the lines in $\mathcal{L}$ that connect points of the arc $(c, d)$ to points of the arc $(d, a)$.*

Now we prove our Theorem 5.

*Proof.* Let $M = ||\lambda||_{\widehat{Th}}$. Given any $\epsilon > 0$, there exists $0 < r < 1$ such that for any geodesic segment $I$ contained in the annulus $W_r = \{z : r < |z| < 1\}$ with hyperbolic length 1, $\lambda(I) \leq M + \epsilon$. Then there exists $\delta > 0$ such that for any quadruple $Q = \{a, b, c, d\}$ with $S_{max}(Q) < \delta$, the geodesic connecting any two points in $Q$ is contained in the annulus $W_r$ and both the common perpendicular geodesic segment between the geodesics $\overline{ab}$ and $\overline{cd}$ and the one between $\overline{bc}$ and $\overline{da}$ are also contained in $W_r$.

Let $C = C_0 + 2C_1C_2$. We show that for any quadruple $Q$ with $cr(Q) = 1$ and $S_{max}(Q) < \delta$,

$$|\log cr(h(Q))| \leq C(M + \epsilon). \tag{5.1}$$

We divide the proof into three cases.

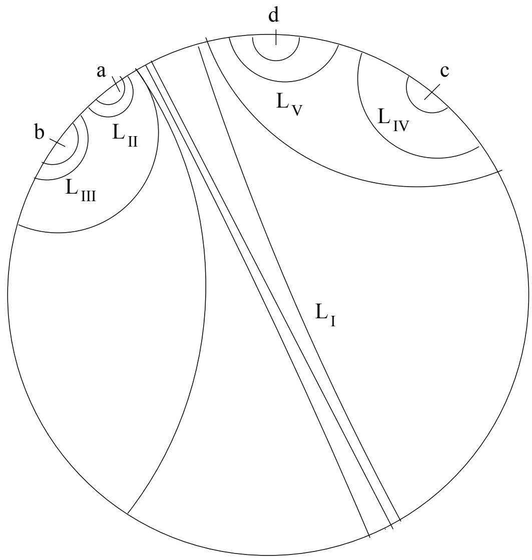


Figure 1: Five subcollections of the leaves of $\mathcal{L}$ described in Corollary 5.

Case 1: The quadruple $Q$ has three points belonging to the same stratum. There are four subcases: either $a, b, c$ or $b, c, d$ or $c, d, a$ or $d, a, b$ belong to the same stratum.

Corollary 3 implies in these four subcases respectively that either

$$0 \leq \log cr(h(Q)) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ed}} \lambda|_{\overline{ed}}(I)$$

or

$$0 \leq \log cr(h(\{b, c, d, a\})) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ea}} \lambda|_{\overline{ea}}(I),$$

$$0 \leq \log cr(h(\{c, d, a, b\})) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{eb}} \lambda|_{\overline{eb}}(I),$$

$$0 \leq \log cr(h(\{d, a, b, c\})) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ec}} \lambda|_{\overline{ec}}(I).$$

Since the values of the previous four suprema are less than or equal to $M + \epsilon$ and since

$$cr(h(\{b, c, d, a\})) = \frac{1}{cr(h(Q))}, \ cr(h(\{c, d, a, b\})) = cr(h(Q)) \text{ and}$$

$$cr(h(\{d, a, b, c\})) = \frac{1}{cr(h(Q))},$$

it follows that

$$|\log cr(h(Q))| \leq C_1 C_2 (M + \epsilon) < C(M + \epsilon).$$

Case 2: The quadruple $Q$ has two opposite points belonging to the same stratum. Then either $a$ and $c$ or $b$ and $d$ belong to the same stratum. By Corollary 4 and a similar fashion in reasoning as in Case 1, we obtain

$$|\log cr(h(Q))| \leq 2C_1 C_2 (M + \epsilon) < C(M + \epsilon).$$

Case 3: The quadruple $Q$ has no opposite points belonging to the same stratum. Then either there exists a geodesic line in $\mathcal{L}$ that separates $a$ and $b$ from $c$ and $d$ or there exists a geodesic line in $\mathcal{L}$ that separates $b$ and $c$ from $d$ and $a$. By Corollary 5 and a similar fashion in reasoning as in Case 1, we obtain

$$|\log cr(h(Q))| \leq (C_0 + 2C_1 C_2)(M + \epsilon) = (M + \epsilon).$$

Therefore no matter which case happens, we obtain that, for any sequence $\{Q_i\}_{i=1}^{\infty}$ of quadruples with $cr(Q_i) = 1$ for all $i$ and $S_{max}(Q_i) \to 0$ as $i \to \infty$,

$$\limsup_{i \to \infty} |\log cr(h(Q_i))| \leq C(M + \epsilon).$$

Thus

$$||h||_{\widehat{\widehat{cr}}} \leq C(M + \epsilon).$$

Since $\epsilon$ is an arbitrarily small positive, it follows that

$$||h||_{\widehat{\widehat{cr}}} \leq CM = C||\lambda||_{\widehat{Th}}.$$

□

## 6. Homeomorphic property of the induced earthquake measure map

In this section, after giving the definition of asymptotically uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$ and some lemmas, we prove Theorem 2.

**Definition 8.** *A sequence* $\{[\lambda_n]\}_{n=1}^{\infty} \subset \mathcal{AML}_b(\mathbb{D})$ *converges to* $[\lambda] \in \mathcal{AML}_b(\mathbb{D})$ *in the asymptotically uniform weak* topology if for any continuous function* $f$ *on* $\mathcal{G}$ *with compact support* $supp(f) \subset B^*$,

$$\sup_{\{B_i\}} \limsup_{i\to\infty} \left| \int_{B^*} f d((\gamma_{B_i})^*(\lambda_n) - (\gamma_{B_i})^*(\lambda)) \right| \to 0 \text{ as } n \to \infty, \tag{6.1}$$

*where the supremum is taken over all degenerating sequences* $\{B_i\}_{i=1}^{\infty}$ *of boxes.*

**Remark 6.** Theorem 1 implies that for each $n$, the value defined by the supremum of the lim sup's in (6.1) is independent of the choices of representatives for $[\lambda_n]$ and $[\lambda]$.

**Theorem C.** [15] Let $h$ be an orientation-preserving homeomorphism of $\mathbb{S}^1$ and $ex(h)$ be the Douady-Earle extension of $h$ to the closed unit disk $\overline{\mathbb{D}}$. Let $p \in \mathbb{S}^1$ and $I_p$ be an open arc on $\mathbb{S}^1$ containing $p$ and symmetric with respect to $p$. If $\|h|_{I_p}\|_{cr} < \infty$, then there exists an open hyperbolic half plane $U_p$ with $p$ at the middle of its boundary on $\mathbb{S}^1$ such that $\log K(ex(h)|U_p) \leq C_1 \|h|_{I_p}\|_{cr} + C_2$ for two universal positive constants $C_1$ and $C_2$, where $K(ex(h)|U_p)$ is the maximal dilatation of $ex(h)$ on $U_p$.

**Remark 7.** It is shown in [14] that there exists a universal constant $C > 0$ such that

$$\log K(ex(h)) \leq C\|h\|_{cr}$$

for any orientation-preserving homeomorphism $h$ of $\mathbb{S}^1$. As a corollary to this result or the previous Theorem C, $ex(h)$ is quasiconformal if $\|h\|_{cr}$ is finite.

**Lemma 5.** *There is a universal constant* $C_0 > 0$ *such that for any measured lamination* $\lambda \in \mathcal{ML}_b(\mathbb{D})$,

$$\frac{1}{C_0} \|\lambda\|_{\widehat{Th}} \leq \sup_{\{B_i\}} \limsup_{i\to\infty} \lambda(B_i) \leq \|\lambda\|_{\widehat{Th}}, \tag{6.2}$$

*where the supremum is taken over all degenerating sequences* $\{B_i\}_{i=1}^{\infty}$ *of boxes.*

*Proof.* This lemma is the asymptotic version of Lemma 2.1 in [17] and the proof is similar. Let $B = [a, b] \times [c, d]$ be a box in $\mathcal{G}$. The measure $\lambda(B)$ is obtained as follows. Without loss of generality, we assume that $a, b, c$ and $d$

lie on $\mathbb{S}^1$ in the counterclockwise order. Let $I$ be the common perpendicular geodesic segment between the hyperbolic geodesics $\overline{ad}$ and $\overline{bc}$. Then any geodesic contained in $B$ intersects $I$. By definition, $\lambda(B)$ is equal to $\lambda(I \cap |\lambda|_B)$, where $|\lambda|_B$ denote the collection of all the geodesics of $|\lambda|$ contained in $B$. If $L(B) = \log 2$, then $I$ is contained in a geodesic segment $I_0$ of hyperbolic length 1. Furthermore, $s(B)$ goes to 0 if and only if Euclidean distance between $I_0$ and $\mathbb{S}^1$ goes to 0. Thus, by definition,

$$\sup_{\{B_i\}} \limsup_{i\to\infty} \lambda(B_i) \leq \|\lambda\|_{\widehat{Th}}.$$

Now we show the other half of the double inequality. We recall a fact used in the proof of Lemma 2.1 in [17]; that is, there is a universal constant $L_0 > 0$ such that for any geodesic segment $I$ of hyperbolic length 1 transversely intersecting a leaf $l$ of $\lambda$, if $J$ is the geodesic segment of length $L_0$ orthogonally intersecting $l$ at $I \cap l$ with the intersecting point at the midpoint of $J$, then any leaf of $\lambda$ intersecting $I$ must intersect $J$. A more general version of this fact can be stated as follows. For any positive number $\alpha$, there exists a positive number $L_0(\alpha)$ such that for any geodesic segment $I$ of hyperbolic length $\alpha$ transversely intersecting a leaf $l$ of $\lambda$, if $J$ is the geodesic segment of length $L_0(\alpha)$ that orthogonally intersects $l$ at the point $I \cap l$ and has $I \cap l$ at its midpoint, then any leaf of $\lambda$ intersecting $I$ must intersect $J$; furthermore, $L_0(\alpha)$ approaches 0 as $\alpha$ goes to 0.

