Pure and Applied Mathematics Quarterly
Volume 11, Number 3, 393–402, 2015

Random Walks on Complete Multipartite Graphs

Xiao Chang and Hao Xu

Abstract: We apply Chung-Yau invariants to calculate the number of spanning trees of a complete multipartite graph. We also give explicit formulas for hitting times of random walks on a complete multipartite graph and prove that it has symmetric hitting times if and only it is vertex-transitive.
Keywords: Random walk, Chung-Yau invariants, complete multipartite graph.

1. Introduction

Let $K_{k_1,k_2,\ldots,k_p}$ denote the complete p-partite graph on vertex set $V = V_1 \cup \cdots \cup V_p$ where $|V_i| = k_i$ for $1 \le i \le p$ and $\cup$ denotes disjoint union. Let $n = k_1 + \cdots + k_p$ be the total number of vertices. Let $\tau(K_{k_1,k_2,\ldots,k_p})$ be the number of spanning trees in $K_{k_1,k_2,\ldots,k_p}$. It is well-known that

$$\tau(K_{k_1,k_2,\ldots,k_p}) = n^{p-2}\prod_{t=1}^{p}(n-k_t)^{k_t-1}. \tag{1}$$

Austin's proof [1] relies on Kirchhoff's matrix-tree theorem. Bijective proofs of (1) were given by Eğecioğlu-Remmel [4] and Lewis [7].

Recall that a *vertex-weighted graph* is an undirected simple graph G equipped with a weight function $w : V(G) \to \mathbb{R}$. Usually w_x at $x \in V(G)$ is the degree of x in some fixed ambient graph of G, hence $d(x) \le w_x \in \mathbb{Z}$. In

Received November 6, 2016.
MSC(2010) 05C81 (05C50 60G50)

[10], we defined *Chung-Yau invariants* $R(G,w)$ and $Z(G,w)$ for a vertex-weighted graph (G,w) by

$$\det B = R(G,w) + Z(G,w)s, \tag{2}$$

where the matrix B is given by

$$B(x,y) = \begin{cases} w_x^2 s + w_x & \text{if } x = y, \\ w_x w_y s - 1 & \text{if } x \sim y, \\ w_x w_y s & \text{otherwise.} \end{cases} \tag{3}$$

Basic properties and applications of Chung-Yau invariants to the computation of hitting times of random walks may be found in [2, 10, 11].

When $w_x = d(x)$ for all $x \in V(G)$, we will denote the weight function by d_G. Then we have $R(G,d_G)$=0 and by Kirchhoff's Matrix-Tree Theorem $R(G-\{x\},d_G) = \tau(G)$ for any $x \in V(G)$.

Let $\mathscr{C}_x(G)$ denote the set of simple closed walks starting at x, i.e., a closed walk with no repetitions of vertices other than the repetition of the starting and ending vertex. Note that if y is adjacent to x, then $(x,y,x) \in \mathscr{C}_x(G)$. We have following recursive equation (see [10, Lem. 2.6])

$$R(G,w) = w_x R(G-\{x\},w) - \sum_{C \in \mathscr{C}_x(G)} R(G-C,w), \tag{4}$$

where a subgraph of a vertex-weighted graph is equipped with the natural inherited weights. It determines $R(G,w)$ uniquely together with the initial value $R(\emptyset) = 1$ for empty graph $\emptyset$.

A *random walk* on G is a time-reversible finite Markov chain $(X_0, X_1, \dots)$ with $X_i \in V$ that at each step it moves to a neighbor of the present vertex x with equal probability $1/d(x)$. The *hitting time* $H(x,y)$ is the expected number of steps to reach y when started from x. See [8] for a very readable discussion of random walks on graphs.

