Pure and Applied Mathematics Quarterly
Volume 11, Number 2, 283–292, 2015

A Note on the Heat Flow of Harmonic Maps Whose Gradients Belong to $L_t^q L_x^p$

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Abstract: For any compact $n-$dimensional Riemannian manifold (M, g) without boundary, a compact Riemannian manifold $N \subset R^k$ without boundary, and $0 < T \leq \infty$, we prove that for $n \geq 3$, if $u : M \times (0, T] \to N$ is a weak solution to the heat flow of harmonic maps such that $\nabla u \in L_x^p L_t^q (M \times (0, T])$ $(n/p + 2/q = 1$ for some $p > n)$, then $u \in C^\infty(M \times (0, T), N)$. For $p = n$, we proved the regularity for the suitable weak solution defined in [1].
Keywords: Heat flow; Suitable solution; Lorentz space; Blow up.

1. Introduction

We adopt the notation and some definitions as in [1] and [2]. For $n \geq 1$, let (M, g) be a smooth, compact $n-$dimensioanl Riemannian manifold without boundary, and $N \subset R^k (k \geq 2)$, be a smooth , closed, oriented $m-$dimensional submanifold without boundary. For $0 < T \leq \infty$, a map $u \in C^2(M \times (0, T), N)$ is a solution to the heat flow of harmonic maps, if

$$\frac{\partial u}{\partial t} = \Delta_g u + \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} A(u) \left(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \right) \text{ in } M \times (0, T), \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator of (M, g), $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of $N \subset \mathbb{R}^k$, and $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ is the inverse of $g = (g_{\alpha,\beta})$. Let us recall the notation of weak solutions of (1.1).

Definition 1.1. *A map $u : M \times [0, T] \to N$ is a weak solution of (1.1), if*
(1) $u_t \in L_x^2 L_t^2 (M \times [0, T])$, $\nabla u \in L_x^2 L_t^\infty (M \times [0, T])$,

Received May 15, 2013.
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(2) u satisfies (1.1) in the distribution sense:

$$\int_0^T \int_M u_t \cdot \phi + \nabla u \cdot \nabla \phi = \int_0^T \int_M A(u)(\nabla u, \nabla u) \cdot \phi,$$

for all $\phi \in C_0^\infty(M \times (0,T), R^k)$.

Our goal in this note is to get

Theorem 1.1. *For* $n \geq 3$*, let* $u : M \times [0,T] \to N$ *be a weak solution of (1.1), with* $\nabla u \in L_x^p L_t^q(M \times [0,T])$ *for some* $n < p \leq \infty$ *satisfying* $\frac{n}{p} + \frac{2}{q} = 1$*. Then* $u \in C^\infty(M \times (0,T], N)$*. Moreover, if* $\nabla u \in L_x^n L_t^\infty$*, then there exists a small number* ε*, such that* $\|\nabla u\|_{L_x^n L_t^\infty(M\times(0,T])} \leq \varepsilon$ *which implies that* $u \in C^\infty(M \times (0,T], N)$*.*

By standard parabolic estimate, Theorem 1.1 can be generalized by Theorem 1.3.

Theorem 1.2. *For* $n = 3$*, let* $u : M \times [0,T] \to N$ *be a weak solution of (1.1) which satisfies the monotonicity inequalities (1.5) and energy inequality (1.6), and* $\nabla u \in L_x^n L_t^\infty(\mathbb{R}^n \times [0,T])$*. Then for any open subset* ω *and for any moment of time* $t_0 \in (0,T)$*, we have*

$$N(t_0, \omega) \leq \varepsilon_0^{-3} \limsup_{r \to 0} \frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_\omega |\nabla u|^3(y,s)dyds.$$

Here, $N(t_0,\omega) = card\{\Sigma(t_0) \cap \omega\}$*; i.e.* $N(t_0,\omega)$ *is the number of points in the set* $\Sigma(t_0) \cap \omega$*.*

