Pure and Applied Mathematics Quarterly
Volume 11, Number 2, 369–392, 2015

An L^2 Injectivity Theorem and Its Application

Chen Zhao

Abstract: We prove an L^2 injectivity theorem which generalizes the injectivity theorems of J. Kollár and I. Enoki. As an application, we give the proof of a Fujino's conjecture under a certain semi-positivity assumption.
Keywords: injectivity theorem, L^2 cohomology, L^2 Dolbeault lemma, vanishing theorem.

Received January 8, 2016.

1. Introduction

Vanishing theorems are very useful and important in the study of both algebraic and complex geometry. Among them, the Kodaira vanishing theorem is the most fundamental result. Besides, there are various generalizations, such as the Andreotti-Grauert vanishing theorem [1], the Grauert-Riemenschneider vanishing theorem [16], the Kawamata-Viehweg vanishing theorem [19, 28], and the Nadel vanishing theorem [21]. The cohomology injectivity theorems which imply various vanishing theorems are also crucial in both complex algebraic geometry and Hodge theory.

Inspired by a cohomology injectivity theorem of S. G. Tankeev [27], J. Kollár [18] gives the following original injectivity theorem:

Theorem 1.1 ([18], Theorem 2.2). *Let X be a smooth projective variety defined over an algebraically closed field of characteristic zero and let L be a semi-ample line bundle on X. Let s be a nonzero holomorphic section of $L^{\otimes k}$ for some $k > 0$. Then*

$$\times s : H^q(X, K_X \otimes L^{\otimes m}) \to H^q(X, K_X \otimes L^{\otimes(m+k)})$$

is injective for every $q \geq 0$ and every $m \geq 1$, where K_X is the canonical line bundle of X. Note that $\times s$ is the homomorphism induced by the tensor product with s.

Later, I. Enoki [9] gives an analytic version of Kollár's injectivity theorem on compact Kähler manifolds:

Theorem 1.2 ([9], Theorem 0.2). *Let X be a compact Kähler manifold and let L be a semi-positive line bundle on X. Then, for any non-zero holomorphic section s of $L^{\otimes k}$ with some positive integer k, the multiplication homomorphism*

$$\times s : H^q(X, K_X \otimes L^{\otimes m}) \to H^q(X, K_X \otimes L^{\otimes(m+k)})$$

is injective for every $q \geq 0$ and every $m \geq 1$.

Enoki's proof only uses the standard results of the theory of harmonic forms on compact Kähler manifolds. T. Ohsawa [23] investigates a curvature condition in a generalized L^2 extension theorem which implies a cohomology injectivity theorem. After that, O. Fujino gives a curvature condition which implies Kollár-type cohomology injectivity theorems by the Ohsawa-Takegoshi twisted version of Nakano's identity in [12, 13].

Recent developments on the mixed Hodge structures on cohomology with compact support lead to generalizations of Kollár's injectivity theorem.

Inspired by these works, Fujino posts the following conjecture:

Conjecture 1.3 ([14], Conjecture 2.4). *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let L be a semi-positive line bundle on X and let s be a nonzero holomorphic section of $L^{\otimes k}$ on X for some positive integer k. Assume that $(s = 0)$ contains no strata of D. Then the multiplication homomorphism*

$$\times s : H^i(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes l}) \to H^i(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes (l+k)})$$

induced by $\otimes s$ is injective for every positive integer l and every i.

Since the 1960s, the L^2-theory for the $\bar{\partial}$-operator has become one of the essential tools in complex analysis on complex manifolds thanks to the fundamental work of Hörmander [17] on L^2-estimates and existence theorems for the $\bar{\partial}$-operator and the related work of Andreotti and Vesentini [2]. The L^2-theory is a cohomology theory which extends the theory of the usual de Rham complex on closed smooth Riemannian manifolds to non-compact manifolds and spaces with singularities. One way of this extension is to restrict to a subcomplex of the de Rham complex, i.e., that of the square integrable differential forms. This leads to the L^2-cohomology. In this paper, we mainly focus on the L^2-cohomology on complex manifolds instead of the general open (possibly incomplete) Riemannian manifolds.

Let (X, g) be a Hermitian manifold of dimension n and let (E, h) be a Hermitian vector bundle over X. Denote by $L_2^{p,q}(X, E)$ the space of L^2-integrable E-valued (p, q)-forms on X with measurable coefficients. Let

$$C_2^{p,q}(X, E) = \{u \in L_2^{p,q}(X, E) \,|\, \bar{\partial}_w u \in L_2^{p,q+1}(X, E)\},$$

where $\bar{\partial}_w$ is the maximal closed extension of the $\bar{\partial}$-operator

$$\bar{\partial}_{cpt} : \mathrm{Dom}\bar{\partial}_{cpt} = A_{cpt}^{p,q}(X, E) \subset L_2^{p,q}(X, E) \to L_2^{p,q+1}(X, E).$$

Here $A_{cpt}^{p,q}(X, E)$ is the set of all smooth sections with compact support. In other words, $\bar{\partial}_w$ is the extended $\bar{\partial}$-operator in the sense of distribution.

Then the corresponding L^2-cohomology $H^{p,q}_{(2)}(X,E)$ is defined to be the cohomology of the complex

$$0 \to C^{p,0}_2(X,E) \xrightarrow{\bar\partial_w} C^{p,1}_2(X,E) \xrightarrow{\bar\partial_w} \cdots \xrightarrow{\bar\partial_w} C^{p,n}_2(X,E) \to 0.$$

The $\bar\partial_{cpt}$-operator has various closed extensions, among which there is another important extension, namely, the minimal extension $\bar\partial_s$, that is the closure of $\bar\partial_{cpt}$ with respect to the graph norm in $L^{p,q}_2(X,E) \times L^{p,q+1}_2(X,E)$. When g is a complete metric, the domains of $\bar\partial_w$ and $\bar\partial_s$ coincide.

By the abstract Hodge theory in [20], we obtain that

$$H^{p,q}_{(2)}(X,E) \xrightarrow{\cong} \mathcal{H}^{p,q}_{(2)}(X,E)$$

when $H^{p,q}_{(2)}(X,E)$ is finite dimensional (See Proposition 3.2). Here $\mathcal{H}^{p,q}_{(2)}(X,E) := \mathrm{Ker}\bar\partial_w \cap \mathrm{Ker}\bar\partial^*_w$.

Following the method in Enoki [9], we obtain our main theorem:

Theorem 1.4. *Let (X,g) be a complete Kähler manifold and L be a semi-positive line bundle on X. Let s be a nonzero holomorphic section of $L^{\otimes k}$ on X for some positive integer k which is bounded in the L^∞-norm. Suppose that for some q, the L^2-cohomologies $H^q_{(2)}(X, L^{\otimes l})$ and $H^q_{(2)}(X, L^{\otimes(l+k)})$ are finite dimensional for some $l > 0$. Then the multiplication homomorphism*

$$\times s : H^q_{(2)}(X, \omega_X \otimes L^{\otimes l}) \to H^q_{(2)}(X, \omega_X \otimes L^{\otimes(l+k)})$$

induced by $\otimes s$ is injective.

