Pure and Applied Mathematics Quarterly
Volume 11, Number 2, 221–237, 2015

# Quasi-minuscule $E_n$ Bundles and Level One $\widehat{E}_n$ Bundles over Rational Surfaces and VOA Structures

Yunxia Chen

**Abstract:** In this paper, we consider a surface with only singularities of type $A$ whose smoothing is a $\mathbb{P}^2$ blown up at 8 (resp. 9) points. we use lines on such smooth surface to construct all the quasi-minuscule (resp. level one fundamental) representation bundles of $E_n$ (resp. affine $E_n$) over them. We also consider the vertex operator algebra structures on these bundles.
**Keywords:** singularity, quasi-minuscule representation bundles, level one fundamental representation bundles.

## 1. Introduction

Let $X_n$ be a blow-up of $\mathbb{P}^2$ at $n$ points and $K$ be the canonical class. When $n \leq 9$, the sub-lattice $\langle K\rangle^{\perp}$ of the Picard lattice is a root lattice of $E_n$ ($E_9 = \widehat{E}_8$ is the affine $E_8$ Lie algebra). Hence we can construct the corresponding $E_n$-Lie algebra bundle over $X_n$. Furthermore, using (possibly reducible) $(-1)$-curves on $X_n$ (those $l \in H^2(X_n, \mathbb{Z})$ satisfying $l^2 = l \cdot K = -1$), we can construct a natural representation bundle of $E_n$ over $X_n$ [3][6][11][12][13][16].

Let $\widetilde{X}_{n,d}$ be $X_n$ together with an $A_d$-chain of $(-2)$-curves given by $\gamma_1, \cdots, \gamma_d$ on $X_n$, after contract the $A_d$-chain, we get a new surface $X_{n,d}$ which has a simple singularity of type $A_d$. When $n \leq 8$, the sub-lattice $\langle \gamma_1, \cdots, \gamma_d, K\rangle^{\perp}$ is a root lattice of some simply-laced Lie algebra from the magic triangle [8][9]. When $n = 9$ and $0 \leq d \leq 5$, the sub-lattice $\langle \gamma_1, \cdots, \gamma_d, K\rangle^{\perp}$ is a root lattice of $\widehat{E}_k$-type ($k = 8 - d$), here

$$\widehat{E}_5 = \widehat{D}_5, \quad \widehat{E}_4 = \widehat{A}_4, \quad \widehat{E}_3 = \widehat{A}_1 \times A_2.$$

Hence we can construct the corresponding (affine) Lie algebra bundle over $\widetilde{X}_{n,d}$ [11]. Since all the divisors corresponding to the line bundle summands

Received January 15, 2014.

of this Lie algebra bundle do not intersect with the $A_d$-chain, this Lie algebra bundle can descend to $X_{n,d}$.

The purpose of this paper is to construct representation bundles from the (possibly reducible) $(-1)$-curves on $\widetilde{X}_{n,d}$ which have certain intersection patterns with the $A_d$-chain.

In section 2, we consider $n = 8$ and $0 \leq d \leq 5$, then on $\widetilde{X}_{8,d}$, the sublattice $\langle \gamma_1, \cdots, \gamma_d, K\rangle^{\perp}$ is a root lattice of $E_k$-type ($k = 8 - d$), here

$$E_5 = D_5, \quad E_4 = A_4, \quad E_3 = A_1 \times A_2.$$

Let $W_k$ and $\Lambda_k$ be the weight lattice and root lattice of $E_k$ respectively, then $W_k/\Lambda_k \cong \{[w_0 = 0], [w_1], \cdots, [w_d]\}$, where $\{w_1, \cdots, w_d\}$ is the set of minuscule weights of $E_k$ [15]. From different intersection patterns of the $(-1)$-curves on $X_8$ with the $A_d$-chain, we can construct all the quasi-minuscule representation bundles of $E_k$ over $\widetilde{X}_{8,d}$. The representation structures are induced from the line configurations.

**Theorem 1.** (*Theorem 9 and Theorem 11*) *Over* $\widetilde{X}_{8,d}$ *with* $0 \leq d \leq 5$ *and* $k = 8 - d$, *we have*

(1) $I_{(w_0)} := \{l \in H^2(\widetilde{X}_{8,d}, \mathbb{Z}) | l^2 = l \cdot K = -1, l \cdot \gamma_1 = \cdots = l \cdot \gamma_d = 0\}$ *is the root system of* $E_k$ *and*

$$\mathcal{L}^k_{(w_0)} := \mathcal{O}(-K)^{\oplus k} \oplus \bigoplus_{l \in I_{(w_0)}} \mathcal{O}(l)$$

*is the adjoint representation bundle over* $\widetilde{X}_{8,d}$.

(2) $I_{(w_i)} := \{l \in H^2(\widetilde{X}_{8,d}, \mathbb{Z}) | l^2 = l \cdot K = -1, l \cdot \gamma_j = \delta_{i,j}$ *for* $1 \leq j \leq d\}$ *for* $1 \leq i \leq d$ *give all the minuscule representations of* $E_k$ *and*

$$\mathcal{L}^k_{(w_i)} = \bigoplus_{l \in I_{(w_i)}} \mathcal{O}(l)$$

*for* $1 \leq i \leq d$ *are the all minuscule representation bundles of* $E_k$ *over* $\widetilde{X}_{8,d}$.

Note all these bundles described above can descend to $X_{8,d}$ after tensoring with some line bundle, as all the divisors corresponding to the line bundle summands of each bundle have the same intersection pattern with the $A_d$-chain. In more detail, $\mathcal{L}^k_{(w_0)}$ itself can descend to $X_{8,d}$, and $\mathcal{L}^k_{(w_i)} \otimes O(-l_{w_i})$ for any $l_{w_i} \in I_{(w_i)}$ can descend to $X_{8,d}$.

