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Loss of Derivatives in the Infinite Type

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Abstract: We prove hypoellipticity with loss of ϵ derivatives for a system of complex vector fields whose Lie-span has a superlogarithmic estimate. In $\mathbb{C} \times R$, the model is $(\bar{L}, \bar{f}^k L)$ where $\bar{f} = \bar{z}h$ for $h \neq 0$ and L is the vector field tangential to the exponentially non-degenerate hypersurface of infinite type defined by $x_2 = e^{-\frac{1}{|z|^\alpha}}$ for $\alpha < 1$.
Keywords: hypoellipticity, loss of derivatives, superlogarithmic estimate, infinite type.

1. Introduction

A system of vector fields $(L_j)_j$ has a subelliptic estimate when it has a gain of $\epsilon > 0$ derivatives in the sense that $\|\Lambda^\epsilon u\|^2 \underset{\sim}{<} \sum_j \|L_j u\|^2 + \|u\|^2$, $u \in C_c^\infty$. Here Λ is the standard elliptic pseudodifferential operator of order 1. A system which has finite bracket type $2m$ is a system whose commutators of order $2m-1$ span the whole tangent space. It is well known that finite type of order $2m$ implies a δ-subelliptic estimate for some $\epsilon \leq \frac{1}{2m}$. If $(L, \bar{L})$, in $\mathbb{C} \times \mathbb{R}$, are identified to the generators of the tangential bundle $T^{1,0}M \oplus T^{0,1}M$ to a pseudoconvex hypersurface $M \subset \mathbb{C}^2$, then $(L, \bar{L})$ has finite type $2m$ if and only if the contact of a complex curve γ with M is at most $2m$. Let the hypersurface M be "rigid", that is, graphed by $\mathsf{Re}\, w = g(z)$ for a real C^∞ function g, and set $g_1 = \partial_z g$, $g_{1\bar{1}} = \partial_z \partial_{\bar{z}} g$ and $t = \mathsf{Im}\, w$. With this notation we have $L = \partial_z - i g_1(z)\partial_t$ and $[L, \bar{L}] = g_{1\bar{1}}\partial_t$. It is assumed that M is pseudoconvex, that is, $g_{1\bar{1}} \geq 0$ (which also motivates the choice of an even type $2m$). In terms of g, the condition of finite type $2m$ means that $g_{1\bar{1}}$ has some non-vanishing derivative of order $2(m-1)$. In particular, this happens if $g_{1\bar{1}} \underset{\sim}{>} |x|^{2(m-1)}$; in this case, according for example to [12], we have a $\frac{1}{2m}$-subelliptic estimate.

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A system has a superlogarithmic estimate if it has logarithmic gain of derivatives with an arbitrarily large constant, that is, for any δ and for suitable c_δ

$$\|\log(\Lambda)u\|^2 \underset{\sim}{<} \delta \sum_j \|L_j u\|^2 + c_\delta \|u\|^2_{-1}, \qquad u \in C^\infty_c. \tag{1.1}$$

A system which satisfies (1.1) is "precisely H^s-hypoelliptic" for any s: u is H^s where the $L_j u$'s are (Kohn [7]). In particular, the system is C^∞-hypoelliptic. Let $L = \partial_z - ig_1(z)\partial_t$ for g of infinite type but exponentially non-degenerate in the sense that for a real curve $S \subset \mathbb{C}$ we have

$$d_S^\alpha |\log g_{1\bar 1}| \searrow 0 \text{ as } d_S \searrow 0 \text{ for } \alpha < 1, \tag{1.2}$$

where d_S denotes the distance to S. Under this assumption, the system $(L, \bar L)$ has a superlogarithmic estimate (cf. [12]). If we consider the perturbed system $(\bar L, \bar f^k L)$ for $\bar f = \bar z h(z)$ with $h \neq 0$ and $k \geq 1$, the system has no more a superlogarithmic estimate, in general; if $k > 1$, a logarithmic loss occurs (Proposition 1.4 below). However, it is worth noticing that $\mathcal{L}ie(\bar L, \bar f^k L)$, the span of commutators of order $\leq k-1$, has a superlogarithmic estimate (since it produces L as a commutator of order $k-1$). We are able to prove below that, in the terminology of Kohn [8], the system $(\bar L, \bar f^k L)$ has an arbitrarily small loss of ϵ derivatives and thus, in particular, is C^∞-, but not exactly H^s-, hypoelliptic. Let ζ_0 and ζ_1 be cut-off functions in a neighborhood of 0 with $\zeta_0 \prec \zeta_1$ in the sense that $\zeta_1 \equiv 1$ over a neighborhood of $\operatorname{supp} \zeta_0$.