Then we give the proof of Fujino's conjecture 1.3 under the assumption that L is semi-positive over X and positive on D (which means that for any point $x \in D$, the curvature at x is positive). In order to obtain the injectivity theorem, we need to establish the corresponding L^2 Dolbeault lemma, i.e., the corresponding L^2 Dolbeault complex on X is a resolution of a certain sheaf complex.

Such an L^2 Dolbeault lemma was first established by S. Zucker in [29] when X is one dimensional. Then a generalization to arbitrary dimensions is given by A. Fujiki [11].

By the method in [11], choosing different metrics leads to different L^2 Dolbeault lemmas, namely, the isomorphisms between certain kinds of geometrically meaningful cohomology and the L^2-cohomology. Thus we can get the corresponding injectivity theorems which have applications in geometry.

The main idea of the proof of our L^2 Dolbeault lemma comes from the discussions in H. Luo's thesis [26]. Let X be a compact Kähler manifold and $D = \sum_{i=1}^{r} D_i$ be a simple normal crossing divisor where D_i are irreducible components of D. Let s_i be the defining section of D_i in $\mathcal{O}(D_i)$. Let (L, h) be a semi-positive line bundle on X which is positive on D. Fix a neighborhood W of D such that $\overline{W}$ is compact and h is positive on $\overline{W}$. We can choose for each $\mathcal{O}(D_i)$ a Hermitian metric $\|\cdot\|_{D_i}$ such that $\|s_i\|_{D_i} = 1$ on $X \setminus W$. We fix a Poincaré-type metric on $X \setminus D$ and perturb the Hermitian metric h to

$$h_L = \prod_{i=1}^{r} \|s_i\|_{D_i}^{2\delta} (\log \epsilon \|s_i\|_{D_i}^2)^{2\alpha} h$$

by small positive constants δ and ϵ. By choosing proper δ and ϵ, the new metric h_L is also semi-positive on $X \setminus D$ and positive on D, and leads to a canonical isomorphism (Proposition 4.3):

$$H^q(X, K_X \otimes L \otimes \mathcal{O}(D)) \simeq H_{(2)}^{n,q}(X \setminus D, L).$$

Hence we prove Fujino's conjecture 1.3 in the case that the line bundle L is semi-positive on X and positive on D:

Theorem 1.5. *Let X be a compact Kähler manifold and let D be a reduced simple normal crossing divisor on X. Let (L, h) be a Hermitian line bundle on X which is semi-positive on X and positive on D (which means the curvature $\Theta(h)_x \geq 0$ for $x \in X$ and $\Theta(h)_x > 0$ for $x \in D$). Let s be a nonzero holomorphic section of $L^{\otimes k}$ on X for some positive integer k. Then the multiplication homomorphism*

$$\times s : H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes l}) \to H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes (l+k)})$$

which is induced by $\otimes s$ is injective for every $q \geq 0$ and $l > 0$. Here ω_X is the canonical line bundle of X and $\times s$ is the homomorphism induced by the tensor product with s.

This theorem gives a proof of Conjecture 1.3 under the assumption that L is semi-positive on X and positive on D. When $D = \emptyset$, this theorem is reduced to Theorem 1.2 of Enoki mentioned above.

When $D \neq \emptyset$, the cohomology in fact vanishes:

Proposition 1.6. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor on X. Suppose that L is a line bundle* *on X and h is a semi-positive Hermitian metric on L which is positive on D. Then*

$$H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L) = 0$$

for every $q > 0$.

When $p = n$, it is a generalization of Norimatsu's vanishing theorem [22]:

Theorem 1.7 ([22], Theorem 1). *Let (X, D) be a compact Kähler manifold and L be an ample invertible sheaf on X. Then*

$$H^q(X, \Omega^p(logD) \otimes L) = 0, \quad for \quad p + q \geq n + 1.$$

Here $n = dimX$ and $\Omega^p(logD)$ denotes the sheaf of logarithmic p-forms of X.

In the sequel, we summarize the contents of this paper. In Section 2, we briefly review some basic knowledge of Kähler geometry. Section 3 is devoted to the L^2 Hodge theory and the L^2-injectivity theorem. In Section 4, we give the proof of Theorem 1.5.

Acknowledgements. The author would like to thank Prof. Kefeng Liu for his encouragement, patience, valuable discussions, and bringing this problem to the author's interest. The author also appreciates Haisheng Liu, Sheng Rao, Xueyuan Wan and Quanting Zhao for their careful reading, helpful discussions and suggestions. The author also thank Yifei Chen, Xiankui Meng, Chunle Huang for their helpful discussions and the anonymous referees for their detailed and helpful comments on the improvement of this paper. In particular, the author would like to point out that Remark 4.7 is due to the anonymous referee.

2. Preliminary

Definition 2.1 (Chern connection and its curvature form). Let X be a complex manifold and let (E, h) be a holomorphic Hermitian vector bundle on X. Then there exists the Chern connection $D = D_{(E,h)}$, which can be split in a unique way as a sum of a $(1, 0)$-connection and a $(0, 1)$-connection, $D = D'_{(E,h)} + D''_{(E,h)}$ with $D'' = D''_{(E,h)} = \bar{\partial}$. We obtain the curvature form $\Theta_h(E) := D^2_{(E,h)}$. The subscripts might be suppressed if there is no danger of confusion.

Definition 2.2 (Inner product). Let (X, g) be an n-dimensional Hermitian manifold. We denote by ω the fundamental form of g. Let (E, h) be a holomorphic Hermitian vector bundle on X, and u, v are E-valued (p, q)-forms with measurable coefficients. We set

$$\|u\|^2 = \int_X |u|^2 dV_\omega, \langle\langle u, v\rangle\rangle = \int_X \langle u, v\rangle dV_\omega,$$

where $|u|$ (resp. $\langle u, v\rangle$) is the pointwise norm (resp. inner product) induced by g and h on $\bigwedge^{p,q} T_X^* \otimes E$, and $dV_\omega = \frac{1}{n!}\omega^n$.

Let $L^{p,q}_{(2)}(X, E)$ be the space of square integrable E-valued (p, q)-forms on X . One can view D' and D'' as closed and densely defined operators on the Hilbert space $L^{p,q}_{(2)}(X, E)$. The formal adjoints D'^* and D''^* also have closed extensions in the sense of distributions, which do not necessarily coincide with the Hilbert space adjoints in the sense of Von Neumann, since the latter ones may have strictly smaller domains. It is well known that the domains coincide if the Hermitian metric of X is complete.

