Pure and Applied Mathematics Quarterly
Volume 11, Number 2, 293–314, 2015

Traveling Wave-Front Solutions with Small Oscillations at Infinity for a KdV6 Equation under a Small Perturbation

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Abstract: This paper studies the traveling wave solutions of a KdV6 equation under a small perturbation. Applying the dynamical system approach, we rigorously prove that this equation has a new wave solution—wave-front solution with small non-decaying oscillations at infinity (called thereafter generalized wave-front solution).
Keywords: KdV6 equation, wave-front solution, homoclinic solution, periodic solution.

1. Introduction

This paper concerns with the travelling wave solutions of a sixth-order nonlinear wave equation (KdV6 equation)

$$\begin{aligned}\phi_{\xi\xi\xi\xi\xi\xi} + a\phi_\xi\phi_{\xi\xi\xi\xi} + b\phi_{\xi\xi}\phi_{\xi\xi\xi} + c\phi_\xi^2\phi_{\xi\xi} + d\phi_{tt} + e\phi_{\xi\xi\xi t}&\\ + f\phi_\xi\phi_{\xi t} + g\phi_t\phi_{\xi\xi} = 0,&\end{aligned} \tag{1}$$

where a, b, c, d, e, f and g are arbitrary parameters. This equation was first introduced by Karasu-Kalkanli *et al.* [7]. With a Painlevé analysis, they derived from (1) four distinct equations under the differential parameters:

Received March 26, 2015.
2010 *Mathematics Subject Classification*. Primary: 34C37, 35B32; Secondary: 34C25, 74J35.
This research was supported by National Natural Science Foundation of China (11371314), Guangdong Natural Science Foundation of China (S2013010015957), High-Level Talent Project of Colleges and Universities in Guangdong Province (QBS201501) and Natural Science Research Training Program of Lingnan Normal University (YL1503).

the bidirectional version of the Sawada-Kotera-Caudrey-Dodd-Gibbon equation [1, 17], the Kaup-Kupershmidt equation [4, 8], the Drinfeld-Sokolov-Hirota-Satsuma equation [3, 16], and a new KdV equation with a self-consistent source. The equation (1) has been studied mathematically and numerically and a lot of interesting results have been discovered. The Lax pair and the Bäcklund self-transformation were given in [7]. The Hamiltonian structures and conservation laws were studied in [9] and [22]. A much nicer form was discussed in [5] and [14]. The complete integrability was examined in [11]. The connection between HH3 system and the new KdV equatoin was given in [24]. The bilinear forms and soliton interactions were investigated in [18]. Multiple solitons and multiple singular soliton solutions were presented in [20] and [23]. The exact solutions and in particular solitary wave solutions and quasi-periodic wave solutions were obtained with different methods such as the aid of two first integrals, the Bäcklund transformation, the Darboux transformation, the Cole-Hopf transformation and the tanh-coth method [5, 7, 12, 13, 15, 21]. It is worthy to point out that all the obtained solitary wave solutions with explicit expressions exponentially approach to a constant at infinity. However, the existence of other forms of solutions of (1) is an interesting open problem. In this paper, we shall rigorously prove the existence of a new wave-front solution with some small non-decaying oscillations at infinity (called a generalized wave-front solution) using a dynamical system method.

Suppose that the travelling solution ϕ of (1) has a form

$$\phi(t,\xi) = \phi(\xi - vt) = \phi(x),$$

where v is the wave speed, and let

$$y = -(\phi_x - \frac{ev}{a}). \tag{2}$$

Integrating (1) with respect to x once, we have (see [13])

$$y_{xxxx} = ayy_{xx} + \frac{b-a}{2}y_x^2 - \frac{c}{3}y^3 - \gamma y^2 + \alpha y + \beta, \tag{3}$$

where

$$\begin{aligned} &\gamma = -\frac{cev}{a} + \frac{v(f+g)}{2}, \quad \alpha = -\left(\frac{ce^2v^2}{a^2} - \frac{ev^2(f+g)}{a} + dv^2\right), \\ &\beta = \frac{cv^3e^3}{3a^3} - \frac{e^2v^3(f+g)}{2a^2} + \frac{dev^3}{a} + \beta_0, \end{aligned} \tag{4}$$

and β_0 is an integral constant. Note that the natural world always has some very small noises or disturbances and so on. Here, we are specially interested in the solutions of the equation (3) under small perturbations, that is, we will focus on the following equation

$$\begin{aligned} y_{xxxx} =& ayy_{xx} + \frac{b-a}{2}y_x^2 - \frac{c}{3}y^3 - \gamma y^2 + \alpha y + \beta \\ & + \epsilon h(\epsilon, y, y_x, y_{xx}, y_{xxx}), \end{aligned} \tag{5}$$

where ϵ is a small parameter and the condition for the smooth function h is given in (6). If the parameters satisfy some conditions (see (7) or Section 2), some equilibrium of the equation (5) is a saddle-center equilibrium, i.e., the linear operator around the equilibrium has a positive eigenvalue, a negative eigenvalue and a pair of purely imaginary eigenvalues. The stable and unstable manifolds are both one dimensional so that it is not obvious that these stable and unstable manifolds will intersect to form a homoclinic solution (After integrating, it often corresponds to a wave-front solution of (1) under small perturbations) that approaches to this equilibrium as $x \to \pm\infty$. In this paper, we shall theoretically prove the existence of a homoclinic solution with small non-decaying oscillations at infinity of (5) (or a generalized homoclinic solution) by applying a dynamical system approach, which corresponds to a generalized wave-front solution of (1) under small perturbations. The main theorem for (5) can be stated as follows.

Theorem 1.1. *Suppose that the smooth function h satisfies*

$$h(\epsilon, 1, 0, 0, 0) = 0, \quad h(\epsilon, y, -y_x, y_{xx}, -y_{xxx}) = h(\epsilon, y, y_x, y_{xx}, y_{xxx}), \tag{6}$$

and

$$\begin{aligned} & a = -a_0^2,\ c = c_1\mu,\ \alpha = 2\gamma + (c_1+1)\mu,\ \beta = -\gamma - (\frac{2}{3}c_1 + 1)\mu, \\ & \gamma \neq 0,\ \epsilon = O(\mu^k) \end{aligned} \tag{7}$$

for a fixed integer $k \geq 2$ where $a_0 > 0$ and c_1 are constants, and $\mu > 0$ is chosen as a small parameter. Let $\varsigma(x)$ be a smooth even cut-off function with $\varsigma(x) = 0$ for $|x| \leq 1$ and $\varsigma(x) = 1$ for $|x| \geq 2$. For any given constant $I_0 > 0$ with $I = I_0\mu^2$, there exists $\mu_0 > 0$ such that for $0 < \mu \leq \mu_0$, there are constants r_1 and θ satisfying

$$|r_1| \leq M\mu, \quad |\theta| \leq M\sqrt{\mu},$$

and the equation (5) has an even generalized homoclinic solution $y(x)$ defined by

$$
\begin{aligned}
y(x) =& \frac{3\mu}{2\gamma}\mathrm{sech}^2\left(\frac{\sqrt{\mu}}{2a_0}x\right) + 1 - 2a_0^3 I\varsigma(x)\cos((a_0+r_1)(x+\theta)) \\
&+ K_1(x) + K_2(x),
\end{aligned} \tag{8}
$$

where the function $K_2(x)$ is periodic in x with periodic $\frac{2\pi}{a_0+r_1}$ and the functions $K_1(x)$ and $K_2(x)$ satisfy uniformly that

$$|K_1(x)| \le M\mu^2 \mathrm{e}^{-\nu|x|}, \quad |K_2(x)| \le M\mu I$$

for $x \in \mathbf{R}$ and any fixed constant $\nu \in (\frac{\sqrt{\mu}}{2a_0}, \frac{\sqrt{\mu}}{a_0})$ and M is a generic positive constant.