Theorem 1.1. *Let $L = \partial_z - ig_1(z)\partial_t$ and assume that 0 is a point of infinite type, i.e. $g_{1\bar 1} = 0^\infty(|z|)$ but not exponentially degenerate, i.e. (1.2) is fulfilled. Then the system $(\bar L, \bar f^k L)$ (any k) has an arbitrarily small loss of ϵ derivatives, that is,*

$$\|\zeta_0 u\|^2_s \underset{\sim}{<} \|\zeta_1 \bar L u\|^2_{s+\epsilon} + \|\zeta_1 \bar f^k L u\|^2_{s+\epsilon} + \|\bar f^k u\|^2_\epsilon + \|u\|^2_0. \tag{1.3}$$

The proof of this theorem and of the two propositions below follows in Section 2. Generally, an estimate of type (1.3) for smooth u does not yield finiteness of $\|\zeta_0 u\|_s$ for a H^ϵ-solution u of $\bar L u = f$, $\bar f^k L u = g$ when $\zeta_1 f$ and $\zeta_1 g$ are in $H^{s+\epsilon}$. However, L has coefficient t-independent and therefore it commutes with the t-derivatives. On the other hand, the t-derivatives describe the full Sobolev norm on the "positively microlocalized" component u^+ (cf. §2 below) which is the only one which needs to be controlled. For this reason, if we use a sequence of pseudodifferential smoothing operators in t, $\chi_\nu(\partial_t) \to \mathrm{id}$ as in [8] and [1], and we remark that

$$\bar{L}(\chi_\nu(\partial_t)u^+) = \chi_\nu(\partial_t)(\bar{L}u^+) + \mathrm{Order}_{-\infty},$$

then, (1.3) applied to $\Lambda^s\Big(\chi_\nu(\partial_t)u^+\Big) = \chi_\nu(\partial_t)\Big(\Lambda^s u^+\Big)$ yields

Corollary 1.2. *In the situation of Theorem 1.1, the system $(\bar{L}, \bar{f}^k Lu)$ is hypoelliptic with loss of ϵ derivatives: $(\bar{L}u, \bar{f}^k Lu) \in H^{s+\epsilon}$, $u \in H^\epsilon$ implies $u \in H^s$.*

For $k = 1$ we have an estimate for local regularity without loss

Proposition 1.3. *In the situation above, assume in addition*

$$|g_1| \underset{\sim}{<} g_{1\bar{1}}^{\frac{1}{2}}; \tag{1.4}$$

then

$$\|\zeta_0 u\|_s^2 \underset{\sim}{<} \|\zeta_1 \bar{L}u\|_s^2 + \|\zeta_1 \bar{f} Lu\|_s^2 + \|u\|_0^2. \tag{1.5}$$

When $k > 1$, a loss must occur

Proposition 1.4. *Assume that $g = e^{-\frac{1}{|z|^\alpha}}$. If*

$$\|\zeta_0 u\|_s^2 \underset{\sim}{<} \|(\log\Lambda)^r \zeta_1 \bar{L}u\|_s^2 + \|(\log\Lambda)^r \zeta_1 \bar{f}^k Lu\|_s^2 + \|\bar{f}^k u\|_\epsilon^2 + \|u\|_0^2, \tag{1.6}$$

then we must have $r \geq \frac{k-(\alpha+1)}{\alpha}$.

As far as we know, this is the first time that the problem of hypoellipticity is discussed for degenerate vector fields $(\bar{L}, \bar{f}^k L)$ obtained from $L = \partial_z - ig_1(z)\partial_t$ of infinite type, that is, satisfying $g_{1\bar{1}} = 0^\infty(|z|)$. However, it is necessary to make further assumptions such as (1.2). This guarantees a superlogarithmic estimate ([12]), and in turn, hypoellipticity according to Kohn [7]. Hypoellipticity with loss of derivatives for $L = \partial_z - i\bar{z}\partial_t$ was discovered by Kohn in [8]. In this case, L is the $(1,0)$ vector field tangential to the strictly pseudoconvex hypersurface $\mathrm{Re}\, w = |z|^2$ and the loss amounts to $\frac{k-1}{2}$. The problem was further discussed by Bove, Derridj, Kohn and Tartakoff in [1] essentially for the vector field $L = \partial_z - i\bar{z}|z|^{2(m-1)}\partial_t$ tangential to the hypersurface $\mathrm{Re}\, w = |z|^{2m}$ and the resulting loss is $\frac{k-1}{2m}$. In both cases the conclusion extends to the sum of squares $L\bar{L} + \bar{L}|z|^{2k}L$ and the loss doubles to $\frac{k-1}{m}$. Moreover, in [1], analytic hypoellipticity has been proved; notice that this cannot be discussed in our framework, since, g having infinite type, it cannot be real analytic. For the vector fields $L = \partial_z - ig_1(z)\partial_t$ tangential to a general pseudoconvex hypersurface of finite type with $g_{1\bar{1}}$ vanishing at order $2(m-1)$ along a real curve), hypoellipticity with loss of $\frac{k-1}{2m}$ derivatives has been proved by the authors in [11]. Under some additional conditions, the result also extends to sums of squares (with double loss $\frac{k-1}{m}$). When the hypersurface has infinite type as in the present paper, it is therefore natural to expect an arbitrarily small loss of derivatives.

