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Errata: "Extensions of truncated discrete valuation rings"

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The purposes of these errata are:

(1) to fill in a gap in the proof of Part (ii) of Proposition 2.2 of [I] (= Proposition 2.1 of [R]), and

(2) to explain the current status of, and wrong points in, the preprint [II] (which will never be published) and the survey paper [R].

We thank Shin Hattori for pointing out the gap (1) and discussions on it, and Takeshi Saito for pointing out a fatal error in [II] and for providing a counterexample to Proposition 3.7 of [II].

1. We use the notation of [I]. The proposition in question is the following:

Proposition. (i) *Let A be a tdvr with residue field k of characteristic $p \geq 0$, and let a be the length of A. Then there exists a cdvr $\mathcal{O}$ such that A is isomorphic to $\mathcal{O}/\mathfrak{m}^a$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. If $pA = 0$, then this $\mathcal{O}$ can be taken to be the power series ring $k[\![\pi]\!]$; if $pA \neq 0$, then $\mathcal{O}$ as above must be finite over a Cohen p-ring ([G], $0_{\mathrm{IV}}, 19.8$) with residue field k. (If $pA = 0$ and $p \neq 0$, then both types of $\mathcal{O}$ are possible.)*

(ii) *Let K be a cdvf and let $A = \mathcal{O}_K/\mathfrak{m}_K^a$ with $a \geq 1$. For any finite extension B/A of tdvr's, there exist a finite separable extension L/K and an isomorphism $\psi : \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L \to B$ such that the diagram*

$$(1) \qquad \begin{array}{ccc} \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L & \xrightarrow{\psi} & B \\ \uparrow & & \uparrow \\ \mathcal{O}_K/\mathfrak{m}_K^a & = & A \end{array}$$

is commutative, where the left vertical arrow is the one induced by $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$.

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The proof in [I] has a gap in proving that L/K can be taken to be separable (the Jacobian criterion applied to the newly taken $g_1, \ldots, g_n$ should have been considered modulo $\mathfrak{q}' = (g_1, \ldots, g_n)$ rather than the original $\mathfrak{q}$). We give here a correct one, including the whole proof (but printing in the tiny font the part which is identical with the original) for the convenience of the reader.

Proof. (i) Let W be a Cohen p-ring with residue field k. The reduction map $W \to k$ lifts by the formal smoothness of W to a local ring homomorphism $W \to A$ ([G], 0_{IV}, 19.8.6).

If $pA = 0$, the map $W \to A$ factors through the residue field k, which makes A a k-algebra. Then there exists a surjective A-algebra homomorphism $k[\![\pi]\!] \to A$ which maps π to π_A, where π_A is a uniformizer of A. Hence A is isomorphic to $k[\![\pi]\!]/(\pi^a)$ (*cf.* [M], Th. 3.1).

In the general case, we can write A as a quotient of the polynomial ring $W[X]$ by sending X to π_A. Then we obtain a surjection onto A from a cdvr $\mathcal{O}$ which is finite over W by the same procedure as in the proof of (ii) below.

(ii) Since B is finite over $A = \mathcal{O}_K/\mathfrak{m}_K^a$, there exists a surjective $\mathcal{O}_K$-algebra homomorphism $\phi : R \to B$ from a polynomial ring $R = \mathcal{O}_K[X_1, \ldots, X_n]$ onto B. Let $\mathfrak{m} = \phi^{-1}(\mathfrak{m}_B)$ and $R_\mathfrak{m}$ the localization of R at the maximal ideal $\mathfrak{m}$. Then $R_\mathfrak{m}$ is a regular local ring of Krull dimension $n+1$ ([G], 0_{IV}, 17.3.7), and ϕ extends to a surjective $\mathcal{O}_K$-algebra homomorphism $\varphi : R_\mathfrak{m} \to B$. By abuse of notation, we denote also by $\mathfrak{m}$ the maximal ideal of $R_\mathfrak{m}$. Put $\mathfrak{n} = \mathrm{Ker}\,(\varphi)$. We identify the residue field k' of $R_\mathfrak{m}$ with that of B via φ. Since $\varphi(\mathfrak{m}^2) = \mathfrak{m}_B^2$, the map φ induces a surjective k'-linear map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ and its kernel is $(\mathfrak{n} + \mathfrak{m}^2)/\mathfrak{m}^2 \simeq \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$. Thus we have an exact sequence

$$0 \;\to\; \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2) \;\to\; \mathfrak{m}/\mathfrak{m}^2 \;\to\; \mathfrak{m}_B/\mathfrak{m}_B^2 \;\to\; 0.$$