Definition 2.3 (Nakano positivity and semi-positivity). Let (E, h) be a Hermitian vector bundle on a complex manifold X. Let Ξ be a $\mathrm{Hom}(E, E)$-valued $(1, 1)$-form such that ${}^t({}^t\bar{\Xi}h) = {}^t\,\Xi h$. The form Ξ is said to be Nakano positive (resp. Nakano semi-positive) if the Hermitian form on $T_X \otimes E$ associated to ${}^t\Xi h$ is positive definite (resp. semi-positive definite). A holomorphic vector bundle (E, h) is said to be Nakano positive (resp. Nakano semi-positive) if $\sqrt{-1}\Theta(E)$ is Nakano positive (resp. Nakano semi-positive).

In particular, a Hermitian line bundle (L, h) over X is said to be positive if

$$\sqrt{-1}\Theta(L) = -\sqrt{-1}\partial\bar{\partial}\log h > 0.$$

Assume that X has a Kähler metric ω.

Consider the curvature form $\sqrt{-1}\Theta(L) \in \Lambda^{1,1}T_X^*$. There exists an ω-orthogonal basis $(\zeta_1, \zeta_2, ..., \zeta_n)$ in T_X which diagonalizes both forms ω and $\sqrt{-1}\Theta(L)$:

$$\omega = \sqrt{-1}\sum_{1\le j\le n} \zeta_j^* \wedge \bar{\zeta}_j^*, \quad \sqrt{-1}\Theta(L)_x = \sqrt{-1}\sum_{1\le j\le n} \gamma_j(x)\zeta_j^* \wedge \bar{\zeta}_j^*, \quad \zeta_j^* \in T_x^*X.$$

Here $\gamma_1(x) \le \cdots \le \gamma_n(x)$ are the eigenvalues of $\sqrt{-1}\Theta(L)_x$ with respect to ω_x at each point $x \in X$. Then

$$\langle[\sqrt{-1}\Theta(L), \Lambda_\omega]u, u\rangle \ge (\gamma_1 + \cdots + \gamma_q - \gamma_{p+1} - \cdots - \gamma_n)|u|^2 \tag{2.1}$$

for any form $u \in \Omega^{p,q}(L)$ ([5], VII §3).

Here we introduce two L^2 estimates which will be used while proving the L^2 Dolbeault lemma (Proposition 4.3).

Theorem 2.4 ([5], VIII, Theorem 6.9). *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex open subset and φ an upper semi-continuous plurisubharmonic function on Ω. For every $\epsilon \in (0, 1]$ and every $g \in L_2^{p,q}(\Omega, loc)$ such that $\bar{\partial} g = 0$ and*

$$\int_\Omega (1 + |z|^2)|g|^2 e^{-\varphi} dV < +\infty, \tag{2.2}$$

we can find a L^2_{loc} form f of type $(p, q-1)$ such that $\bar{\partial} f = g$ and

$$\int_\Omega (1 + |z|^2)^{-\epsilon} |f|^2 e^{-\varphi} dV \leq \frac{4}{q\epsilon^2} \int_\Omega (1 + |z|^2)|g|^2 e^{-\varphi} dV < +\infty.$$

Moreover, f can be chosen smooth if g and φ are smooth.

Remark 2.5. In particular, when $\Omega = \Delta^n$ is the product of n polydisks and $\varphi = 0$, then for any $g \in L_2^{p,q}(\Omega, loc)$ such that $\bar{\partial} g = 0$, there exists $f \in L_2^{p,q-1}(\Omega, loc)$ such that $\bar{\partial} f = g$.

The following famous L^2-estimate is essentially due to Hörmander [17] and Andreotti-Vesentini [2].

Theorem 2.6 ([2],[17]). *Let (X, ω) be a complete Kähler manifold of dimension n. Let E be a holomorphic Hermitian vector bundle of rank r over X. Assume the curvature operator $A_{E,\omega} = [\sqrt{-1}\Theta(E), \Lambda_\omega]$ is positive definite everywhere on $\Omega^{p,q}(X, E)$, $q \geq 1$. Then for any form $g \in L_2^{p,q}(X, E)$ satisfying*

$$\bar{\partial} g = 0 \text{ and } \int_X \langle A_{E,\omega}^{-1} g, g \rangle \omega^n < +\infty,$$

there exists $f \in L_2^{p,q-1}(X, E)$ such that

$$\bar{\partial} f = g \text{ and } \int_X |f|^2 \omega^n \leq \int_X \langle A_{E,\omega}^{-1} g, g \rangle \omega^n.$$

To finish this section, let's recall the Bochner-Kodaira-Nakano identity [7, Theorem 4.5], which is useful in the proof of our main theorem.

Proposition 2.7 (Bochner-Kodaira-Nakano identity). *Let (X,ω) be a Kähler manifold and (E,h) be a Hermitian vector bundle on X. The complex Laplace operators Δ' and Δ'' acting on E-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [\sqrt{-1}\Theta(E), \Lambda],$$

where $\Delta' = D'D'^* + D'^*D'$ and $\Delta'' = D''D''^* + D''^*D''$.

3. L^2 Hodge theory and injectivity theorem

In this section we review the the L^2-Hodge theory and prove an L^2-cohomology injectivity theorem.

Let (X,g) be a Hermitian manifold of dimension n and let (E,h) be a Hermitian vector bundle over X. Denote by $L_2^{p,q}(X,E)$ the space of L^2-integrable E-valued (p,q)-forms on X with measurable coefficients. Let

$$C_2^{p,q}(X,E) = \{u \in L_2^{p,q}(X,E) \,|\, \bar\partial_w u \in L_2^{p,q+1}(X,E)\}, \tag{3.1}$$

where $\bar\partial_w$ is the maximal closed extension of the $\bar\partial$-operator

$$\bar\partial_{cpt} : \mathrm{Dom}\bar\partial_{cpt} = A_{cpt}^{p,q}(X,E) \subset L_2^{p,q}(X,E) \to L_2^{p,q+1}(X,E).$$

Here $A_{cpt}^{p,q}(X,E)$ is the set of all smooth sections with compact support. In other words, $\bar\partial_w$ is the extended $\bar\partial$-operator in the sense of distribution.

Then the corresponding L^2-cohomology $H_{(2)}^{p,q}(X,E)$ is defined to be the cohomology of the complex

$$0 \to C_2^{p,0}(X,E) \xrightarrow{\bar\partial_w} C_2^{p,1}(X,E) \xrightarrow{\bar\partial_w} \cdots \xrightarrow{\bar\partial_w} C_2^{p,n}(X,E) \to 0.$$

The $\bar\partial_{cpt}$-operator has various closed extensions, among which there is another important extension, the minimal extension $\bar\partial_s$, i.e., the closure of the graph of $\bar\partial_{cpt}$ in $L_2^{p,q}(X,E) \times L_2^{p,q+1}(X,E)$. When g is a complete metric, the domains of $\bar\partial_w$ and $\bar\partial_s$ coincide. In such case, we simply write $\bar\partial$ since there is no danger of confusion.

Let $\mathcal{C}_{2,E}^{p,q}$ be the sheaf defined by

$$\mathcal{C}_{2,E}^{p,q}(U) = \{\omega \in L_{2,loc}^{p,q}(U,E) \,|\, \bar\partial\omega \in L_{2,loc}^{p,q+1}(U,E)\},$$

for any open subset U of X.