Note that the scaling invariant norm for ∇u is $\nabla u \in L_x^p L_t^q(M \times [0,T])$ for some $p \in [n, \infty)$ and $q \in [2,\infty)$ satisfying

$$\frac{n}{p} + \frac{2}{q} = 1. \tag{1.2}$$

The scaling invariant space $L_x^p L_t^q$ with (p,q) satisfying (1.2) has played an important role in the regularity issue of Navier-Stokes equation for the Leray-Hopf weak solution. It is well known that both uniqueness and smoothness for the class of weak solutions v of the Navier-Stokes Equation in which $v \in L_x^p L_t^q$ $(\mathbb{R}^3 \times (0,\infty))$ for some $p \in (3,\infty]$ and $q \in [2,\infty)$ satisfying Serrin's condition (1.2), have been established through works by Prodi [3], Serrin [4], and Ladyzhenskaya [5] in 1960s. On the other hand, for the end point case $p = 3, q = \infty$, only until recently Escauriaza et al. [5,6] proved the smoothness for weak solutions $v \in L_x^3 L_t^\infty, 0 < T < \infty$.

Motivated by these results for the Navier-Stokes equation, Wang [2] considered the class of weak solutions $u : M \times [0, T] \to N$ of (1.1) with $\nabla u \in L_x^p L_t^q (M \times [0, T])$ for some $p \in [n, +\infty]$ and $q \in [2, +\infty]$ satisfying Serrin's condition (1.2). it is stated in [2] that

(i) if $n \geq 4$, and u is a weak solution of (1.1) with $\nabla u \in L_x^n L_t^\infty$, then $u \in C^\infty (M \times \ (0, T], N)$.

(ii) If $n = 3$, they get the blow up criteria.

(iii) Either $n \geq 4$ or $2 \leq n < 4$ and $p \geq 4$, Theorem 1.1 is true with $\nabla u \in L_x^p L_t^q$.

for some $p > n, q \geq 2$ satisfying $n/p + 2/q = 1$.

Our Theorem 1.1 extends their result (iii) to all p, q with $p > n, q \geq 2$ satisfying (1.2).

Since the regularity is a local property, for the sake of simplicity, we will present our proofs in the case where $M = \mathbb{R}^n$. The general case is essentially the same, but technically a little more complicated. Here we shall consider the weak solutions of

$$\frac{\partial u}{\partial t} - \Delta u = A(u)(\nabla u, \nabla u), \text{ in } Q \tag{1.3}$$

where $Q = \Omega \times (0, T)$, Ω is a domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary, $0 < T < \infty$. For any weak solution $u : \mathbb{R}^n \times (0, T] \to N$ of (1.3), define

$$\Sigma = \{z_0 = (x_0, t_0) \in \mathbb{R}^n \times (0, T];\ u \text{ is not continuous at } z_0\},$$

and

$$\Sigma(t_0) = \Sigma \cap \{t_0\}, \text{ for } t_0 \in (0, T].$$

The proof in [2] depends on the fact that for $n \geq 4$, u satisfies the monotonicity inequalities ([2, (2.4)]) (which is stronger then (1.5)) and the energy inequality (1.6) under the assumption of $\nabla u \in L_x^n L_t^\infty$ (see [2, Lemma 2.4 and Lemma 2.2]). So the case $n = 3, q = \infty$ and the case $4 > p > n = 3$ are not considered in their paper. Note that in [7], the author consider the interior regularity for the distribution solution of one kind parabolic system. It help us to deal with the case $n = 3, q = \infty$ and the case $4 > p > n = 3$. In Navier-Stokes equation, It is shown in [9], from the assumption $v \in L^{3,\infty}$ one can define the associated pressure $\widetilde{p}$ such that $(v, \widetilde{p})$ is a suitable weak solution of Navier-Stokes Equation. So the regularity for the weak solution $v \in L^{3,\infty}$ is just the regularity for the suitable weak solution in some sense.

In fact, if we denote $Q_r = Q_r(x_0, t_0)$ is a parabolic ball centered at $(x_0, t_0) \in Q$:

$$Q_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R};\ |x - x_0| < r, -r^2 < t - t_0 < 0\}$$

such that $Q_r \subset Q$ and $B_r(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < r\}$. where Ω is a domain in $\mathbb{R}^n$ with smooth boundary and $0 < T < \infty$. Using the results of [7], we have the $\varepsilon-$regularity for all p, q with $p \geq n, q \geq 2$ satisfying (1.2).