Assume $a \geq 2$, as the case $a = 1$ can be treated similarly and more easily. Then $\dim_{k'}(\mathfrak{m}_B/\mathfrak{m}_B^2) = 1$ and $\dim_{k'}(\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)) = n$. Choose a regular system of parameters $(w, f_1, \ldots, f_n)$ of $R_\mathfrak{m}$ such that $\varphi(w)$ gives a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$ and $f_1, \ldots, f_n \in \mathfrak{n}$ give a basis of $\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$. Let $\mathfrak{p}$ be the ideal of $R_\mathfrak{m}$ generated by $f_1, \ldots, f_n$. Then by [G], 0_{IV}, 17.1.7, the quotient ring $\mathcal{O} = R_\mathfrak{m}/\mathfrak{p}$ is a regular local ring of dimension 1 and hence a discrete valuation ring. It contains $\mathcal{O}_K$ since φ maps π_K to a non-zero non-unit in B, and is finite over $\mathcal{O}_K$. Hence it is a cdvr. Since $\mathfrak{n} \supset \mathfrak{p}$, the map φ factors through $\mathcal{O}$. Thus we see the diagram (1) commutes (with $\mathcal{O}$ in place of $\mathcal{O}_L$). Since B is flat over A, the induced homomorphism ψ is bijective.

To make the fraction field L of $\mathcal{O}$ separable over K, we "deform" $\mathcal{O}$ if necessary. Let L_0 be the separable closure of K in L. Then L/L_0 is purely inseparable and we can find a series of extensions $L_0 \subset L_1 \subset \cdots \subset L_s = L$ such that

$$L_{i+1} = L_i(\alpha_i^{1/p}) \quad \text{with some} \quad \alpha_i \in L_i^\times \smallsetminus (L_i^\times)^p.$$

For each i, the ramification index e_{i+1} of L_{i+1}/L_i is either p or 1. If $e_{i+1} = p$, then we can take α_i to be a prime element of $\mathcal{O}_i := \mathcal{O}_{L_i}$. If $e_{i+1} = 1$, then L_{i+1}/L_i has inseparable residual extension of degree p and hence we can take α_i to be a unit of $\mathcal{O}_i$ whose image in the residue field is not a p-th power. In either case, $\mathcal{O}_{i+1}$ is then generated by $\alpha_i^{1/p}$ as an $\mathcal{O}_i$-algebra and hence we have

$$\mathcal{O}_{i+1} \;\simeq\; \mathcal{O}_i[Y]/(Y^p - \alpha_i).$$

To deform the $\mathcal{O}_i$'s inductively, we adapt the following
Recipe: In general, if M is a finite extension of K and $\alpha \in \mathcal{O}_M$ has the same property as α_i above (*i.e.* prime or unit which is residually non-p-th power), then for any non-zero $\beta \in \mathfrak{m}_K^a \mathcal{O}_M$, the polynomial $Y^p + \beta Y - \alpha \in \mathcal{O}_M[Y]$ is separable and irreducible over M. In fact, it is Eisenstein if α is a prime element, and otherwise it gives rise to an inseparable extension of degree p of the residue field. Hence $\mathcal{O}_{\alpha,\beta} := \mathcal{O}_M[Y]/(Y^p + \beta Y - \alpha)$ is a complete

discrete valuation ring whose fraction field is separable over M. Note also that the $\mathcal{O}_M$-algebras $\mathcal{O}_{\alpha,\beta} \otimes_{\mathcal{O}_K} A$ are canonically isomorphic for all $\alpha \in \mathcal{O}_M$ in a fixed class mod $\mathfrak{m}_K^a \mathcal{O}_M$ and all $\beta \in \mathfrak{m}_K^a \mathcal{O}_M$.