In section 3, we consider $n = 9$ and $0 \leq d \leq 5$, then on $\widetilde{X}_{9,d}$, the sublattice $\langle \gamma_1, \cdots, \gamma_d, K\rangle^{\perp}$ is a root lattice of $\widehat{E}_k$-type ($k = 8 - d$). Similar to $n = 8$ cases, we have

**Theorem 2.** (*Theorem 16, Lemma 19 and Theorem 21*) *Over* $\widetilde{X}_{9,d}$ *with* $0 \leq d \leq 5$ *and* $k = 8 - d$, *we have*

(1) $J_{(w_0)} := \{l \in H^2(\widetilde{X}_{9,d}, \mathbb{Z}) | l^2 = l \cdot K = -1, l \cdot \gamma_1 = \cdots = l \cdot \gamma_d = 0\}$ *is the root lattice of* $E_k$ *and*

$$\widehat{\mathcal{L}}^k_{(w_0)} := S^{\cdot}(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus k}) \otimes (\bigoplus_{l \in J_{(w_0)}} \mathcal{O}(l))$$

*is the basic representation bundle of* $\widehat{E}_k$ *over* $\widetilde{X}_{9,d}$.

(2) $J_{(w_i)} := \{l \in H^2(\widetilde{X}_{9,d}, \mathbb{Z}) | l^2 = l \cdot K = -1, l \cdot \gamma_j = \delta_{i,j}$ *for* $1 \leq j \leq d\}$ *for* $0 \leq i \leq d$ *give all the level one fundamental representations of* $\widehat{E}_k$ *and*

$$\widehat{\mathcal{L}}^k_{(w_i)} := S^{\cdot}(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus k}) \otimes (\bigoplus_{l \in J_{(w_i)}} \mathcal{O}(l))$$

*for* $0 \leq i \leq d$ *are the all level one fundamental representation bundles of* $\widehat{E}_k$ *over* $\widetilde{X}_{9,d}$.

Similar to the $n = 8$ cases, all above bundles can descend to $X_{9,d}$ after tensor with some line bundle.

In section 4, we show that the $\widehat{\mathcal{L}}^k_{(w_0)}$ has vertex operator algebra structures such that $\widehat{\mathcal{L}}^k_{(w_i)}$ for $1 \leq i \leq d$ have VOA-module structures.

**Theorem 3.** (*Theorem 24 and Theorem 25*) *Over* $\widetilde{X}_{9,d}$ *with* $0 \leq d \leq 5$ *and* $k = 8 - d$, *we have*

(1) *for any* $(-1)$*-curve* $l_0 \in J_{(w_0)}$, $\widehat{\mathcal{L}}^k_{(w_0)}(-l_0)$ *is a VOA bundle over* $\widetilde{X}_{9,d}$.

(2) *for* $1 \leq i \leq d$, $\widehat{\mathcal{L}}^k_{(w_i)}$ *is a representation bundle of* $\mathcal{L}^k_{(w_0)}(-l_0)$ *over* $\widetilde{X}_{9,d}$.

**Acknowledgements**. I am grateful to Professor Conan Leung and Professor Yongchang Zhu for many useful comments and discussions. The author is supported by the National Natural Science Foundation of China (no. 11501201), the China Postdoctoral Science Foundation (no. 2015M570334) and the Fundamental Research Funds for the Central Universities under project no. 222201514322.

## 2. Quasi-minuscule $E_k$ bundles over $X_{8,d}$

### 2.1. Quasi-minuscule representations

**Definition 4.** *A minuscule (resp. quasi-minuscule) representation of a semi-simple Lie algebra is an irreducible representation such that the Weyl group acts transitively on all the weights (resp. non-zero weights).*

Minuscule representations are always fundamental representations and quasi-minuscule representations are either minuscule or adjoint representations. For the simply-laced Lie algebras, their minuscule representations are listed below:

| $\mathfrak{g}$ | Miniscule representations |
|---|---|
| $A_n = sl\,(n+1)$ | $\wedge^k \mathbb{C}^{n+1}$ for $k = 1, 2, \cdots, n$ |
| $D_n = o\,(2n)$ | $\mathbb{C}^{2n}$, $\mathcal{S}^+$, $\mathcal{S}^-$ |
| $E_6$ | $\mathbb{C}^{27}$, $\overline{\mathbb{C}^{27}}$ |
| $E_7$ | $\mathbb{C}^{56}$ |

Note there is no minuscule representation for $E_8$.

The dual representation of a minuscule representation is still minuscule. For example, for $E_7$, the only minuscule representation $\mathbb{C}^{56}$ is self-dual; for $E_6$, the two minuscule representations $\mathbb{C}^{27}$ and $\overline{\mathbb{C}^{27}}$ are dual to each other; for $D_5$, $\mathbb{C}^{10}$ is self-dual and $\mathcal{S}^+$ is dual to $\mathcal{S}^-$; for $A_4$, $\mathbb{C}^5$ is dual to $\wedge^4\mathbb{C}^5$ and $\wedge^2\mathbb{C}^5$ is dual to $\wedge^3\mathbb{C}^5$.

Now we want to construct minuscule representations from $(-1)$-curves on some particular surfaces.

**Definition 5.** *A $(-1)$-curve in a surface $X$ is a genus zero (possibly reducible) curve $l$ in $X$ with $l \cdot l = -1$.*

Note the genus zero condition can be replaced by $l \cdot K = -1$ by the genus formula, where $K$ is the canonical class of $X$.

Let $X$ be a surface which has divisors $C_1, C_2, \cdots C_n$ whose dual graph is an $ADE$ Dynkin diagram of type $\mathfrak{g}$, i.e. the intersection matrix of these $C_i$'s is a Cartan matrix of type $\mathfrak{g}$. Suppose $C_k$ is one of the divisors whose corresponding fundamental representation $V_k$ is a minuscule representation and we can find a $(-1)$-curve $C_0^k$ in $X$ such that $C_0 \cdot C_i = \delta_{i,k}$, then we consider $I_{C_0^k} = \{l = C_0^k + \sum_{i=1}^{i=n} a_i C_i | l \cdot l = -1,\, a_i \in \mathbb{Z}\}$.

**Lemma 6.** *$V_{C_0^k} := \mathbb{C}\langle I_{C_0^k}\rangle$ is a representation of $\mathfrak{g}$ which is dual to $V_k$ .*

*Proof.* See Proposition 21 of [2]. ∎

### 2.2. Quasi-minuscule $E_8$ bundle over $X_8$

Let $X_8$ be the blow-up of $\mathbb{P}^2$ at 8 points $x_1, ..., x_8$. The Picard group $Pic(X_8) \cong H^2(X_8, \mathbb{Z})$ is a rank 9 lattice with generators $h, l_1, \cdots, l_8$, where $h$ is the class of lines in $\mathbb{P}^2$ and $l_i$ is the exceptional class of the blow-up at $x_i$. So $h^2 = 1 = -l_i^2$ and $h \cdot l_i = 0 = l_i \cdot l_j$, $i \neq j$. Thus $H^2(X_8, \mathbb{Z}) \cong \mathbb{Z}^{1,8}$. The canonical class is $K = -3h + l_1 + \cdots + l_8$.