Let $\gamma$ be a Möbius transformation from $\mathbb{H}$ onto $\mathbb{D}$ such that $\gamma^{-1}(J) = [1, e^{L_0(\alpha)}]i$ and $\gamma^{-1}(l) = \{z : |z| = e^{L_0(\alpha)/2}\} \cap \mathbb{H}$. Consider the box

$$B_0 = [-e^{3L_0(\alpha)/2}, -e^{-L_0(\alpha)/2}] \times [e^{-L_0(\alpha)/2}, e^{3L_0(\alpha)/2}].$$

Through an elementary calculation, we can see that if a leaf of the pullback $\gamma^*(\lambda)$ of $\lambda$ under $\gamma$ intersects $\gamma^{-1}(J)$, then it is contained in the box $B_0$. Let $B(I) = \gamma(B_0)$. Then any leaf of $\lambda$ intersecting $J$ is contained in $B(I)$. It follows that for any geodesic segment $I$ of hyperbolic length $\alpha$,

$$\lambda(I) \leq \lambda(J) \leq \lambda(B(I)). \tag{6.3}$$

One can also see that the Euclidean distance from $I$ to $\partial\mathbb{D}$ goes to 0 if and only if the Euclidean distance from $J$ to $\partial\mathbb{D}$ goes to 0. Furthermore, from the constructions of $J$ and $B(I)$, it is also true that the Euclidean distance from $J$ to $\partial\mathbb{D}$ goes to 0 if and only if $s(B(I))$ goes to 0. Now we set $\alpha$ at a value $\alpha_0$ such that

$$L(B_0) = 2\log\cosh(L_0(\alpha_0)/2) = \log 2.$$

Following (6.3), we conclude that for generating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes,

$$\|\lambda\|_{\widehat{Th}} \leq ([\frac{1}{\alpha_0}]+1)\sup_{\{B_i\}}\limsup_{i\to\infty}\lambda(B_i),$$

where $[\frac{1}{\alpha_0}]$ stands for the integral part of $\frac{1}{\alpha_0}$. We complete the proof. □

**Lemma 6.** *If a sequence $\{[\lambda_n]\}_{n=1}^{\infty}\subset \mathcal{AML}_b(\mathbb{D})$ converges to $[\lambda]\in \mathcal{AML}_b(\mathbb{D})$ in the asymptotically uniform weak* topology, then there is a sequence $\{\lambda_n'\}_{n=1}^{\infty}\subset \mathcal{ML}_b(\mathbb{D})$ such that $[\lambda_n']=[\lambda_n]$ for all $n$ and $\{\|\lambda_n'\|_{Th}\}_{n=1}^{\infty}$ is bounded.*

*Proof.* We show first that if $\{[\lambda_n]\}_{n=1}^{\infty}\subset \mathcal{AML}_b(\mathbb{D})$ converges to $[\lambda]\in \mathcal{AML}_b(\mathbb{D})$ in the asymptotically uniform weak* topology, then $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded. Let us follow the same notation and reasoning given in the last two paragraphs of the proof of the previous lemma. Now we set $\alpha$ at a value $\alpha_0'$ such that

$$L(B_0) = 2\log\cosh(L_0(\alpha_0')/2) = \frac{1}{3}\log 2.$$

Then for any measured lamination $\lambda$ and any geodesic segment $I$ of hyperbolic length $\alpha_0'$, there is a box $B(I)$ of Liouville measure $\frac{1}{3}\log 2$ such that every leaf of $\lambda$ intersecting $I$ is contained in $B(I)$. It follows that $\lambda(B(I))\geq\lambda(I)$.

Now suppose that $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is not bounded. Then there exists a sequence $\{I_n\}_{n=1}^{\infty}$ of geodesic segments of hyperbolic length $\alpha_0'$ approaching $\partial\mathbb{D}$ such that $\{\lambda_n(I_n)\}_{n=1}^{\infty}$ is not bounded. Passing to a subsequence, we may assume that

$$\lambda_n(I_n)\to\infty \text{ as } n\to\infty.$$

For each $n$, $L(B(I_n))=\frac{1}{3}\log 2$. Then $B(I_n)$ sits in the interior of a larger box of Liouville measure $\log 2$, denote it by $B_n$.

Let $\gamma_{B_n}$ be the Möbius transformation mapping the standard box $B^*$ to $B_n$. Then $B_m^*=\gamma_{B_n}^{-1}(B(I_n))$ sits in the interior of $B^*$ with Liouville measure $\frac{1}{3}\log 2$. Now let $f$ be a continuous real function between 0 and 1 with

$supp(f) \subset B^*$ and taking value 1 on $B_m^*$. Then

$$\int_{B^*} fd((\gamma_{B_n})^*\lambda_n) \geq ((\gamma_{B_n})^*\lambda_n)(B_m^*) = \lambda_n(\gamma_{B_n}(B_m^*)) = \lambda_n(B(I_n)) \geq \lambda_n(I_n).$$

Thus, $\int_{B^*} fd((\gamma_{B_n})^*\lambda_n) \to \infty$ as $n \to \infty$. On the other hand,

$$\int_{B^*} fd((\gamma_{B_n})^*\lambda) \leq \lambda(B_n).$$

Using Lemma 2.1 of [17], we know $\lambda(B_n) \leq ||\lambda||_{Th}$; or using the previous lemma, we know

$$\limsup_{n\to\infty} \lambda(B_n) \leq ||\lambda||_{\widehat{Th}}.$$

Therefore, $\lambda(B_n)$ has to be bounded. The above two estimates make a contradiction to the assumption that $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology. Therefore, $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded.

Let $h_n = E^{\lambda_n}|_{\mathbb{S}^1}$ and $ex(h_n)$ be the Douady-Earle extension. Since $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded, Theorem 5 and Theorem C together imply that the boundary dilatation $H(ex(h_n)) = \inf_E K(ex(h_n)|_{\mathbb{D}\setminus E})$ is bounded by a positive constant independent of $n$, where the infimum is taken over all compact subsets $E$ of $\mathbb{D}$. Then there exists $M' > 0$ such that $d_{AT}([[h_n]], [[0]]) < M'$ for all $n$. Thus for each $n$, there exists $h_n' \in [[h_n]]$ such that $d_T([h_n'], [0]) < M'$. Let $\lambda_n' = \mathcal{EM}([h_n'])$. Then $\lambda_n' \in [\lambda_n]$. It follows that $[\lambda_n'] = [\lambda_n]$ for each $n$ and $\{||\lambda_n'||_{Th}\}_{n=1}^{\infty}$ is a bounded sequence. We complete the proof. □

Similar to the proof of Lemma 2, one can show the following lemma.

**Lemma 7.** *Let $[[h]] \in AT(\mathbb{D})$ and $\{[[h_n]]\}_{n=1}^{\infty}$ be a sequence of points in $AT(\mathbb{D})$. Then $[[h_n]]$ converges to $[[h]]$ in the asymptotic Teichmüller topology on $AT(\mathbb{D})$ provided that*

$$\sup_{\{B_i\}} \limsup_{i\to\infty} |L(h_n(B_i)) - L(h(B_i))| \to 0 \quad as\ n \to \infty, \tag{6.4}$$

*where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes.*

*Proof.* Suppose that $[[h_n]]$ does not converge to $[[h]]$ in the asymptotic Teichmüller topology on $AT(\mathbb{D})$. Let $ex(h_n)$ and $ex(h)$ be the Douady-Earle extensions of $h_n$ and $h$ respectively, where $n \in \mathbb{N}$. From the definition given by (2.2), it follows that the boundary dilatation of $ex(h_n) \circ (ex(h))^{-1}$ does not converge to 0 as $n \to \infty$. Passing to a subsequence, we may assume that the boundary dilatation of $ex(h_n) \circ (ex(h))^{-1}$ is greater than a positive number $\epsilon$ for all $n$. Thus there exists a sequence of hyperbolic disks $\{D_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ of diameter 1 with the Euclidean distance from $D_n$ to $\mathbb{S}^1$ approaching 0 as $n \to \infty$ such that

$$\|Belt(ex(h_n)|_{D_n}) - Belt(ex(h)|_{D_n})\|_{L^\infty} \geq \epsilon, \tag{6.5}$$

for all $n$. Let $D_0$ be the hyperbolic disk on $\mathbb{D}$ of diameter 1 and centered in 0, and assume that $\gamma_n \in M\ddot{o}b(\mathbb{D})$ and $\gamma_n(D_0) = D_n$. Let $A_{1,n}$ and $A_{2,n}$ $\in M\ddot{o}b(\mathbb{D})$ such that $A_{1,n} \circ h_n \circ \gamma_n$ and $A_{2,n} \circ h \circ \gamma_n$ fix $1, -1, i$ for all $n$. Using the assumption (6.4) and applying Lemma 1 to $\gamma_n$, for any box $B$ with $L(B) = \log 2$, we obtain

$$\lim_{n\to\infty} |L(A_{1,n} \circ h_n \circ \gamma_n(B)) - L(A_{2,n} \circ h \circ \gamma_n(B))| = 0. \tag{6.6}$$