If X is compact, then $\mathcal{C}_{2,E}^{p,q}(X) = C_2^{p,q}(X,E)$.

Here we would like to introduce an abstract Hodge decomposition theorem in [20]. Let H_1, H_2 and H_3 be three Hilbert spaces. Let $T : H_1 \to H_2$ and $S : H_2 \to H_3$ be densely defined closed operators. We assume

$$\mathrm{Im}(T) \subset \mathrm{Dom}(S)$$

and

$$STx = 0 \quad \text{for} \quad x \in \mathrm{Dom}(T).$$

Then their Hilbert space adjoint operators $S^* : H_3 \to H_2$ and $T^* : H_2 \to H_1$ satisfy the same conditions, namely,

$$\mathrm{Im}(S^*) \subset \mathrm{Dom}(T^*),$$

and

$$T^*S^*x = 0 \quad \text{for} \quad x \in \mathrm{Dom}(S^*).$$

If we define $\Delta : H_2 \to H_2$ by

$$\mathrm{Dom}(\Delta) = \{x \in \mathrm{Dom}(S) \cap \mathrm{Dom}(T^*); T^*x \in \mathrm{Dom}(T), Sx \in \mathrm{Dom}(S^*)\},$$

$$\Delta x = S^*Sx + TT^*x \quad \text{for} \quad x \in \mathrm{Dom}(\Delta).$$

Then we have the following existence theorem for the Green operator for the operator Δ, which is Theorem A.2.2 in [20]. We summarize it here for the readers' convenience.

Theorem 3.1. *Let S, T and Δ be as above, and suppose that both Im(S) and Im(T) are closed. Denoting Ker(Δ) by $\mathcal{H}$, we find the following:*

1. Δ is self-adjoint, that is, $\Delta = \Delta^*$ holds.

2. $\mathcal{H} = Ker(\Delta) = Ker(T^*) \cap Ker(S)$ and $\mathcal{H}^\perp = Im(\Delta)$.

3. Denoting by $P_{\mathcal{H}}$ the projection operator $: H_2 \to \mathcal{H}$, the operator $(\Delta|_{\mathcal{H}^\perp})^{-1}(I - P_{\mathcal{H}})$ is well-defined and bounded.

By the above abstract Hodge theory, we have the following L^2 Hodge decomposition theorem:

Proposition 3.2. *Let (X, g) be a complete Hermitian manifold of dimension n and let (E, h) be a Hermitian vector bundle over X. Fix $0 \leq p \leq n$. Suppose that $H^{p,q}_{(2)}(X, E)$ are finite dimensional for all q. Then*

1. (Hodge theorem) The (unbounded) operator $\bar\partial : L_2^{p,q}(X,E) \to L_2^{p,q+1}(X,E)$ *and its Hilbert adjoint* $\bar\partial^*$ *have closed images, and there is an orthogonal decomposition for each* q,

$$L_2^{p,q}(X,E) = Im\bar\partial \oplus Im\bar\partial^* \oplus \mathcal{H}_{(2)}^{p,q}(X,E),$$

where $\mathcal{H}_{(2)}^{p,q}(X,E) := Ker\bar\partial \cap Ker\bar\partial^*$.

There is consequently an isomorphism

$$H_{(2)}^{p,q}(X,E) \xrightarrow{\cong} \mathcal{H}_{(2)}^{p,q}(X,E)$$

induced in the usual way.

2. (Green operator) Denoting H *be the projection operator* $H : L_{(2)}^{p,q}(X,E) \to \mathcal{H}_{(2)}^{p,q}(X,E)$, *then the Green operator* $G = (\Delta_{\bar\partial}|_{\mathcal{H}_{(2)}^{p,q}(X,E)^\perp})^{-1}(I-H)$ *is well-defined and bounded. Then, we have the following identity*

$$\Delta_{\bar\partial} G = G\Delta_{\bar\partial} = Id - H, HG = GH = 0.$$

Proof. See Appendix in [20]. □

Before we prove our L^2 injectivity theorem, we need the following lemma:

Lemma 3.3. *If the manifold* (M,g) *is complete. Let* E *be a Hermitian vector bundle on* M. *Let* $\Delta'_E = D'_E D'^*_E + D'^*_E D'_E$ *and* $\Delta''_E = D''_E D''^*_E + D''^*_E D''_E$ *be the complex Laplacians acting on* E*-valued forms calculated in the sense of distributions. For any* L^2*-integrable smooth* E*-valued form* $u \in C^\infty(M, \Lambda^\bullet T_M^* \otimes E) \cap L^2(M, \Lambda^\bullet T_M^* \otimes E)$, *one has*

$$\langle\langle u, \Delta'_E u\rangle\rangle \geq \|D'_E u\|^2 + \|D'^*_E u\|^2$$

and

$$\langle\langle u, \Delta''_E u\rangle\rangle \geq \|D''_E u\|^2 + \|D''^*_E u\|^2,$$

where every term is allowed to be infinity.

Proof. Since $u \in C^\infty(M, \Lambda^\bullet T_M^* \otimes E)$, $\Delta' u$, $D'_E u$, $D'^*_E u \in L^2_{\mathrm{loc}}$.

By the Hopf-Rinow lemma of complete manifolds in ([8], Part I, 12.1), there exists an exhaustive sequence (K_ν) of compact sets and functions

$\theta_\nu \in C^\infty(M, \mathbb{R})$ such that

$$\theta_\nu = 1 \quad \text{on a neighbourhood of } K_\nu, \quad \text{Supp } \theta_\nu \subset K^\circ_{\nu+1},$$

$$0 \leq \theta_\nu \leq 1 \quad \text{and} \quad |d\theta_\nu|_g \leq 2^{-\nu}.$$

We can apply integration by parts as needed, after multiplying the respective forms by C^∞ functions θ_ν with compact support. some calculations then give

$$\begin{aligned}
&\|\theta_\nu D'_E u\|^2 + \|\theta_\nu D'^*_E u\|^2 \\
&= \langle\langle \theta_\nu^2 D'_E u, D'_E u\rangle\rangle + \langle\langle u, D'_E(\theta_\nu^2 D'^*_E u)\rangle\rangle \\
&= \langle\langle D'_E(\theta_\nu^2 u), D'_E u\rangle\rangle + \langle\langle u, \theta_\nu^2 D'_E D'^*_E u\rangle\rangle - 2\langle\langle \theta_\nu \partial\theta_\nu \wedge u, D'_E u\rangle\rangle \\
&\quad + 2\langle\langle u, \theta_\nu \partial\theta_\nu \wedge D'^*_E u\rangle\rangle \\
&= \langle\langle \theta_\nu^2 u, \Delta'_E u\rangle\rangle - 2\langle\langle \partial\theta_\nu \wedge u, \theta_\nu D'_E u\rangle\rangle + 2\langle\langle u, \partial\theta_\nu \wedge (\theta_\nu D'^*_E u)\rangle\rangle \\
&\leq \langle\langle \theta_\nu^2 u, \Delta'_E u\rangle\rangle + 2^{-\nu}(2\|\theta_\nu D'_E u\|\|u\| + 2\|\theta_\nu D'^*_E u\|\|u\|) \\
&\leq \langle\langle \theta_\nu^2 u, \Delta'_E u\rangle\rangle + 2^{-\nu}(\|\theta_\nu D'_E u\|^2 + \|\theta_\nu D'^*_E u\|^2 + 2\|u\|^2).
\end{aligned}$$