The paper is organized as follows. In Section 2, the equation (5) is transformed into a four dimensional system. Under the assumptions (6) and (7), a saddle-center equilibrium is obtained. Then a normal form analysis is applied so that a homoclinic solution $H(x)$ of its dominant system can be easily computed. Section 3 regards the first Fourier coefficient $I > 0$ as a very small parameter and gives the existence of periodic solutions $X_p(x)$ for the full system. Section 4 changes the problem of the existence of the generalized homoclinic solutions near $H(x)$ into the one for an integral equation with respect to the small perturbation term $Z(x)$. Then some estimates are presented. The fixed point theorem shows the existence of $Z(x)$ for $x \in [0,\infty)$. Activating the phase shift θ of the periodic solution $X_p(x)$ and using the reversibility, we extend $Z(x)$ to $(-\infty,\infty)$ in Section 5, which yields a smooth generalized homoclinic solution exponentially tending to $X_p(x)$. Thus, the main theorem is proved. Integrating this solution gives the existence of a generalized wave-front solution of (1) under small perturbations.

Throughout this paper, M stands for a generic positive constant and $B = O(C)$ means that $|B| \le M|C|$.

2. Formulation of the problem

In order to have a saddle-center equilibrium of (5), we adjust the constants and assume that (7) is valid. Since ϵ is arbitrary, for the sake of simplicity, we consider ϵ as a function of μ and then suppose that $\epsilon = \mu^k$ for any integer $k \ge 2$ (see the assumption (7)). It is easy to check that $y = 1$ is an equilibrium of the equation (5) for $\mu = 0$. In order to move this equilibrium

$y = 1$ to the origin, let $u = y - 1, u_1 = u_x$, $u_2 = u_{xx}$ and $u_3 = u_{xxx}$, which changes the equation (5) into a system

$$\dot{U} = LU + L_\mu U + N_2(U) + \mu N_1(U) + \mu^k N_3(\mu, U), \tag{9}$$

where $U = (u, u_1, u_2, u_3)^T$, the dot stands for the derivative with respect to x and

$$LU = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ -a_0^2 u_2 \end{pmatrix}, \; L_\mu U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu u \end{pmatrix}, \; N_1(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{c_1}{3}u^3 - c_1 u^2 \end{pmatrix},$$

$$N_2(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\gamma u^2 - a_0^2 u u_2 + \frac{b+a_0^2}{2} u_1^2 \end{pmatrix}, \; N_3(\mu, U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ h_0(\mu, U) \end{pmatrix},$$

$$h_0(\mu, U) = h(\mu^k, u + 1, u_1, u_2, u_3). \tag{10}$$

The assumption (6) for the function h shows that the function h_0 satisfies

$$h_0(\mu, 0, 0, 0, 0) = 0, \quad h_0(\mu, u, -u_1, u_2, -u_3) = h_0(\mu, u, u_1, u_2, u_3). \tag{11}$$

Hence, the system (9) is reversible with a reverser S defined by

$$S(u, u_1, u_2, u_3) = (u, -u_1, u_2, -u_3), \tag{12}$$

that is, $SU(-x)$ is also a solution whenever $U(x)$ is. A solution $U(x)$ is reversible if $SU(-x) = U(x)$. This implies that $u(x)$ and $u_2(x)$ are even functions and $u_1(x)$ and $u_3(x)$ are odd functions.

The linear operator L has a double eigenvalue 0 and a pair of purely imaginary eigenvalues $\pm ia_0$, which corresponding eigenvectors and generalized eigenvectors are respectively given by

$$\begin{aligned} &U_1 = (1, 0, 0, 0)^T, && U_2 = (0, 1, 0, 0)^T, \\ &U_3 = a_0^3 \left(-1, -ia_0, a_0^2, ia_0^3\right)^T, && U_4 = \bar{U}_3 = a_0^3 \left(-1, ia_0, a_0^2, -ia_0^3\right)^T. \end{aligned} \tag{13}$$

Moreover,

$$SU_1 = U_1, \quad SU_2 = -U_2, \quad SU_3 = \bar{U}_3, \quad SU_4 = U_3. \tag{14}$$

Remark 2.1. If add the perturbation operator L_μ to the operator L, one easily obtains that the linear operator $L + L_\mu$ has a positive eigenvalue, a negative eigenvalue and a pair of purely imaginary eigenvalues, which is what we need.

Since the system (9) is real, we assume that $U = AU_1 + BU_2 + CU_3 + \bar{C}\bar{U}_3$ where A, B are real and C is complex. Then the system (9) is equivalent to the following system

$$\begin{aligned}
&\dot{A} = B,\\
&\dot{B} = \frac{\mu}{a_0^5}(a_0^3 A - C - \bar{C}) + h_1 + \frac{\mu^k}{a_0^2} f(\mu, A, B, C, \bar{C}),\\
&\dot{C} = ia_0 C - \frac{i\mu}{2a_0^3}(a_0^3 A - C - \bar{C}) - \frac{ia_0^2}{2} h_1 - \frac{i\mu^k}{2} f(\mu, A, B, C, \bar{C}),\\
&\dot{\bar{C}} = -ia_0 \bar{C} + \frac{i\mu}{2a_0^3}(a_0^3 A - C - \bar{C}) + \frac{ia_0^2}{2} h_1 + \frac{i\mu^k}{2} f(\mu, A, B, C, \bar{C}),
\end{aligned} \tag{15}$$

where the real function $f(\mu, A, B, C, \bar{C}) = h_0(\mu, U)$ and the real function h_1 is defined by

$$\begin{aligned}
h_1(\mu, A, B, C, \bar{C}) = \frac{1}{{a_0}^2}\Bigg[&\frac{b + a_0^2}{2}\left(B - \frac{i(C - \bar{C})}{{a_0}^2}\right)^2\\
&+ \frac{c_1\mu\left(-{a_0}^3 A + C + \bar{C}\right)^3}{3{a_0}^9} - \frac{(c_1\mu + \gamma)\left(-{a_0}^3 A + C + \bar{C}\right)^2}{{a_0}^6}\\
&- \frac{(C + \bar{C})\left({a_0}^3 A - C - \bar{C}\right)}{{a_0}^2}\Bigg].
\end{aligned} \tag{16}$$

