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Some Results of Deformations on Compact H-twisted Generalized Calabi-Yau Manifolds

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Abstract: In this paper, we prove several formulas related to Hodge theory, and using them to prove the deformations of a compact H-twisted generalized Calabi-Yau manifold are unobstructed and L^2 convergence in a fixed neighbourhood in another power series. And if we assume that the deformation is smooth in a fixed neighbourhood, and assume the existence of a global canonical family of deformation, we also construct the global canonical family of the deformations of generalized Kähler manifolds.
Keywords: Deformations of complex structures, Hodge theory, Hermitian and Kählerian manifolds, Calabi-Yau manifolds.

1. Introduction

The generalized complex geometry is introduced by N. Hitchin and developed by M. Gualtieri and G. R. Cavalcanti and many others in [2, 3, 5, 6, 10]. This new geometry provides an indeed broad platform for the people working in both mathematics and physics. The theory of deformations of complex structures can be dated back to Riemann, and extensively studied by K. Kodaira, D. C. Spencer, N. Nirenberg, M. Kuranishi and many other great mathematicians in [11, 15]. The deformation theory of generalized complex geometry is first studied by M. Gualtieri, R. Goto and so on in in [2, 9]. The concept of H-twisted was introduced by P. Ševera and A. Weinstein in [16]. Yi Li in [14] has proved that the deformations of a compact H-twisted Generalized Calabi-Yau manifold are unobstructed in an infinitesimal neighborhood by using Kodaira-Spencer-Kuranishis method.

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In this paper, we re-state the deformations of a compact H-twisted generalized Calabi-Yau manifold are unobstructed and L^2 convergence in a fixed neighbourhood by using Hodge theory. And if we assume that the deformation is smooth in a fixed neighbourhood, we also construct the global canonical family of the deformations by using the parallel method in [12].

This paper is organized as follows. In Section 2 and Section 3, we introduce some basic definitions and some propositions of compact H-twisted generalized Calabi-Yau manifolds which we will use. In Section 4, by using the Hodge theory, we prove the following two propositions which will be used in constructing the deformations.

Lemma 1.1. *On compact generalized Kähler manifold M, For any $\rho \in \wedge^* T_M^*$, we have*

$$\begin{aligned} ||\overline{\partial}_H^* G\rho||^2 &\leq (\rho, G\rho), \\ ||\overline{\partial}_H^* G\partial_H\rho||^2 &\leq ||\rho||^2, \end{aligned}$$

where $\overline{\partial}_H^*$ is the adjoint operator of $\overline{\partial}_H$, G is the Green operator corresponding to the harmonic operator $\triangle_{\overline{\partial}_H}$, $\overline{\partial}_H, \partial_H$ is definite in Definition 2.5 and Definition 2.10, and the norms $||\cdot||$ is definite in Definition 2.7 in Section 2 in details.

Proposition 1.2. *Let (M, J) be a compact H-twisted generalized Kähler manifold. Then for any $\rho \in \wedge^* T_M^*$,*

$$s = \overline{\partial}_H^* G\partial_H\rho$$

is a solution to the equation $\overline{\partial}_H s = \partial_H\rho$ with condition $\overline{\partial}_H\partial_H\rho = 0$, such that

$$||s||^2 \leq (\partial_H\rho, G\partial_H\rho),$$

where $(\cdot,\cdot)$ is the Born-Infeld inner product definite in Definition 2.7. Moreover, if $\mathbb{H}s = 0$ and $\overline{\partial}_H^* s = 0$, s is uniquely determined.

In Section 5, we re-state that the deformations of a compact H-twisted generalized Calabi-Yau manifold are unobstructed and L^2 convergence in a fixed neighbourhood by another power series. In details, we have

Theorem 1.3. *Let (M, J) be a compact H-twisted generalized Calabi-Yau manifold. Then there exists a globally L^2 convergent power series which determines the deformation in $t < \frac{1}{4ac}$,*

$$\epsilon(t)\rho_0 \quad = \quad \sum_{i=1}^{N} \epsilon_i \rho_0 t^i + \sum_{k\geq 2} \sum_{k_1+\cdots+k_N=k, k_i\geq 0} \epsilon_{k_1\cdots k_N} \rho_0 (t^1)^{k_1} \cdots (t^N)^{k_N},$$

which satisfies:
(1) $\overline{\partial}_H \epsilon(t)\rho_0 + \frac{1}{2}[\epsilon(t), \epsilon(t)]_H \rho_0 = 0$;
(2) $\epsilon_{k_1\cdots k_N}\rho_0$ is $\overline{\partial}^*_H$-closed, and ∂_H-exact for any $k_1 + \cdots + k_N = k \geq 2$;
(3) $\epsilon(t)\rho_0$ is L^2 convergence in $t < \frac{1}{4ac}$.