Denote

$$\Lambda_8 = \{\alpha \in H^2(X_8, \mathbb{Z}) | \alpha \cdot K = 0\}.$$

$$R_8 = \{\alpha \in H^2(X_8, \mathbb{Z}) | \alpha \cdot K = 0, \quad \alpha^2 = -2\}.$$

It is well-known that $\Lambda_8$ is a root lattice of type $E_8$ and $R_8$ is a root system of type $E_8$ with a system of simple roots $\alpha_1 = l_1 - l_2$, $\alpha_2 = l_2 - l_3$,...., $\alpha_5 = l_5 - l_6$, $\alpha_6 = h - l_8 - l_7 - l_6$, $\alpha_7 = l_6 - l_7$, $\alpha_8 = l_7 - l_8$ (see Mannin's book [14]). The corresponding Dynkin diagram is as follows:

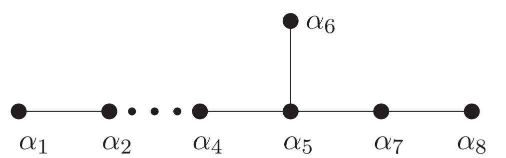


Figure 1. The Dynkin diagram of $E_8$

Since we have a root system of type $E_8$ attached to $X_8$, inspired by the Cartan decomposition of a complex simple Lie algebra, we can construct a Lie algebra bundle over $X_8$ as follows:

$$\mathcal{E}_8 := \mathcal{O}^{\oplus 8} \oplus \bigoplus_{\alpha \in R_8} \mathcal{O}(\alpha).$$

We can define a fiberwise Lie algebra structure on $\mathcal{E}_8$ which is compatible with any trivialization (see [2]), i.e. $\mathcal{E}_8$ is an $E_8$-Lie algebra bundle over $X_8$.

Denote

$$I_8 = \{l \in H^2(X_8, \mathbb{Z}) | l \cdot l = -1 = l \cdot K\},$$

then all the divisors in $I_8$ are effective (See Lemma 4 of [11]), i.e. they are $(-1)$-curves in $X_8$.

Inspired by the bijection $I_8 \to R_8$ given by $l \mapsto l + K$, we construct the vector bundle $\mathcal{L}_8$ using the $(-1)$-curves on $X_8$ as follows:

$$\mathcal{L}_8 = \bigoplus_{l \in I_8} \mathcal{O}(l) \oplus \mathcal{O}(-K)^{\oplus 8} = \mathcal{E}_8 \otimes \mathcal{O}(-K),$$

then $\mathcal{L}_8$ is the adjoint representation bundle, i.e. we have a globally defined action:

$$\mathcal{E}_8 \otimes \mathcal{L}_8 \to \mathcal{L}_8.$$

Note the above action is related to the line configurations.

### 2.3. Quasi-minuscule $E_k$ bundles over $X_{8,d}$

In this subsection, we consider $\widetilde{X}_{8,d}$ with $0 \leq d \leq 5$, i.e. $X_8$ together with an $A_d$-chain given by $\gamma_1, \cdots, \gamma_d \in R_8$, which means that the intersection matrix of the $\gamma_i$, $i = 1, \cdots, d$, is the negative of Cartan matrix of $A_d$-type. Namely, $\gamma_i \cdot \gamma_j = -2$ if $|i - j| = 0$; $\gamma_i \cdot \gamma_j = 1$ if $|i - j| = 1$; $\gamma_i \cdot \gamma_j = 0$ if $|i - j| \geq 2$. Let $k = 8 - d$.

**Lemma 7.** *The sub-lattice $\langle \gamma_1, \cdots, \gamma_d, K \rangle^{\perp}$ is a root lattice of $E_k$-type and $\langle K \rangle^{\perp} / \langle \gamma_1, \cdots, \gamma_d \rangle$ is a weight lattice of $E_k$-type.*

*Proof.* See Lemma 17 of [11]. ∎

Let $W_k$ and $\Lambda_k$ be the weight lattice and root lattice of $E_k$, then $W_k/\Lambda_k \cong \{[w_0 = 0], [w_1], \cdots, [w_d]\}$, where $\{w_1, \cdots, w_d\}$ is the set of minuscule weights of $E_k$. Denote $I_{(r_1, \cdots, r_d)} = \{l \in I_8 | l \cdot \gamma_j = r_j \text{ for } j = 1, \cdots, d\}$ and $I_{(w_i)} = I_{(0, \cdots, 1, \cdots 0)} = \{l \in I_8 | l \cdot \gamma_j = \delta_{i,j} \text{ for } j = 1, \cdots, d\}$, then

**Lemma 8.** *$I_{(w_0)} + K$ is a root system of $E_k$-type.*

*Proof.* It is directly from the bijection $I_8 \to R_8 : l \mapsto l + K$ and Lemma 7. ∎

From the above lemma, we can construct the adjoint representation of $E_k$ using $I_{(w_0)}$:

**Theorem 9.** *$V_{(w_0)} := \mathbb{C}^k \oplus \mathbb{C}\langle I_{(w_0)} \rangle$ is the adjoint representation of $E_k$.*

Now we consider other $I_{(w_i)}$'s.

**Lemma 10.** *(i) For $d = 1$, $|I_{(w_1)}| = 56$;*
*(ii) For $d = 2$, $|I_{(w_1)}| = |I_{(w_2)}| = 27$;*

*(iii) For* $d = 3$, $|I_{(w_1)}| = |I_{(w_3)}| = 16$, $|I_{(w_2)}| = 10$;
*(iv) For* $d = 4$, $|I_{(w_1)}| = |I_{(w_4)}| = 10$, $|I_{(w_2)}| = |I_{(w_3)}| = 5$.
*(v) For* $d = 5$, $|I_{(w_1)}| = |I_{(w_5)}| = 6$, $|I_{(w_2)}| = |I_{(w_4)}| = 3$, $|I_{(w_3)}| = 2$.