In this case, the reverser S is given by

$$S(A, B, C, \bar{C}) = (A, -B, \bar{C}, C). \tag{17}$$

Now we use the norm form theory to determine the terms in (15) that are essential in the dynamical and bifurcation behaviors. The normal form theorem (see Exercise I.22 on page 60 in [6]) yields that there exists a change of variables from $(A, B, C, \bar{C})^T$ to Y which is close to identity, and transforms the system (15) into

$$\dot{Y} = LY + \mathcal{P}(\mu, Y) + O(|Y|\,|(\mu, Y)|^n), \tag{18}$$

where $\mathcal{P}$ is a polynomial with degree $\leq n$, $\mathcal{P}(0,0) = 0$ and $D\mathcal{P}(0,0) = 0$. Here n is an arbitrary positive integer but fixed. Moreover, $\mathcal{P}$ satisfies

$$S\mathcal{P}(Y) = -\mathcal{P}(SY). \tag{19}$$

For the notation simplicity, we still use $(A, B, C, \bar{C})^T$ to denote Y. Thus, if set $n \geq 2$, (18) can be written as

$$\begin{aligned} \dot{A} &= B + R_1(\mu, A, B, C, \bar{C}), \\ \dot{B} &= P(\mu, A, |C|^2) + R_2(\mu, A, B, C, \bar{C}), \\ \dot{C} &= ia_0C + iCQ(\mu, A, |C|^2) + R_3(\mu, A, B, C, \bar{C}), \\ \dot{\bar{C}} &= -ia_0\bar{C} - i\bar{C}Q(\mu, A, |C|^2) + \overline{R_3(\mu, A, B, C, \bar{C})}, \end{aligned} \tag{20}$$

where P and Q are real polynomials of their arguments with degree n and $n-1$, and

$$R_k = O\big(|(A, B, C, \bar{C})||(\mu, A, B, C, \bar{C})|^n\big), \quad k = 1, 2, 3, 4. \tag{21}$$

Note that R_1 can be always chosen equal to 0. This can be done by a change of coordinates of the type $\tilde{B} = B + R_1(\mu, A, B, C, \bar{C})$. Thus, (20) can be written as

$$\begin{aligned} \dot{A} &= B, \\ \dot{B} &= P(\mu, A, |C|^2) + R_2(\mu, A, B, C, \bar{C}), \\ \dot{C} &= ia_0C + iCQ(\mu, A, |C|^2) + R_3(\mu, A, B, C, \bar{C}), \\ \dot{\bar{C}} &= -ia_0\bar{C} - i\bar{C}Q(\mu, A, |C|^2) + \overline{R_3(\mu, A, B, C, \bar{C})}. \end{aligned} \tag{22}$$

In order to find the homoclinic solutions of the system (22), we need the coefficients of some important terms in P. Suppose that

$$P(\mu, A, |C|^2) = \mu p_1 A + p_2 A^2 + p_3|C|^2 + P_1(\mu, A, |C|^2), \tag{23}$$

where

$$P_1(\mu, A, |C|^2) = O\left((\mu + |A|)|C|^2 + |C|^4 + |(A, |C|^2)|\,|(\mu, A, |C|^2)|^2\right). \tag{24}$$

A direct calculation shows that (More details can be found in [6])

$$p_1 = \frac{1}{a_0^2}, \quad p_2 = -\frac{\gamma}{a_0^2}, \quad p_3 = \frac{3}{a_0^4} + \frac{b}{a_0^6} - \frac{2\gamma}{a_0^8}. \tag{25}$$

Let $v_1 = \frac{1}{2}(C+\bar{C})$ and $v_2 = \frac{i}{2}(C-\bar{C})$, which changes the complex system (22) into a real system

$$\dot{X} = F(\mu, X) + F_1(\mu, X) + \tilde{R}(\mu, X), \tag{26}$$

where $X = (A, B, v_1, v_2)^T$, and

$$F(\mu, X) = \begin{pmatrix} B \\ \frac{\mu}{a_0^2}A - \frac{\gamma}{a_0^2}A^2 \\ a_0 v_2 \\ -a_0 v_1 \end{pmatrix},$$

$$F_1(\mu, X) = \begin{pmatrix} 0 \\ p_3(v_1^2+v_2^2) + P_1(\mu, A, v_1^2+v_2^2) \\ v_2 Q(\mu, A, v_1^2+v_2^2) \\ -v_1 Q(\mu, A, v_1^2+v_2^2) \end{pmatrix},$$

$$\begin{aligned} \tilde{R}(\mu, X) &\triangleq \begin{pmatrix} 0 \\ \tilde{R}_2(\mu, X) \\ \tilde{R}_3(\mu, X) \\ \tilde{R}_4(\mu, X) \end{pmatrix} = \begin{pmatrix} 0 \\ R_2(\mu, A, B, v_1 - iv_2, v_1 + iv_2) \\ \mathrm{Re}(R_3(\mu, A, B, v_1 - iv_2, v_1 + iv_2)) \\ -\mathrm{Im}(R_3(\mu, A, B, v_1 - iv_2, v_1 + iv_2)) \end{pmatrix} \\ &= O\big(|(A, B, v_1, v_2)||(\mu, A, B, v_1, v_2)|^n\big). \end{aligned} \tag{27}$$

The reverser S is given by

$$S(A, B, v_1, v_2) = S(A, -B, v_1, -v_2). \tag{28}$$

The dominant system of (26)

$$\dot{X} = F(\mu, X) \tag{29}$$

has a homoclinic solution

$$\begin{aligned} H(x) &\triangleq (H_A(x), H_B(x), 0, 0)^T, \\ &= \left(\frac{3\mu}{2\gamma}\mathrm{sech}^2\Big(\frac{\sqrt{\mu}}{2a_0}x\Big), -\frac{3\mu^{3/2}}{2a_0\gamma}\mathrm{sech}^2\left(\frac{\sqrt{\mu}}{2a_0}x\right)\tanh\left(\frac{\sqrt{\mu}}{2a_0}x\right),\right. \\ &\qquad \left. 0, 0\right)^T, \end{aligned} \tag{30}$$

which satisfies

$$SH(-x) = H(x), \qquad |H(x)| \leq M\mu e^{-\frac{\sqrt{\mu}}{a_0}|x|} \quad \text{for} \quad x \in \mathbf{R}. \tag{31}$$

Since the equation (5) is equivalent to the system (26), in what follows, we only study the solutions of (26). More precisely, we will show that (26) has a solution near the homoclinic solution $H(x)$, which exponentially approaches to a periodic solution given in the next section. If this can be done, Theorem 1.1 will be proved.

3. Periodic solutions

In this section, we use the Fourier series to find the periodic solutions of (26). Their estimates will play an important role for the proof of existence of generalized homoclinic solutions of (26).

Let $\tilde{x} = (a_0 + r_1)x$ where r_1 is a small real constant to be determined later. The system (26) is transformed to

$$\begin{aligned}
A' &= \frac{1}{a_0 + r_1} B, \\
B' &= \frac{1}{a_0 + r_1}\Big(\frac{\mu}{a_0^2} A - \frac{\gamma}{a_0^2} A^2 + p_3(v_1^2 + v_2^2) + P_1(\mu, A, v_1^2 + v_2^2) \\
&\qquad\qquad + \tilde{R}_2(\mu, X)\Big), \\
v_1' &= \frac{1}{a_0 + r_1}\Big(a_0 v_2 + v_2 Q(\mu, A, v_1^2 + v_2^2) + \tilde{R}_3(\mu, X)\Big), \\
v_2' &= \frac{1}{a_0 + r_1}\Big(- a_0 v_1 - v_1 Q(\mu, A, v_1^2 + v_2^2) + \tilde{R}_4(\mu, X)\Big),
\end{aligned} \tag{32}$$

where the prime denotes the derivative with respect to $\tilde{x}$. To solve the above equations for a periodic solution with period 2π, we define some spaces. Use $H_p^m(0, 2\pi)$ to denote the space of periodic functions of $\tilde{x}$ with period 2π whose derivatives up to order m are in $L^2(0, 2\pi)$ with the norm given by

$$\|f\|_m^2 = \sum_{n \in \mathbb{Z}} (1 + n^{2m})|f_n|^2,$$

and $f = \sum_n f_n \mathrm{e}^{in\tilde{x}} \in H_p^m(0, 2\pi)$. Define

$$\tilde{H}_1^m(0, 2\pi) = \left\{ f(\tilde{x}) = \sum_n f_n \mathrm{e}^{in\tilde{x}} \in H_p^m(0, 2\pi) \;\middle|\; f_1 = 0 \right\}$$

and

$$\mathcal{H}_p^m(0, 2\pi) = H_p^m(0, 2\pi) \times H_p^m(0, 2\pi) \times \tilde{H}_1^m(0, 2\pi) \times H_p^m(0, 2\pi).$$