The convergence of the deformations is proved by using the power series in Lemma 3.4 as follows:

Lemma 1.4 (See Lemma 4.1 in [12]). *Let $x_1 = a$ be a constant, and $x_k = c\sum_{i=1}^{k-1} x_i x_{k-i}$, where c is a constant, then $\sum_{i=1}^{\infty} x_i t^i$ converge on $|t| \leq \frac{1}{4ac}$.*

In Section 6, if we assume the existence of a global canonical family of deformation, we also construct the global canonical family of the deformations of generalized Kähler manifolds. We have the theorem as follows:

Proposition 1.5. *On compact generalized Kähler manifold M, if we assume that $\epsilon(t)$ smooth with convergence radius $\frac{1}{4ac}$ exists. Then*

$$\epsilon(t)\rho_0 \in H^2_{\overline{\partial}_H}(M, \wedge^* T^*_M),$$

where $H^2_{\overline{\partial}_H}(M, \wedge^* T^*_M)$ is the cohomology group of the complex $(\wedge^* T^*_M, \overline{\partial}_H)$, and there exists

$$\rho_t \quad := \quad \rho_0 + \sum_{K, |K|\geq 1} \rho_K t^K \in U_0,$$

where $t^K := (t^1)^{k_1} \cdots (t^N)^{k_N}, |K| := k_1 + \cdots + k_N$, such that
(1) $\rho^c_t := e^{-\epsilon(t)}\rho_t$ holomorphic with respect to $J_{\epsilon(t)}$,
(2) ρ_K is ∂_H -exact and $\overline{\partial}^*_H$-closed, ($|K| \geq 1$)
(3) ρ_t converges with radius $\frac{1}{16a^2c^2}$.

And we also give the representations of the global canonical family of the deformations:

Proposition 1.6.

$$\rho_t^c := e^{-\epsilon(t)}\rho_t,$$

and

$$[\rho_t^c] = [\rho_0] + \sum_{i=1}^{N}[\mathbb{H}(-\epsilon_i\rho_0)]t + O(|t|^2),$$

where $[\rho_0]$ *means a representation in* $H^0_{\bar{\partial}_H}(M, \wedge^* T^*_M)$*, and* $O(|t|^2)$ *denotes the terms in*
$H^4_{\bar{\partial}_H}(M, \wedge^* T^*_M) \bigoplus \cdots \bigoplus H^{2n}_{\bar{\partial}_H}(M, \wedge^* T^*_M)$ *of order at least* 2 *in* t*.*

The method we use is parallel to the method in [12].

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2. Some basic definitions

In this section, we review some basic definitions of compact H-twisted generalized Calabi-Yau manifolds. We refer the reader to [1–3, 8, 10] for details.

We define a pair on smooth manifold M similar to the pair between T_M and T^*_M.

Definition 2.1. *Let* M^n *be a smooth manifold with* $dim_{\mathbb{R}} = n$*. We define the following pair on* $(T_M \oplus T^*_M) \otimes_{\mathbb{R}} \mathbb{C}$*:*

$$< A, B > \quad := \quad < X + \xi, Y + \eta > := \frac{1}{2}(\xi(Y) + \eta(X)),$$

where $A := X + \xi, B := Y + \eta \in (T_M \oplus T^*_M) \otimes_{\mathbb{R}} \mathbb{C},\ X, Y \in T_M \otimes_{\mathbb{R}} \mathbb{C},\ \xi, \eta \in T^*_M \otimes_{\mathbb{R}} \mathbb{C}$*.*

We can define the generalized complex structures similar to complex structures on M.