2. The number of spanning trees in terms of Chung-Yau invariants

Take a complete multipartite graph $K_{k_1,k_2,\dots,k_p}$ on vertex set $V = V_1 \cup \cdots \cup V_p$ where $|V_i| = k_i$. For $1 \le i \le p$, we assign a weight $d_i \in \mathbb{Z}$ to each of the k_i vertices in V_i. Denote by $r^{d_1,d_2,\dots,d_p}_{k_1,k_2,\dots,k_p}$ the R-invariant of the resulting vertex-weighted graph.

Let $r^{d_1,d_2,\dots,d_p}_{k_1,k_2,\dots,k_p} = d_1^{k_1-1}\cdots d_p^{k_p-1}F(k_1,\dots,k_p)$. Note that $F(k_1,\dots,k_p)$ also depends on d_i, $1\le i\le p$, which are dropped for simplicity of notation. Let $x\in V_1$. By (4), we have a recursive equation for $F(k_1,\dots,k_p)$,

$$
\begin{aligned}
F(k_1,\dots,k_p) =& F(k_1-1,k_2,\dots,k_p)\\
&- \sum_{C\in\mathscr{C}_x(G)} \frac{F(k_1-\phi_1(C),\dots,k_p-\phi_p(C))}{d_1^{\phi_1(C)}\cdots d_p^{\phi_p(C)}},
\end{aligned}
\tag{5}
$$

where $\phi_i(C)=|V(C)\cap V_i|$.

Theorem 2.1. *Let $K_{k_1,k_2,\dots,k_p}$ be a complete multipartite graph on vertex set $V=V_1\cup\cdots\cup V_p$ where $|V_i|=k_i$. For $1\le i\le p$, vertices in V_i are assigned a weight $d_i\in\mathbb{Z}$. Then*

$$
F(k_1,\dots,k_p) = d_1\cdots d_p - \sum_{t=2}^{p}(t-1)\sum_{\substack{I\coprod J=[p]\\|I|=t}}\prod_{i\in I}k_i\prod_{j\in J}d_j,
\tag{6}
$$

where $[p]=\{1,\dots,p\}$.

Proof. We only need to verify that $F(k_1,\dots,k_p)$ given by (6) satisfy the recursion (4). Namely,

$$
\begin{aligned}
&\sum_{t=2}^{p}(t-1)\sum_{\substack{I\coprod J=[p]\setminus\{1\}\\|I|=t-1}}\prod_{i\in I}k_i\prod_{j\in J}d_j\\
&= \sum_{C\in\mathscr{C}_x(G)}\frac{1}{d_1^{\phi_1(C)}\cdots d_p^{\phi_p(C)}}\left(\prod_{i=1}^{p}d_i - \sum_{t=2}^{p}(t-1)\sum_{\substack{I\coprod J=[p]\\|I|=t}}\prod_{i\in I}(k_i-\phi_i(C))\prod_{j\in J}d_j\right).
\end{aligned}
\tag{7}
$$

Note that the left-hand side is equal to $F(k_1-1,\dots,k_p)-F(k_1,\dots,k_p)$.

First we prove the polynomial part on the right-hand side, which is

$$
\sum_{t=2}^{p}\sum_{\substack{I\coprod J=[p]\setminus\{1\}\\|I|=t-1}}(t-1)!\prod_{i\in I}k_i\left(\prod_{j\in J}d_j - \sum_{\substack{L_1\coprod L_2=[p]\setminus(\{1\}\cup I)\\|L_1|\ge 2}}(|L_1|-1)\prod_{i\in L_1}k_i\prod_{j\in L_2}d_j\right),
$$

is equal to the left-hand-side of (7). Given any $I\coprod J=[p]$ with $|I|\geq 1$, the coefficient of $\prod_{i\in I}k_i\prod_{j\in J}d_j$ on the right-hand side of (7) is equal to

$$\begin{aligned}|I|!-\sum_{\substack{I_1\coprod I_2=I\\|I_1|\geq1,|I_2|\geq2}}|I_1|!(|I_2|-1)&=|I|!-\sum_{i=1}^{|I|-2}\binom{|I|}{i}i!(|I|-i-1)\\&=|I|!-(|I|!-|I|)\\&=|I|,\end{aligned}$$

which is just the coefficient of $\prod_{i\in I}k_i\prod_{j\in J}d_j$ on the left-hand side of (7). We still remain to prove the fractional part on the right-hand side is zero which is not difficult to check. We omit the details. □