In order to find the reversible periodic solutions of (32) (see the definition (28) of the reverser S), we assume that

$$A(\tilde{x}) = \sum_{n=0}^{\infty} A_n \cos(n\tilde{x}), \qquad B(\tilde{x}) = \sum_{n=1}^{\infty} B_n \sin(n\tilde{x}),$$

$$v_1(\tilde{x}) = I\cos(\tilde{x}) + v_1^0(\tilde{x}), \quad v_1^0(\tilde{x}) = \sum_{n=0, n\neq 1}^{\infty} v_{1,n}\cos(n\tilde{x}),$$

$$v_2(\tilde{x}) = \sum_{n=1}^{\infty} v_{2,n}\sin(n\tilde{x}), \tag{33}$$

where $I > 0$ is a small constant. For simplicity, we consider I as a smooth function of μ and assume that

$$I = O(\mu^2). \tag{34}$$

Inserting (33) into (32) and making each Fourier coefficient in the equation equal zero, we obtain

$$\begin{aligned}
A_n &= \frac{a_0^2}{-n^2a_0^2(a_0+r_1)^2-\mu}\Big[-\frac{\gamma}{a_0^2}A^2 + p_3(v_1^2+v_2^2) \\
&\qquad + P_1(\mu, A, v_1^2+v_2^2) + \tilde{R}_2(\mu, X)\Big]_n, \quad n \geq 0, \\
B_n &= \frac{na_0^2(a_0+r_1)}{n^2a_0^2(a_0+r_1)^2+\mu}\Big[-\frac{\gamma}{a_0^2}A^2 + p_3(v_1^2+v_2^2) \\
&\qquad + P_1(\mu, A, v_1^2+v_2^2) + \tilde{R}_2(\mu, X)\Big]_n, \quad n \geq 1, \\
v_{1n} &= \frac{a_0}{a_0^2 - n^2(a_0+r_1)^2}\Big[\frac{n(a_0+r_1)}{a_0}\Big(v_2Q(\mu, A, v_1^2+v_2^2) + \tilde{R}_3(\mu, X)\Big) \\
&\qquad - v_1Q(\mu, A, v_1^2+v_2^2) + \tilde{R}_4(\mu, X)\Big]_n, \quad n \neq 1, \\
v_{2n} &= \frac{n(a_0+r_1)}{n^2(a_0+r_1)^2 - a_0^2}\Big[\frac{n(a_0+r_1)}{a_0}\Big(v_2Q(\mu, A, v_1^2+v_2^2) + \tilde{R}_3(\mu, X)\Big) \\
&\qquad - v_1Q(\mu, A, v_1^2+v_2^2) + \tilde{R}_4(\mu, X)\Big]_n \\
&\qquad - \frac{1}{a_0}\Big[v_2Q(\mu, A, v_1^2+v_2^2) + \tilde{R}_3(\mu, X)\Big]_n, \quad n \neq 1, \\
v_{21} &= -\frac{(a_0+r_1)}{a_0}I - \frac{1}{a_0}\Big[v_2Q(\mu, A, v_1^2+v_2^2) + \tilde{R}_3(\mu, X)\Big]_1, \\
r_1 &= -\frac{r_1^2}{2a_0} - \frac{1}{2I}\Big[\frac{a_0+r}{a_0}\Big(v_2Q(\mu, A, v_1^2+v_2^2) + \tilde{R}_3(\mu, X)\Big)
\end{aligned}$$

$$- v_1 Q(\mu, A, v_1^2 + v_2^2) + \tilde{R}_4(\mu, X)\Big]_1, \tag{35}$$

where $f[k]$ denotes the k-th component of f. Using (24), (27), (34) and (35), together with the fixed point theorem, one can solve for $A(\tilde{x}), B(\tilde{x}), v_1^0(\tilde{x})$, $v_2(\tilde{x}) \in \mathcal{H}_p^m(0, 2\pi)$ and the real constant r_1 as smooth functions in terms of small (μ, I) (More details can be found in [2] or [10]), i.e.,

$$(A, B, v_1^0, v_2)(\tilde{x}) = (A, B, v_1^0, v_2)(\tilde{x}; \mu, I), \quad r_1 = r_1(\mu, I),$$

and

$$\begin{aligned} &(A, B, v_1^0, v_2)(\tilde{x}; \mu, 0) = 0 \ \text{ for all } \tilde{x} \in [0, 2\pi], \qquad |r_1(\mu, I)| \leq M\mu, \\ &\|A\|_m + \|B\|_m \leq M\mu I, \qquad \|v_1^0\|_m + \|v_2\|_m \leq MI \end{aligned} \tag{36}$$

for $\mu \in (0, \mu_1]$ and $I \in (0, I_1]$ where μ_1 and I_1 are small positive constants. Using the relationship $\tilde{x} = (\lambda_0 + r_1)x$, we define

$$\begin{aligned} X_p(x) =& \Big(A_p(x), B_p(x), v_{1p}(x), v_{2p}(x)\Big)^T \\ =& \Big(A\big((a_0 + r_1)x; \mu, I\big), B\big((a_0 + r_1)x; \mu, I\big), \\ & I \cos\big((a_0 + r_1)x\big) + v_1^0\big((a_0 + r_1)x; \mu, I\big), v_2\big((a_0 + r_1)x; \mu, I\big)\Big)^T. \end{aligned}$$

It is clear that $X_p(x)$ is a periodic solution of (26) and satisfies, by (36), that

$$\|A_p(x)\|_m + \|B_p(x)\|_m \leq M\mu I, \quad \|v_{1p}(x)\|_m + \|v_{2p}(x)\|_m \leq MI. \tag{37}$$

The Sobolev embedding theorem implies that (37) is also valid under the $C_B^m(\mathbb{R})$-norm, where $C_B^m(\mathbb{R})$ is the space of continuously differentiable functions up to order m with a supremum norm. We will need the dominant terms of $v_{1p}(x)$ and $v_{2p}(x)$ later. A direct calculation gives

$$\begin{aligned} v_{1p}(x) &= I \cos((a_0 + r_1)x) + O(\mu I), \\ v_{2p}(x) &= -I \sin((a_0 + r_1)x) + O(\mu I). \end{aligned} \tag{38}$$

4. Existence of solutions on $[0, \infty)$

This section proves the existence of the homoclinic solutions of the system (26) near the obtained solution $H(x)$, which exponentially tends to the periodic solution $X_p(x)$.