Definition 2.2. *Let* M^n *be a smooth manifold with* $dim_{\mathbb{R}} = n$*. If there exists an endomorphism* J *on* $(T_M \oplus T^*_M) \otimes_{\mathbb{R}} \mathbb{C}$*, satisfying*

$$J^2 = -1,$$

and

$$< A, B >=< JA, JB >,$$

J is called a generalized almost complex structure on M.

Since $J^2 = -1$, we may decompose $(T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}$ into the $\pm i$ -eigenvalue subspaces of J:

$$(T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C} = E + \bar{E},$$

where E be the $+i$-eigenvalue subspace.

If there is an $H \in H^3(M, \mathbb{R})$, we can define H-twisted Courant bracket as follow:

Let $A = X + \xi, B = Y + \eta \in (T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}$, where $X, Y \in T_M \otimes_{\mathbb{R}} \mathbb{C}$, $\xi, \eta \in T_M^* \otimes_{\mathbb{R}} \mathbb{C}$,

$$\begin{aligned} [A, B]_H &:= [X + \xi, Y + \eta]_H \\ &:= [X, Y] + L_X \eta - L_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) + \iota_Y \iota_X H, \end{aligned}$$

where $[X, Y] := XY - YX, L_X \eta := d\iota_X \eta + \iota_X d\eta, \iota_X$ means the contraction by the vector field X.

If J is an almost generalized complex structure, $[E, E]_H \subset E$,

J is integrable, and J is called the generalized complex structure. See Definition 4.36 in [2]. From Proposition 4.5 in [2], we have also known that the generalized complex manifold has even real dimension, so from now on, we assume that $dim_{\mathbb{C}} M = n$. We have also known the the generalized complex manifold has even real dimension, so we assume that $dim_{\mathbb{C}} M = n$ in the following.

This Courant bracket restricts to a Lie bracket on $E^* \cong \bar{E}$ with respect to the pair in Definition 2.1, and this Lie bracket can be extended into a Schouten bracket on $\wedge^* T_M^*$, which we continue to denote by $[\cdot, \cdot]_H$; it is defined by

$$[A, B]_H := \sum_{i,j} [A_i, B_j]_H \wedge A_1 \wedge \cdots \wedge \hat{A}_i \wedge \cdots \wedge A_p \wedge B_1 \wedge \cdots \wedge \hat{B}_j \wedge \cdots \wedge B_q,$$

where $A = A_1 \wedge \cdots \wedge A_p \in \wedge^p E^*$, $B = B_1 \wedge \cdots \wedge B_q \in \wedge^q E^*$, $\hat{A}_i$ means omit A_i.

Specially, if $\epsilon \in \wedge^2 E^*$, $[\epsilon, \epsilon]_H \in \wedge^3 E^*$ by definition.

Next, we define Clifford actions on $\wedge^* T_M^*$.

Definition 2.3. *Let $\alpha \in \wedge^* T_M^*$ and $X + \xi \in (T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}$, where $X \in T_M \otimes_{\mathbb{R}} \mathbb{C}$, $\xi \in T_M^* \otimes_{\mathbb{R}} \mathbb{C}$, we can define Clifford actions on $\wedge^* T_M^*$:*

$$(X + \xi)\alpha := \iota_X \alpha + \xi \wedge \alpha$$

Since J is integrable, we have $< \bar{E}, \bar{E} >= 0$, so

$$AB\alpha + BA\alpha \quad = \quad 0,$$

where $A, B \in E^*$. More generally,

$$AB\alpha \quad = \quad (-1)^{pq} BA\alpha,$$

where $A \in \wedge^p E^*,\ B \in \wedge^q E^*$.

Definition 2.4. *In complex case, we have the Hodge decomposition. In generalized case, we can also define*

$$\begin{aligned} U_0 &:= \{\rho \in \wedge^* T_M^* \mid E\rho = 0\}, \\ U_k &:= \wedge^k E^* \cdot U_0. \end{aligned}$$

One can show that U_0 is a complex line bundle in $\wedge^* T_M^*$. See Definition 20 in Page 34 [4]. we call it the canonical bundle of J. And U_k is the $(n-k)i$-eigenvalue subspace of J.

Thus we can get the decomposition of $\wedge^* T_M^*$:

$$\wedge^* T_M^* = U_0 \oplus U_1 \oplus ... \oplus U_{2n},$$

where $n = dim_{\mathbb{C}} M$. See P13 in [1].