Using Remark 4, one can show that the condition (6.4) implies that there exists a constant $M > 1$ such that for each $n$,

$$\frac{1}{M} \leq \sup_{\{B_i\}} \limsup_{i\to\infty} cr(h_n(B_i)) \leq M,$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes. Using the definition of $\|h\|_{\widehat{\widehat{cr}}}$, we obtain for each $n$,

$$\|h_n\|_{\widehat{\widehat{cr}}} \leq \sup_{\{B_i\}} \limsup_{i\to\infty} |\log cr(h_n(B_i))| \leq \log M,$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes. Theorem C implies that the sequence $\{H(ex(h_n))\}_{n=1}^{\infty}$ of the boundary dilatation of $ex(h_n)$ is bounded, which means that $H(ex(h_n)) < M'$ for each $n$ and some constant $M' > 1$. For each $n$, there exists $0 < r_n < 1$ such that the maximal dilatation $K(ex(h_n))(z) < M'$ for any $z$ with $r_n < |z| < 1$. Let $\mu_n$ be defined as

$$\mu_n(z) = \begin{cases} Belt(ex(h_n))(z) & \text{if } r_n \leq |z| < 1, \\ 0 & \text{if } |z| < r_n. \end{cases}$$

For each $n$, let $f_n$ be the normalized (i.e., 1, $-1$ and $i$ are fixed) quasiconformal homeomorphism of $\mathbb{D}$ with the Beltrami coefficient $\mu_n$, and let $\widetilde{h}_n = f_n|_{\mathbb{S}^1}$. By definiion, $\widetilde{h}_n \in [[h_n]]$ and $K(f_n) < M'$ for each $n$. Using Theorem 2 and Remark (1) after that theorem or Proposition 7 in [1], we know that $\{K(ex(\widetilde{h}_n))\}_{n=1}^{\infty}$ is bounded. Therefore, we conclude that one can replace the representatives of $[[h_n]]$'s such that $\{K(ex(h_n))\}_{n=1}^{\infty}$ is bounded. For simplicity of notation, we continue to denote these representatives by $h_n$'s.

Let $ex(A_{1,n} \circ h_n \circ \gamma_n)$ and $ex(A_{2,n} \circ h \circ \gamma_n)$ be the Douady-Earle extensions of $A_{1,n} \circ h_n \circ \gamma_n$ and $A_{2,n} \circ h \circ \gamma_n$ respectively. These quasiconformal mappings fix three common points. Using the conformal naturality of Douady-Earle extensions, their maximal dilatations are uniformly bounded. Passing to subsequences, we may assume that $A_{1,n} \circ h_n \circ \gamma_n$ and $A_{2,n} \circ h \circ \gamma_n$ converge uniformly to quasisymmetric homeomorphisms $\widehat{\widehat{h}}$ and $\widehat{h}$ on $\mathbb{S}^1$ respectively. Using (6.4) and the convergence, we obtain

$$L(\widehat{\widehat{h}}(B)) = L(\widehat{h}(B))$$

for any box $B$ with $L(B) = \log 2$. Thus, $\widehat{\widehat{h}} = \widehat{h}$. Again using the convergence properties of Douady-Earle extensions, $Belt(ex(A_{1,n} \circ h_n \circ \gamma_n))$ and $Belt(ex(A_{2,n} \circ h \circ \gamma_n))$ converge to $Belt(ex(\widehat{\widehat{h}}))$ and $Belt(ex(\widehat{h}))$ uniformly on $D_0$; that is,

$$||Belt(ex(A_{1,n} \circ h_n \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2,n} \circ h \circ \gamma_n)|_{D_0})||_{L^\infty} \to 0 \quad (6.7)$$

as $n \to \infty$. On the other hand, by the conformal naturality of Douady-Earle extensions,

$$\begin{aligned} & ||Belt(ex(A_{1,n} \circ h_n \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2,n} \circ h \circ \gamma_n)|_{D_0})||_{L^\infty} \\ = \; & ||Belt(ex(h_n)|_{D_n}) - Belt(ex(h)|_{D_n})||_{L^\infty} \geq \epsilon. \end{aligned}$$

This is a contradiction to (6.7). Therefore, the conclusion of the lemma has to hold. □

Now we prove Theorem 2.

*Proof of Theorem 2.* We first show that the induced earthquake measure map $\widehat{\mathcal{EM}}$ is continuous. Assume that $d_{AT}([[h_n]], [[h]]) \to 0$ as $n \to \infty$. Let $[\lambda_n] = \widehat{\mathcal{EM}}([h_n])$ and $[\lambda] = \widehat{\mathcal{EM}}([h])$. We need to prove $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology. Since $d_{AT}([[h_n]], [[h]]) \to 0$, there exist $h'_n \in [[h_n]]$ and $h' \in [[h]]$ such that $d_T([h'_n], [h']) \to 0$ as $n \to \infty$. Let $\lambda'_n = \mathcal{EM}([h'_n])$ and $\lambda' = \mathcal{EM}([h'])$. By Theorem 1, $\lambda'_n \in [\lambda_n]$ and $\lambda' = [\lambda]$. Using the continuity of $\mathcal{EM}$, we know that $\lambda'_n$ converges to $\lambda'$ in the uniform weak* topology. Then for any continuous function $f$ on $\mathcal{G}$ with compact support $\mathrm{supp}(f) \subset B^*$ and any box $B$ with the Liouville measure $L(B) = \log 2$,

$$\lim_{n\to\infty} [\sup_{B} | \int_{B^*} f d((\gamma_B)^* \lambda_n' - (\gamma_B)^* \lambda')|] = 0.$$

Clearly, for any degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes,

$$\sup_{\{B_i\}} \limsup_{i\to\infty} | \int_{B^*} f d((\gamma_{_i})^*(\lambda_n) - (\gamma_{B_i})^*(\lambda))| \leq \sup_{B} | \int_{B^*} f d((\gamma_B)^* \lambda_n' - (\gamma_B)^* \lambda')|.$$

Therefore

$$\sup_{\{B_i\}} \limsup_{i\to\infty} | \int_{B^*} f d((\gamma_{B_i})^*(\lambda_n) - (\gamma_{B_i})^*(\lambda))| \to 0 \ (n \to \infty).$$

By Definition 8, $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak$^*$ topology. Therefore $\widehat{\mathcal{EM}}$ is continuous.

Next we show that the inverse $\widehat{\mathcal{EM}}^{-1}$ is continuous. Suppose not, then there exists a sequence $\{[\lambda_n]\}_{n=1}^{\infty}$ of points in $\mathcal{AML}_b(\mathbb{D})$ such that $[\lambda_n]$ converges to a point $[\lambda]$ of $\mathcal{AML}_b(\mathbb{D})$ in the asymptotic uniform weak$^*$ topology but $[[h_n]] = \widehat{\mathcal{EM}}^{-1}([\lambda_n])$ does not converge to $[[h]] = \widehat{\mathcal{EM}}^{-1}([\lambda])$ in the asymptotic Teichmüller topology on $\mathcal{AT}(\mathbb{D})$.

Using Lemma 6 to replace representatives if necessary, we may assume that $\{\|\lambda_n\|_{Th}\}_{n=1}^{\infty}$ is bounded.

Since $[[h_n]]$ does not converge to $[[h]]$ in the asymptotic Teichmüller topology, applying Lemma 7 and passing to a subsequence, we may assume that there exist $\epsilon > 0$ and a degenerating sequences $\{B_n\}_{n=1}^{\infty}$ of boxes such that

$$\lim_{n\to\infty} |L(h(B_n)) - L(h_n(B_n))| \geq \epsilon. \tag{6.8}$$

Now we show that if $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak$^*$ topology, then $\{(\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda\}_{n=1}^{\infty}$ converges to 0 in the weak$^*$ topology. It suffices to show that for any box $B$ with $L(B) = \log 2$ and any continuous function $f$ on $\mathcal{G}$ with a compact support $supp(f) \subset B$,

$$\lim_{n\to\infty} \int_{B} f d((\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda) = 0.$$

Let $\gamma \in M\ddot{o}b(\mathbb{D})$ such that $B = \gamma(B^*)$. Then

$$\begin{aligned}\int_B f d((\gamma_{B_n})^*\lambda_n - (\gamma_{B_n})^*\lambda) &= \int_{B^*} f \circ \gamma d(\gamma^*((\gamma_{B_n})^*\lambda_n - (\gamma_{B_n})^*\lambda)) \\ &= \int_{B^*} f \circ \gamma d((\gamma_{B_n} \circ \gamma)^*\lambda_n - (\gamma_{B_n} \circ \gamma)^*\lambda)).\end{aligned}$$

By Lemma 1, we know $s(\gamma_{B_n} \circ \gamma(B^*)) \to 0$ as $n \to \infty$. Then the definition of $[\lambda_n]$ converging to $[\lambda]$ in the asymptotically uniform weak$^*$ implies that the last integral in the previous expression converges to 0 as $n \to \infty$. Thus, $(\gamma_{B_n})^*\lambda_n - (\gamma_{B_n})^*\lambda$ converges to 0 in the weak$^*$ topology as $n \to \infty$.