Since the system (26) is reversible, we will first focus on the existence of its solutions for $x \in [0, \infty)$ and then use the reversibility to extend the solutions to $x \in (-\infty, \infty)$. Assume that the system (26) has a solution $\mathcal{X}(x)$ with the form

$$\mathcal{X}(x) = H(x) + Z(x) + \varsigma(x) X_p(x + \theta) \tag{39}$$

for $x \in [0, \infty)$ where the constant $\theta \in S^1 = [-\pi, \pi]$ is a phase shift, $Z(x)$ is a small perturbation term to be determined later and exponentially approaches to 0 as $x \to \infty$, and the smooth even cut-off function $\varsigma(x)$ satisfies $0 \le \varsigma(x) \le 1$ and

$$\varsigma(x) = \begin{cases} 1, & |x| \ge 2, \\ 0, & |x| \le 1. \end{cases}$$

Substituting (39) into (26) yields

$$\dot{Z} = \mathcal{L}(x) Z + \mathcal{F}(x, \mu, Z), \tag{40}$$

where $\mathcal{L}(x) = dF(\mu, H)$, d means taking the Fréchet derivative and

$$\begin{aligned} \mathcal{F}(x, \mu, Z) &= F(\mu, H(x) + Z(x) + \varsigma(x) X_p(x + \theta)) - F(\mu, H(x)) \\ &\quad - dF[\mu, H(x)] Z(x) - \varsigma(x) F(\mu, X_p(x + \theta)) \\ &\quad + F_1(\mu, H(x) + Z(x) + \varsigma(x) X_p(x + \theta)) - \varsigma(x) F_1(\mu, X_p(x + \theta)) \\ &\quad + \tilde{R}(\mu, H(x) + Z(x) + \varsigma(x) X_p(x + \theta)) - \varsigma(x) \tilde{R}(\mu, X_p(x + \theta)) \\ &\quad - \varsigma'(x) X_p(x + \theta). \end{aligned} \tag{41}$$

Lemma 4.1. *Under the assumption (34), if $|Z| + |Z_1| + |Z_2| \le M_0$ with some positive constant M_0, then the function $\mathcal{F}(x, \mu, Z)$ satisfies for $x \ge 0$ and small $\mu > 0$, $I > 0$*

$$\begin{aligned} |\mathcal{F}[j](x, \mu, Z)| &\le M\big(|1 - \varsigma(x)|\mu I + (\mu^3 + \mu I) \mathrm{e}^{-\frac{\sqrt{\mu}}{a_0} x} \\ &\qquad + (\mu^2 + I)|Z| + |Z|^2\big), \quad j = 1, 2, \\ |\mathcal{F}[k](x, \mu, Z)| &\le M\big(|1 - \varsigma(x)| I + (\mu^3 + \mu I) \mathrm{e}^{-\frac{\sqrt{\mu}}{a_0} x} \\ &\qquad + (\mu + I)|Z| + |Z|^2\big), \quad k = 3, 4, \\ |\mathcal{F}(x, \mu, Z_1) - \mathcal{F}(x, \mu, Z_2)| &\le M \left(\mu + I + |Z_1| + |Z_2|\right) |Z_1 - Z_2|, \end{aligned} \tag{42}$$

where $f[j]$ denotes the j-th component of f.

Proof. It is easy to get that from (31) and (37) the the terms including the function F is bounded by

$$M(|1-\varsigma(x)|I^2+\mu I\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}x}+I|Z|+|Z|^2).$$

Note that the second component of F_1 is dominated by

$$v_1^2+v_2^2+A(\mu^2+A^2)$$

and the third and the fourth components of F_1 are respectively governed by

$$v_2(\mu+A+v_1^2+v_2^2),\quad v_1(\mu+A+v_1^2+v_2^2)$$

(ignoring the uniform constant M). Thus, we have

$$\begin{aligned}
&|F_1[2](\mu,H(x)+Z(x)+\varsigma(x)X_p(x+\theta))-\varsigma(x)F_1[2](\mu,X_p(x+\theta))|\\
&\quad\leq M\left(|1-\varsigma(x)|I^2+\mu^3\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}x}+(\mu^2+I)|Z|+|Z|^2\right), \qquad (43)\\
&|F_1[k](\mu,H(x)+Z(x)+\varsigma(x)X_p(x+\theta))-\varsigma(x)F_1[k](\mu,X_p(x+\theta))|\\
&\quad\leq M\left(|1-\varsigma(x)|I^2+\mu I\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}x}+(\mu+I)|Z|+|Z|^2\right),\quad k=3,4. \qquad (44)
\end{aligned}$$

The function $\tilde{R}(\mu,X)$ is dominated by

$$(A+B+v_1+v_2)(\mu+A+B+v_1+v_2)^2$$

(again ignoring the uniform constant M). This implies that

$$\begin{aligned}
&|\tilde{R}(\mu,H(x)+Z(x)+\varsigma(x)X_p(x+\theta))-\varsigma(x)\tilde{R}(\mu,X_p(x+\theta))|\\
&\quad\leq M\left(|1-\varsigma(x)|I^2+(\mu^3+\mu^2I)\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}x}+(\mu^2+I)|Z|+|Z|^2\right). \qquad (45)
\end{aligned}$$

The above estimates, together with

$$\begin{aligned}
&|\varsigma'(x)A_p(x+\theta)|+|\varsigma'(x)B_p(x+\theta)|\leq M|1-\varsigma(x)|\mu I,\\
&|\varsigma'(x)v_{1p}(x+\theta)|+|\varsigma'(x)v_{2p}(x+\theta)|\leq M|1-\varsigma(x)|I,
\end{aligned}$$

yield the first two inequalities in (42). Similarly, the rest inequality in (42) can be obtained. The proof is completed. □

In order to prove the existence of the solution $Z(x)$ on $[0,\infty)$ by the fixed point theorem, we change this problem into one of an integral equation.

Note that the linear system of (40)

$$\dot{Z} = \mathcal{L}(x)Z \tag{46}$$

has four linearly independent solutions

$$\begin{aligned}
s_1(x) &= -\frac{6\mu^{3/2}4{a_0}^2\gamma\Big(a_0\tanh(\frac{\sqrt{\mu}}{2a_0}x)\mathrm{sech}^2(\frac{\sqrt{\mu}}{2a_0}x)}{,}\\
&\quad - 6\sqrt{\mu}\tanh^2(\frac{\sqrt{\mu}}{2a_0}x)\mathrm{sech}^2(\frac{\sqrt{\mu}}{2a_0}x) + 3\mu^2\mathrm{sech}^4(\frac{\sqrt{\mu}}{2a_0}x), 0, 0\Big)^T,\\
s_2(x) &= \frac{1}{96\mu^2}\Big(8a_0\gamma(-2a_0\cosh(\frac{\sqrt{\mu}}{a_0}x)\\
&\quad + 15(2a_0 - \sqrt{\mu}x\tanh(\frac{\sqrt{\mu}}{2a_0}x))\mathrm{sech}^2(\frac{\sqrt{\mu}}{2a_0}x)) - 96{a_0}^2\gamma,\\
&\quad - \sqrt{\mu}\gamma\mathrm{sech}^4(\frac{\sqrt{\mu}}{2a_0}x)\big(185a_0\sinh(\frac{\sqrt{\mu}}{a_0}x) + 4a_0\sinh(\frac{2\sqrt{\mu}}{a_0}x)\\
&\quad + a_0\sinh(\frac{3\sqrt{\mu}}{a_0}x) - 60\sqrt{\mu}x\cosh(\frac{\sqrt{\mu}}{a_0}x) + 120\sqrt{\mu}x\big), 0, 0\Big)^T,\\
s_3(x) &= \Big(0, 0, \cos(a_0x), -\sin(a_0x)\Big)^T,\\
s_4(x) &= \Big(0, 0, \sin(a_0x), \cos(a_0x)\Big)^T,
\end{aligned} \tag{47}$$

which satisfy

$$\begin{aligned}
&|s_1(x)| \le M\mu^{3/2}\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}x},\ |s_2(x)| \le M\mu^{-2}\mathrm{e}^{\frac{\sqrt{\mu}}{a_0}x},\ |s_3(x)| + |s_4(x)| \le M,\\
&s_1(0) = \Big(0, -\frac{3\mu^2}{4a_0^2\gamma}, 0, 0\Big)^T, \quad s_2(0) = \Big(\frac{4a_0^2\gamma}{3\mu^2}, 0, 0, 0\Big)^T,\\
&s_3(0) = \Big(0, 0, 1, 0\Big)^T, \qquad s_4(0) = \Big(0, 0, 0, 1\Big)^T.
\end{aligned} \tag{48}$$