2. Technical preliminaries and Proof

Our ambient space is $\mathbb{C} \times \mathbb{R}$ identified with $\mathbb{R}^3$ endowed with coordinates $(z, \bar{z}, t)$ or $(\mathsf{Re}\, z, \mathsf{Im}\, z, t)$. We denote by $\xi = (\xi_{\mathsf{Re}\, z}, \xi_{\mathsf{Im}\, z}, \xi_t)$ the variables dual to $(\mathsf{Re}\, z, \mathsf{Im}\, z, t)$, by Λ^s_ξ the standard symbol $(1 + |\xi|^2)^{\frac{s}{2}}$, and by Λ^s the pseudodifferential operator with symbol Λ^s_ξ; this is defined by $\Lambda^s(u) = \mathcal{F}^{-1}(\Lambda^s_\xi \mathcal{F}(u))$ where $\mathcal{F}$ is the Fourier transform. We also consider the partial symbol $\Lambda^s_{\xi_t}$ and the associate pseudodifferential operator Λ^s_t. We denote by $\|u\|_s := \|\Lambda^s u\|_0$ (resp. $\|u\|_{\mathbb{R},\, s} := \|\Lambda^s_t u\|_0$) the full (resp. totally real) s-Sobolev norm. We use the notation $\gtrsim$ and $\lesssim$ to denote inequalities up to multiplicative constants; we denote by $\sim$ the combination of $\gtrsim$ and $\lesssim$. In $\mathbb{R}^3_\xi$, we consider a conical partition of the unity $1 = \psi^+ + \psi^- + \psi^0$ where $\psi^\pm$ have support in a neighborhood of the axes $\pm\xi_t$ and ψ^0 in a neighborhood of the plane $\xi_t = 0$, and introduce a decomposition of the identity $\mathrm{id} = \Psi^+ + \Psi^- + \Psi^0$ by means of $\Psi^{\overset{\pm}{0}}$, the pseudodifferential operators with symbols $\psi^{\overset{\pm}{0}}$; we accordingly write $u = u^+ + u^- + u^0$. Since $|\xi_{\mathsf{Re}\, z}| + |\xi_{\mathsf{Im}\, z}| \lesssim \xi_t$ over $\operatorname{supp} \psi^+$, then $\|u^+\|_{\mathbb{R},\, s} \sim \|u^+\|_s$.

We carry on the discussion by describing the properties of commutation of the vector fields L and $\bar{L}$ for $L = \partial_z - ig_1(z)\partial_t$. The crucial equality is

$$\|Lu\|^2 = ([L, \bar{L}]u, u) + \|\bar{L}u\|^2, \quad u \in C^\infty_c, \tag{2.1}$$

which is readily verified by integration by parts. Since $\sigma(\partial_t)$, the symbol of ∂_t, is dominated by $\sigma(L)$ and $\sigma(\bar{L})$ in the "elliptic region" (the support of ψ^0) and since L can be controlled by $\bar{L}$ with an additional $\epsilon\partial_t$ (because of (2.1)), then $\|u^0\|^2_1 \lesssim \|\bar{L}u\|^2_0 + \|u\|^2_0$. As for u^-, recall that $[L, \bar{L}] = g_{1\bar{1}}\partial_t$

and hence $g_{1\bar{1}}\sigma(\partial_t) \leq 0$ over suppψ^-. Thus (2.1) yields $\|Lu\|^2 \underset{\sim}{<} \|\bar{L}u\|^2$. It follows that, if L and $\bar{L}$ have superlogarithmic estimate as in our application, then

$$\|\log(\Lambda)u^-\|^2 \leq \delta\|\bar{L}u^-\|^2 + c_\delta\|u\|^2.$$

In conclusion, only estimating u^+ is relevant. We note here that, over $\operatorname{supp}\Psi^+$, we have $g_{1\bar{1}}\xi_t \geq 0$; thus

$$\begin{aligned} \|g_{1\bar{1}}^{\frac{1}{2}}u^+\|^2_{\frac{1}{2}} &= |([L,\bar{L}]u^+,u^+)| \\ &\leq \|Lu^+\|^2 + \|\bar{L}u^+\|^2. \end{aligned} \tag{2.2}$$