Since $\{(\gamma_{B_n})^*\lambda_n\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^*\lambda\}_{n=1}^{\infty}$ are uniformly Thurston bounded, it follows that $\{(\gamma_{B_n})^*\lambda_n\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^*\lambda\}_{n=1}^{\infty}$ contain a pair of converging subsequences in the weak$^*$ topology. Then the weak$^*$ limits of the converging subsequences are the same. For simplicity of notation, we continue to denote such subsequences by $\{(\gamma_{B_n})^*\lambda_n\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^*\lambda\}_{n=1}^{\infty}$. Furthermore, using Theorem B, we know there exist $\{A_n\}_{n=1}^{\infty}$ and $\{C_n\}_{n=1}^{\infty}$ in $M\ddot{o}b(\mathbb{D})$ such that the two sequences $\{E^{(\gamma_{B_n})^*\lambda_n}|_{\mathbb{S}^1} = A_n \circ h_n \circ \gamma_{B_n}\}_{n=1}^{\infty}$ and $\{E^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1} = C_n \circ h \circ \gamma_{B_n}\}_{n=1}^{\infty}$ converge to the same quasisymmetric map pointwise on $\mathbb{S}^1$. Thus

$$\lim_{n\to\infty} |L(h(B_n)) - L(h_n(B_n))|$$

$$= \lim_{n\to\infty} |L(A_n \circ h \circ \gamma_{B_n}(B^*)) - L(C_n \circ h_n \circ \gamma_{B_n}(B^*))| = 0,$$

which is a contradiction to (6.8). Therefore $[[h_n]]$ converges to $[[h]]$ in the asymptotic Teichmüller topology. Thus, $\widehat{\mathcal{EM}}^{-1}$ is continuous. □

## 7. Induced infinitesimal earthquake measure map and asymptotic cross-ratio norm topology

In this section, we consider the infinitesimal version of the induced earthquake measure map. As pointed out in Section 2.1, the universal Teichmüller space $T(\mathbb{D})$ is the quotient space $M\ddot{o}b(\mathbb{D})\backslash QS$, where $QS$ is the collection of quasisymmetric homeomorphisms of $\mathbb{S}^1$. The tangent space of $T(\mathbb{D})$ at a base point is characterized by the space $\mathcal{Z}(\mathbb{S}^1)$ (or $\mathcal{Z}(\mathbb{D})$) of Zygmund bounded continuous tangent vector fields on $\mathbb{S}^1$ (resp. $\mathbb{D}$). By developing infinitesimal versions of Beurling-Ahlfors extensions, Gardiner and Sullivan showed in [9] that the tangent space of $T_0(\mathbb{D})$ at a base point is characterized by a subspace $\mathcal{Z}_0(\mathbb{S}^1)$ (or $\mathcal{Z}_0(\mathbb{D})$) of $\mathcal{Z}(\mathbb{S}^1)$ (resp. $\mathcal{Z}(\mathbb{D})$). It follows that the tangent space of $AT(\mathbb{D})$ is the quotient space $\mathcal{Z}(\mathbb{S}^1)/\mathcal{Z}_0(\mathbb{S}^1)$ (resp. $\mathcal{Z}(\mathbb{D})/\mathcal{Z}_0(\mathbb{D})$), denoted by $\mathcal{AZ}(\mathbb{S}^1)$ (resp. $\mathcal{AZ}(\mathbb{D})$). The work of this section is to introduce a topology on $\mathcal{AZ}(\mathbb{S}^1)$ under which the infinitesimal earthquake measure map is a homeomorphism between $\mathcal{AZ}(\mathbb{S}^1)$ and $\mathcal{AML}_b(\mathbb{D})$ with respect to the asymptotic uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$.

### 7.1. Zygmund space and infinitesimal earthquake measure map

A continuous tangent vector field $V$ on $\mathbb{S}^1$ can be viewed as a continuous function from $\mathbb{S}^1$ to the complex plane $\mathbb{C}$. It is said to be Zygmund bounded if

$$|V(e^{2\pi i(x+t)}) + V(e^{2\pi i(x-t)}) - 2V(e^{2\pi ix})| \leq M|t|, \tag{7.1}$$

for all $x \in [0,1)$, $0 < t < \frac{1}{2}$ and some $M > 0$.

Let $Q$ be a quadruple consisting of four points $a, b, c, d$ on the unit circle arranged in the counter-clockwise direction, denoted by $Q = \{a, b, c, d\}$. It is defined in [8] that

$$V[Q] = \frac{V(b) - V(a)}{b - a} + \frac{V(d) - V(c)}{d - c} - \frac{V(c) - V(b)}{c - b} - \frac{V(d) - V(a)}{d - a}$$

and

$$\|V\|_{cr} = \sup_{Q} |V[Q]|,$$

where the supremum is taken over all quadruples $Q$ with $cr(Q) = 1$.

One can show that $\|V\|_{cr} = 0$ if and only if $V$ is a quadratic polynomial. Furthermore, it is true that $V$ is Zygmund bounded if and only if $\|V\|_{cr}$ is finite. We let $\mathcal{Z}(\mathbb{S}^1)$ be the space of Zygmund bounded tangent vector fields on $\mathbb{S}^1$ modulo quadratic polynomials.

Let $\lambda \in \mathcal{ML}_b(\mathbb{D})$. For each $t \geq 0$, let $h_t$ be a quasisymmetric homeomorphism of $\mathbb{S}^1$ defined by an earthquake map on $\mathbb{D}$ inducing $t\lambda$. Suppose that $h_t$ fixes three common points for all $t \geq 0$, which is called an earthquake curve determined by $t\lambda, t \geq 0$. It is shown in [7] that $h_t(z)$ is differentiable on $t$ at each point $z \in \mathbb{S}^1$ and furthermore

$$\frac{d}{dt}h_t(z)|_{t=0} = \int_{\mathcal{G}} E_{ab}(z)d\lambda(a,b) \text{ modulo a quadratic polynomial,}$$

where

$$E_{ab}(z) = \begin{cases} 0 & \text{for } z \text{ outside of } [a,b], \\ \frac{(z-a)(z-b)}{a-b} & \text{for } z \in [a,b]. \end{cases}$$

Here we have an agreement that for each geodesic line connecting $a, b$ in $\mathcal{L}$, $[a, b]$ denotes the short arc on $\mathbb{S}^1$ between $a$ and $b$ and in the counter-clockwise direction.

We denote by

$$V_\lambda = \dot{E}^\lambda|_{\mathbb{S}^1} = \int_{\mathcal{G}} E_{ab}(z) d\lambda(a, b).$$

Then the integral introduces an injective map $\dot{E}$ from $\mathcal{ML}_b(\mathbb{D})$ into $\mathcal{Z}(\mathbb{S}^1)$ [6], and $\dot{E}^\lambda$ is called an infinitesimal earthquake map determined by $\lambda$. Conversely, Gardiner [6] showed that for any $V \in \mathcal{Z}(\mathbb{S}^1)$, there exists a $\lambda_V \in \mathcal{ML}_b(\mathbb{D})$ such that

$$V(z) = \int_{\mathcal{G}} E_{ab}(z) d\lambda_V(a, b) \text{ modulo a quadratic polynomial.}$$

Furthermore, if two $V$'s differ by a quadratic polynomial, then the corresponding $\lambda$'s are the same. Therefore, $\dot{E}$ is a bijection between $\mathcal{ML}_b$ and $\mathcal{Z}(\mathbb{S}^1)$. The inverse of $\dot{E}$ is often called the infinitesimal earthquake measure map, and it is denoted as

$$\dot{\mathcal{EM}} : \mathcal{Z}(\mathbb{S}^1) \to \mathcal{ML}_b(\mathbb{D}) : V \mapsto \lambda_V.$$

### 7.2. Pointwise Convergence of infinitesimal earthquake maps

The infinitesimal version of Theorem B is also proved in [17], which can be improved by the following proposition.

**Proposition 4.** *Let $\lambda \in \mathcal{ML}_b(\mathbb{D})$ and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathcal{ML}_b(\mathbb{D})$ with uniformly bounded Thurston norms. Then $\lambda_n$ converges to $\lambda$ in the weak* topology if and only if $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ converges to $\dot{E}^\lambda|_{\mathbb{S}^1}$ pointwise on $\mathbb{S}^1$ when all $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$ are properly normalized.*

*Proof.* It is proved in [17] that if $\{\lambda_n\}_{n=1}^\infty$ converges to $\lambda$ in the weak* topology, then $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ converges to $\dot{E}^\lambda|_{\mathbb{S}^1}$ pointwise on $\mathbb{S}^1$ when all $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$ are properly normalized. We only need to prove the other direction.