The adjoint equation of (46) has four linearly independent solutions given by

$$\begin{aligned}
s_1^*(x) &= \frac{1}{96\mu^2}\Big(- \sqrt{\mu}\gamma\mathrm{sech}^4(\frac{\sqrt{\mu}}{2a_0}x)(185a_0\sinh(\frac{\sqrt{\mu}}{a_0}x)\\
&\quad + 4a_0\sinh(\frac{2\sqrt{\mu}}{a_0}x) + a_0\sinh(\frac{3\sqrt{\mu}}{a_0}x) - 60\sqrt{\mu}x\cosh(\frac{\sqrt{\mu}}{a_0}x)\\
&\quad + 120\sqrt{\mu}x), 8a_0\gamma(2a_0\cosh(\frac{\sqrt{\mu}}{a_0}x)
\end{aligned}$$

$$
\begin{aligned}
&+ 15(\sqrt{\mu}x \tanh(\frac{\sqrt{\mu}}{2a_0}x) - 2a_0)\mathrm{sech}^2(\frac{\sqrt{\mu}}{2a_0}x) + 12a_0), 0, 0\Big)^T,\\
s_2^*(x) &= \frac{\mu^{3/2}}{4{a_0}^2\gamma}\Big(6\sqrt{\mu}\mathrm{sech}^4(\frac{\sqrt{\mu}}{2a_0}x) - 3\mu^2\cosh(\frac{\sqrt{\mu}}{2a_0}x)\mathrm{sech}^4(\frac{\sqrt{\mu}}{2a_0}x),\\
&- 6a_0\tanh(\frac{\sqrt{\mu}}{2a_0}x)\mathrm{sech}^2(\frac{\sqrt{\mu}}{2a_0}x), 0, 0\Big)^T,\\
s_3^*(x) &= \Big(0, 0, \cos(a_0x), -\sin(a_0x)\Big)^T,\\
s_4^*(x) &= \Big(0, 0, \sin(a_0x), \cos(a_0x)\Big)^T, \qquad (49)
\end{aligned}
$$

which satisfy

$$
|s_1^*(x)| \le M\mu^{-2}\mathrm{e}^{\frac{\sqrt{\mu}}{a_0}x}, \ |s_2^*(x)| \le M\mu^{3/2}\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}x}, \ |s_3^*(x)| + |s_4^*(x)| \le M, \qquad (50)
$$

and for all $x \in \mathbf{R}$

$$
\langle s_j(x), s_k^*(x)\rangle = \begin{cases} 1, & \text{for } j = k,\\ 0, & \text{for } j \ne k, \end{cases} \qquad j, k = 1, 2, 3, 4, \qquad (51)
$$

where $\langle\ \cdot\ \rangle$ denotes the Euclidean inner product on $\mathbf{R}^4$.

Thus, the solution of (40) that approaches to zero at infinity can be found as

$$
\begin{aligned}
Z(x) &= \int_0^x \langle\mathcal{F}(t, \mu, Z), s_1^*(t)\rangle dt s_1(x) - \sum_{k=2}^{4}\int_x^\infty \langle\mathcal{F}(t, \mu, Z), s_k^*(t)\rangle dt s_k(x)\\
&\triangleq \mathcal{P}(\mu, Z)(x). \qquad (52)
\end{aligned}
$$

In order to apply the fixed point theorem to the above equation, we choose a Banach space

$$
\mathcal{B} = \Big\{Z \in C\big([0, \infty) \times S^1\big) \mid \sup_{x\in[0,\infty)}\{|Z(x, \theta)|\mathrm{e}^{\nu x}\} < \infty\Big\} \qquad (53)
$$

for a fixed constant $\nu \in (\frac{\sqrt{\mu}}{2a_0}, \frac{\sqrt{\mu}}{a_0})$ with the norm

$$
\|Z\| = \sup\big\{|Z(x, \theta)|\ \mathrm{e}^{\nu x} \mid x \in [0, \infty), \theta \in S^1\big\}. \qquad (54)
$$

From this definition, it is obvious that $Z(x)$ exponentially tends to 0 as $x \to \infty$.

Lemma 4.2. *Under the assumption (34), if* $\|Z\| + \|Z_1\| + \|Z_2\| \leq M_0$ *with some positive constant* M_0*, then the map* $\mathcal{P}$ *satisfies for* $x \geq 0$ *and small* $\mu > 0$*,* $I > 0$

$$\begin{aligned}
&\|\mathcal{P}(\mu, Z)\| \leq M\left(\mu^2 + I + (\sqrt{\mu} + \mu^{-1}I)\|Z\| + \mu^{-1}\|Z\|^2\right), \\
&\|\mathcal{P}(\mu, Z_1) - \mathcal{P}(\mu, Z_2)\| \leq M\mu^{-1}\left(\mu^{3/2} + I + \|Z_1\| + \|Z_2\|\right)\|Z_1 - Z_2\|.
\end{aligned} \tag{55}$$

Proof. Using (48), (50) and Lemma 4.1, we have

$$\begin{aligned}
&\Big|\int_0^x \langle \mathcal{F}(t, \mu, Z), s_1^*(t)\rangle dt s_1(x)\Big| \mathrm{e}^{\nu x} \\
&\quad \leq \frac{M}{\sqrt{\mu}} \int_0^x \Big(|1 - \varsigma(t)|\mu I + (\mu^3 + \mu I)\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}t} \\
&\qquad\qquad + (\mu^2 + I)|Z| + |Z|^2\Big)\mathrm{e}^{\frac{\sqrt{\mu}}{a_0}t} dt \mathrm{e}^{-(\frac{\sqrt{\mu}}{a_0} - \nu)x} \\
&\quad \leq M\sqrt{\mu} I + \frac{M}{\sqrt{\mu}} \int_0^x \Big(\mu^3 + \mu I + (\mu^2 + I)\|Z\| \\
&\qquad\qquad + \|Z\|^2\Big)\mathrm{e}^{(\frac{\sqrt{\mu}}{a_0} - \nu)t} dt \mathrm{e}^{-(\frac{\sqrt{\mu}}{a_0} - \nu)x} \\
&\quad \leq \frac{M}{\mu}\left(\mu^3 + \mu I + (\mu^2 + I)\|Z\| + \|Z\|^2\right), \\
&\Big|\int_x^\infty \langle \mathcal{F}(t, \mu, Z), s_2^*(t)\rangle dt s_2(x)\Big| \mathrm{e}^{\nu x} \\
&\quad \leq \frac{M}{\sqrt{\mu}} \int_x^\infty \Big(|1 - \varsigma(t)|\mu I + (\mu^3 + \mu I)\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}t} \\
&\qquad\qquad + (\mu^2 + I)|Z| + |Z|^2\Big)\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}t} dt \mathrm{e}^{(\frac{\sqrt{\mu}}{a_0} + \nu)x} \\
&\quad \leq M\sqrt{\mu} I + \frac{M}{\sqrt{\mu}} \int_x^\infty \Big(\mu^3 + \mu I + (\mu^2 + I)\|Z\| \\
&\qquad\qquad + \|Z\|^2\Big)\mathrm{e}^{-(\frac{\sqrt{\mu}}{a_0} + \nu)t} dt \mathrm{e}^{(\frac{\sqrt{\mu}}{a_0} + \nu)x} \\
&\quad \leq \frac{M}{\mu}\left(\mu^3 + \mu I + (\mu^2 + I)\|Z\| + \|Z\|^2\right), \\
&\Big|\int_x^\infty \langle \mathcal{F}(t, \mu, Z), s_k^*(t)\rangle dt s_k(x)\Big| \mathrm{e}^{\nu x} \\
&\quad \leq M \int_x^\infty \Big(|1 - \varsigma(t)| I + (\mu^3 + \mu I)\mathrm{e}^{-\frac{\sqrt{\mu}}{a_0}t} + (\mu + I)|Z| + |Z|^2\Big) dt \mathrm{e}^{\nu x}
\end{aligned}$$