Theorem 1.3. *If u is a weak solution of (1.1) in Q with $u_t \in L^2_x L^2_t(Q)$, $\nabla u \in L^2_x L^\infty_t(Q)$, then there is a positive constant $\varepsilon < 1$ such that $\|\nabla u\|_{L^{p,q}(Q_r)} < \varepsilon$ which implies*
(a) $\nabla u \in L^\infty_x L^\beta_t(Q_{r/2})$ for all $2 \leq \beta < \infty$ when $p > n$.
(b) $\nabla u \in L^\alpha_x L^\beta_t(Q_{r/2})$ for all $2 \leq \alpha, \beta < \infty$ when $p = n$.
Here $\varepsilon = \varepsilon(n, m, p, \beta)$ if $p > n$ and $\varepsilon = \varepsilon(n, m, \alpha, \beta)$ if $p = n$.

We recall the *weak* $- L^{(q)}$ space for $1 < q < \infty$:

$$L^{(q)}(0, T) = \left\{ f \in L^1(0, T); [f]_{L^q(0,T)} < \infty \right\},$$

where

$$[f]_{L^q(0,T)} = \sup_{s>0} s \left(\mu\{t \in (0, T) : |f(t)| > s\}\right)^{1/q}.$$

By Theorem 1.3, we also can get

Theorem 1.4. *Let u be a weak solution of (1.1) in Q with $u_t \in L^2_x L^2_t(Q)$, $\nabla u \in L^2_x L^\infty_t(Q)$. Suppose that $1 \leq p, q \leq \infty$ satisfies $n/p + 2/q = 1$ and $p > n$. Then there exists a positive constant $\varepsilon = \varepsilon(n, m, p, \beta) < 1$ such that*

$$\|\nabla u\|_{L^{p,(q)}(Q_r)} \leq \varepsilon \tag{1.4}$$

which implies $\nabla u \in L^\infty_x L^\beta_t(Q_{r/2})$ for all $\beta > 2$.

Remark 1.5. *The condition (1.4) is fulfilled if , for example,*

$$\|\nabla u(t)\|_{L^p(B_{r(x_0)})} \leq \frac{\varepsilon}{(t_0 - t)^{1/q}} \ for\ t \in (-r^2 + t_0, t_0).$$

Definition 1.2. *We call a map $u : M \times (0, T] \to N$ is a suitable weak solution of (1.1), if it is a weak solution of (1.1), and satisfy the following monotonicity inequalities (1.5) and the energy inequality (1.6).*

We adopt the notation as in [1] and [2]. Denote by $z = (x, t)$ a point in $M \times \mathbb{R}$. For a distinguished point $z_0 = (x_0, t_0)$, $r > 0$, let

$$P_r(z_0) = \{z = (x, t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_0| < r,\ |t - t_0| < r^2\}$$

and

$$T_r(z_0) = \{z = (x, t) \in \mathbb{R}^n \times \mathbb{R} : t_0 - 4r^2 < t < t_0 - r^2\}.$$

Denote the fundamental solution to the (backward) heat equation $(\frac{\partial}{\partial t} + \Delta)f(x, t) = 0$ on $\mathbb{R}^m \times \mathbb{R}$ by

$$G_{z_0}(z) = \frac{1}{(4\pi(t - t_0))^{m/2}} \exp\left(-\frac{(x - x_0)^2}{4(t_0 - t)}\right),\ t < t_0.$$

We also denote by δ the parabolic distance function

$$\delta((x, t), (y, s)) = \max\{|x - y|, \sqrt{|t - s|}\}.$$

Let $\beta > 0$ be any fixed constant. For any $z_1 \in \mathbb{R}^n \times \mathbb{R}_+$, define, for $R \in (0, \sqrt{t_1}/2\beta)$,

$$\Psi_\beta(R, u, z_1) = \frac{1}{2} \int_{T_{\beta R}(z_1)} |\nabla u|^2 G_{z_1} \phi_\beta^2 dxdt,$$

where $\phi_\beta(x) = \phi((x - x_1)/\beta)$ and $\phi \in C_0^\infty(B_{1/2}(0))$ is a cut-off function such that $0 \le \phi \le 1$ and $\phi \equiv 1$ on $B_{1/4}(0)$. It is proved in [6] that the regular solution of (1.1) satisfy:

Monotonicity inequalities: There exists a constant $C > 0$, depending only on m, such that for any $z_1 \in \mathbb{R}^n \times \mathbb{R}_+$ and any $0 < R_1 < R_2 \le \min(1/2, \frac{\sqrt{t}}{2\beta})$,

$$\Psi_\beta(R_1, u, z_1) \le e^{C(R_2 - R_1)} \Psi_\beta(R_2, u, z_1) + C(R_2 - R_1)\beta^{-n} \int_{P_{\beta/2}(z_1)} |\nabla u|^2 dxdt. \tag{1.5}$$

Energy inequality: For any $\phi \in C_0^\infty(\mathbb{R}^n)$ and a.e. $0 \le t_1 \le t_2 < \infty$, it is true that

$$\begin{aligned} &2 \int_{\mathbb{R}^n} \int_{t_1}^{t_2} |u_t|^2 \phi^2 + \int_{\mathbb{R}^n} \phi^2 |\nabla u|^2 (x, t_2) \\ &\le \int_{\mathbb{R}^n} \phi^2 |\nabla u|^2 (x, t_1) + c(n) \int_{\mathbb{R}^n} \int_{t_1}^{t_2} |\nabla u|^2 |\nabla^2 \phi^2|. \end{aligned} \tag{1.6}$$

2. Proof of Theorem 1.3

Proof of Theorem 1.3: Assume that $N \subset \mathbb{R}^k$ has an orthonormal frame field ν_l, $1 \le l \le k - m$, for the normal bundle to N. By [8], the equation

(1.3) may equivalently be written as

$$u_t^i - \Delta u^i = \Omega^{i,j} \cdot \nabla u^j, \tag{2.1}$$

where $\Omega \in L^2(Q; so(k) \times \Lambda^1 \mathbb{R}^k)$, with components locally given by

$$\Omega^{i,j} = \sum_{l=1}^{k-m} \omega_l^i d\omega_l^j - \omega_l^j d\omega_l^i, \tag{2.2}$$

for $1 \leq i, j \leq k$ and $\omega_l = \nu_l \circ u$. If u is a weak solution of (1.3), then $v = \nabla u = (v^{i,\alpha})_{i=\overline{1,k}}^{\alpha=\overline{1,n}} \in L_x^2 L_t^2(Q; \mathbb{R}^{k+n})$ satisfying the following parabolic system

$$v_t - \Delta v = \nabla(\Omega v) \tag{2.3}$$

in Q in the distribution sense:

$$\int \int_Q (\partial_t \phi + \Delta \phi - div\phi\Omega) v dx dt = 0$$

for any $\phi = (\phi^{i,\alpha})_{i=\overline{1,k}}^{\alpha=\overline{1,n}} \in C_0^\infty(Q; \mathbb{R}^{k+n})$. Here

$$div\phi\Omega = \left(\sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha} \phi^{1,\alpha}, \dots \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha} \phi^{k,\alpha} \right) \begin{pmatrix} \Omega^{1,1} & \dots & \Omega^{1,k} \\ \Omega^{2,1} & \dots & \Omega^{2,k} \\ \dots & \dots & \dots \\ \Omega^{k,1} & \dots & \Omega^{k,k} \end{pmatrix}.$$

Then by [7, Theorem 2.1 and Theorem 2.2], we can get Theorem 1.3 and Theorem 1.4.