Definition 2.5. *Like the de Rham differential in complex case, we define the twisted de Rham differential d_H on $\wedge^* T_M^*$ by*

$$\begin{aligned} d_H : \wedge^* T_M^* &\rightarrow \wedge^* T_M^*, \\ \alpha &\mapsto d\alpha - H \wedge \alpha \end{aligned}$$

where $H \in H^3(M, \mathbb{R})$.

We can also define the twisted Dolbeault operator ∂_H *and* $\overline{\partial}_H$ *by*

$$\begin{aligned} \partial_H := \pi_{k-1} \circ d_H : U_k &\rightarrow U_{k-1}, \\ \overline{\partial}_H := \pi_{k+1} \circ d_H : U_k &\rightarrow U_{k+1}, \end{aligned}$$

where π_K *is the projection onto* U_k*.*

Then J is integrable if and only if $d_H = \partial_H + \overline{\partial}_H$. See P51 in [2].

Now, we define the compact generalized Kähler manifolds and the compact generalized Calabi-Yau manifold.

Definition 2.6. *Let* (M, J) *be a compact generalized complex manifold, if there exists another generalized complex structure* I*, such that* $IJ = JI$*, then we call* (M, J, I) *a compact* H*-twisted generalized manifold. And also exists a global nowhere zero* $\rho_0 \in U_0$*, satisfying*

$$d_H \rho_0 = 0,$$

we call (M, J, I) *a compact* H*-twisted generalized Calabi-Yau manifold, and* U_0 *is a trivial line bundle.*

We still can define the inner product similar to the complex case which we call it as Born-Infeld inner products.

Definition 2.7. *See P3 in [3]. Let* M *be a compact H-twisted Generalized Kähler manifold, we can define a positive-definite metric on* $(T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}$ *by*

$$\begin{aligned} ((T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}) \otimes ((T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}) &\rightarrow C^\infty(M), \\ A, B &\mapsto < GA, B > \end{aligned}$$

where $GA := -IJA, \quad A \in (T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}$*. And one can show that* $G^2 = 1$*.*

The restriction of this metric to the the sub-bundle T_M *is the Riemannian metric* $g - bg^{-1}b$*, where* g *is a Riemannian metric and* b *is a 2-form. And the volume element induced by this metric is*

$$vol_G = \sqrt{det(g - bg^{-1}b)} = \frac{det(g+b)}{\sqrt{detg}}.$$

Next, we define the $*$ *operator by*

$$* = A_1 A_2 ... A_n,$$

which is a product of an oriented orthonormal basis for C_+*, where* C_+ *is the* $+1$*-eigenvalue subspace of* G *on* $(T_M \oplus T_M^*) \otimes_{\mathbb{R}} \mathbb{C}$*, since* $G^2 = 1$*. And one can also show that* C_+ *is a real space,* $A_i = \bar{A}_i$*. So* $*$ *is a real operator, i.e.* $* = \bar{*}$*.*

Then we can get

$$\begin{aligned} *^2 &:= A_1 A_2 ... A_n A_1 A_2 ... A_n \\ &= (-1)^{n(n-1)/2} A_n A_{n-1} ... A_1 A_1 A_2 ... A_n \\ &= (-1)^{n(n-1)/2}, \end{aligned}$$

where $n = dim_{\mathbb{C}} M$*. The 2nd equality holds since* $A_i A_j = -A_j A_i$ *and, the 3rd equality holds since* A_i *be orthonormal with respect to the pair* $< \cdot, \cdot >$ *definite in Definition 2.1.*

We now define the positive-definite Hermitian inner product on $\Gamma(M, \wedge^* T_M^*)$ *which we call the Born-Infeld inner product by*

$$\begin{aligned} (\alpha, \beta) &:= \int_M < \alpha, \sigma(*)\bar{\beta} >, \\ ||\alpha||^2 &:= (\alpha, \alpha), \end{aligned}$$

where $< \alpha, \sigma(*)\bar{\beta} >$ *is definite by*

$$\begin{aligned} < \alpha, \sigma(*)\bar{\beta} > &= G(\alpha, \beta) < 1, \sigma(*)1 > \\ &= G(\alpha, \beta) vol_G \\ &= g(e^{-b}\alpha, e^{-b}\beta) vol_g, \end{aligned}$$