We normalize $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$, by subtracting quadratic polynomials, such that $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$ vanish at three common points of $\mathbb{S}^1$. Assume on the contrary that $\lambda_n$ does not converge to $\lambda$ in the weak* topology. By the same argument in [20], there exists a subsequence $\{\lambda_{n_j}\}_{j=1}^\infty$ of $\{\lambda_n\}_{n=1}^\infty$ weakly converging to $\kappa \in \mathcal{ML}_b(\mathbb{D})$. For simplicity, we rename $\{\lambda_{n_j}\}_{j=1}^\infty$ to be

$\{\lambda_n\}_{n=1}^{\infty}$. The assumption that $\lambda_n$ does not converge to $\lambda$ in the weak* topology implies $\kappa \neq \lambda$. Normalizing $\dot{E}^{\kappa}|_{\mathbb{S}^1}$ as the same as $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$, it follows that $\dot{E}^{\lambda_n}|_{\mathbb{S}^1} \to \dot{E}^{\kappa}|_{\mathbb{S}^1}$ pointwise as $n \to \infty$. By the assmption that $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ pointwise converges to $\dot{E}^{\lambda}|_{\mathbb{S}^1}$, we know that $\dot{E}^{\kappa}|_{\mathbb{S}^1} = \dot{E}^{\lambda}|_{\mathbb{S}^1}$. Thus $\kappa = \lambda$, which is a contradiction. □

### 7.3. Asymptotic cross-ratio vanishing equivalence on $\mathcal{Z}(\mathbb{S}^1)$

The tangent space of $AT(\mathbb{D})$ at a base point is the quotient space $\mathcal{Z}(\mathbb{S}^1)/\mathcal{Z}_0(\mathbb{S}^1)$, where an element $V$ of $\mathcal{Z}(\mathbb{S}^1)$ belongs to $\mathcal{Z}_0(\mathbb{S}^1)$ provided that the constant $M$ in (7.1) converges to 0 independent of $x$ as $|t| \to 0$. In this subsection, we first introduce an alternative definition of the elements in $\mathcal{Z}_0(\mathbb{S}^1)$ by using $V[Q]$.

In the previous sections, we have used two cross ratios. One is used to define the Liouville measure of a box $B = [a,b] \times [c,d]$ and the other is used to define the cross-ratio distortion norm. They are used in different situations based on different purposes and the quantitative results are different. In the following, one can see that their infinitesimal versions only differ up to multiplication by 2.

**Definition 9.** *Let $B = [a,b] \times [c,d]$ be a box of geodesics, where $a, b, c, d$ lie on $\mathbb{S}^1$ in the counter-clockwise direction. For any $V \in \mathcal{Z}(\mathbb{S}^1)$, we set*

$$V_L[B] = \frac{V(a) - V(c)}{a - c} + \frac{V(b) - V(d)}{b - d} - \frac{V(a) - V(d)}{a - d} - \frac{V(b) - V(c)}{b - c},$$

*and the cross-ratio norm $\|V\|_{cr_L}$ of $V$ is defined by*

$$\|V\|_{cr_L} = \sup_B |V_L[B]|,$$

*where the supremum is taken over all $B$ with Liouville measure $L(B) = \log 2$.*

**Proposition 5.** *For any $Q = \{a, b, c, d\}$ with $cr_L(Q) = 2$,*

$$V[Q] = 2V_L[Q].$$

*Proof.* Let $V \in \mathcal{Z}(\mathbb{S}^1)$. Given any $Q = \{a, b, c, d\}$ with $cr_L(Q) = 2$, we consider that $a$, $b$, $c$ and $d$ are temporarily fixed. Let $f_t(z) = z + tV$. Then $f_t(Q)$ is a quadruple of four distinct points when $|t|$ is sufficiently small.

Clearly, $cr_L(Q) = 2$ if and only if $cr(Q) = 1$. By the definitions of $V[Q]$ and $V_L[Q]$, we obtain

$$\begin{aligned} V_L[Q] &= \frac{d}{dt}\ln cr_L(f_t(Q))|_{t=0} = \frac{d}{dt}\ln(1 + cr(f_t(Q)))|_{t=0} \\ &= \frac{\frac{d}{dt}cr(f_t(Q))}{1 + cr(f_t(Q))}|_{t=0} = \frac{1}{2}\frac{d}{dt}cr(f_t(Q))|_{t=0} \\ &= \frac{1}{2}\frac{d}{dt}e^{\ln cr(f_t(Q))}|_{t=0} = \frac{1}{2}e^{\ln cr(f_t(Q))}\frac{d}{dt}\ln cr(f_t(Q))|_{t=0} \\ &= \frac{1}{2}V[Q]. \end{aligned}$$

Therefore, $V[Q] = 2V_L[Q]$. □

Now we introduce an alternative characterization of the elements in $\mathcal{Z}_0(\mathbb{D})$.

**Lemma 8 ([12]).** *If two tangent vector fields $V$ and $\tilde{V}$ satisfy $\tilde{V}(x) = \frac{V(\gamma(x))}{\gamma'(x)}$ for an element $\gamma \in \text{Möb}(\mathbb{D})$, then for any quadruple $Q$ of four points $a, b, c, d$ on the unit circle in the counter-clockwise order,*

$$\tilde{V}[Q] = V[\gamma(Q)] \ \ (\text{or} \ \ V[Q] = \tilde{V}[\gamma^{-1}(Q)]).$$

Similar to the definition of a degenerating sequence of boxes in $\mathcal{G}$, we define a degenerating sequence of quadruples to be a sequence $\{Q_n\}_{n=1}^{\infty}$ of quadruples $Q_n$ such that $cr(Q_n) = 1$ for all $n$ and $s(Q_n) \to 0$ as $n \to \infty$, where $s(Q)$ is the minimum scale of $Q = \{a, b, c, d\}$; that is,

$$s(Q) = \min\{|a - b|, |b - c|, |c - d|, |d - a|\}.$$

**Proposition 6.** *A continuous tangent vector field $V$ on $\mathbb{S}^1$ belongs to $\mathcal{Z}_0(\mathbb{S}^1)$ if and only if*

$$\sup_{\{B_n\}} \limsup_{n\to\infty} V_L[B_n] = 0 \ \ \text{or} \ \ \sup_{\{Q_n\}} \limsup_{n\to\infty} V[Q_n] = 0, \tag{7.2}$$

*where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes or all degenerating sequences $\{Q_i\}_{i=1}^{\infty}$ of quadruples.*

*Proof.* Let $V$ be a continuous tangent vector field on $\mathbb{S}^1$ and $p \in \mathbb{S}^1$, and let $\gamma_p$ be an orientation-preserving Möbius transformation from $\mathbb{H}$ onto $\mathbb{D}$ mapping $i$ to the origin, 0 to $p$ and $\infty$ to $-p$. Assume that $V_p(z) = V(\gamma_p(z))/\gamma_p'(z)$. We note that $V \in \mathcal{Z}_0(\mathbb{S}^1)$ if and only if $V_p \in \mathcal{Z}_0(\mathbb{R})$ for any $p \in \mathbb{S}^1$, where $\mathcal{Z}_0(\mathbb{R})$ denotes the space of all continuous functions $V_p$ defined on $\mathbb{R}$ satisfying

$$\frac{V_p(x+t)+V_p(x-t)-2V_p(x)}{t}=\delta(x,t)$$

for any points $x$ and $t$ on $\mathbb{R}$, and $\delta(x,t)$ converges to 0 uniformly on $x$ as $t \to 0$.

In the following, we first show that if $V$ satisfies the condition (7.2) then $V \in \mathcal{Z}_0(\mathbb{S}^1)$. By the above note, it suffices to show that $V_p \in \mathcal{Z}_0(\mathbb{R})$ for any $p \in \mathbb{S}^1$.

Let $p$ be a point on $\mathbb{S}^1$. Using a lemma in [12] (see Lemma 10 in Subsection 7.5), we see that $V_p[Q]=V[\gamma_p(Q)]$. Given any quadruple $Q=\{x-t,x,x+t,\infty\}$,

$$\frac{V_p(x+t)+V_p(x-t)-2V_p(x)}{t}=V_p[Q]=V[\gamma_p(Q)].$$

Applying the condition (7.2) to $V$ on the quadruples $\gamma_p(Q)$ and using proof by contradiction, we can show that $V_p \in \mathcal{Z}_0(\mathbb{R})$.

Conversely, assuming that $V \in \mathcal{Z}_0(\mathbb{S}^1)$, we want to show it satisfies condition (7.2). Suppose not, it follows that there exist $\epsilon>0$ and a degenerating sequence $\{Q_n\}_{n=1}^{\infty}$ of quadruples such that

$$|V[Q_n]|>\epsilon$$

for each $n$.