Consequently,

$$\|\theta_\nu D'_E u\|^2 + \|\theta_\nu D'^*_E u\|^2 \leq \frac{1}{1 - 2^{-\nu}}(\langle\langle \theta_\nu^2 u, \Delta'_E u\rangle\rangle + 2^{1-\nu}\|u\|^2).$$

Letting ν tend to $+\infty$, one obtains that $\langle\langle u, \Delta'_E u\rangle\rangle \geq \|D'_E u\|^2 + \|D'^*_E u\|^2$.

The same argument applies to the second inequality. □

Now we turn to the proof of the L^2 injectivity theorem.

Theorem 3.4. *Let (X, g) be a complete Kähler manifold and L be a semipositive line bundle on X. Let s be a nonzero holomorphic section of $L^{\otimes k}$ on X for some positive integer k which is bounded in the L^∞-norm. Suppose that for some q, the L^2-cohomologies $H^q_{(2)}(X, L^{\otimes l})$ and $H^q_{(2)}(X, L^{\otimes(l+k)})$ are finite dimensional for some $l > 0$. Then the multiplication homomorphism*

$$\times s : H^q_{(2)}(X, \omega_X \otimes L^{\otimes l}) \to H^q_{(2)}(X, \omega_X \otimes L^{\otimes(l+k)})$$

induced by $\otimes s$ is injective.

Proof. By Proposition 3.2, because $H^q_{(2)}(X, L^{\otimes l})$ and $H^q_{(2)}(X, L^{\otimes(l+k)})$ are finite dimensional, we have the isomorphisms

$$H^{n,q}_{(2)}(X, L^{\otimes l}) \simeq \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes l})$$

and

$$H^{n,q}_{(2)}(X, L^{\otimes(l+k)}) \simeq \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes(l+k)}).$$

We claim that the multiplication map

$$\times s : \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes l}) \longrightarrow \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes(l+k)})$$

is well-defined.

If the claim is true, then the theorem is obvious. In fact, assume $su = 0$ in $\mathcal{H}^{n,q}_{(2)}(X, L^{\otimes(l+k)})$. Since s is holomorphic over X, the codimension of $\{s = 0\}$ is at least one, thus the locus $\{s \neq 0\}$ is dense in X. Hence $u = 0$ for $u \in \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes l})$ since u is smooth over X. This implies the desired injectivity. Thus, it is sufficient to prove the above claim.

Take an arbitrary $L^{\otimes l}$-valued (n,q)-form $u \in \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes l})$, u and $\bar\partial u$ are both L^2-integrable over X, i.e., $\|u\|_{L^2} < \infty$ and $\|\bar\partial u\|_{L^2} < \infty$. Since $\|s\|_{L^\infty} < \infty$,

$$\|su\|_{L^2} \leq \|s\|_{L^\infty}\|u\|_{L^2} < \infty, \|\bar\partial(su)\|_{L^2} = \|s\bar\partial u\|_{L^2} \leq \|s\|_{L^\infty}\|\bar\partial u\|_{L^2} < \infty.$$

Therefore, su and $\bar\partial(su)$ are L^2-integrable, i.e., $su \in L^{n,q}_2(X, L^{\otimes(l+k)})$.

By the Bochner-Kodaria-Nakano identity (2.7),

$$\Delta'' u = \Delta' u + [\sqrt{-1}\Theta_{h^l}(L^{\otimes l}), \Lambda]u, \tag{3.2}$$

where Λ is the adjoint of $\omega \wedge \cdot$ and ω is the fundamental form of g.

Because $u \in \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes l})$, we have

$$\Delta'' u = 0 \text{ and } \Delta' u + [\sqrt{-1}\Theta_{h^l}(L^{\otimes l}), \Lambda]u = 0.$$

Since $\sqrt{-1}\Theta_{h^l}(L^{\otimes l}) = \sqrt{-1}l\Theta_h(L)$ is a smooth semi-positive $(1,1)$-form on X, we get

$$\langle\langle[\sqrt{-1}\Theta_{h^l}(L^{\otimes l}), \Lambda]u, u\rangle\rangle \geq 0.$$

By Lemma 3.3, we have

$$\langle\langle u, \Delta' u\rangle\rangle \geq \|D' u\|^2 + \|D'^* u\|^2 \geq 0.$$

Hence

$$\langle\langle \Delta' u, u\rangle\rangle + \langle\langle [\sqrt{-1}\Theta_{h^l}(L^{\otimes l}), \Lambda]u, u\rangle\rangle = \langle\langle \Delta'' u, u\rangle\rangle = 0.$$

Applying Lemma 3.3 again, we get

$$\begin{aligned} 0 &= \langle\langle \Delta' u, u\rangle\rangle + \langle\langle [\sqrt{-1}\Theta_{h^l}(L^{\otimes l}), \Lambda]u, u\rangle\rangle \\ &\geq \|D'^* u\|^2 + \langle\langle [\sqrt{-1}\Theta_{h^l}(L^{\otimes l}), \Lambda]u, u\rangle\rangle. \end{aligned}$$

Thus we obtain that

$$\|D'^* u\|^2 = \langle\langle \sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle\rangle = 0.$$

Therefore,

$$D'^* u = 0 \text{ and } \langle \sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle_{h^l} = 0,$$

where $\langle , \rangle_{h^l}$ is the pointwise inner product with respect to h^l and g.

Since $u \in \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes l})$ and s is holomorphic over X,

$$D''(su) = \bar{\partial}(su) = 0$$

by the Leibnitz rule. We know that

$$D'^*(su) = - * \bar{\partial} * (su) = s D'^* u = 0$$

since s is a holomorphic $L^{\otimes k}$-valued $(0,0)$-form and $D'^* u = 0$, where $*$ is the Hodge star operator with respect to g. Combined with the fact that $D'(su) = 0$, we obtain that $\Delta'(su) = 0$.

Applying the Bochner-Kodaria-Nakano identity to su and by the same discussions as above, we have

$$\langle\langle \sqrt{-1}\Theta_{h^{(l+k)}}(L^{\otimes(l+k)})\Lambda su, su\rangle\rangle = \langle\langle \Delta'' su, su\rangle\rangle \geq \|D''^*_{(L^{\otimes(l+k)}, h^{(l+k)})} su\|^2.$$

Note that

$$\langle \sqrt{-1}\Theta_{h^{l+k}}(L^{\otimes(l+k)})\Lambda su, su\rangle_{h^{l+k}} = \frac{l+k}{k}|s|^2_{h^k}\langle \sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle_{h^l} = 0,$$

where $\langle , \rangle_{h^{l+k}}$ (resp. $|s|_{h^k}$) is the pointwise inner product (resp. the pointwise norm of s) with respect to h^{l+k} and g.