*Proof.* Direct computations. ∎

From Lemma 6, 8 and Lemma 10, we have the following result:

**Theorem 11.** *$V_{(w_i)} := \mathbb{C}\langle I_{(w_i)}\rangle$ for $1 \le i \le d$ are the all minuscule representations of $E_k$.*

*Proof.* For $d = 1$, without loss of generality, we can assume $\gamma_1 = l_7 - l_8$, then $\langle\gamma_1\rangle^{\perp}$ is the root lattice of $E_7$, we can take $\{\beta_1 = h - l_1 - l_7 - l_8, \beta_2 = l_1 - l_2, \beta_3 = l_2 - l_3, \beta_4 = l_3 - l_4, \beta_5 = h - l_1 - l_2 - l_3, \beta_6 = l_4 - l_5, \beta_7 = l_5 - l_6\}$ as its basis. For $E_7$, there is one minuscule representation, i.e. $\mathbb{C}^{56}$. In the above basis $\{\beta_1, \cdots, \beta_7\}$, the fundamental representation corresponding to $\beta_1$ is the minuscule representation $\mathbb{C}^{56}$. If we can find a $(-1)$-curve $C_0^1$ such that $C_0^1 \cdot \beta_j = \delta_{1,j}$ and $C_0^1 \cdot \gamma_1 = 1$, then we have $I_{C_0^1} = \{l = C_0^1 + \sum_{i=1}^{i=7} a_i\beta_i | l \cdot l = -1,\ a_i \in \mathbb{Z}\}$ is a subset of $I_{(w_1)}$. Moreover, from Lemma 6 and Lemma 10, we have $|I_{C_0^1}| = |I_{(w_1)}| = 56$, hence $I_{C_0^1} = I_{(w_1)}$ and $V_{(w_1)} = V_{C_0^1}$ is the minuscule representation $\mathbb{C}^{56}$. By direct computations, such a $C_0^1$ exists and hence unique, $C_0^1 = l_8$. And we have $I_{(w_1)} = -2K - \gamma_1 - I_{(w_1)}$, that is the representation $V_{(w_1)}$ is self-dual.

For $d = 2, 3, 4, 5$, the proofs are similar. Note here for $d = 5$, $E_3 = A_1 \times A_2$, since $A_1$ has one minuscule representation $U$ and $A_2$ has two minuscule representations $V_1$ and $V_2$, we say the five representations $U$, $V_1$, $V_2$, $U \otimes V_1$ and $U \otimes V_2$ are minuscule representations of $E_3$. ∎

We can also use branching rules to prove the above theorem: For $E_8$, $V := \mathbb{C}^8 \oplus \mathbb{C}\langle l + K : l \in I_8\rangle$ is the $E_8$ Lie algebra, $V$ acts on itself by the adjoint action. Suppose we have an $A_d$-chain given by $\{\gamma_1, \cdots, \gamma_d\}$ for $0 \le d \le 5$ , then $V_{(w_0)} := \mathbb{C}^k \oplus \mathbb{C}\langle l + K : l \in I_{(w_0)}\rangle$ is the $E_k$ Lie algebra, where $k = 8 - d$. Since $V_{(w_0)}$ is a Lie sub-algebra of $V$, $V_{(w_0)}$ also acts on $V$ and we can decompose $V$ as sum of irreducible representations of $V_{(w_0)}$. From the adjoint action of $V$ on $V$ itself, we know $V_{(w_0)}$ maps $V_{(w_i)}$ to $V_{(w_i)}$ for any $i \in [1, d]$, i.e. $V_{(w_i)}$ is a representation of $V_{(w_0)}$. From the branching rule of $E_8$ to $E_k$ and Lemma 10, we know $V_{(w_i)}$ for any $i \in [1, d]$ is a minuscule representation of $V_{(w_0)}$. For these minuscule representations, if two of them have the same dimension, then we can show that they are dual to each other. Hence, $V_{(w_i)}$ for $1 \le i \le d$ are the all minuscule representations of $E_k$.

Over $\widetilde{X}_{8,d}$ with $0 \le d \le 5$ and $k = 8 - d$, we have

$$\mathcal{E}_k := \mathcal{O}^{\oplus k} \oplus \bigoplus_{l \in I_{(w_0)}} \mathcal{O}(l + K)$$

is the $E_k$-Lie algebra bundle and

$$\mathcal{L}^k_{(w_i)} = \bigoplus_{l \in I_{(w_i)}} \mathcal{O}(l)$$

for $1 \le i \le d$ are the all minuscule representation bundles of $E_k$, i.e. we have a globally defined action:

$$\mathcal{E}_k \otimes \mathcal{L}^k_{(w_i)} \to \mathcal{L}^k_{(w_i)}.$$

The reason is that all these bundles are sub-bundles of $\mathcal{E}_8$ or $\mathcal{L}_8$, and all the actions are induced from $\mathcal{E}_8 \otimes \mathcal{L}_8 \to \mathcal{L}_8$.

Note that the $A_d$ singularity is a rational singularity, hence a vector bundle over $\widetilde{X}_{8,d}$ can descend to $X_{8,d}$ if and only if its restriction to the $A_d$-chain is trivial [6]. In our cases, all these bundles described above can descend to $X_{8,d}$ after tensoring with some line bundle, as all the divisors corresponding to the line bundle summands of each bundle have the same intersection pattern with the $A_d$-chain.

## 3. Level one $\widehat{E}_k$ bundles over $X_{9,d}$

### 3.1. Level one fundamental representations

In this subsection, we give a brief review of level one fundamental representations of affine $ADE$ Lie algebras. For more details, please refer to Frenkel and Kac's paper [4][5] and Tsukada's book [15].

Let's first recall the definition of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated with a complex finite dimensional simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, $\Lambda$ be the root lattice in $\mathfrak{h}^*$. Let $\langle , \rangle$ denote the killing form on $\mathfrak{g}$, normalized in such a way that the square length of a long root is 2. We identify $\mathfrak{h}^*$ and $\mathfrak{h}$ by the form $\langle , \rangle$. The affine Lie algebra $\widehat{\mathfrak{g}}$ is the complex vector space:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\langle c \rangle$$

provided with the bracket

$$[xt^n + \lambda c, yt^m + \mu c] := [x, y]_0 t^{n+m} + n\delta_{n+m,0}\langle x, y \rangle c,$$

where $x, y \in \mathfrak{g}$, $[,]_0$ denotes the Lie bracket induced from $\mathfrak{g}$, $\lambda, \mu \in \mathbb{C}$ [10]. We will give the explicit construction of all the level one fundamental representations of affine $ADE$ Lie algebra without proof.