Passing to a subsequence, we may assume that $a_n$, $b_n$, $c_n$ and $d_n$ converge to $a$, $b$, $c$ and $d$ on $\mathbb{S}^1$ respectively. Using the conditions that $cr(Q_n)=1$ for each $n$ and $s(Q_n)\to 0$ as $n\to\infty$, we conclude that the set $\{a,b,c,d\}$ contains at most two distinct points, namely $a$ and $d$. Now let $p$ be a point on $\mathbb{S}^1$ such that $-p$ is different from $a$ and $d$. Then $V_p(z)=V(\gamma_p(z))/\gamma_p'(z)\in\mathcal{Z}_0(\mathbb{R})$. Now we apply the infinitesimal Beurling-Ahlfors extension of $V_p$ to the upper half plane $\mathbb{H}$ introduced by Gardiner and Sullivan in [9]. It is shown there that if $V_p\in\mathcal{Z}_0(\mathbb{R})$, then $\mu=\bar\partial V_p$ is a Beltrami coefficient vanishing when approaching the boundary $\mathbb{R}$. Note that if $n$ is big enough, $Q_n$ is outside a neighborhood of $-p$ on $\mathbb{S}^1$, and hence $\gamma_p^{-1}(Q_n)$ is contained in a compact subset $K$ of $\mathbb{R}$. Denote by $Q_n'=\gamma_p^{-1}(Q_n)=\{a_n',b_n',c_n',d_n'\}$. Now applying the measurable Riemann mapping theorem to $t\mu$, the approximation

$$f^{t\mu}(z)=z+tV_p(z)+o(|t|)$$

is uniform on every compact subset of $\mathbb{R}$, where $|t| < 1$ and $f^{t\mu}$ is normalized to fix 0, 1 and $\infty$ for all $t$. Furthermore,

$$\begin{aligned} V_p(z) &= -\frac{z(z-1)}{\pi}\int\int \frac{\mu(\xi)d\zeta d\eta}{\xi(\xi-1)(\xi-z)} \\ &= \frac{1}{\pi}\int\int \mu(\xi)(\frac{z}{\xi-1} - \frac{z-1}{\xi} - \frac{1}{\xi-z})d\zeta d\eta, \end{aligned}$$

where $\xi = \zeta + i\eta$. Then

$$\begin{aligned} V[Q_n] &= V_p[Q'_n] \\ &= -\frac{1}{\pi}\int\int \frac{(c'_n - a'_n)(d'_n - b'_n)}{(\xi - a'_n)(\xi - b'_n)(\xi - c'_n)(\xi - d'_n)}\mu(\xi)d\zeta d\eta. \end{aligned}$$

By arranging $-p$ at a point on the open circular arc on $\mathbb{S}^1$ from $d$ to $a$ in the counterclockwise direction and using the corresponding $\gamma_p$, we may assume that $a'_n < b'_n < c'_n < d'_n$ for each sufficiently large $n$. Passing to a subsequence and without loss of generality, we assume further that $s(Q'_n) = c'_n - b'_n$ for all sufficiently large $n$. For each sufficiently large $n$, substituting $\xi$ by $\xi = (c'_n - b'_n)w + b'_n$ we obtain

$$V_p[Q'_n] = -\frac{1}{\pi}\int\int \frac{\frac{c'_n-a'_n}{c'_n-b'_n}\frac{d'_n-b'_n}{c'_n-b'_n}}{(w-a''_n)w(w-1)(w-d''_n)}\mu((c'_n - b'_n)w + b'_n)dudv,$$

where $w = u + iv$ and $a''_n = -\frac{b'_n-a'_n}{c'_n-b'_n}$ and $d''_n = \frac{d'_n-b'_n}{c'_n-b'_n}$.

Since $c'_n - b'_n$ approaches 0 as $n \to \infty$ and $\mu$ vanishes near the real line $\mathbb{R}$, it follows that $\mu((c'_n - b'_n)w + b'_n)$ converges to 0 pointwise at almost every $w$ as $n \to \infty$. According to the relation between the maximal and minimal scales of $Q'_n$, we consider the following two cases. In one case, the ratios of the maximal scales over the minimal ones of $Q'_n$'s are bounded, and in the other, the ratios converge to $\infty$ as $n \to \infty$ (by passing to a subsequence if necessary). Using the condition $cr(Q'_n) = 1$ for each $n$, we obtain in the first case, $a''_n$'s, $\frac{c'_n-a'_n}{c'_n-b'_n}$'s and $\frac{d'_n-b'_n}{c'_n-b'_n}$'s are all bounded; in the other case, either $a'_n$ converges to $-1$, $\frac{c'_n-a'_n}{c'_n-b'_n}$ converges to 2 and $\frac{d'_n-b'_n}{c'_n-b'_n}$ converges to $\infty$ as $n \to \infty$ or $a'_n$ converges to $-\infty$, $\frac{c'_n-a'_n}{c'_n-b'_n}$ converges to $\infty$ and $\frac{d'_n-b'_n}{c'_n-b'_n}$ converges to 2 as $n \to \infty$. Applying Lebesgue's dominating convergence theorem, we conclude that in either case, $V[Q_n] = V_p[Q'_n]$ converges to 0 as $n \to \infty$. This is a contradiction to the assumption that $V[Q_n] > \epsilon > 0$ for each $n$. □

**Definition 10.** *Given two elements $V, V' \in \mathcal{Z}(\mathbb{S}^1)$, we say that $V$ is equivalent to $V'$, denoted by $V \sim V'$, if*

$$\sup_{\{B_n\}} \limsup_{n\to\infty} |V_L[B_n] - V'_L[B_n]| = 0 \text{ or } \sup_{\{Q_n\}} \limsup_{n\to\infty} |V[Q_n] - V'[Q_n]| = 0$$

*where the supremum is taken over all degenerating sequences $\{B_n\}_{n=1}^{\infty}$ of boxes or all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.*

*For each $V \in \mathcal{Z}(\mathbb{S}^1)$, we denote by $[V]$ the equivalence class of $V$ in $\mathcal{Z}(\mathbb{S}^1)$. Define*

$$\mathcal{AZ}(\mathbb{S}^1) = \mathcal{Z}(\mathbb{S}^1)/\sim .$$

Using Proposition 6, the following corollary is obvious.

**Corollary 6.** $\mathcal{AZ}(\mathbb{S}^1) = \mathcal{Z}(\mathbb{S}^1)/\mathcal{Z}_0(\mathbb{S}^1)$.

### 7.4. Proof of Theorem 3

*Proof.* We first show that if $[\lambda] = [\lambda']$, then $[V] = [V']$. Suppose that $[V] \neq [V']$. Then there exist $\epsilon > 0$ and a degenerating sequence of $\{B_n\}_{n=1}^{\infty}$ boxes such that, for all $n$

$$|\dot{E}^{\lambda}|_{\mathbb{S}^1}[B_n] - \dot{E}^{\lambda'}|_{\mathbb{S}^1}[B_n]| \geq \epsilon > 0.$$

Then for all $n$,

$$|\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B^*] - \dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}[B^*]| = |V[B_n] - V'[B_n]| \geq \epsilon > 0. \tag{7.3}$$

Since $\|(\gamma_{B_n})^*\lambda\|_{Th} = \|\lambda\|_{Th}$ and $\|(\gamma_{B_n})^*\lambda'\|_{Th} = \|\lambda'\|_{Th}$, $\{(\gamma_{B_n})^*\lambda\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^*\lambda'\}_{n=1}^{\infty}$ are uniformly Thurston bounded. Therefore there exist convergent subsequences of $\{(\gamma_{B_n})^*\lambda\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^*\lambda'\}_{n=1}^{\infty}$ in the weak$^*$ topology. For simplicity, we denote them by the same notation. In the proof of the sufficiency of Theorem 1, we have shown that the condition $[\lambda] = [\lambda']$ implies that the limit of $(\gamma_{B_n})^*\lambda$ equals to the limit of $(\gamma_{B_n})^*\lambda'$. We normalize $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$, by adding quadratic polynomials, such that $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$ take value 0 at three fixed points on $\mathbb{S}^1$. By Proposition 4, the (pointwise) limits of the two sequences $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$ are the same. This is a contradiction to (7.3).

Now we show that if $[V] = [V']$, then $[\lambda] = [\lambda']$. Assume on the contrary that $[\lambda] \neq [\lambda']$, then there exist a degenerating subsequence $\{B_n\}_{n=1}^{\infty}$

of boxes, and a continuous function $f$ on $\mathcal{G}$ with compact support contained in $B^*$ such that

$$\lim_{n\to\infty} \int_{B^*} fd((\gamma_{B_n})^*\lambda - (\gamma_{B_n})^*\lambda') \geq \epsilon > 0. \tag{7.4}$$

Since they are uniformly Thurston bounded, there exist convergent subsequences of $\{(\gamma_{B_n})^*\lambda\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^*\lambda'\}_{n=1}^{\infty}$ in the weak* topology, which we denote by the same notation for simplicity. Using the assumption (7.4), we know that the weak* limit of $(\gamma_{B_n})^*\lambda$ does not equals to the limit of $(\gamma_{B_n})^*\lambda'$. By Proposition 4, the pointwise limits of the two sequences $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$ are not the same even though they vanish at three common points on $\mathbb{S}^1$. Thus there exists a box $B_0 \in \mathcal{G}$ with $L(B_0) = \log 2$ such that

$$\lim_{n\to\infty} |\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B_0]| \geq \delta > 0. \tag{7.5}$$

On the other hand, using the assumption that $V \sim V'$ and $\lim_{n\to\infty} s(B_n) = 0$, we obtain

$$\lim_{n\to\infty} |\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B_0]|$$

$$= \lim_{n\to\infty} |\dot{E}^{\lambda'}|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)] - \dot{E}^{\lambda}|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)]| \leq \limsup_{s(B)\to 0} |V'[B] - V[B]| = 0.$$

This is a contradiction to (7.5). It follows that $[\lambda] = [\lambda']$. □

## 7.5. Asymptotic Thurston's norm and asymptotic cross-ratio norm

It is shown in [11] and [12] with two different methods that the cross-ratio norm $||V||_{cr}$ of a vector field $V \in \mathcal{Z}(\mathbb{S}^1)$ and the Thurston's norm $\|\lambda_V\|_{Th}$ of $\lambda_V$ are equivalent. Now we define the following.