Thus we obtain that $D''^*_{(L^{\otimes(l+k)},h^{l+k})}su = 0$ and consequently $\Delta''_{(L^{\otimes(l+k)},h^{l+k})}(su) = 0$. It implies that $su \in \mathcal{H}^{n,q}_{(2)}(X, L^{\otimes(l+k)})$. We finish the proof of the claim. This implies the desired injectivity.

$\square$

This proof is parallel to the proof in Enoki [9].

4. Application

In this section we study the case of spaces with the Poincaré-type metric and prove a weak version of Fujino's conjecture using our L^2 injectivity theorem.

4.1. Poincaré-type Metric

Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n$ and $D \subset X$ be a divisor with simple normal crossings. Then we can define the Poincaré-type metric ω_P over $X - D$ as follows:

Definition 4.1 (Poincaré-type metric). A Kähler metric ω_P on $X - D$ is said to have Poincaré-type growth near the divisor D if for every point $p \in D$, there is a coordinate neighborhood $U_p \subset X$ of p with $U_p \cap (X - D) \cong (\Delta^*)^k_r \times \Delta^{n-k}_r, 1 \leq k \leq n$, such that in these coordinates, ω_P is quasi-isometric to a product of k copies of the Poincaré metric on Δ^*_r and $n-k$ copies of the Euclidean metric on Δ_r, i.e., near D,

$$\omega_P \sim \sqrt{-1} \sum_{1 \leq j \leq k} \frac{2dz_j \wedge d\bar{z}_j}{|z_j|^2 \log^2 |z_j|^2} + \sqrt{-1} \sum_{k+1 \leq j \leq n} dz_j \wedge d\bar{z}_j.$$

Here $\Delta_r = \{z \in \mathbb{C} \,|\, |z| < r\}, \Delta^*_r = \Delta_r \setminus \{0\}$. We say two metrics γ_1, γ_2 are quasi-isometric, i.e., $\gamma_1 \sim \gamma_2$ if there exists a constant $C > 0$ such that $C^{-1}\gamma_1 \leq \gamma_2 \leq C\gamma_1$.

The Poincaré-type metric always exists when X is compact Kähler. It has the following properties:

Theorem 4.2. *Let X be a compact Kähler manifold, D be a union of smooth divisors on X with only normal crossings. Then there exists a Kähler metric ω_P in $X - D$ which has the Poincaré-type growth near the divisor D. Furthermore, this metric has the following properties:*

1. *it is a complete Kähler metric,*
2. *it has finite volume,*

3. the curvature tensor and its covariant derivatives have bounded lengths.

For more details, refer to [3] or [29].

4.2. L^2 Dolbeault lemma for Poincaré-type metric

Let X be a compact Kähler manifold of complex dimension n, Y be the complement of an effective normal crossing divisor D. We fix a Poincaré-type metric ω_P on Y. Let (L, h) be a Hermitian line bundle on X. Let $D = \sum_{i=1}^{r} D_i$ where each D_i is an irreducible component of D and let s_i be the defining section of D_i in $\mathcal{O}(D_i)$. We can choose for each $\mathcal{O}(D_i)$ a Hermitian metric $\|\cdot\|_{D_i}$ such that $\|s_i\|_{D_i} = 1$ on $X \setminus W$. We give L a new Hermitian metric h_L over X defined by

$$h_L = \prod_{i=1}^{r} \|s_i\|_{D_i}^{2\delta} (\log(\epsilon \|s_i\|_{D_i}^2))^{2\alpha} h,$$

where $\delta > 0$, $\epsilon > 0$ and $\alpha > 0$ are constants to be determined later. We will choose α to be very large, while we will choose δ and ϵ to be small.

Using the metrics ω_P and h_L, we can define the L^2-integrable L-valued (p, q)-forms and the corresponding L^2-cohomology.

Theorem 4.3. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor on X. Choose the metrics as above, then there is a canonical isomorphism*

$$H^q(X, K_X \otimes L \otimes \mathcal{O}(D)) \simeq H^{n,q}_{(2)}(X \setminus D, L),$$

where $H^{n,q}_{(2)}(X \setminus D, L)$ is the L^2-cohomology with respect to ω_P and h_L.

Proof. It suffices to establish the following L^2 Dolbeault lemma:

There is a resolution of $K_X \otimes L \otimes \mathcal{O}(D)$ by fine sheaves on X given by

$$0 \to K_X \otimes L \otimes \mathcal{O}(D) \to \mathcal{C}^{n,*}_{2,L}. \tag{4.1}$$

Recall that $\mathcal{C}^{n,*}_{2,L}$ are fine sheaves defined by

$$\mathcal{C}^{n,q}_{2,L}(U) = \{\omega \in L^{n,q}_{2,loc}(U, L) \mid \bar{\partial}\omega \in L^{n,q+1}_{2,loc}(U, L)\},$$

for any open subset U of X.

On $X - D$, the Poincaré metric ω_P is locally quasi-isometric to the Euclidean metric, therefore the exactness of (4.1) on $X \setminus D$ is due to Remark 2.5. It suffices to verify the exactness at D.

Step 1: (Exactness at $\mathcal{C}^{n,0}_{2,L}$)

Let $(U; z_1, \cdots, z_n)$ be the local coordinate chart of X such that $U = \Delta^k_{\frac{1}{2}} \times \Delta^l_{\frac{1}{2}}$ and $U^* = U \cap Y = \Delta^{*k}_{\frac{1}{2}} \times \Delta^l_{\frac{1}{2}}$, and let e be a trivializing section of L on U. Denote ξ_i to be

$$\xi_i = \begin{cases} \dfrac{1}{z_i} dz_i, & 1 \le i \le k, \\ dz_i, & k+1 \le i \le n. \end{cases}$$

Let $s \in \Omega^{n,0}(U^*, L)$. If $\bar{\partial}(s) = 0$, then s is holomorphic. Hence

$$s = \lambda(z)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) \otimes e,$$

where $\lambda(z)$ is a holomorphic function on U^*.