We define

$$V_\Lambda = S^{\cdot}(\mathfrak{s}_-)\otimes\mathbb{C}(\Lambda),$$

$$V_W = S^{\cdot}(\mathfrak{s}_-)\otimes\mathbb{C}(W),$$

$$V_{(w)} = S^{\cdot}(\mathfrak{s}_-)\otimes\mathbb{C}(\Lambda + w),$$

where $\mathfrak{s}_- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$, $S^{\cdot}(\mathfrak{s}_-)$ is the symmetric algebra of the space $\mathfrak{s}_-$, $\mathbb{C}(\Lambda)$ is the group algebra of the lattice $\Lambda$, $W$ is the dual lattice of $\Lambda$, i.e. the weight lattice, and $w \in W/\Lambda$. If we fix representing elements $w_0, \cdots w_k$ of $W/\Lambda$ (we set $w_0 = 0$ and take $\{w_1, \cdots w_k\}$ as the set of minuscule weights), then

$$V_W \cong V_{(w_0)} \oplus \cdots \oplus V_{(w_k)}.$$

We define a grading on $V_W$ with the degree defined as:

$$\deg(h_1^{-n_1} h_2^{-n_2} \cdots h_N^{-n_N} e^\alpha) := n_1 + n_2 + \cdots + n_N + \frac{\langle \alpha, \alpha \rangle}{2}.$$

In particular, the subspace $V_\Lambda$ is graded by $\mathbb{Z}$ and we have

$$V_\Lambda = \sum_{n=0}^{n=\infty} V_n,\ V_n = \{v \in V_\Lambda | \deg(v) = n\}.$$

For every element $v \in V_\Lambda$ and $z \in \mathbb{C} - \{0\}$, we can define the vertex operator

$$Y(v, z) : V_W \to V_W^*,$$

where $V_W^*$ is the algebraic dual of $V_W$. However, developing this operator by power of $z$ we obtain:

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n},$$

where each $v(n)$ maps $V_W$ into itself.

Define a product $[u, v] = u(0) \cdot v$ on the degree=1 subspace $V_1$, then $V_1$ is the Lie algebra $\mathfrak{g}$. The affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\langle c\rangle$ acts on $V_W$ via the vertex operators: $\pi(u \otimes t^n) = u(n)$ and $\pi(c) = Id$. Each $V_{(w)}$ is irreducible under this action. Since we have

$$\{\text{minuscule weights}\} \cup \{0\} \cong W/\Lambda,$$

it follows that $\{V_{(w)}|w \in W/\Lambda\}$ is the set of all level one fundamental representations of $\widehat{\mathfrak{g}}$. In particular, $V_{(w_0)} = V_\Lambda$ is the basic representation of $\widehat{\mathfrak{g}}$.

**Remark 12.** *Level $k$ representation means the center $c$ acts as $kId$. Note $c = \sum_{i=0}^{i=r} n_i h_i$ for some integers $n_i$'s, where $n_i = 1$ if and only if the corresponding base root $C_i$ can be treated as the extended root, if and only if the fundamental representation of the corresponding finite Lie algebra corresponding to $C_i$ is minuscule. Hence, we have the correspondence between the minuscule weights and the level one fundamental representations.*

### 3.2. Basic representation bundle over $X_9$

Let $X_9$ be the blow-up of $\mathbb{P}^2$ at 9 points $x_1, ..., x_9$. The Picard group $Pic(X_9) \cong H^2(X_9, \mathbb{Z})$ is a rank 10 lattice with generators $h, l_1, \cdots, l_9$, where $h$ is the class of lines in $\mathbb{P}^2$ and $l_i$ is the exceptional class of the blowup at $x_i$. So $h^2 = 1 = -l_i^2$ and $h \cdot l_i = 0 = l_i \cdot l_j$, $i \neq j$. Thus $H^2(X_n, \mathbb{Z}) \cong \mathbb{Z}^{1,9}$. The canonical class is $K = -3h + l_1 + \cdots + l_9$.

Denote $R_9$ as before, i.e.

$$R_9 = \{\alpha \in H^2(X_9, \mathbb{Z}) | \alpha \cdot K = 0,\ \alpha^2 = -2\}.$$

It is well-known that the set $R_9 \cup \{m(-K)|m \neq 0, m \in \mathbb{Z}\}$ forms a root system of (untwisted) affine $E_8$-type (that is, $\widehat{E}_8$ type), with real roots $\Delta^{re} = R_9$ and imaginary roots $\Delta^{im} = \{m(-K)|m \neq 0, m \in \mathbb{Z}\}$ [7][10][11]. Here the system of simple roots of $R_9$ is $\{\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, ...., \alpha_6 = l_6 - l_7, \alpha_7 = h - l_9 - l_8 - l_7, \alpha_8 = l_7 - l_8, \alpha_9 = l_8 - l_9\}$.

Inspired by this, we can construct the $\widehat{E}_8$-bundle $\mathcal{E}_9$ over $X_9$ as follows:

$$\mathcal{E}_9 := (\mathcal{O}^{\oplus 8} \oplus \mathcal{O}) \oplus \bigoplus_{\alpha \in \Delta^{re}} \mathcal{O}(\alpha) \oplus \bigoplus_{\beta \in \Delta^{im}} \mathcal{O}(\beta)^{\oplus 8}.$$

We can define a fiberwise affine Lie algebra structure on $\mathcal{E}_9$ which is compatible with any trivialization (see [11]), i.e. $\mathcal{E}_9$ is an $\widehat{E}_8$-bundle over $X_9$.

Denote

$$I_9 = \{l \in H^2(X_9, \mathbb{Z}) | l \cdot l = -1 = l \cdot K\},$$

then $I_9$ is an infinite set and all the divisors in $I_9$ are effective (See Lemma 5 of [11]), i.e. they are $(-1)$-curves in $X_9$.