**Definition 11.** *Given a vector field $V \in \mathcal{Z}(\mathbb{S}^1)$, the asymptotic cross-ratio norm $\|V\|_{\widehat{cr}}$ of $V \in \mathcal{Z}(\mathbb{S}^1)$ is defined to be*

$$\|V\|_{\widehat{cr}} = \sup_{\{Q_n\}} \limsup_{n\to\infty} |V[Q_n]|,$$

*where the supremum is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.*

In this subsection, we prove Theorem 6, which is viewed as an asymptotic version of the result

$$\|\lambda_V\|_{Th} \leq C\|V\|_{cr}$$

for a universal positive constant $C$ in [12]. In the next subsection, we apply this theorem to prove the continuity of the inverse of the induced infinitesimal earthquake measure map $\widehat{\mathcal{E}\dot{\mathcal{M}}}$.

The strategy of the proof of Theorem 6 is similar to the one used to prove the inequality $||\lambda_V||_{Th} \leq C||V||_{Th}$ in [12], but extra effort has to be made in order to have the scales $s(Q)$ of selected quadruples $Q$ approach 0 as disks $D$ of hyperbolic diameter $\leq \frac{1}{2}$ approach the boundary $\mathbb{S}^1$ of $\mathbb{D}$. We first recall three technical lemmas developed in [12].

Let $\lambda$ denote a Thurston bounded measured lamination and $V = V_\lambda$. Let $B$ be an orientation-preserving Möbius transformation from the upper half plane $\mathbb{H}$ or the unit open disk $\mathbb{D}$ onto $\mathbb{D}$, and $\tilde{\lambda} = (B^*\lambda)$ be the pullback of $\lambda$ by $B$ (or the pushforward of $\lambda$ by $B^{-1}$). And define

$$\tilde{V}(x) = V_{\tilde{\lambda}}(x) = \dot{E}^{\tilde{\lambda}}(x) = \int_{\mathcal{G}} E_{ab}(x)d\tilde{\lambda}(a,b),$$

where $E_{ab}(x)$ is defined by the same formula given in Section 7.1.

**Lemma 9 ([12]).** *The vector fields $V$ and $\tilde{V}$ satisfy*

$$\tilde{V}(x) = \frac{V(B(x))}{B'(x)} \text{ modulo a quadratic polynomial.}$$

**Lemma 10 ([12]).** *Let $B$ be a Möbius transformation from $\mathbb{D}$ or $\mathbb{H}$ onto $\mathbb{D}$ or $\mathbb{H}$. Assume that two continuous tangent vector fields $\tilde{V}$ and $V$ on $\mathbb{S}^1$ or $\mathbb{R}$ satisfy the condition in the previous lemma. Then for any quadruple $Q$ of four points,*

$$\tilde{V}[Q] = V[B(Q)].$$

**Lemma 11 ([12]).** *Assume $\rho > 0$, $-\infty \leq a < b < c < d$, and $c \leq s \leq d \leq t$. Let $V(x) = \rho E_{st}(x)$ and $Q = \{a,b,c,d\}$. Consider $V[Q]$ as a function of $s$ and $t$. Then $V[Q] \geq 0$ and $V[Q]$ is an increasing function on $t$ for each fixed $s$ and a decreasing function on $s$ for each fixed $t$.*

**Lemma 12 ([12]).** *Assume $\rho > 0$, $-\infty \leq a < b < c < d \leq \infty$, and $b \leq s \leq c$ and $t \geq d$. Let $V(x) = \rho E_{st}(x)$ and $Q = \{a,b,c,d\}$. Consider $V[Q]$ as a function of $s$ and $t$. Then $V[Q] \geq 0$ and $V[Q]$ is increasing on $s$ for each fixed $t$ and also increasing on $t$ for each fixed $s$.*

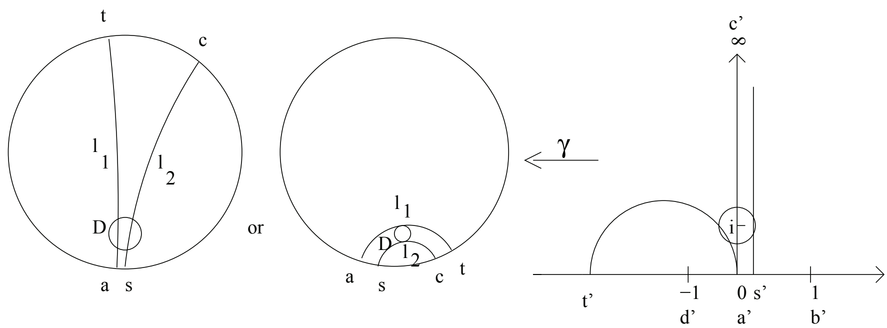


Figure 2: Illustration of the quadruples $Q$ and $Q'$ in the proof of Theorem 6.

Now we prove Theorem 6.

*Proof.* Let $r_0$ be a constant between 0 and 1, which will be selected later. Let $D$ denote a closed disk in $\mathbb{D}$ of hyperbolic diameter $\leq r_0$. It suffices to show that there exists a universal positive constant $C$ such that the measure of the leaves of the lamination $\mathcal{L}$ intersecting $D$ is less than or equal to $C||V||_{\widehat{cr}}$.

Suppose that $D$ is near the boundary. Let $l_1$ and $l_2$ denote the lines in the lamination $\mathcal{L}$ of $\lambda$ which bound all the lines in $\mathcal{L}$ intersecting $D$. We label the endpoints of $l_1$ and $l_2$ by $a$, $s$, $c$ and $t$ in the counter-clockwise order such that $a$ and $t$ are the endpoints of one leaf and $s$ and $c$ are the endpoints of the other, and furthermore the length of the arc between $a$ and $s$ is less than or equal to the length of the arc between $c$ and $t$. In the special case that $l_1$ and $l_2$ share one endpoint, then $a = s$ or $c = t$; in the one that they share both endpoints, then $a = s$ and $c = t$.

Let $p$ be a point on the intersection of $D$ with the geodesic connecting $a$ and $c$. Now we let $B$ be an orientation-preserving Möbius transformation from $\mathbb{H}$ onto $\mathbb{D}$ such that $B^{-1}(p) = i$, $B^{-1}(a) = 0$ and $B^{-1}(c) = \infty$. It also follows that $t' = B^{-1}(t) < 0$, $s' = B^{-1}(s) \geq 0$, and $B(D)$ is a disk containing of $i$ and with hyperbolic diameter $\leq r_0$. Now it is easy to see that if $r_0$ is small enough, then $t' < -2$ and $0 \leq s' < 1$. Furthermore, we can see that as soon as $r_0$ is small enough, the properties $t' < -2$ and $0 \leq s' < 1$ hold universally in the sense that for any disk $D$ of diameter $r_0$ in the hyperbolic metric and any geodesic lamination $\mathcal{L}$, the two variables $t'$ and $s'$ resulting from the previous process satisfy those two inequalities. We thereby choose $r_0$ to be such a positive constant.

Let $Q' = \{0, 1, \infty, -1\}$, which is a quadruple on the extended line $\mathbb{R} \cup \{\infty\}$ with $cr(Q') = -1$. Let $Q = B(Q')$. By using a similar idea to that in the proof of Lemma 1, one can show that $s(Q)$ goes to 0 if $D$ approaches the boundary $\mathbb{S}^1$.