$G(\alpha, \beta)$ *is a positive-definite metric on* $\wedge^* T_M^*$ *satisfying* $G(1,1) = 1$*,* $\sigma(A_1 ... A_n) := A_n ... A_1 = (-1)^{n(n-1)/2} A_1 ... A_n$*, and* g *is the Riemannian metric* b *is some 2-form definite above.*

Remark 2.8. *If M is a Kähler manifold, we get that $b=0$, $*$ is just the ordinary Hodge $*$-operator and $(\cdot,\cdot)$ is the ordinary positive-definite inner product on differential forms definite in [15].*

Lemma 2.9 ($\partial_H\overline{\partial}_H$-Lemma). *We say (M,J) satisfies the $\partial_H\overline{\partial}_H$-Lemma if*

$$Im(\partial_H)\cap Ker(\overline{\partial}_H)=Im(\overline{\partial}_H)\cap Ker(\partial_H)=Im(\partial_H\overline{\partial}_H).$$

One can show that a compact H-twisted generalized Kähler manifold (M,J)satisfies the $\partial_H\overline{\partial}_H$-Lemma. See Corollary 4.2 in Page 6 [3].

From now on, we only discuss about the compact H-twisted generalized Kähler manifold (M,J) which satisfies the $\partial_H\overline{\partial}_H$-Lemma.

We also have some elliptic operator similar to the complex case.

Definition 2.10. *We now let ∂_H^*, $\overline{\partial}_H^*$, d_H^* be the dual operators of ∂_H, $\overline{\partial}_H$, d_H with respect to the inner product $(\cdot,\cdot)$ respectively, i.e.*

$$\begin{aligned}(\partial_H\alpha,\beta) &= (\alpha,\partial_H^*\beta),\\ (\overline{\partial}_H\alpha,\beta) &= (\alpha,\overline{\partial}_H^*\beta),\\ (d_H\alpha,\beta) &= (\alpha,d_H^*\beta).\end{aligned}$$

And we also define the Laplacian operators by

$$\begin{aligned}\triangle_{d_H} &:= d_Hd_H^*+d_H^*d_H,\\ \triangle_{\partial_H} &:= \partial_H\partial_H^*+\partial_H^*\partial_H,\\ \triangle_{\overline{\partial}_H} &:= \overline{\partial}_H\overline{\partial}_H^*+\overline{\partial}_H^*\overline{\partial}_H.\end{aligned}$$

Then one can show that $\triangle_{d_H}$, $\triangle_{\partial_H}$, $\triangle_{\overline{\partial}_H}$ are self-duality operators with respect to inner product $(\cdot,\cdot)$ and

$$\begin{aligned}\triangle_{d_H} &= 2\triangle_{\partial_H}=2\triangle_{\overline{\partial}_H},\\ id &= \mathbb{H}+G\triangle_{\overline{\partial}_H}=\mathbb{H}+\triangle_{\overline{\partial}_H}G,\end{aligned}$$

where $\mathbb{H}$ is the projection onto $H^*_{\overline{\partial}_H}(M,\wedge^*T_M^*)$, and G is the Green operator corresponding to $\triangle_{\overline{\partial}_H}$. See P6 in [3].

We can also get the Hodge decomposition in twisted cohomology:

$$H^{even/odd}_{d_H}(M,\wedge^*T_M^*)=H^0_{\overline{\partial}_H}(M,\wedge^*T_M^*)\oplus H^2_{\overline{\partial}_H}(M,\wedge^*T_M^*)\oplus \cdots\oplus H^{2n}_{\overline{\partial}_H}(M,\wedge^*T_M^*).$$

3. Some lemmas

In this section, we review some basic propositions of compact H-twisted generalized Calabi-Yau manifolds which will be used in this paper.