From the above subsection, to construct the basic representation of $\widehat{E}_8$, we need to find the $E_8$ root lattice $\Lambda_8$.

**Lemma 13.** *Fixing any $l_0 \in I_9$, $\Lambda_8 \cong \langle K, l_0 \rangle^{\perp} \subset H^2(X_9, \mathbb{Z})$.*

*Proof.* Fix any $l_0 \in I_9$, if we contract this $l_0$, then we will get $X_8$. Over this $X_8$, we know $\langle K \rangle^{\perp}$ is a root lattice of $E_8$ type. But now $\langle K \rangle^{\perp}$ is the same with $\langle K, l_0 \rangle^{\perp}$, hence $\Lambda_8 \cong \langle K, l_0 \rangle^{\perp}$. ∎

The relationship between $I_9$ and $\Lambda_8$ is given by the following lemma.

**Lemma 14.** *Fixing any $l_0 \in I_9$, there is a bijection between $I_9$ and $\Lambda_8$.*

*Proof.* Define $f : I_9 \to \Lambda_8$ by $l \mapsto l - l_0 + (1 + l \cdot l_0)K$. It is obvious that $f$ is injective. For any $\alpha \in \Lambda_8$, we have $(\alpha + l_0 + \frac{\alpha^2}{2}K) \in I_9$ and $f(\alpha + l_0 + \frac{\alpha^2}{2}K) = \alpha$. Hence $f$ is also surjective. ∎

Linearly extending this $f$ in the above lemma to $\mathbb{C}\langle I_9 \rangle \to \mathbb{C}\langle \Lambda_8 \rangle$, then we have a bijection between $\mathbb{C}\langle I_9 \rangle$ and $\mathbb{C}\langle \Lambda_8 \rangle$. Inspired by the above lemmas, we construct a bundle $\mathcal{L}_9$ over $X_9$ as follows:

$$\mathcal{L}_9 := S^{\cdot}\Big(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus 8}\Big) \otimes \Big(\bigoplus_{l \in I_9} \mathcal{O}(l)\Big).$$

Compare the definition of the basic representation and the vector bundle $\mathcal{L}_9$, we know each fiber $L_9$ of $\mathcal{L}_9$ is a basic representation of $\widehat{E}_8$ under the following action:

$$\rho : \widehat{E}_8 \times L_9 \to L_9,$$

$$\rho(x, v) := (id \otimes f^{-1}) \circ \pi(x, (id \otimes f) \cdot v).$$

Note that we will sometimes write $\rho(x, v)$ as $\rho(x) \cdot v$, and similarly for $\pi$.

Take a trivialization open subset $U$ for both $\mathcal{E}_9$ and $\mathcal{L}_9$,then we have the action

$$\rho_U : \mathcal{E}_9|_U \times \mathcal{L}_9|_U \to \mathcal{L}_9|_U$$

induced from $\rho : \widehat{E}_8 \times L_9 \to L_9$.

**Lemma 15.** *$\rho_U : \mathcal{E}_9|_U \times \mathcal{L}_9|_U \to \mathcal{L}_9|_U$ satisfies $\mathcal{O}_U(x) \times \mathcal{O}_U(v) \to \bigoplus \mathcal{O}_U(x+v)$ for any direct summand $\mathcal{O}(x)$ of $\mathcal{E}_9$ and $\mathcal{O}(v)$ of $\mathcal{L}_9$.*

*Proof.* See Lemma 7 of [1]. ∎

From the above lemma and the relationship between the transition functions of these direct summand line bundles, we know that the fiberwise action $\rho$ is compatible with any trivialization of $\mathcal{E}_9$ and $\mathcal{L}_9$, i.e.

**Theorem 16.** *$\mathcal{L}_9$ is the basic representation bundle of $\widehat{E}_8$ over $X_9$.*

**Remark 17.** *Note that if we use the root lattice $\Lambda_8$ instead of $I_9$ to construct the bundle, i.e.*

$$\mathcal{V} := S^{\cdot}(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus 8}) \otimes (\bigoplus_{\alpha \in \Lambda_8} \mathcal{O}(\alpha)),$$

*though each fiber of $\mathcal{V}$ is a basic representation of $\widehat{E}_8$, the fiberwise action is not compatible with different trivializations of $\mathcal{E}_9$ and $\mathcal{V}$.*

### 3.3. Level one $\widehat{E}_k$ bundles over $X_{9,d}$

In this subsection, we consider $\widetilde{X}_{9,d}$ with $0 \le d \le 5$, i.e. $X_9$ together with an $A_d$-chain given by $\gamma_1, \cdots, \gamma_d \in R_8$. Let $k = 8 - d$.

**Lemma 18.** *The sub-lattice $\Lambda(\widehat{E}_k) := \langle \gamma_1, \cdots, \gamma_d, K \rangle^{\perp}$ is a root lattice of $\widehat{E}_k$-type $(k = 8-d)$ with the real root system $\Delta_k^{re} = \{\alpha \in \Lambda(\widehat{E}_k) | \alpha^2 = -2\}$ and the imaginary roots $\Delta_k^{im} = \{m(-K) | m \neq 0, m \in \mathbb{Z}\}$.*

*Proof.* See Theorem 20 of [11]. ∎

Since $\Lambda_8 \cong \langle K, l_0 \rangle^{\perp} \cong W_8$ for any fixed $l_0 \in I_9$, we have the root lattice of $E_k$ is $\Lambda_k \cong \langle K, l_0, \gamma_1, \cdots, \gamma_d \rangle^{\perp}$ and the weight lattice of $E_k$ is $W_k \cong \langle K, l_0 \rangle^{\perp} / \langle \gamma_1, \cdots, \gamma_d \rangle$. Then $W_k/\Lambda_k \cong \{[w_0 = 0], [w_1], \cdots, [w_d]\}$, where we can take $\{w_1, \cdots, w_d\}$ as the set of minuscule weights of $E_k$. Denote $J_{(r_1,\cdots,r_d)} = \{l \in I_9 | l \cdot \gamma_j = r_j \text{ for } j = 1, \cdots, d\}$ and $J_{(w_i)} = J_{(0,\cdots,1,\cdots 0)} = \{l \in I_9 | l \cdot \gamma_j = \delta_{i,j} \text{ for } j = 1, \cdots, d\}$, then

**Lemma 19.** *$J_{(w_0)} \cong \Lambda_k$.*

*Proof.* It is directly obtained from Lemma 14. ∎

From the above lemma, we can construct the basic representation of $E_k$ using $I_{(w_0)}$. Now we consider the other $I_{(w_i)}$'s. Under the same map from $J_{(w_0)}$ to $\Lambda_k$, we have an isomorphism between $J_{(w)} = \{l \in I_9 | l \cdot \gamma_j = w \cdot \gamma_j$ for $j = 1, \cdots, d\}$ and $\Lambda_k + w$ for any $w \in \langle K, l_0 \rangle^\perp$.