Lemma 2.2. *Let $n=k_1+\cdots+k_p$. Then*

$$\sum_{t=2}^{p}(t-1)\sum_{\substack{I\coprod J=[p]\setminus\{1\}\\|I|=t-1}}\prod_{i\in I}k_i\prod_{j\in J}(n-k_j)=(n-k_1)n^{p-2}. \tag{8}$$

Proof. The left-hand side is equal to

$$\begin{aligned}&\sum_{s=0}^{p-2}n^s\sum_{\substack{I\coprod J=[p]\setminus\{1\}\\1\leq|I|\leq p-1-s}}\left(|I|\prod_{i\in I}k_i\sum_{\substack{J_1\coprod J_2=J\\|J_1|=s}}\prod_{j\in J_2}(-k_j)\right)\\=&\sum_{s=0}^{p-2}n^s\sum_{\substack{I\coprod J=[p]\setminus\{1\}\\|J|=s}}\left(\sum_{j=1}^{|I|}(-1)^{|I|-j}\binom{|I|}{j}j\right)\prod_{i\in I}k_i\\=&(n-k_1)n^{p-2}.\end{aligned}$$

Note that in the last equation, we used

$$\sum_{j=1}^{|I|}(-1)^{|I|-j}\binom{|I|}{j}j=\begin{cases}1 & \text{if } |I|=1,\\0 & \text{if } |I|>1.\end{cases} \tag{9}$$

We conclude the proof. □

Lemma 2.3. *Let $n=k_1+\cdots+k_p$ and $d_i=n-k_i$ for $1\leq i\leq p$. Then $F(k_1,k_2,\ldots,k_p)=0$ and $F(k_1-1,k_2,\ldots,k_p)=(n-k_1)n^{p-2}$. In general, we have $F(k_1-s,k_2,\ldots,k_p)=s(n-k_1)n^{p-2}$.*

Proof. Under the above conditions, $r^{d_1,d_2,\ldots,d_p}_{k_1,k_2,\ldots,k_p}$ is equal to the determinant of the Laplacian matrix of $K_{k_1,k_2,\ldots,k_p}$, which is zero. So $F(k_1,k_2,\ldots,k_p)=0$.

The second identity follows from

$$F(k_1-1,k_2,\ldots,k_p) = F(k_1,k_2,\ldots,k_p) + \sum_{t=2}^{p}(t-1)\sum_{\substack{I\coprod J=[p]\setminus\{1\}\\ |I|=t-1}}\prod_{i\in I}k_i\prod_{j\in J}(n-k_j)$$

and Lemma 2.2. The last identity can be derived by induction. □

Now we can give a new proof about the number of spanning trees in a complete multipartite graph.

Theorem 2.4. *Let $n=k_1+\cdots+k_p$. Then the number of spanning trees in a complete p-partite graph $K_{k_1,k_2,\ldots,k_p}$ is equal to*

$$\tau(K_{k_1,k_2,\ldots,k_p}) = n^{p-2}\prod_{t=1}^{p}(n-k_t)^{k_t-1}.$$

Proof. By Kirchhoff's Matrix-Tree Theorem, $\tau(K_{k_1,k_2,\ldots,k_p}) = r^{d_1,d_2,\ldots,d_p}_{k_1-1,k_2,\ldots,k_p}$, where $d_i=n-k_i$ for $1\le i\le p$. We need only show that

$$F(k_1-1,k_2,\ldots,k_p) = (n-k_1)n^{p-2},$$

which is proved in Lemma 2.3. □

3. Hitting times of random walks on complete multipartite graphs

Lemma 3.1. *Let $n=k_1+\cdots+k_p$. Then*

$$\sum_{t=2}^{p}(t-1)\sum_{\substack{I\coprod J=[p]\setminus\{1,2\}\\ |I|=t-2}}\prod_{i\in I}k_i\prod_{j\in J}(n-k_j) = (2n-k_1-k_2)n^{p-3}. \tag{10}$$