Following Kohn [7], we introduce a microlocal modification of Λ^s, denoted by R^s; this is the pseudodifferential operator with symbol $R^s_\xi := (1+|\xi|^2)^{\frac{s\sigma(x)}{2}}$, $\sigma \in C^\infty_c$, that is, $R^s(u) = \mathcal{F}^{-1}(R^s_\xi\mathcal{F}(u))$. Often, what is used is in fact the partial operator in t with symbol $R^s_{\xi_t}$. We denote it by the same symbol R^s and observe that, f being independent of t, we have

$$[R^s, f] = 0. \tag{2.3}$$

The relevant property of R^s is

$$\|\Lambda^s\zeta_0 u\|^2 \underset{\sim}{<} \|R^s\zeta_0 u\|^2 + \|\zeta_0 u\|^2 \qquad \text{if } \zeta_0 \prec \sigma.$$

Thus, R^s is equivalent to Λ^s over functions supported in the region where $\sigma \equiv 1$. In addition, $\zeta_1 R^s$ better behaves with respect to commutation with L; in fact, Jacobi equality yields

$$[\zeta_1 R^s, L] \sim \dot{\zeta}_1 R^s + \zeta_1\log(\Lambda)R^s. \tag{2.4}$$

Thus, on one hand we have the disadvantage of the additional $\log(\Lambda)$ in the second term, but we gain much in the cut-off because

$$\dot{\zeta}_1 R^s \text{ is of order 0 if } \operatorname{supp}\dot{\zeta}_1 \cap \operatorname{supp}\sigma = \emptyset. \tag{2.5}$$

Property (2.5) is crucial in localizing regularity in presence of superlogarithmic estimate.

Proof of Theorem 1.1. As it has already been noticed, it suffices to prove (1.3) only for u^+ and for $\|\cdot\|_{\mathbb{R},\,s}$; thus we write for simplicity u and $\|\cdot\|_s$ but mean u^+ and $\|\cdot\|_{\mathbb{R},\,s}$. Moreover, we can use a cut-off $\zeta = \zeta(t)$ in t only. In fact, for a cut-off $\zeta = \zeta(z)$ we have $[L, \zeta(z)] = \dot{\zeta}$ and $\dot{\zeta} \equiv 0$ at $z = 0$. On the other hand, $\bar{f}^k L \sim L$ outside $z=0$ which yields gain of derivatives, instead of loss. We call "good" a term in the right side (upper bound) of an estimate we wish to prove and "absorbable" a term which comes as a fraction (small constant or sc) of a formerly encountered term. We take cut-off functions in a neighborhood of 0: $\zeta_0 \prec \sigma \prec \zeta_1$; we have for $u \in C^\infty$

$$\begin{aligned} \|\zeta_0 u\|_s^2 &= \|\zeta_0\zeta_1 u\|_s^2 \\ &\lesssim \|R^s\zeta_0\zeta_1 u\|^2 + \|u\|_0^2 \\ &\lesssim \|\zeta_0 R^s\zeta_1 u\|_0^2 + \|[R^s,\zeta_0]\zeta_1 u\|_0^2 + \|u\|_0^2 \\ &\lesssim \|R^s\zeta_1 u\|_0^2 + \|u\|_0^2 \\ &\lesssim \|\zeta_1 R^s\zeta_1 u\|_0^2 + \|u\|_0^2, \end{aligned} \tag{2.6}$$

where the inequality in the fourth line follows from interpolation in Sobolev spaces and the last from $\mathrm{supp}(1-\zeta_1)\cap \mathrm{supp}\sigma = \emptyset$. We have

$$\begin{aligned} \|\zeta_0 u\|_s^2 &\underset{\text{by (2.6)}}{\lesssim} \underbrace{\|\zeta_1 R^s\zeta_1 u\|^2}_{(a)} + \|u\|^2 \\ &\underset{(*)}{\leq} \delta\left(\|L(\zeta_1 R^s\zeta_1)u\|^2 + \|\bar{L}(\zeta_1 R^s\zeta_1)u\|^2\right) + c_\delta\|u\|^2, \end{aligned} \tag{2.7}$$

where the inequality marked by $(*)$ follows from compactness which is a byproduct of superlogarithmic estimate. In the last line, we leave aside the central term and attack the first. Using integration by parts, we have

$$L \lesssim \bar{L} + [L,\bar{L}] \quad \text{microlocally on supp } \psi^+. \tag{2.8}$$

We rewrite the commutator. For this we recall an easy result about interpolation in Sobolev spaces. For positive ϵ, r, n_1, n_2 with n_1 and n_2 integers satisfying $0 < n_1 \leq r$ and $n_2 > 0$,

$$\|h^r u\|_{\frac12}^2 \leq sc\|h^{r-n_1}u\|^2_{\frac12 - n_1\epsilon} + lc\|h^{r+n_2}u\|^2_{\frac12+n_2\epsilon}. \tag{2.9}$$