Lemma 3.1. *The complex $(\wedge^* T_M^*, \overline{\partial}_H)$ is definite in Definition 2.3. We now introduce the complex $(\wedge^p E^*, d_E)$:*
$d_E : \wedge^k E^* \to \wedge^{k+1} E^*$ is the Lie derivation define by

$$\begin{aligned} d_E A(X_0, ...X_k) \quad := \quad & \sum_i (-1)^i a(X_i) A(X_0, ..., \hat{X}_i, ..., X_k) \\ & + \sum_{i<j} (-1)^{i+j} A([X_i, X_j]_H, X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k), \end{aligned}$$

where $A \in \wedge^k E^*, X_i \in E, a : E \to T_M$ is the projection which is called the anchor.

On compact generalized Kähler manifold M, we have

$$\overline{\partial}_H (A\rho) = (d_E A)\overline{\rho} + (-1)^p A \overline{\partial}_H \rho,$$

where $A \in \wedge^p E^*, \rho \in \wedge^* T_M^*$.

So, on compact generalized Calabi-Yau manifold M, we have the relationship between two complex $(\wedge^* T_M^*, \overline{\partial}_H)$ and $(\wedge^p E^*, d_E)$. See P8 in [14], or [2].

Then we have the two complex are isomorphism:

$$(\wedge^p E^*, d_E) \cong (\wedge^* T_M^*, \overline{\partial}_H),$$

$$(d_E A)\rho_0 = \overline{\partial}_H (A\rho_0),$$

where $A \in \wedge^p E^*,\ \rho_0 \in U_0$ is the global nowhere zero d_H-closed section which is fixed and we call it the pure spinor for J.

Next, we have the following Clifford actions on compact generalized Kähler manifold M. See [13, 14].

Lemma 3.2. *Let $\alpha \in \wedge^* T_M^*$ be a smooth differential form , $A \in \wedge^p E^*,\ B \in \wedge^q E^*$, we have*

$$
\begin{aligned}
d_H(AB\alpha) &= (-1)^p A d_H(B\alpha) + (-1)^{(p-1)q} B d_H(A\alpha) + (-1)^{p-1}[A,B]_H\alpha \\
&\quad +(-1)^{p+q+1} AB d_H\alpha, \\
\partial_H(AB\alpha) &= (-1)^p A \partial_H(B\alpha) + (-1)^{(p-1)q} B \partial_H(A\alpha) + (-1)^{p-1}[A,B]_H\alpha \\
&\quad +(-1)^{p+q+1} AB \partial_H\alpha, \\
\overline{\partial}_H(AB\alpha) &= (-1)^p A \overline{\partial}_H(B\alpha) + (-1)^{(p-1)q} B \overline{\partial}_H(A\alpha) \\
&\quad +(-1)^{p+q+1} AB \overline{\partial}_H\alpha.
\end{aligned}
$$

Further more, if $d_H\rho_0 = \partial_H\rho_0 = \overline{\partial}_H\rho_0 = 0$ *and* $\partial_H(A\rho_0) = \partial_H(B\rho_0) = 0,\ A, B \in \wedge^2 E^*$*, we have*

$$\partial_H(AB\rho_0) = -[A,B]_H\rho_0. \tag{3.1}$$

The following inequivalent we will need to prove the convergence.

Lemma 3.3. *On compact generalized Kähler manifold* M*. Let* $s \geq 2 \in \mathbb{N}$, $\alpha \in \mathbb{R}, 0 < \alpha < 1$ *is fixed. Then there exist positive constants* c_1, c_2, c_3 *which only depend on* s, α, H, *and the manifold* M *itself, such that for any* $A, B \in \wedge^* E^*$,

$$
\begin{aligned}
||[A,B]_H||_{s+\alpha} &\leq c_1||A||_{s+1+\alpha} \cdot ||B||_{s+1+\alpha}, \\
||d_E^* A||_{s+\alpha} &\leq c_2||B||_{s+1+\alpha}, \\
||G_{d_E} A||_{s+\alpha} &\leq c_3||A||_{s-2+\alpha},
\end{aligned}
$$

where G_{d_E} *is the Green operator corresponding to* $\triangle_{d_E}$*, and* $||\cdot||_{s+\alpha}$ *means the Hölder norms definite in [11] and we consider the norms of* $A \in \wedge^p E$ *the same as the norms of* $A\rho_0 \in \wedge^* T_M^*$ *since Lemma 3.1.*

So we have

$$
\begin{aligned}
||d_E^* G_{d_E}[A,B]_H||_{s+\alpha} &\leq c_1c_2c_3||A||_{s+\alpha}||B||_{s+\alpha} \\
&:= (2C)||A||_{s+\alpha}||B||_{s+\alpha}.
\end{aligned}
$$

On compact generalized Calabi-Yau manifold M. If $\partial_H(A\rho_0) = \partial_H(B\rho_0) = 0$, we have

$$(3.2) \qquad ||\overline{\partial}_H^* G\partial_H(AB\rho_0)||_{s+\alpha} \leq (2c)||A\rho_0||_{s+\alpha}||B\rho_0||_{s+\alpha}$$

by using the formula (3.1) in Lemma 3.2.