Proof. The proof is similar to that of Lemma 2.2. The left-hand side is equal to

$$\sum_{s=0}^{p-2} n^s \sum_{\substack{I \coprod J=[p]\setminus\{1,2\} \\ |J|=s}} \left(\sum_{j=0}^{|I|} (-1)^{|I|-j} \binom{|I|}{j} (j+1) \right) \prod_{i\in I} k_i.$$

Then the lemma follows from

$$\sum_{j=0}^{|I|} (-1)^{|I|-j} \binom{|I|}{j} (j+1) = \begin{cases} 1 & \text{if } 0 \leq |I| \leq 1, \\ 0 & \text{if } |I| > 1. \end{cases} \tag{11}$$

Namely only terms corresponding to $s = p-2$ and $s = p-3$ are nonzero. □

Lemma 3.2. *Let $n = k_1 + \cdots + k_p$ and $d_i = n - k_i$ for $1 \leq i \leq p$. Then*

$$F(k_1 - 1, k_2 - 1, \ldots, k_p) = (2n - k_1 - k_2)n^{p-3}(n-1).$$

Proof. We have

$$\begin{aligned} &F(k_1 - 1, k_2 - 1, \ldots, k_p) = F(k_1 - 1, k_2, \ldots, k_p) \\ &+ \sum_{t=2}^{p} (t-1) \sum_{\substack{I \coprod J=[p]\setminus\{1\} \\ |I|=t-1}} \prod_{i\in I} k_i \prod_{j\in J} d_j - \sum_{t=2}^{p} (t-1) \sum_{\substack{I \coprod J=[p]\setminus\{1,2\} \\ |I|=t-2}} \prod_{i\in I} k_i \prod_{j\in J} d_j. \end{aligned}$$

So our identity follows from Lemma 2.2, Lemma 2.3 and Lemma 3.1. □

A connected graph G is called *reversible* if $H(x, y) = H(y, x)$ holds for any $x, y \in V(G)$. See [2, 5] for a discussion on reversible graphs. We know [10, Cor. 4.2] that G is reversible if and only if $Z(G - \{x\})$ is independent of $x \in V(G)$.

Lemma 3.3. *A graph G is reversible if and only if $\sum_{\substack{u\in V(G) \\ u\neq x}} d(u)R(G - \{x, u\})$ is independent of $x \in V(G)$.*

Proof. By [2, Lem. 3.5], we have $R(G - \{x, u\}) = \tau(G/\{x, u\})$. So the assertion of the lemma follows from [2, Prop. 3.12], [9, Cor. 4] or [6]. □

Given a complete p-partite graph $G = K_{k_1,k_2,\ldots,k_p}$ on vertex set $V = V_1 \cup \cdots \cup V_p$, let $n = k_1 + \cdots + k_p$ and $d_i = n - k_i$ for $1 \leq i \leq p$. For $x \in V_1$, we have

$$(12) \quad \sum_{\substack{u\in V(G)\\ u\neq x}} d(u)R(G-\{x,u\})$$

$$\begin{aligned}
=&d_1(k_1-1)r^{d_1,\ldots,d_p}_{k_1-2,k_2,\ldots,k_p}+\sum_{u=2}^{p} d_u k_u r^{d_1,\ldots,d_p}_{k_1-1,k_2,\ldots,k_{u-1},k_u-1,k_{u+1},\ldots,k_p}\\
=&d_1^{k_1-2}\prod_{i=2}^{p} d_i^{k_i-1}\Bigg((k_1-1)F(k_1-2,k_2,\ldots,k_p)\\
&\qquad+\sum_{u=2}^{p} k_u F(k_1-1,k_2,\ldots,k_{u-1},k_u-1,k_{u+1},\ldots,k_p)\Bigg).
\end{aligned}$$