Restricted on the chart U, h_L is quasi-isometric to

$$\tilde{h} = \prod_{i=1}^{k} |z_i|^{2\delta} (\log|z_i|^2)^{2\alpha} h.$$

Here h is a smooth Hermitian metric of L and we may assume $\|e(z)\|_h \equiv 1$. Now s is L^2-integrable with respect to ω_P and h_L if and only if it is L^2-integrable with respect to ω_P and $\tilde{h}$. So we will replace h_L by $\tilde{h}$ when we talk about L^2-integrability. By direct calculation,

$$\|s\|^2_{L^2(U^*)} = \int_{U^*} |\lambda(z)|^2 \prod_{j=1}^{k} |z_j|^{2\delta} (\log|z_j|^2)^{2\alpha+2} \omega_P^n. \tag{4.2}$$

Assume the Laurent series representation of $\lambda(z)$ on U^* is given by

$$\lambda(z) = \sum_{\beta=-\infty}^{\infty} c_\beta(z_{k+1}, \cdots, z_n) z_1^{\beta_1} \cdots z_k^{\beta_k}.$$

Here $c_\beta(z_{k+1,\cdots,z_n})$ is a holomorphic function on $\Delta^l_{\frac{1}{2}}$. Taking $0 < \delta < 1$, the integral (4.2) is finite if and only if s has at most log poles on the divisor D.

Hence we obtain the exactness at $\mathcal{C}^{n,0}_{2,L}$.

Step 2: (Exactness at $\mathcal{C}^{n,q}_{2,L}$, $q > 0$)

Fix $0 < r < \frac{1}{2}$, pick a point p_0 on D with multiple k. We need to show that for any $g \in L_2^{n,q}(U_r^*, L)$ with respect to ω_P and $\tilde{h}$ if $\bar{\partial} g = 0$ then on some $U_\epsilon^* = \Delta_\epsilon^{*k} \times \Delta_\epsilon^{n-k}$ $(0 < \epsilon \leq r)$ we can find f such that

$$\bar{\partial} f = g|_{U_\epsilon^*}, \quad \text{and} \quad f \in L_2^{n,q-1}(U_\epsilon^*, L). \tag{4.3}$$

The strategy is to deform the metric ω_P to be a complete Kähler metric on $U_r^* = \Delta_r^{*k} \times \Delta_r^{n-k}$ and then apply L^2 estimate (2.6) to solve (4.3).

We deform ω_P to be a new Kähler metric $\tilde{\omega}_P$ on $U_r^* = \Delta_r^{*k} \times \Delta_r^{n-k}$ given by

$$\tilde{\omega}_P = \omega_P + \sqrt{-1}\partial\bar{\partial}(\psi_1 + \cdots \psi_n), \tag{4.4}$$

where ψ_i is a function given by $\psi_i(z) = \frac{1}{r^2 - |z_i|^2}$.

Then

$$\begin{aligned} \tilde{\omega}_P = & \sqrt{-1}\sum_{i=1}^{k}\left(\frac{1}{|z_i|^2(\log|z_i|^2)^2} + \frac{r^2+|z_i|^2}{(r^2-|z_i|^2)^3}\right) dz_i \wedge d\bar{z}_i \\ & + \sqrt{-1}\sum_{i=k+1}^{n}\left(1 + \frac{r^2+|z_i|^2}{(r^2-|z_i|^2)^3}\right) dz_i \wedge d\bar{z}_i. \end{aligned} \tag{4.5}$$

Define a new Hermitian metric $\tilde{h}_L$ on L by

$$\tilde{h}_L = \exp(-\sum_{i=1}^{n}(\alpha|z_i|^2 + \alpha\psi_i))\tilde{h}, \tag{4.6}$$

explicitly,

$$\tilde{h}_L = \prod_{i=1}^{k}|z_i|^{2\delta}(\log|z_i|^2)^2 \prod_{i=1}^{n}\exp(-\alpha|z_i|^2 - \alpha\psi_i)h.$$

Notice that $\tilde{h}_L$ decays to zero exponentially when $|z_i|$ goes to r. Since $g \in L_2^{n,q}(U_r^*, L)$ with respect to ω_P and $\tilde{h}$, from (4.4) and (4.6) we get $g \in L_2^{n,q}(U_r^*, L)$ with respect to $\tilde{\omega}_P$ and $\tilde{h}_L$.

Denote the curvature of $\tilde{h}_L$ to be $\Theta(\tilde{h}_L)$. Recall that $\|e(z)\|_h \equiv 1$ for the generating holomorphic section e of L on U_r. By an easy calculation, we have

$$\Theta(\tilde{h}_L) = \sum_{i=1}^{k} \frac{2\alpha}{|z_i|^2(\log|z_i|^2)^2} dz_i \wedge d\bar{z}_i + \sum_{i=1}^{n} \alpha \left(1 + \frac{r^2 + |z_i|^2}{(r^2 - |z_i|^2)^3}\right) dz_i \wedge d\bar{z}_i + \Theta(h).$$

Compared with (4.5),

$$\sqrt{-1}\Theta(\tilde{h}_L) \geq \alpha \cdot \tilde{\omega}_P$$

where $\alpha > 1$ is chosen very large.

Let $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$ be the eigenvalues of $\sqrt{-1}\Theta(\tilde{h}_L)$ with respect to $\tilde{\omega}_P$ at each point $x \in X$, then $\lambda_i(x) \geq 1, \forall i = 1, ..., n$. By the estimate (2.1),

$$\langle[\sqrt{-1}\Theta(\tilde{h}_L), \Lambda]u, u\rangle \geq (\lambda_1 + \cdots + \lambda_q)|u|^2$$

for any form $u \in \Omega^{n,q}(L)$. As a consequence, all eigenvalues of $A = [\sqrt{-1}\Theta(\tilde{h}_L), \Lambda]$ are ≥ 1. Let $\lambda(x)$ the smallest eigenvalue of A, then

$$\int_{U_r^*} \langle A^{-1}g, g\rangle \tilde{\omega}_P^n \leq \int_{U_r^*} \lambda(x)^{-1} \langle g, g\rangle \tilde{\omega}_P^n \leq \int_{U_r^*} \langle g, g\rangle \tilde{\omega}_P^n \leq +\infty.$$

Hence by Theorem 2.6, there exists an L-valued $(n, q-1)$-form f such that $\bar{\partial} f = g$ and f is L^2-integrable with respect to $\tilde{\omega}_P$ and $\tilde{h}_L$. By the construction we see that $\tilde{\omega}_P$ (resp. $\tilde{h}_L$) is quasi-isometric to ω_P (resp. h_L) on $U^*_{\frac{r}{2}}$. Hence f is L^2-integrable with respect to ω_P and h_L on $U^*_{\frac{r}{2}}$. Hence (4.1) is exact at $\mathcal{C}^{n,p}_{2,L}$, $\forall p > 0$.

Therefore we conclude that

$$H^q(X, K_X \otimes L \otimes \mathcal{O}(D)) \simeq H^{n,q}_{(2)}(X \setminus D, L), \quad \forall q \geq 0.$$

$\square$

4.3. A partial result on Fujino's conjecture and a vanishing theorem

Theorem 4.4. *Let X be a compact Kähler manifold and let D be a reduced simple normal crossing divisor on X. Let (L, h) be a Hermitian line bundle on X which is semi-positive on X and positive on D. Let s be a nonzero* *holomorphic section of $L^{\otimes k}$ on X for some positive integer k. Then the multiplication homomorphism*

$$\times s : H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes l}) \to H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes(l+k)})$$

which is induced by $\otimes s$ is injective for every $q \geq 0$ and $l > 0$. Here ω_X is the canonical line bundle of X and $\times s$ is the homomorphism induced by the tensor product with s.