**Lemma 20.** *$(i) d = 1$, $[w] = [w'] \in W_7/\Lambda_7 \Leftrightarrow w \cdot \gamma_1 \equiv w' \cdot \gamma_1$ mod 2;*
*$(ii) d = 2$, $[w] = [w'] \in W_6/\Lambda_6 \Leftrightarrow w \cdot (\gamma_1 + 2\gamma_2) \equiv w' \cdot (\gamma_1 + 2\gamma_2)$ mod 3;*
*$(iii) d = 3$, $[w] = [w'] \in W_5/\Lambda_5 \Leftrightarrow w \cdot (\gamma_1 + 2\gamma_2 + 3\gamma_3) \equiv w' \cdot (\gamma_1 + 2\gamma_2 + 3\gamma_3)$ mod 4;*
*$(iv) d = 4$, $[w] = [w'] \in W_4/\Lambda_4 \Leftrightarrow w \cdot (\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4) \equiv w' \cdot (\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4)$ mod 5.*
*$(v) d = 5$, $[w] = [w'] \in W_3/\Lambda_3 \Leftrightarrow w \cdot (\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5) \equiv w' \cdot (\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5)$ mod 6.*

*Proof.* $(i)$ Since $\gamma_1^2 = -2$, we have for any $[w], [w'] \in W_7$ with $w, w' \in \langle K, l_0 \rangle^\perp$, if $[w] = [w']$ in $W_7$, then $w \cdot \gamma_1 \equiv w' \cdot \gamma_1$ mod 2. Consider $W_7/\Lambda_7$, for any $[w], [w'] \in W_7/\Lambda_7$, if $[w] = [w']$ in $W_7/\Lambda_7$, then $[w] = [w' + \sum_{i=1}^{i=7} a_i \beta_i]$ in $W_7$, where $\{\beta_1, \cdots, \beta_7\}$ is a basis of $\Lambda_7$ and $a_i$'s are some integers. Hence $w \cdot \gamma_1 \equiv (w' + \sum_{i=1}^{i=7} a_i \beta_i) \cdot \gamma_1$ mod 2, that is $w \cdot \gamma_1 \equiv w' \cdot \gamma_1$ mod 2. Conversely, for any $w, w' \in \langle K, l_0 \rangle^\perp$, if $w \cdot \gamma_1 \equiv w' \cdot \gamma_1$ mod 2, then $(w - w' + a\gamma_1) \cdot \gamma_1 = 0$ for some integer $a$, that means $w - w' + a\gamma_1 \in \Lambda_7$, hence $[w] = [w']$ in $W_7/\Lambda_7$.

The proofs of $(ii), (iii), (iv)$ and $(v)$ are similar to $(i)$. ∎

From the above lemma, we have the following result.

**Theorem 21.** *We can take $[w_0 = 0], [w_1], \cdots, [w_d] \in W_k/\Lambda_k$ such that $W_k/\Lambda_k \cong \{[w_0], [w_1], \cdots, [w_d]\}$ and $w_i \cdot \gamma_j = \delta_{i,j}$ for every $0 \leq i \leq d$.*

*Proof.* For any $(r_1, \cdots, r_d) \in \mathbb{Z}^d$, we can find $w \in \langle K, l_0 \rangle^\perp$ such that $w \cdot \gamma_i = r_i$. Together with Lemma 20, we have the results. ∎

For the computation of $\{[0], [w_1], \cdots, [w_d]\}$, we can first take $w_i$'s as the minuscule weights of $E_k$, then adjust them using $\sum_{j=1}^{j=d} a_j \gamma_j$ to get $w_i \cdot \gamma_j = \delta_{i,j}$.

Over $\widetilde{X}_{9,d}$ with $0 \leq d \leq 5$ and $k = 8 - d$, we have

$$\widehat{\mathcal{E}}_k := (\mathcal{O}^{\oplus k} \oplus \mathcal{O}) \oplus \bigoplus_{\alpha \in \Delta_k^{re}} \mathcal{O}(\alpha) \oplus \bigoplus_{\beta \in \Delta_k^{im}} \mathcal{O}(\beta)^{\oplus k}.$$

is the $\widehat{E}_k$-Lie algebra bundle and

$$\widehat{\mathcal{L}}^k_{(w_i)} := S^{\cdot}(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus k}) \otimes (\bigoplus_{l \in J_{(w_i)}} \mathcal{O}(l))$$

for $0 \leq i \leq d$ are the all level one fundamental representation bundles of $\widehat{E}_k$, i.e. we have a globally defined action:

$$\widehat{\mathcal{E}}_k \otimes \widehat{\mathcal{L}}^k_{(w_i)} \to \widehat{\mathcal{L}}^k_{(w_i)}.$$

The reason is that all these bundles are sub-bundles of $\mathcal{E}_9$ or $\mathcal{L}_9$, and all the actions are induced from $\mathcal{E}_9 \otimes \mathcal{L}_9 \to \mathcal{L}_9$. In particular, $\widehat{\mathcal{L}}^k_{(w_0)}$ is the basic representation bundle of $\widehat{E}_k$ over $\widetilde{X}_{9,d}$. Note that all these bundles can descend to $X_{9,d}$.

## 4. Vertex operator algebra structures

It is well-known that the basic representations of affine Lie algebras admit vertex operator algebra structures [15][17], i.e. the basic representation $V$ of $\widehat{E}_k$ together with the vertex operators $Y(v,z)$ is a VOA.