By Lemma 2.3 and Lemma 3.2, we have

$$\begin{aligned}
F(k_1-2,k_2,\ldots,k_p)=&2(n-k_1)n^{p-2},\\
F(k_1-1,k_2,\ldots,k_{u-1},k_u-1,k_{u+1},\ldots,k_p)=&(2n-k_1-k_u)n^{p-3}(n-1).
\end{aligned}$$

Substituting into (12), we get

$$\sum_{\substack{u\in V(G)\\ u\neq x}} d(u)R(G-\{x,u\})$$

$$=d_1^{k_1-2}\prod_{i=2}^{p} d_i^{k_i-1}\left(2(k_1-1)(n-k_1)n^{p-2}+\sum_{u=2}^{p} k_u(2n-k_1-k_u)n^{p-3}(n-1)\right).$$

Similarly for $y\in V_p$, we have

$$\sum_{\substack{u\in V(G)\\ u\neq y}} d(u)R(G-\{y,u\})$$

$$=d_p^{k_p-2}\prod_{i=1}^{p-1} d_i^{k_i-1}\left(2(k_p-1)(n-k_p)n^{p-2}+\sum_{u=1}^{p-1} k_u(2n-k_p-k_u)n^{p-3}(n-1)\right).$$

It is not difficult to check that

(13)

$$\sum_{\substack{u\in V(G)\\ u\neq x}} d(u)R(G-\{x,u\})-\sum_{\substack{u\in V(G)\\ u\neq y}} d(u)R(G-\{y,u\})$$

$$=2(k_1-k_p)n^{p-3}d_1^{k_1-2}d_p^{k_p-2}\prod_{i=2}^{p-1}d_i^{k_i-1}$$
$$\times\left(\sum_{i=1}^{p}k_i^2(n-k_i)+3\sum_{1\le i<j<m\le p}k_ik_jk_m+\sum_{i=2}^{p-1}k_i^2+\sum_{2\le i<j\le p-1}k_ik_j\right).$$

Theorem 3.4. *A complete p-partite graph $K_{k_1,\dots,k_p}$ is reversible if and only $k_1=k_2=\cdots=k_p$, i.e., $K_{k_1,\dots,k_p}$ is vertex-transitive.*

Proof. By Lemma 3.3 and (13), we have that for $x\in V_i$ and $y\in V_j$ where $i\neq j$, $H(x,y)=H(y,x)$ if and only if $k_i=k_j$. □

Theorem 3.5. *Let $G=K_{k_1,\dots,k_p}$ be a complete p-partite graph on vertex set $V=V_1\cup\cdots\cup V_p$. Let $n=k_1+\cdots+k_p$ and $d_i=n-k_i$ for $1\le i\le p$. Then for $x\in V_1$, $y\in V_p$,*

$$H(x,y)=\frac{(2n-k_1-k_p)(n-1)}{2n(n-k_1)(n-k_p)}\sum_{i=1}^{p}k_i(n-k_i)+\frac{k_p-k_1}{n(n-k_1)(n-k_p)}$$
$$\times\left(\sum_{i=1}^{p}k_i^2(n-k_i)+3\sum_{1\le i<j<m\le p}k_ik_jk_m+\sum_{i=2}^{p-1}k_i^2+\sum_{2\le i<j\le p-1}k_ik_j\right).$$

Let $x\neq y\in V_1$. Then

$$H(x,y)=\frac{1}{n-k_1}\sum_{i=1}^{p}k_i(n-k_i).$$

Proof. Recall Tetali's electrical formula [9] for hitting times,

$$H(x,y)=|E(G)|R_{xy}+\frac{1}{2}\sum_{z\in V(G)}d(z)(R_{yz}-R_{xz}),\tag{14}$$

where $E(G)$ is the set of edges of G and R_{xy} is the effective resistance between x and y. Note that for $x\neq y$, we have $R_{xy}=\frac{\tau(G/\{x,y\})}{\tau(G)}=\frac{R(G-\{x,y\})}{\tau(G)}$. Obviously $|E(G)|=\frac{1}{2}\sum_{i=1}^{p}k_i(n-k_i)$.