Proof. We put $n = \dim_{\mathbb{C}} X$. Fix a Poincaré-type metric ω_P on the complement $Y := X \setminus D$. Since h is a semi-positive Hermitian metric on L which is positive on D (it means that for an arbitrary point $x \in D$, the curvature at x is positive). Fix a neighborhood W of D such that $\overline{W}$ is compact and h is positive on $\overline{W}$.

Let $D = \sum_{i=1}^r D_i$ where each D_i is an irreducible component of D and let s_i be the defining section of D_i in $\mathcal{O}(D_i)$. We can choose for each $\mathcal{O}(D_i)$ a Hermitian metric $\| \cdot \|_{D_i}$ such that $\|s_i\|_{D_i} = 1$ on $X \setminus W$. We give L a new Hermitian metric h_L over X defined by

$$h_L = \prod_{i=1}^r \|s_i\|_{D_i}^{2\delta} (\log \epsilon \|s_i\|_{D_i}^2)^{2\alpha} h,$$

where $\delta > 0$, $\epsilon > 0$ and $\alpha > 0$ are constants to be determined later. We will choose α to be very large, while we will choose δ and ϵ to be small.

Notice that this metric coincides with $(\log \epsilon)^{2\alpha} h$ on $X \setminus W$. The curvature of h_L is

$$\begin{aligned}\Theta(h_L) = \Theta(h) - \sum_{i=1}^r \delta \cdot \partial\bar{\partial} \log \|s_i\|_{D_i}^2 - \sum_{i=1}^r \frac{\alpha \cdot \partial\bar{\partial} \log \|s_i\|_{D_i}^2}{\log(\epsilon \|s_i\|_{D_i}^2)} \\ + \sum_{i=1}^r \frac{\alpha \cdot \partial(\log \|s_i\|_{D_i}^2) \wedge \bar{\partial}(\log \|s_i\|_{D_i}^2)}{(\log(\epsilon \|s_i\|_{D_i}^2))^2}.\end{aligned}$$

Given an arbitrary $\alpha > 0$, since $\overline{W}$ is compact and h is positive on $\overline{W}$, $\Theta(h_L)$ is positive on W if ϵ and δ are small positive constants.

Using the metrics ω_P and h_L, we can define the L^2-integrable L-valued (p, q)-forms and the corresponding L^2-cohomology.

By Theorem 4.3, there are canonical isomorphisms:

$$H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes l}) \simeq H^{n,q}_{(2)}(X \setminus D, L^{\otimes l})$$

and

$$H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes(l+k)}) \simeq H^{n,q}_{(2)}(X \setminus D, L^{\otimes(l+k)}).$$

Notice that the sheaf cohomology of coherent sheaves on a compact complex manifold is finite dimensional. Let $s \in H^0(X, L^{\otimes k})$ be a holomorphic section, then $\|s\|_{L^\infty}$ is bounded since it is defined on the compact space X. Hence by the same discussions in Theorem 3.4, we obtain the desired injectivity theorem:

$$\times s : H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes l}) \to H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes(l+k)}).$$

□

When $D = \emptyset$, this theorem is reduced to Theorem 1.2 of Enoki mentioned in the introduction.

However, when $D \neq \emptyset$, it is a pity that the cohomology in fact vanishes, which may be a key point for the further research.

Proposition 4.5. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor on X. Suppose that L is a line bundle on X and h is a semi-positive Hermitian metric on L which is positive on D. Then*

$$H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L) = 0$$

for every $q > 0$.

Proof. Choose the same Poincaré-type metric ω_P on X and the Hermitian metric h_L on L as in Theorem 4.4. By the discussions in Theorem 4.4, h_L is semi-positive on X and positive on D. Then we can define the corresponding L^2-cohomology $H^{n,q}_{(2)}(X, L)$ with respect to ω_P and h_L.

There is the canonical isomorphism (Theorem 4.3):

$$H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L) \simeq H^{n,q}_{(2)}(X \setminus D, L), \quad \forall q \geq 0.$$

Notice that the sheaf cohomology of coherent sheaves on a compact complex manifold is finite dimensional. Proposition 3.2 leads to

$$H^{n,q}_{(2)}(X, L) \simeq \mathcal{H}^{n,q}_{(2)}(X, L),$$

where $\mathcal{H}^{n,q}_{(2)}(X, L)$ is the space of L^2-integrable harmonic (n, q)-forms on $X \setminus D$.

Taking any element $u \in \mathcal{H}^{n,q}_{(2)}(X,L)$, by the same arguments in Theorem 3.4, we have

$$\langle \sqrt{-1}\Theta_{h_L}(L)\Lambda u, u\rangle = 0,$$

where Λ is the adjoint of $\omega \wedge \cdot$ and ω is the fundamental form of ω_P.

Since (L, h_L) is strictly positive in a neighborhood U of D, $u = 0$ on U, which implies that $u = 0$ on $X \setminus D$ ([5], VII, Lemma 2.4). Thus $\mathcal{H}^{n,q}_{(2)}(X,L) = 0$. Combined with the two isomorphisms above, we obtain the proposition. □

The above vanishing theorem is a generalization of Norimatsu's vanishing theorem 1.7 when $p = n$.

Remark 4.6. When $L|_D$ is ample, the vanishing theorem may fail even if L is generated by global sections on X. For example, let's consider $S = \mathbb{P}^1 \times E$ where E is an elliptic curve. Denote by $p : S \to \mathbb{P}^1$ the projection and ω_S the canonical bundle of S, then $\omega_S = p^*\mathcal{O}_{\mathbb{P}^1}(-2)$. We put $L = p^*\mathcal{O}_{\mathbb{P}^1}(2)$, then

$$H^1(S, \omega_S \otimes L) = H^1(S, \mathcal{O}_S) = \mathbb{C}.$$

Let $\pi : S' \to S$ be the blowup at a point on S and denote by D the exceptional divisor. We define $L' = \pi^*L \otimes \mathcal{O}_{S'}(-D)$. Then L' is generated by global sections and $L'|_D$ is ample since $L' \cdot D = 1$. Because $R^1\pi_*\omega_{S'} = 0$,

$$H^1(S', \omega_{S'} \otimes \mathcal{O}_{S'}(D) \otimes L') = H^1(S, \omega_S \otimes L) = \mathbb{C}.$$

Remark 4.7. The assumption that L is semi-positive on X and positive on D impies that L is big. Because of the existence of the big line bundle, X is Moishezon which is Kähler and hence is projective. The divisor $L + \epsilon D$ is nef and big where ϵ is a small positive rational number. Hence Proposition 4.5 follows directly from the Kawamata-Viehweg vanishing theorem.

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Chen Zhao
Center of Mathematical Sciences
Zhejiang University, 310027
Hangzhou, P. R. China
E-mail: lancyzhao@gmail.com