3. Proof of Theorem 1.2

First, we have the following lemma,

Lemma 3.1. *[2, Lemma 2.1] For $n \geq 2$ and $0 < T \leq +\infty$, suppose that $u : M \times [0, T] \to N$ is a weak solution of (1.1) with $\nabla u \in L_x^n L_t^\infty(M \times [0, T])$. Then $u \in C([0, T], L^n(M))$, and*

$$\|\nabla u(t)\|_{L^n(M)} \leq \|\nabla u(t)\|_{L_x^n L_t^\infty(M \times [0,T])}, \ \forall t \in [0, T]. \tag{3.1}$$

For $z \in \mathbb{R}^n \times \mathbb{R}_+$ and $0 < r < \sqrt{t}$ define

$$E(r, u, z) = r^{-n} \int_{P_r(z)} |\nabla u|^2 dyds.$$

In fact, if we let $\omega \subset \mathbb{R}^n$ be an open domain, we have the following Lemma which is given in [10] for stability solution of (1.3).

Lemma 3.2. *Let $u \in H^1_{loc}(\omega \times (0, \infty), N)$ be a weak solution of (1.3) satisfying the (1.5) and (1.6). Then for any parabolic cylinder $P_{r_0}(z_0) \subset \omega \times (0, \infty)$ and for any $z_1 = (x_1, t_1) \in P_{ar_0}(z_0)$ and $0 < r < br_0$, where a and b are two positive constants satisfying $a + 2b \leq 1/2$, we have*

$$\begin{aligned} &r^{2-n} \left(\int_{P_r(z_1)} |\partial_t u|^2 dz + \int_{B_r(x_1)\times\{t_1\}} |\nabla u|^2 dx \right) \\ &\leq C r_0^{-n} \int_{P_{r_0}(z_0)} |\nabla u|^2 dz. \end{aligned}$$

Proof: As the argument in [1, Lemma 2.2, Lemma 2.3](Although only the case $N = S^k$ is considered there, these two lemmas are true without this restriction), we can show there exists $K > 0$, such that

$$E(r, u, z_1) \leq K E(r_0, u, z_0)$$

for any $z_1 = (x_1, t_1) \in P_{ar_0}(z_0)$ and $0 < r < br_0$, where a and b are two positive constants satisfying $a + 2b \leq 1/2$. By Fubini's theorem we may choose $\alpha \in (1/2, 7/8)$ such that

$$\int_{B_r(x_1)} |\nabla u|^2(y, t_1 - \alpha^2 r^2) dy \leq C r^{-2} \int_{P_r(z_1)} |\nabla u|^2 dyds. \tag{3.2}$$

Choose a smooth function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi = 1$ in $B_{\alpha r}(x)$, $\phi = 0$ outside $B_r(x_0)$, $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq C/r$. It follows from (1.6) and (3.2) that

$$r^{2-n} \int_{P_{r/2}(z_1)} |\partial_t u|^2 dyds \leq C E(r_0, u, z_0)$$

for any $z_1 = (x_1, t_1) \in P_{ar_0}(z_0)$ and $0 < r < br_0$. On the other hand, use (1.6) with $t_2 = t_1$ and $t_1 = t_0 - \alpha r_0^2$, we can also have

$$r^{2-n} \int_{B_r(x_1)\times\{t_1\}} |\nabla u|^2 dx \leq C E(r_0, u, z_0).$$

By Lemma 3.2, [10, Theorem 2], and the argument in [10], we can get

Lemma 3.3. *Let $n=3$ or $n=4$, and let $u \in H^1(P_r(z_0), N)$ be a weak solution of (1.3) satisfying (1.5) and (1.6). If*

$$r^{-n}\int_{P_r(z_0)} |\nabla u|^2 \le \varepsilon_0^2$$

for a sufficiently small number $\varepsilon_0 > 0$, then u is smooth in $P_{r/2}(z_0)$.

Proof of Theorem 1.2: For simplicity, we can assume that $M = \mathbb{R}^3$. Suppose that $\Sigma(t_0) \neq \emptyset$. We can follow the blow up argument in [2] to get a map $v : \mathbb{R}^{n+1}_- \to N$ is a weak solution of (1.1), with $\nabla v \in L^3_x L^\infty_t(\mathbb{R}^{n+1}_-)$, and

$$R^{-3}\int_{P_R} |\nabla v|^2 \ge \varepsilon_0^2,\ \forall R > 0. \tag{3.3}$$

For the completeness of our theorem, we show it here. By Lemma 3.3, we have that $x_0 \in \Sigma(t_0)$ implies that

$$r^{-3}\int_{P_r(x_0,t_0)} |\nabla u|^2 \ge \varepsilon_0^2,\ \forall r > 0. \tag{3.4}$$