When $x\in V_1$, $y\in V_p$, by Theorem 2.4 and Lemma 3.2, we get $R_{xy}=\frac{(k_1+k_p)(1-n)}{nd_1d_p}$. The second term in the right-hand side of (14) has been computed in (13).

When $x\neq y\in V_1$, by Theorem 2.4 and Lemma 2.3, we get $R_{xy}=\frac{2(n-k_1)}{d_1^2}$. The second term in the right-hand side of (14) is zero. □

Corollary 3.6. *Let $K_{a,b}$ be a complete bipartite graph on vertex set $V = V_1 \cup V_2$ where $|V_1| = a$, $|V_2| = b$. Then*

$$H(x,y) = \begin{cases} 2b-1 & \text{if } x \in V_1,\ y \in V_2, \\ 2a & \text{if } x \neq y \in V_1. \end{cases}$$

Corollary 3.7. *Let K'_n, $n \geq 3$ be a graph obtained by deleting an edge $e = x_1x_2$ from the complete graph K_n. Then $H(x_1, x_2) = n+1$. Let $y \in V(K'_n) - \{x_1, x_2\}$. Then $H(x_1, y) = n - 1 - \frac{3}{n}$ and $H(y, x_1) = n$. Let $y_1 \neq y_2 \in V(K'_n) - \{x_1, x_2\}$. Then $H(y_1, y_2) = n - 1 - \frac{2}{n}$.*

Proof. First note that K'_n is a complete $(n-1)$-partite graph on vertex set $V = V_1 \cup \cdots \cup V_{n-1}$ where $|V_1| = 2$ and $|V_i| = 1$ for $2 \leq i \leq n-1$. We then apply Theorem 3.5. □

References

[1] T. L. Austin, *The enumeration of point labelled chromatic graphs and tress*, Canad. J. Math. **12** (1960) 535–545.

[2] X. Chang and H. Xu, *Chung-Yau invariants and graphs with symmetric hitting times*, J. Graph Theory (to appear).

[3] Fan R. K. Chung and S.-T. Yau, *Discrete green's functions*, J. Combin. Theory Ser. A **91** (2000), 191–214.

[4] Ö. Eğecioğlu and J. B. Remmel, *A bijection for spanning trees of complete multipartite graphs*, Congr. Numer. **100** (1994), 225–243.

[5] A. Georgakopoulos, *On walk-regular graphs and graphs with symmetric hitting times*, arXiv:1211.5689.

[6] A. Georgakopoulos and S. Wagner, *Hitting times, Cover cost, and the Wiener index of a tree*, arXiv:1302.3212.

[7] R. P. Lewis, *The number of spanning trees of a complete multipartite graph*, Discrete Math. **197/198** (1999), 537–541.

[8] L. Lovász, *Random walks on graphs: A survey*, in Combinatorics, Paul Erdös Is Eighty, Bolyai Soc. Math. Stud. 2 (1993), 353–397.

[9] P. Tetali, *Random walks and the effective resistance of networks*, J. Theoret. Probab. **4** (1991), 101–109.

[10] H. Xu and S.-T. Yau, *Discrete Green's functions and random walks on graphs*, J. Combin. Theory Ser. A, **120** (2013), 483–499.

[11] H. Xu and S.-T. Yau, *An explicit formula of hitting times for random walks on graphs*, Pure Appl. Math. Q. **10** (2014), 567–581.

Xiao Chang
Department of Mathematics, University of Pittsburgh,
301 Thackeray Hall, Pittsburgh, PA 15260, USA
E-mail: xic58@pitt.edu

Hao Xu
Department of Mathematics, University of Pittsburgh,
301 Thackeray Hall, Pittsburgh, PA 15260, USA
E-mail: mathxuhao@gmail.com