We apply (2.9) for $h = g_{1\bar{1}}^{\frac{1}{2r}}$, $n_1 = r$, $\epsilon = \frac{1}{2r}$, $n_2 = 1$ (and note that h needs not to be smooth because h is a function of z whereas Sobolev norms are taken with respect to t). We observe that, since g has infinite order, then

$g_{1\bar{1}}^{\frac{1}{2r}} \lesssim |f|^k$ for any r and any k. It follows

$$
\begin{aligned}
\||[L,\bar{L}]|^{\frac{1}{2}}\zeta_1 R^s\zeta_1 u\|^2 &= \|g_{1\bar{1}}^{\frac{1}{2}}\zeta_1 R^s\zeta_1 u\|^2_{\frac{1}{2}} \\
&\lesssim \text{sc}\|\zeta_1 R^s\zeta_1 u\|_0^2 + \text{lc}\|g_{1\bar{1}}^{\frac{1}{2}} g_{1\bar{1}}^{\frac{1}{2r}}\zeta_1 R^s\zeta_1 u\|^2_{\frac{1}{2}+\frac{1}{2r}} \\
&\lesssim \text{sc}\|\zeta_1 R^s\zeta_1 u\|_0^2 + \text{lc}\|g_{1\bar{1}}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\bar{f}^k\zeta_1 R^s\zeta_1 u\|^2_{\epsilon} \qquad (2.10)\\
&= \text{sc}\|\zeta_1 R^s\zeta_1 u\|_0^2 + \text{lc}\|[L,\bar{L}]^{\frac{1}{2}}\bar{f}^k\zeta_1 R^s\zeta_1 u\|^2_{\epsilon} \\
&\lesssim \|\zeta_1 R^s\zeta_1 u\|_0^2 + \|L\bar{f}^k(\zeta_1 R^s\zeta_1)u\|^2_{\epsilon} + \|\bar{L}\bar{f}^k(\zeta_1 R^s\zeta_1)u\|^2_{\epsilon}.
\end{aligned}
$$

We wish to first estimate the second term in the bottom of (2.10) in which we also replace $L\bar{f}^k$ by $\bar{f}^k L$. In doing so, we encounter an error term $\|\bar{f}^k(\zeta_1 R^s\zeta_1)u\|^2_{\epsilon}$ that we will estimate later on; (in fact, $[L,\bar{f}^k] \sim \bar{f}^k$ since $\bar{f} = \bar{z}h$ and $[L,\bar{z}] = 0$). After this, we recall Jacobi identity, observe that $[\bar{f}^k L, \zeta_1 R^s\zeta_1]$ has order arbitrarily close to $s-1$ (because of a logarithmic extra term), that is

$$
\begin{aligned}
[\bar{f}^k L, \zeta_1 R^s\zeta_1] &= [L,\zeta_1]R^s\zeta_1\bar{f}^k + \zeta_1[\bar{L},R^s]\zeta_1\bar{f}^k + \zeta_1 R^s[\bar{L},\zeta_1]\bar{f}^k \\
&\sim \underbrace{\dot{\zeta}_1 R^s\zeta_1}_{\text{0-order by (2.5)}}\ \bar{f}^k + \underbrace{\zeta_1\log(\Lambda)R^s\zeta_1}_{\text{by (2.4)}}\bar{f}^k + \underbrace{\zeta_1 R^s\dot{\zeta}_1}_{\text{0-order by (2.5)}}\ \bar{f}^k. \qquad (2.11)
\end{aligned}
$$

Thus we can commutate $\bar{f}^k L$ with $\zeta_1 R^s\zeta_1$ up to an error as described in (2.11) which yields

$$
\begin{aligned}
\|L\bar{f}^k(\zeta_1 R^s\zeta_1)u\|^2_{\epsilon} &\lesssim \|\bar{f}^k L(\zeta_1 R^s\zeta_1)u\|^2_{\epsilon} + \|\bar{f}^k(\zeta_1 R^s\zeta_1)u\|^2_{\epsilon} \\
&\lesssim \|(\zeta_1 R^s\zeta_1)\bar{f}^k Lu\|^2_{\epsilon} + \|(\zeta_1\log(\Lambda)R^s\zeta_1)\bar{f}^k u\|^2_{\epsilon} \\
&\quad + \|\bar{f}^k u\|^2_{\epsilon} + \|\bar{f}^k(\zeta_1 R^s\zeta_1)u\|^2_{\epsilon}.
\end{aligned}
$$