For $r_i \downarrow 0$, define $v_i(x,t) = u(x_0 + r_i x, t_0 + r_i^2 t) : \mathbb{R}^3 \times (-r_i^2, 0] \to N$. Then we can show $v_i(x,t)$ is a weak solution of (1.3), and v_i satisfies

$$\|\nabla v_i\|_{L^3_x L^\infty_t(\mathbb{R}^3\times[-r_i^{-2}t_0,0])} = \|\nabla u\|_{L^3_x L^\infty_t(\mathbb{R}^3\times[0,t_0])} < \infty, \tag{3.5}$$

and

$$R^{-n}\int_{P_R} |\nabla v_i|^2 = (Rr_i)^{-3}\int_{P_{Rr_i}(x_0,t_0)} |\nabla u|^2 \ge \varepsilon_0^2,\ \forall R > 0. \tag{3.6}$$

Moreover, by (3.5), we have

$$\begin{aligned}
\sup_i \int_{P_R} |\nabla v_i|^2 &\le \sup_i R^{5/3}\left[\int_{P_R} |\nabla v_i|^3(x,t)dxdt\right]^{2/3} \\
&\le \sup_i R^3 \|\nabla v_i\|_{L^3_x L^\infty_t(P_R)} \\
&\le \sup_i R^3 \sup_{t_0-(Rr_i)^2<s<t_0} \int_{B_R(x_0)} |\nabla u|^3(y,s)dy \\
&\le CR^3,\ \forall R > 0.
\end{aligned} \tag{3.7}$$

From Lemma 3.2, we have

$$\sup_i \int_{P_R} |(v_i)|_t^2 \le CR^{-2}\sup_i \int_{P_{2R}} |\nabla v_i|^2 \le CR,\ \forall R > 0. \tag{3.8}$$

It follows from (3.7) and (3.8) that $\{v_i\} \subset H^1_{loc}(\mathbb{R}^{n+1}_-)$ is a bounded sequence. Thus there exists a map $v : \mathbb{R}^{n+1}_- \to N$ such that $\nabla v_i \to \nabla v$ weakly in $L^2_{loc}(\mathbb{R}^{n+1}_-)$, and $v_i \to v$ strongly in $L^2_{loc}(\mathbb{R}^{n+1}_-)$. Note that

$$(v_i)_t - \Delta v_i = A(v_i)(\nabla v_i, \nabla v_i) \text{ in } \mathbb{R}^n \times (-r_i^{-2} t_0, 0], \tag{3.9}$$

and

$$|A(v_i)(\nabla v_i, \nabla v_i)| \leq C|\nabla v_i|^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}^{n+1}_-) \tag{3.10}$$

Thus

$$r^{-2} \int_{P_r(x_0,t_0)} |\nabla v|^3 \geq \varepsilon_0^3, \ \forall r > 0. \tag{3.11}$$

For any finite subset $\{x_1, ..., x_l\} \subset \Sigma(t_0) \cap \omega$, let $r_0 > 0$ be small enough so that $\{B_r(x_j)\}_{j=1}^l$ are mutually disjoint for any $0 < r \leq r_0$ and $B_r(x_j) \subset \omega$ for all $j = 1, ..., l$. By (3.11), for any $0 < r \leq r_0$, we have

$$l\varepsilon_0^3 \leq r^{-2} \sum_{i=1}^{l} \int_{P_r(x_i,t_0)} |\nabla v|^3 \leq r^{-2} \int_{t_0-r^2}^{t_0} \int_{\omega} |\nabla u|^3.$$

Then the proof is over.

Acknowledgments.

This study was sponsored by the Natural Science Foundation of Zhejiang Province of China (NSFZC) under Grant No. LY14F020025, the Foundation of Zhejiang Educational Committee of China (FZECC) under Grant No. Y201326931, and supported by the Fundamental Research Funds for the Central Universities; and the National Natural Science Foundation of China (NSFC) under Grant No. 10931001 and 11371316. Any opinion, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of the NSFZC, the FZECC, or the NSFC.

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