Now assume that $\tilde{\lambda}$ and $\tilde{V}$ are as the same as introduced in this subsection. Clearly, $cr(Q) = cr(Q') = -1$. By Lemma 8,

$$\tilde{V}[Q'] = V[Q].$$

Let $\tilde{\mathcal{L}} = B(\mathcal{L})$, where $\mathcal{L}$ is the lamination for $\lambda$. In order to estimate $\tilde{V}[Q']$, we divide the lines in the lamination $\tilde{\mathcal{L}}$ that affect the value of $V[Q']$ into three groups. Let $\mathcal{L}_m$ denote the collection of the lines in $\tilde{\mathcal{L}}$ intersecting $B(D)$, $\mathcal{L}_b$ denote the collection of the lines in $\tilde{\mathcal{L}} \setminus \mathcal{L}_m$ connecting points in $(s', 1)$ to points in $(1, \infty)$, and $\mathcal{L}_d$ the collection of the lines in $\tilde{\mathcal{L}} \setminus \mathcal{L}_m$ connecting points in $(t', -1)$ to points in $(-1, 0)$. Denote by $\lambda_k = \tilde{\lambda}|_{\mathcal{L}_k}$ and $V_i = \dot{E}^{\lambda_k}$ for $k = m, b, d$. By the linearity of the operator $\dot{E}$, we obtain

$$\tilde{V}[Q'] = V_m[Q'] + V_b[Q'] + V_d[Q'].$$

Note that the order required on the four points of a quadruple in the assumptions of Lemmas 11 and 12 should be viewed as the counterclockwise order on the extended line $\mathbb{R} \cup \{\infty\}$; that is, the conclusions of Lemmas 11 and 12 hold as soon as the four points of the quadruple are arranged in the counterclockwise order on $\mathbb{R} \cup \{\infty\}$. Now we denote $Q'$ by $\{a', b', c', d'\}$. Using Lemma 11, we obtain

$$V_d[Q'] = V_d[\{a', b', c', d'\}] \geq 0 \;\text{ and }\; V_b[\{c', d', a', b'\}] \geq 0.$$

Then

$$V_b[Q'] = V_b[\{a', b', c', d'\}] = -V_b[\{b', c', d', a'\}] = V_b[\{c', d', a', b'\}] \geq 0.$$

Therefore

$$\tilde{V}[Q'] \geq V_m[Q'].$$

In the next, we use Lemma 12 to obtain an explicit lower bound for $V_m[Q']$, which enables us to complete the proof. By Lemma 12, if we move the weights of the geodesic lines in the lamination $\mathcal{L}_m$ to the geodesic line connecting $t'$ to $s'$, then the value of $V_m[\{b', c', d', a'\}]$ is possibly increased, and hence the value of $V_m[Q'] = -V_m[\{b', c', d', a'\}]$ is possibly decreased. Therefore

$$V_m[Q'] \geq (\rho E_{t's'})[Q'],$$

where $\rho = \tilde{\lambda}(\mathcal{L}_m)$. It is easy to check

$$E_{t's'}[Q'] = \frac{2E_{t',s'}(d') - E_{t',s'}(a')}{a' - d'}$$

$$= \frac{2 + 2t' + 2s' + t's'}{t' - s'} = \frac{2[-(1+t')] + s'[-(2+t')]}{s' - t'}.$$

Since $0 \leq s' < 1$ and $t' < -2$, $2[-(1+t')] \geq 2$, $s'[-(2+t')] > 0$, and $s' - t' \leq 1 - t'$. It follows that

$$\frac{2[-(1+t')] + s'[-(2+t')]}{s' - t'} \geq \frac{2[-(1+t')]}{1 - t'} = 2\frac{t'+1}{t'-1}.$$

Clearly, $2\frac{t'+1}{t'-1}$ attains its minimal value $\frac{2}{3}$ on the interval $(-\infty, -2]$. Thus

$$V_m[Q'] \geq \frac{2}{3}\rho.$$

In summary,

$$V[Q] = \tilde{V}[Q'] \geq V_m[Q'] \geq \frac{2}{3}\rho,$$

where $\rho = \tilde{\lambda}(\mathcal{L}_m)$, which is equal to the $\lambda$ measure of the lines of $\mathcal{L}$ intersecting $D$, $D$ is a closed disk in $\mathbb{D}$ of hyperbolic diameter $\leq r_0$ and $0 < r_0 < 1$, and $cr(Q) = cr(Q') = -1$.

Now let $Q_1' = \{-1, 0, 1, \infty\}$ and $Q_1 = B(Q_1')$. Then $cr(Q_1') = cr(Q_1) = 1$, $s(Q_1) = s(Q)$, and furthermore

$$V[Q_1] = -V[Q] \leq -\frac{2}{3}\rho.$$

Thus

$$|V[Q_1]| \geq \frac{2}{3}\rho.$$

It follows that there exists a universal positive constant $C$ such that

$$||\lambda||_{\widehat{Th}} \leq C||V||_{\widehat{cr}}.$$

□

### 7.6. Proof of Theorem 4

For any $V \in \mathcal{Z}(\mathbb{S}^1)$, define

$$\|V\|_{\widehat{cr_L}} = \sup_{\{B_n\}} \limsup_{n\to\infty} |V[B_n]|,$$

where the supremum is taken over all degenerating sequences $\{B_n\}_{n=1}^{\infty}$ of boxes.

By Proposition 5, the following corollary is obvious.

**Corollary 7.** *For any* $V \in \mathcal{Z}(\mathbb{S}^1)$, $\|V\|_{\widehat{cr}} = 2\|V\|_{\widehat{cr_L}}$.

Now we prove Theorem 4.

*Proof.* We first show that $\widehat{\mathcal{E}\dot{\mathcal{M}}}^{-1}$ is continuous. Assume that $[\lambda_n] \to [\lambda]$ in the asymptotically uniform weak* topology, we prove $[V_n] = \widehat{\mathcal{E}\dot{\mathcal{M}}}^{-1}([\lambda_n])$ goes to $[V] = \widehat{\mathcal{E}\dot{\mathcal{M}}}^{-1}([\lambda])$ in the $\|\cdot\|_{\widehat{cr_L}}$ norm. By Corollary 2, since $[\lambda_n] \to [\lambda]$ in the asymptotically uniform weak* topology, there exist $\lambda'_n \in [\lambda_n]$ converging to $\lambda' \in [\lambda]$ in the uniform weak* topology. Let $V'_n = \mathcal{E}\dot{\mathcal{M}}^{-1}(\lambda'_n)$ and $V' = \mathcal{E}\dot{\mathcal{M}}^{-1}(\lambda')$. By the homeomorphic property of $\mathcal{E}\dot{\mathcal{M}}$ ([17]), $V'_n$ converges to $V'$ in the $\|\cdot\|_{cr_L}$ norm. Since $\|V\|_{cr_L} \geq \|V\|_{\widehat{cr_L}}$ for any $V \in \mathcal{Z}(\mathbb{S}^1)$, it follows that $[V'_n] = [V_n]$ converges to $[V'] = [V]$ in the $\|\cdot\|_{\widehat{cr_L}}$ norm.

It remains to show that $\widehat{\mathcal{E}\dot{\mathcal{M}}}$ is continuous. Assume that $[V_n] \to [V]$ in the asymptotically uniform weak* topology. Let $[\lambda_n] = \widehat{\mathcal{E}\dot{\mathcal{M}}}([V_n])$ and $[\lambda] = \widehat{\mathcal{E}\dot{\mathcal{M}}}([V])$. We prove $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology.

Since $[V_n] \to [V]$ in the asymptotic cross-ratio norm $\|\cdot\|_{\widehat{cr_L}}$, $\{\|V_n\|_{\widehat{cr_L}}\}_{n=1}^{\infty}$ is bounded. By Corollary 7, $\|V_n\|_{\widehat{cr}} = 2\|V_n\|_{\widehat{cr_L}}$. Hence $\{\|V_n\|_{\widehat{cr}}\}_{n=1}^{\infty}$ is bounded. By Theorem 6, $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded. Using Lemma 6 to replace the representatives, we may assume that $\{\lambda_n\}_{n=1}^{\infty}$ is uniformly Thurston bounded.

Suppose on the contrary that $[\lambda_n]$ does not converges to $[\lambda]$ in the asymptotically uniform weak* topology. Then there exist a degenerating sequence $\{B_n\}_{n=1}^{\infty}$ of boxes and a continuous function $f$ on $\mathcal{G}$ with compact support contained in $B^*$ such that

$$\lim_{n\to\infty} \int_{B^*} f d((\gamma_{B_n})^*\lambda_n - (\gamma_{B_n})^*\lambda) > \epsilon > 0. \tag{7.6}$$

Since $\|(\gamma_{B_n})^*\lambda_n\|_{Th} = \|\lambda_n\|_{Th}$, $\|(\gamma_{B_n})^*\lambda\|_{Th} = \|\lambda\|_{Th}$ and $\{\|\lambda_n\|_{Th}\}_{n=1}^{\infty}$ is bounded, it follows that $(\gamma_{B_n})^*\lambda_n$ and $(\gamma_{B_n})^*\lambda$ are uniformly Thurston bounded. Therefore there exist convergent subsequences of $\{(\gamma_{B_n})^*\lambda_n\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^*\lambda\}_{n=1}^{\infty}$ in the weak* topology, which we denote by the same notation for simplicity. By the assumption (7.6), we see that the weak* limit of $(\gamma_{B_n})^*\lambda_n$ does not equal to the limit of $(\gamma_{B_n})^*\lambda$. By Proposition 4, the pointwise limits of the two sequences $\dot{E}^{(\gamma_{B_n})^*\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ are not the same even they vanish at the three common points on $\mathbb{S}^1$. Thus there exists a box $B_0 \in \mathcal{G}$ with $L(B_0) = \log 2$ such that

$$\lim_{n\to\infty} |\dot{E}^{(\gamma_{B_n})^*\lambda_n}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B_0]| > \epsilon' > 0.$$

By Lemma 1, $s(\gamma_{B_n}(B_0)) \to 0$ as $n \to \infty$. Using the assumption that $[V_n] \to [V]$ in the asymptotically uniform weak* topology, we obtain

$$\lim_{n\to\infty} |\dot{E}^{(\gamma_{B_n})^*\lambda_n}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B_0]|$$

$$= \lim_{n\to\infty} |\dot{E}^{\lambda_n}|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)] - \dot{E}^{\lambda}|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)]| \leq \limsup_{s(B)\to 0} |V_n[B] - V[B]| = 0.$$

This is a contradiction to (7.6). It follows that $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology. □

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