Fix any $l_0 \in I_9$, we define a vector bundle $\mathcal{L}_9(-l_0)$ over $X_9$ as follows:

$$\mathcal{L}_9(-l_0) := S^{\cdot}(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus 8}) \otimes (\bigoplus_{l \in I_9} \mathcal{O}(l-l_0)),$$

i.e. $\mathcal{L}_9(-l_0)= \mathcal{L}_9 \otimes \mathcal{O}(-l_0)$. We know that each fiber of $\mathcal{L}_9(-l_0)$ admits a VOA structure (through the map $f: I_9 \to \Lambda_8$) [1].

For any trivialization open subset $U$ of $\mathcal{L}_9(-l_0)$, we have a linear map

$$\mathcal{L}_9(-l_0)|_U \times \mathcal{L}_9(-l_0)|_U \to \bigoplus_n \mathcal{L}_9(-l_0)|_U \otimes \mathcal{O}_U(nK)$$

(here we view $z^n$ as a section of $\mathcal{O}_U(nK)$) induced from the vertex operator $Y: V \otimes V \to V((z))$.

**Lemma 22.** *([1]) $\mathcal{L}_9(-l_0)|_U \times \mathcal{L}_9(-l_0)|_U \to \bigoplus_n \mathcal{L}_9(-l_0)|_U \otimes \mathcal{O}_U(nK)$ satisfies $\mathcal{O}_U(x) \times \mathcal{O}_U(y) \to \bigoplus \mathcal{O}_U(x+y)$ for any two direct summands $\mathcal{O}(x)$ and $\mathcal{O}(y)$ of $\mathcal{L}_9(-l_0)$.*

From the above lemma and the relationship between the transition functions of these direct summand line bundles, we know that the fiberwise VOA structure on $\mathcal{L}_9(-l_0)$ is compatible with any trivialization of $\mathcal{L}_9(-l_0)$, i.e.

**Theorem 23.** ([*1*]) *$\mathcal{L}_9(-l_0)$ is a vertex operator algebra bundle over $X_9$.*

When $X_9$ admits an $A_d$-chain with $0 \leq d \leq 5$, i.e. $X_9 = \widetilde{X}_{9,d}$, fixing any $l_0 \in J_{(w_0)}$, $\widehat{\mathcal{L}}^k_{(w_0)}(-l_0) \subset \mathcal{L}_9(-l_0)$ is a VOA sub-bundle where $k = 8 - d$, i.e.

**Theorem 24.** *$\widehat{\mathcal{L}}^k_{(w_0)}(-l_0)$ is a vertex operator algebra bundle over $\widetilde{X}_{9,d}$.*

For all the other level one fundamental representations, they are irreducible representations of the corresponding vertex operator algebra. Since we have a globally defined action

$$\widehat{\mathcal{L}}^k_{(w_0)}(-l_0) \otimes \widehat{\mathcal{L}}^k_{(w_i)} \to \bigoplus_n \widehat{\mathcal{L}}^k_{(w_i)} \otimes \mathcal{O}(nK)$$

induced from

$$\mathcal{L}_9(-l_0) \times \mathcal{L}_9(-l_0) \to \bigoplus_n \mathcal{L}_9(-l_0) \otimes \mathcal{O}(nK),$$

we have

**Theorem 25.** *$\widehat{\mathcal{L}}^k_{(w_i)}$ for $1 \leq i \leq d$ are the VOA representation bundles over $\widetilde{X}_{9,d}$.*

Note that all above bundles can descend to $X_{9,d}$.

## References

[1] Y.X. Chen, *Affine $E_8$ basic representation bundles over rational surfaces with $c_1^2 = 0$*, The Asian Journal of Mathematics (in press).

[2] Y. X. Chen and N. C. Leung, *ADE bundles over surfaces with ADE singularities*, Int. Math. Res. Not. (2013), doi: 10.1093/imrn/rnt065.

[3] Y. X. Chen and N. C. Leung, *Affine ADE bundles over complex surfaces with $p_g = 0$*, arXiv:1303.5578.

[4] I.B. Frenkel, *Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory*, J. Funct. Anal. 44 (3) (1981), 259-327.

[5] I.B. Frenkel and V.G. Kac, *Basic Representations of Affine Lie Algebras and Dual Resonance Models*, inventiones math. 62 (1980), 23-66.

[6] R. Friedman and J. W. Morgan, *Exceptional groups and del Pezzo surfaces*, Contemporary mathematics 312, Symposium in Honor of C.H. Clemens (2000), 101-116.

[7] G. Heckman and E. Looijenga, *The moduli space of rational elliptic surfaces*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math, Vol. 36. Math. Soc. Japan, Tokyo (2002), 185-248,.

[8] P. Henry-Labordere, B. Julia, L. Paulot, *Borcherds symmetries in M-theory*, JHEP 0204 (2002) 049 (or preprint, arxiv:hep-th/0203070).

[9] P. Henry-Labordere, B. Julia, L. Paulot, Real Borcherds Superalgebras and M-theory, JHEP 0304 (2003) 060 (or preprint, arxiv:hep-th/0212346).

[10] V. G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press (1994), Third edition.

[11] N. C. Leung, M. Xu and J. J. Zhang, *Kac–Moody* $\widetilde{E_k}$*-bundles over elliptic curves and del Pezzo surfaces with singularities of type A*, Math. Ann. (2012) 352:805–828.

[12] N. C. Leung and J. J. Zhang, *Moduli of bundles over rational surfaces and elliptic curves I: simply laced cases*, J. London Math. Soc. (2009), 750-770.

[13] N. C. Leung and J. J. Zhang, *Moduli of bundles over rational surfaces and elliptic curves II: non-simply laced cases*, Int. Math. Res. Not. 24 (2009), 4597–4625.

[14] Y. I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetic*, North-holland Mathematical library (1986).

[15] H. Tsukada, *String Path Integral Realization of Vertex Operator Algebras*, Memoirs of AMS, Volume 91, Number 444, (1991).

[16] M. Xu and J. J. Zhang, *G-bundles over elliptic curves for non-simply laced Lie groups and configurations of lines in rational surfaces*, Pacific J.Math, Vol. 261, No. 2 (2013), 497-510.

[17] Y.C. Zhu, *Vertex Operator Algebras associated to Representations of Affine and Virasoro Algebras*, Duke Math. J. Vol. 66, No. 1, (1992).

Yunxia Chen
School of Science, East China University of Science and Technology,

Meilong Road 130, Shanghai, China
E-mail: yxchen76@ecust.edu.cn