$$
\begin{aligned}
&\leq MI + M\int_x^{\infty}\left(\mu^3+\mu I+(\mu+I)\|Z\|+\|Z\|^2\right)\mathrm{e}^{-\nu t}dt\mathrm{e}^{\nu x}\\
&\leq MI + \frac{M}{\sqrt{\mu}}\left(\mu^3+(\mu+I)\|Z\|+\|Z\|^2\right), \quad k=3,4,
\end{aligned}
$$

which yield the first inequality of (55). Similarly, the second one can be obtained. □

Suppose

$$I = I_0\mu^2 \tag{56}$$

for any fixed positive constant I_0 and take a closed ball $\bar{B}(0)$ in the space $\mathcal{B}$ with a radius

$$r = O(\mu^{3/2}). \tag{57}$$

Clearly, the assumption (34) holds. Lemma 4.2 shows that $\mathcal{P}$ is a contraction map on $\bar{B}(0)$ for small $\mu > 0$. Thus, the equation (52) has a unique solution $Z(x;\mu,\theta,I)$ satisfying with a subtle estimate

$$\|Z\| \leq M\mu^2. \tag{58}$$

Note that the smoothness of $Z(x;\mu,\theta,I)$ in its arguments can also be obtained by using an extension of a contraction mapping principle [19]. Hence, (26) has a smooth solution $\mathcal{X}(x;\mu,\theta,I)$ for $x \in [0,\infty)$.

5. Existence of generalized homoclinic solution

In the previous section, we proved the existence of the solution $\mathcal{X}(x;\mu,\theta,I)$ of (26) for $x \in [0,\infty)$. The reversibility yields that $S\mathcal{X}(-x;\mu,\theta,I)$ is also a solution of (26). To obtain a reversible solution of (26) for $x \in (-\infty,\infty)$, we need to solve the following equation

$$(\mathcal{I} - S)\mathcal{X}(0;\mu,\theta,I) = 0, \tag{59}$$

where $\mathcal{I}$ is an identity map. If the above equation holds, we can define

$$\mathcal{X}_1(x) = \begin{cases} \mathcal{X}(x;\mu,\theta,I) & \text{for } x \geq 0,\\ S\mathcal{X}(-x;\mu,\theta,I) & \text{for } x \leq 0. \end{cases} \tag{60}$$

The uniqueness of the solution for an initial value problem implies that $\mathcal{X}_1(x)$ is a smooth solution of (26) for $x \in (-\infty,\infty)$, which exponentially tends to the periodic solution $X_p(x+\theta)$ as $x \to \infty$ and the periodic solution $SX_p(-x+\theta)$ as $x \to -\infty$. Moreover, $S\mathcal{X}_1(-x) = \mathcal{X}_1(x)$, which gives that $\mathcal{X}_1(x)$ is a reversible generalized homoclinic solution of (26).

In what follows, we pay our attention on the equation (59). The definition of the reverser S shows that the equation (59) is equivalent to

$$B(0) = 0, \tag{61}$$
$$v_2(0) = 0. \tag{62}$$

(48) and (52) imply that the equation (61) automatically holds. In order to make the equation (62) valid, we need to find an unknown constant. Here we choose the phase shift θ.

Lemma 5.1. *If (56) is valid, then the equation (62) is equivalent to*

$$\theta = \sqrt{\mu}\,\Xi(\mu,\theta), \tag{63}$$

where $\Xi(\mu,\theta)$ is differentiable with respect to its arguments, and Ξ and its derivative with respect to θ are uniformly bounded for small $\mu > 0$.

Proof. Let $x = 0$ in (52) which shows that (62) is changed into

$$\int_0^\infty \langle \mathcal{F}(t,\mu,Z), s_4^*(t)\rangle dt = 0. \tag{64}$$

Note that the first two components of $s_4^*(x)$ in (49) are zero. Obviously, by (44) and (45),

$$\begin{aligned}\mathcal{F}[3](x,\mu,Z) &= -\varsigma'(x)v_{1p}(x+\theta) + \Phi_3(x,\mu,Z),\\ \mathcal{F}[4](x,\mu,Z) &= -\varsigma'(x)v_{2p}(x+\theta) + \Phi_4(x,\mu,Z),\end{aligned}$$

where $\Phi_3(\mu,\theta)$ and $\Phi_4(\mu,\theta)$ uniformly satisfy

$$\begin{aligned}|\Phi_3(x,\mu,Z)| + |\Phi_4(x,\mu,Z)| \le M\Big(&|1-\varsigma(x)|I^2 + (\mu^3+\mu I)e^{-\frac{\sqrt{\mu}}{a_0}x}\\ &+ (\mu+I)|Z| + |Z|^2\Big).\end{aligned}$$

Using (38), (49) and (58), we can transform the equation (64) into

$$-I\int_1^2 \varsigma'(t)\Big(\cos((a_0+r_1)(t+\theta))\sin(a_0 t)$$

$$- \sin((a_0 + r_1)(t+\theta)) \cos(a_0 t)\Big) dt = \Phi_5(\mu, \theta),$$

or

$$I \int_1^2 \varsigma'(t) \sin(a_0\theta + r_1(t+\theta)) dt = \Phi_5(\mu, \theta), \tag{65}$$

where $\Phi_5(\mu, \theta)$ uniformly satisfies

$$\begin{aligned} |\Phi_5(\mu, \theta)| &\leq M\Big(\mu^{5/2} + \sqrt{\mu} I + \mu^{-1/2}((\mu + I)\|Z\| + \|Z\|^2)\Big) \\ &\leq M\mu^{5/2}. \end{aligned}$$

(36) and (56) imply that the equation (65) is equivalent to

$$I_0 \sin(a_0\theta) = \Phi_6(\mu, \theta), \tag{66}$$

where $\Phi_6(\mu, \theta)$ uniformly satisfies

$$|\Phi_6(\mu, \theta)| \leq M\sqrt{\mu}.$$

It is consequently obtained that

$$\theta = \sqrt{\mu}\, \Xi(\mu, \theta), \tag{67}$$

where $\Xi(\mu, \theta) = \frac{1}{a_0\sqrt{\mu}} \arcsin(\Phi_6(\mu, \theta)/I_0)$. It is easy to check that $\Xi(\mu, \theta)$ is differentiable with respect to its arguments, and Ξ and its derivative with respect to θ are uniformly bounded for small $\mu > 0$. Hence the proof is completed. □

By this lemma, we can choose a closed interval $[-\theta_0, \theta_0]$ where $\theta_0 > 0$ and $\theta_0 = O(\mu^{1/4})$. It is then straightforward to show that θ is a contraction mapping on $[-\theta_0, \theta_0]$ for small $\mu > 0$. Therefore, (63) has a unique solution θ satisfying $|\theta| \leq M\sqrt{\mu}$. This implies that (59) holds by choosing θ that has been derived. Hence, $\mathcal{X}_1(x)$ defined by (60) is a reversible generalized homoclinic solution of (26).