On the other hand, since $[\zeta_1,\log(\Lambda)]R^s$ has order 0, then

$$
\begin{aligned}
\|(\zeta_1\log(\Lambda)R^s\zeta_1)\bar{f}^k u\|^2_{\epsilon} &\lesssim \|(\log(\Lambda)(\zeta_1 R^s\zeta_1)\bar{f}^k u\|^2_{\epsilon} + \|\bar{f}^k u\|^2_{\epsilon} \\
&\underset{\text{suplog estimate}}{\lesssim} \underbrace{\delta\left(\|L(\zeta_1 R^s\zeta_1)\bar{f}^k u\|^2_{\epsilon} + \|\bar{L}(\zeta_1 R^s\zeta_1)\bar{f}^k u\|^2_{\epsilon}\right)}_{\text{absorbed by the last line of (2.10)}} + \|\bar{f}^k u\|^2_{\epsilon},
\end{aligned}
$$

where we are using the equality $[\Lambda^{\epsilon}_t, L] = 0$ as well as $[\Lambda^{\epsilon},\log(\Lambda)] = 0$. In the same way, using again (2.11), we estimate the central term in the last line of (2.7) which was left aside, that is,

$$\|\bar{L}(\zeta_1 R^s \zeta_1)u\|^2 \lesssim \|(\zeta_1 R^s \zeta_1)\bar{L}u\|^2 + \|u\|^2.$$

What remains, is to estimate the last term in the bottom of (2.10) (together with the error term $\|\bar{f}^k(\zeta_1 R^s \zeta_1)u\|^2_\epsilon$). First, from Jacobi identity we get

$$[\bar{L}\bar{f}^k, \zeta_1 R^s \zeta_1] \sim (0\text{-order})\bar{f}^k + \zeta_1 \log(\Lambda) R^s \zeta_1 \bar{f}^k + (0\text{-order})\bar{f}^k,$$

so that we are eventually reduced to estimate $\|(\zeta_1 R^s \zeta_1)\bar{L}\bar{f}^k u\|^2$. This is the most difficult operation. We have (by the identity $[\bar{L}, \bar{f}^k] \sim \bar{f}^{k-1}$)

$$\|(\zeta_1 R^s \zeta_1)\bar{L}\bar{f}^k u\|^2_\epsilon \lesssim \underbrace{\|(\zeta_1 R^s \zeta_1)\bar{f}^k \bar{L}u\|^2_\epsilon}_{\text{good}} + \|(\zeta_1 R^s \zeta_1)\bar{f}^{k-1}u\|^2_\epsilon.$$

Next we estimate the last term in the line above which also serves as an estimate for the term $\|\bar{f}^k(\zeta_1 R^s \zeta_1)u\|^2_\epsilon$ that was encountered before. We have

$$\begin{aligned}\underbrace{\|(\zeta_1 R^s \zeta_1)\bar{f}^{k-1}u\|^2_\epsilon}_{(c)} &\sim (\underbrace{(\zeta_1 R^s \zeta_1)\bar{f}^{k-1}u}_{*}, (\zeta_1 R^s \zeta_1)[\bar{L}, \bar{f}^k]u)_\epsilon \\ &= -(*, (\zeta_1 R^s \zeta_1)\bar{f}^k \bar{L}u)_\epsilon + (*, (\zeta_1 R^s \zeta_1)\bar{L}\bar{f}^k u)_\epsilon.\end{aligned}$$

Now,

$$\begin{cases} \left|(*, (\zeta_1 R^s \zeta_1)\bar{f}^k \bar{L}u)_\epsilon\right| \lesssim sc\|*\|^2_\epsilon + \underbrace{\|(\zeta_1 R^s \zeta_1)\bar{f}^k \bar{L}u\|^2_\epsilon}_{\text{good}} \\ \left|(*, (\zeta_1 R^s \zeta_1)\bar{L}\bar{f}^k u)_\epsilon\right| \lesssim \left|(\underbrace{(\zeta_1 R^s \zeta_1)\bar{f}^{k-1} f L u}_{\text{good}}, \underbrace{(\zeta_1 R^s \zeta_1)\bar{f}^{k-1}u}_{\text{absorbed by (c)}})_\epsilon\right| \\ \qquad\qquad + 2\left|(\underbrace{\quad * \quad}_{\text{absorbed by (c)}}, \underbrace{[\bar{L}, (\zeta_1 R^s \zeta_1)]\bar{f}^k u}_{(d)})_\epsilon\right|. \end{cases}$$

We estimate (d). We notice that

$$[\bar{L}, (\zeta_1 R^s \zeta_1)] \sim \zeta_1 \log(\Lambda) R^s \zeta_1 + (0\text{-order}). \tag{2.12}$$

We also remark that

$$\begin{cases} [\Lambda^\epsilon \zeta_1, \log(\Lambda)]R^s \text{ has order } 0 \quad (i) \\ [\zeta_1, \Lambda^\epsilon]R^s \text{ has order } 0 \quad (ii) \\ [L, \Lambda^\epsilon] = 0 \quad (iii). \end{cases} \tag{2.13}$$