Now choose any non-zero $\beta \in \mathfrak{m}_K^a \mathcal{O}_0$. Set $\mathcal{O}'_0 := \mathcal{O}_0$. For $i \geq 0$, suppose that we have a finite extension of complete discrete valuation rings $\mathcal{O}'_i/\mathcal{O}'_0$ such that $\mathrm{Frac}(\mathcal{O}'_i)/K$ is separable, and also an isomorphism of $\mathcal{O}_K$-algebras $\mathcal{O}'_i \otimes_{\mathcal{O}_K} A \simeq \mathcal{O}_i \otimes_{\mathcal{O}_K} A$. Choose $\alpha'_i \in \mathcal{O}'_i$ such that the images of α'_i and α_i in these rings correspond via this isomorphism. Note that α'_i is a prime element (resp. unit which is residually non-p-th power) if α_i is so. Then the ring

$$\mathcal{O}'_{i+1} \ := \ \mathcal{O}'_i[Y]/(Y^p + \beta Y - \alpha'_i).$$

is a finite extension of complete discrete valuation rings over $\mathcal{O}'_i$, the extension $\mathrm{Frac}(\mathcal{O}'_{i+1})/K$ is separable and we also have an isomorphism of $\mathcal{O}_K$-algebras $\mathcal{O}'_{i+1} \otimes_{\mathcal{O}_K} A \simeq \mathcal{O}_{i+1} \otimes_{\mathcal{O}_K} A$. Repeating this, we obtain a desired lift of B whose fraction field is separable over K. □

2. The theorem numbers of this section are those of [II]. The purpose of [II] was to show that, for a truncated discrete valuation ring A of length $\geq m$, the category* $\mathcal{FFP}_A^{<m}$ of finite flat principal A-algebras with "ramification bounded by m" can be constructed with no reference to a particular lift of A to a complete discrete valuation ring (in particular, it is independent of such a lift). After [II] was posted in the `arXiv`, however, Takeshi Saito found that there was a counterexample to Proposition 3.7 and that there was a serious error in the proof of Lemma 3.10, which was used in the proof of Proposition 3.7.

The counterexample is as follows: Let $S = k[\![X, Y]\!]$, where k is an algebraically closed field of characteristic $\neq 2$, and let $\mathfrak{p}$ be the height 1 prime ideal $(Y^2 - X)$ of S. Then S is normal, integral and $\mathfrak{p}$-adically complete. Let $\mathbb{B} := S[Z]/(Z^2 - X)$, which is $\mathfrak{p}$-adically complete and flat over S. The residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ can be identified with the power series field $k(\!(Y)\!)$, and we have $\mathbb{B} \otimes_S \kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{p}) \times \kappa(\mathfrak{p})$ (so that $\pi_0(\mathbb{B} \otimes_S \kappa(\mathfrak{p}))$ consisits of two points). On the other hand, the fraction field C of S is $k(\!(X, Y)\!)$ and $\mathbb{B} \otimes_S C = k(\!(Y, Z)\!)$ (so that $\pi_0(\mathbb{B} \otimes_S C)$ consists of one point).

The error in the proof of Lemma 3.10 is that, in applying the Henselian property, we did not (and in fact cannot) check that $s^b \bar{g}(x/s)$ and $s^c \bar{h}(x/s)$ are coprime modulo I.

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*In [I], we used the notation $\mathcal{FFP}_A^{\leq m}$ to denote this category. It was pointed out by M. Yoshida that the strict inequality "$< m$" was more suitable in view of the meaning of the category, and we adopted the notation $\mathcal{FFP}_A^{<m}$ in [II] and [R].

Thus the main “results” of [II], as well as Corollary 1.2 of [R], remain to be a “conjecture”, while Theorem 1.1 of [R] is correct as long as the category $\mathcal{FFP}_A^{<m}$ is defined by using a lift $\mathcal{O}_K \to A$ (Note that Corollary 1.2 follows from Theorem 1.1 only if the category $\mathcal{FFP}_A^{<m}$ is independent of the choice of such a lift.)

A large part of the “conjecture” (in the case where A is of p-torsion) has been proved by Hattori [H] by using the theory of perfectoid spaces.

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