Recall the relationships among (5), (9) and (26), and focus on only the first component. Then, we have

$$y(x) = H_A(x) + A(x) + \varsigma(x)A_p(x+\theta) - 2a_0^3(v_1(x) + \varsigma(x)v_{1p}(x+\theta)) + 1,$$

which yield the expression of $y(x)$ given in (8). Thus, Theorem 1.1 in the Introduction is proved.

Remark 5.1. *From (2) we have*

$$\phi(x) = \int \left(-y(t) + \frac{ev}{a}\right) dt. \tag{68}$$

Note that $y(x)$ *in Theorem 1.1 has a constant* $y_0 = 1 + [K_2(x)]_0 = 1 + O(\mu I)$ *where* $[K_2(x)]_0$ *stands for the constant term in the Fourier series of the periodic function* $K_2(x)$*. We can always choose appropriate constants in (1) satisfying*

$$\frac{ev}{a} = y_0$$

so that the constant in the integrand $-y(x) + \frac{ev}{a}$ *vanishes and (7) holds. Thus, if we integrate* $-y(x) + \frac{ev}{a}$*, there is no any polynomial in* x *appearing. Then we set*

$$\begin{aligned}\phi(x) &= \int_0^x \left(-y(x) + \frac{ev}{a}\right) dx \\ &= \frac{3\sqrt{\mu}a_0}{\gamma} \tanh\left(\frac{\sqrt{\mu}}{2a_0}x\right) + \mathcal{K}_1(x) + \mathcal{K}_2(x),\end{aligned} \tag{69}$$

where the function $\mathcal{K}_2(x)$ *is periodic in* x *with periodic* $\frac{2\pi}{a_0+r_1}$ *and the functions* $\mathcal{K}_1(x)$ *and* $\mathcal{K}_2(x)$ *satisfy uniformly that*

$$|\mathcal{K}_1(x)| \leq M\mu^{3/2}\mathrm{e}^{-\nu|x|}, \quad |\mathcal{K}_2(x)| \leq MI$$

for $x \in \mathbf{R}$*. Therefore,* $\phi(x)$ *given in (69) is an odd generalized wave-front solution of (1) under a small perturbation.*

Acknowledgments. The authors are grateful to Prof. Boling Guo, Prof. Shu-Ming Sun and Prof. Chongchun Zeng for many helpful conversations during the course of this work.

References

[1] P. J. Caudrey, R. K. Dodd and J. D. Gibbon, A new hierarchy of Korteweg-de Vries equations, *Proc. R. Soc. Lond. A*, **351** (1976), 407-422.

[2] S. Deng and B. Guo, Generalized homoclinic solutions of a coupled Schrödinger system under a small perturbation, *J. Dyn. Diff. Equat.*, **24** (2012), 761-776.

[3] V. G. Drinfeld and V. V. Sokolov, Equations of Korteweg-de Vries type, and simple Lie algebras, *Dokl. Akad. Nauk SSSR*, **258** (1981), 11-16.

[4] A. P. Fordy and J. Gibbons, Some remarkable nonlinear transformations, *Phys. Lett. A*, **75** (1980), 325-325.

[5] C. A. Gómez and A. Salas, The Cole-Hopf transformation and improved tanh-coth method applied to new integrable system (KdV6), *Appl. Math. Comput.*, **204** (2008), 957-962.

[6] G. Iooss and M. Adelmeyer, *Topics in Bifurcation Theory and Applications*, World Scientific, 1992.

[7] A. Karasu-Kalkanli, A. Karasu, A. Sakovich, S. Sakovich and R. Turhan, A new integrable generalization of the Korteweg-de Vries equation, *J. Math. Phys.*, **49** (2008), Article ID 073516, 10 pages.

[8] D. J. Kaup, On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6q\psi_x + 6r\psi = \lambda\psi$, *Stud. Appl. Math.*, **62** (1980), 189-216.

[9] P. Kersten, I. S. Krasil'shchik, A. M. Verbovetsky and R. Vitolo, Hamiltonian structures for general PDEs, *Differ. Geom. Dyn. Syst.*, **5** (2009), 187-198.

[10] H. Kielhöfer, *Bifurcation Theory: An Introduction with Applications to PDEs*, Springer-Verlag, 2003.

[11] B. A. Kupershmidt, KdV6: an integrable system, *Phys. Lett. A*, **372** (2008), 2634-2639.

[12] J. Li and Y. Zhang, Exact solitary wave and quasi-periodic wave solutions for four fifth-order nonlinear wave equations, *Discrete Contin. Dyn. Syst. Ser. B*, **13** (2010), 623-631.

[13] J. Li and Y. Zhang, The exact traveling wave solutions to two integrable KdV6 equations, *Chin. Ann. Math.*, **33B** (2012), 179-190.

[14] A. Ramani, B. Grammaticos and R. Willox, Bilinearize and solutions of the KdV6, *Anal. Appl.*, **6** (2008), 401-412.

[15] A. H. Salas and C. A. Gómez, Exact solutions for the KdV6 and mKdV6 equations via tanh-coth and sech methods, *Appl. Appl. Math.*, **5** (2010), 1504-1510.

[16] J. Satsuma and R. Hirota, A coupled KdV equation is one case of the four-reduction of the KP hierarchy, *J. Phys. Soc. Jpn.*, **51** (1982), 3390-3397.

[17] K. Sawada and T. Kotera, A method for finding n-soliton solutions of the K.d.V. equation and K.d.V.-like equation, *Prog. Theor. Phys.*, **51** (1974), 1355-1367.

[18] Y. Sun, Y. Gao, G. Meng, X. Yu, Y. Shen and Z. Sun, Bilinear forms and soliton interactions for two generalized KdV equations for nonlinear waves, *Nonlinear Dyn.*, **78** (2014), 349-357.

[19] W. Walter, *Gewöhnliche Differentialgleichungen*, Springer-Verlag, 1972.

[20] A. M. Wazwaz, The integrable KdV6 equations: multiple solitons and multiple singular soliton solutions, *Appl. Math. Comput.*, **204** (2008), 963-972.

[21] X. Wen, Y. Gao and L. Wang, Darboux transformation and explicit solutions for the integrable sixth-order KdV equation for nonlinear waves, *Appl. Math. Comput.*, **218** (2011), 55-60.

[22] Y. Yao and Y. Zeng, The bi-Hamiltonian structure and new solutions of KdV6 equation, *Lett. Math. Phys.*, **86** (2008), 193-208.

[23] Y. Zhang, X. Cai and H. Xu, A note on the integrable KdV6 equation multiple soliton solutions and multiple singular soliton solutions, *Appl. Math. Comput.*, **214** (2009), 1-3.

[24] J. Zhao and R. Conte, A connection between HH3 and Korteweg-de Vries with one source, *J. Math. Phys.*, **51** (2010), 033511, 6 pages.

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