Hence

$$
(2.14)\quad
\begin{aligned}
\|(d)\|^2_\epsilon &\underset{\text{by (2.12)}}{\lesssim} \|(\zeta_1\log(\Lambda)R^s\zeta_1)\bar f^k u\|^2_\epsilon + \|\bar f^{k-1}u\|^2_\epsilon + \|u\|^2_0 \\
&\underset{\text{by (2.13) (i) and (ii)}}{\le} \|(\log(\Lambda)\zeta_1\Lambda^\epsilon R^s\zeta_1)\bar f^k u\|^2_0 + \|\bar f^{k-1}u\|^2_\epsilon + \|u\|^2_0 \\
&\underset{\text{by suplog estimate}}{\le} \delta\left(\|L(\zeta_1\Lambda^\epsilon R^s\zeta_1)\bar f^k u\|^2 + \|\bar L(\zeta_1\Lambda^\epsilon R^s\zeta_1)\bar f^k u\|^2\right) \\
&\qquad + \underbrace{c_\delta\|(\zeta_1\Lambda^\epsilon R^s\zeta_1)\bar f^k u\|^2}_{\text{absorbed by (c)}} + \|\bar f^{k-1}u\|^2_\epsilon + \|u\|^2_0,
\end{aligned}
$$

where the last absorption occurs because $\bar f^k = sc\, \bar f^{k-1}$.

Finally, the term with δ is absorbed by the last term in (2.10) (after we transform Λ^ϵ into $\|\cdot\|_\epsilon$ to fit into (2.10) and use the fact that $[\bar L\zeta_1, \Lambda^\epsilon] \sim \dot\zeta_1\Lambda^{\epsilon-1}\bar L$ and $[L\zeta_1, \Lambda^\epsilon] \sim \dot\zeta_1\Lambda^{\epsilon-1}L$). This concludes the proof of (1.3).

Proof of Proposition 1.3. As above, we stay in the positive microlocal cone, the support of ψ^+, and consider only derivatives and cut-off with respect to t. From the trivial identity $[L, f] \sim 1$, and from $[L, \zeta_0] \sim \dot\zeta_0 g_1$, we get

$$
\begin{aligned}
\|\zeta_0 u\|^2_s &= ([L, f]\zeta_0 u, \zeta_0 u)_s \\
&\lesssim \|\bar f\zeta_0\bar L u\|^2_s + \|\bar f\zeta_0 L u\|^2_s + \|\bar f g_1\zeta_1 u\|^2_s + sc\|\zeta_0 u\|^2_s.
\end{aligned}
$$

Now, the last term is absorbed. As for the term before

$$
\begin{aligned}
\|\bar f g_1\zeta_1 u\|^2_s &\underset{\text{by (1.4)}}{\le} \|\bar f g_{1\bar 1}^{\frac12}\Lambda^{\frac12}\zeta_1 u\|^2_{s-\frac12} \\
&\underset{\text{by (2.2)}}{\le} \|\bar f L\zeta_1 u\|^2_{s-\frac12} + \|\bar f\bar L\zeta_1 u\|^2_{s-\frac12} + \|\zeta_1 u\|^2_{s-\frac12} \\
&\lesssim \|\zeta_1\bar f L u\|^2_{s-\frac12} + \|\bar f\bar L\zeta_1 u\|^2_{s-\frac12} + \|\bar f\zeta_2 u\|^2_{s-\frac12} + \|\zeta_1 u\|^2_{s-\frac12},
\end{aligned}
$$

for $\zeta_2 \succ \zeta_1$. Now, $\|\bar f\zeta_2 u\|^2_{s-\frac12}$ and $\|\zeta_1 u\|^2_{s-\frac12}$ are not absorbable by $\|\zeta_0 u\|^2_s$, but can be estimated by the 0-norm using induction over j such that $\frac{j}{2} \ge s$.

Proof of Proposition 1.4. As always, we stay in the positive microlocal cone and take derivatives and cut-off only in t. We prove the result for s replaced by 0 and ϵ replaced by $-\eta$. The conclusion for general s follows from the fact that ∂_t commutes with L and $\bar{L}$. We define

$$v_\lambda = e^{-\lambda(e^{-\frac{1}{|z|^\alpha}}-it+(e^{-\frac{1}{|z|^\alpha}}-it)^2)} \qquad \lambda >> 0.$$

We denote by $-\lambda A$ the term at exponent and note that $\mathsf{Re}\,\lambda A \sim \lambda(e^{-\frac{1}{|z|^\alpha}} + t^2)$. For $L = \partial_z + ig_1(z)\partial_t$, we have $\bar{L}v_\lambda = 0$ (which is the key point) and moreover, since $|\bar{f}|^k \sim |z|^k$

$$|\bar{f}^k L v_\lambda| \sim \lambda|z|^{k-(\alpha+1)}e^{-\lambda(e^{-\frac{1}{|z|^\alpha}}+t^2)}e^{-\frac{1}{|z|^\alpha}}.$$

We set

$$\lambda(e^{-\frac{1}{|z|^\alpha}}, t) = (\theta_1, \frac{1}{\sqrt{\lambda}}\theta_2).$$

Under this change we have, over $\operatorname{supp}\zeta_0$ and $\operatorname{supp}\zeta_1$ which implies $\theta_1 << \lambda$,

$$|z|^{k-(\alpha+1)} = \frac{1}{(\log\lambda - \log\theta_1)^{\frac{k-(\alpha+1)}{\alpha}}}.$$

Hence we interchange

$$|\bar{f}^k L v_\lambda| \dashrightarrow \frac{1}{(\log\lambda)^{\frac{k-(\alpha+1)}{\alpha}}}\left(\frac{\theta_1+\theta_2^2}{\left(1-\frac{\log\theta_1}{\log\lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}}\right)e^{-(\theta_1+\theta_2^2)}.$$

Notice that $\theta_1 << \lambda$ and hence, for suitable positive c_1 and c_2, we have $c_1 < \frac{\theta_1+\theta_2^2}{\left(1-\frac{\log\theta_1}{\log\lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}} < c_2$, uniformly over λ. We also interchange

$$v_\lambda \dashrightarrow e^{-(\theta_1+\theta_2^2)}.$$

Taking L^2 norms yields

$$\|\bar{f}^k L v_\lambda\|^2 \sim \frac{1}{(\log\lambda)^{2\frac{k-(\alpha+1)}{\alpha}}}\|v_\lambda\|^2.$$

So, the effect on L^2 norm of the action of $\bar{f}^k L$ over v_λ is comparable to $\frac{1}{(\log\lambda)^{\frac{k-(\alpha+1)}{\alpha}}}$. We describe now the effect of the pseudodifferential operator

$\log(\Lambda_t)$. We claim that

$$\|\log(\Lambda_t)e^{-\lambda t^2}\|^2 \sim (\log\lambda)^2\|e^{-\lambda t^2}\|^2. \tag{2.15}$$

This is a consequence of

$$\log(\Lambda_t)e^{-\lambda t^2} \sim \log\lambda e^{-\lambda t^2} + \Big(\log(\Lambda_{\tilde{t}})e^{-\tilde{t}^2}\Big)\Big|_{\tilde{t}=\sqrt{\lambda}t}, \tag{2.16}$$

that we go to prove now. Using the coordinate change $\tilde{\theta} = \sqrt{\lambda}\theta,\ \tilde{\xi} = \frac{\xi}{\sqrt{\lambda}}$, we get

$$\begin{aligned}\int e^{it\xi}\log(\Lambda_\xi)\Big(\int e^{-i\xi\theta}e^{-\lambda\theta^2}d\theta\Big)d\xi \\ &= \int e^{it\sqrt{\lambda}\tilde{\xi}}\Big(\log(\frac{1}{\lambda}+|\tilde{\xi}|^2)^{\frac12}+\log(\sqrt{\lambda})\Big)\Big(\int e^{i\tilde{\xi}\tilde{\theta}-\tilde{\theta}^2}d\tilde{\theta}\Big)d\tilde{\xi} \\ &= \log(\sqrt{\lambda})e^{-\lambda t^2} + \Big(\log(\Lambda^{\lambda}_{\tilde{t}})e^{-\tilde{t}^2}\Big)\Big|_{\tilde{t}=\sqrt{\lambda}t},\end{aligned}$$

where $\log(\Lambda^{\lambda}_{\tilde{t}})$ is the operator with symbol $\log(\frac{1}{\lambda}+|\tilde{\xi}|^2)^{\frac12}$. This proves (2.16) and in turn the claim (2.15). In the same way, we can check that $\|\Lambda_t^{-\eta}e^{-\lambda t^2}\|^2 \sim \lambda^{-2\eta}\|e^{-\lambda t^2}\|^2$.

We combine now the effect over v_λ of $\bar{f}^kL$ with that of $\log(\Lambda_t)$. If

$$\|\zeta_0 v_\lambda\|^2 \lesssim \|\zeta_1(\log\Lambda_t)^r\bar{f}^kLv_\lambda\|^2 + \|v_\lambda\|^2_{-\eta},$$

then, since the right side is estimated from above by

$$\Big((\log\lambda)^{2r}(\log\lambda)^{-2\frac{k-(\alpha+1)}{\alpha}} + \lambda^{-2\eta}\Big)\|v_\lambda\|^2,$$

we must have that the logarithmic term is not infinitesimal which forces $r \geq \frac{k-(\alpha+1)}{\alpha}$.

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