Proof. (1) Locally, $A|_U = A^I_J \frac{\partial}{\partial x^I} \wedge dx^J$. Then $||A||^U_{s+\alpha} := ||A^I_J||_{s+\alpha}$ just as the definition in Calabi-Yau case.

By the definition of the d_E, it is sufficient to prove the inequality holds for $A = X + \varphi, B = Y + \eta \in E^*$. Since by direct computation, we have

$$\begin{aligned}
\iota_{Y_0} d\eta_1 &= \iota_{Y_0^i \frac{\partial}{\partial x^i}} \frac{\partial \eta_{1j}}{\partial x^p} dx^p \wedge dx^j \\
&= \iota_{Y_0^i \frac{\partial}{\partial x^i}} \frac{1}{2}(\frac{\partial \eta_{1j}}{\partial x^p} - \frac{\partial \eta_{1p}}{\partial x^j}) dx^p \wedge dx^j \\
&= \frac{1}{2}(\frac{\partial \eta_{1j}}{\partial x^p} - \frac{\partial \eta_{1p}}{\partial x^j}) Y_0^p dx^j - \frac{1}{2}(\frac{\partial \eta_{1j}}{\partial x^p} - \frac{\partial \eta_{1p}}{\partial x^j}) Y_0^j dx^p \\
&= \frac{1}{2}(\frac{\partial \eta_{1p}}{\partial x^j} - \frac{\partial \eta_{1j}}{\partial x^p}) Y_0^j dx^p - \frac{1}{2}(\frac{\partial \eta_{1j}}{\partial x^p} - \frac{\partial \eta_{1p}}{\partial x^j}) Y_0^j dx^p \\
&= (\frac{\partial \eta_{1p}}{\partial x^j} - \frac{\partial \eta_{1j}}{\partial x^p}) Y_0^j dx^p,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} d \circ \iota_{Y_0} \eta_1 &= \frac{1}{2} d(Y_0^i \eta_{1i}) \\
&= \frac{1}{2}(\frac{\partial Y_0^i}{\partial x^p} \eta_{1i} + Y_0^i \frac{\partial \eta_{1i}}{\partial x^p}) dx^p,
\end{aligned}$$

$$\begin{aligned}
\iota_{Y_1} \iota_{Y_0} H &= \iota_{Y_1} \iota_{Y_0} (\frac{1}{3!} H_{ijk} dx^i \wedge dx^j \wedge dx^k) \\
&= \iota_{Y_1} (\frac{1}{3!} Y_0^p (H_{pij} - H_{ipj} + H_{ijp}) dx^i \wedge dx^j) \\
&= \iota_{Y_1} (\frac{1}{2!} Y_0^p H_{ijp} dx^i \wedge dx^j) \\
&= \frac{1}{2!} Y_0^p Y_1^q (H_{qjp} - H_{jqp}) dx^j \\
&= Y_0^p Y_1^q H_{qjp} dx^j \\
&= Y_0^p Y_1^q H_{pqj} dx^j,
\end{aligned}$$

$$
\begin{aligned}
[A,B]_H &:= [Y_0+\eta_0, Y_1+\eta_1]_H \\
&:= [Y_0,Y_1]+L_{Y_0}\eta_1-L_{Y_1}\eta_0-\frac{1}{2}d(\iota_{Y_0}\eta_1-\iota_{Y_1}\eta_0)+\iota_{Y_1}\iota_{Y_0}H \\
&= [Y_0^i\frac{\partial}{\partial x^i}, Y_1^j\frac{\partial}{\partial x^j}]+d\circ\iota_{Y_0}\eta_1+\iota_{Y_0}d\eta_1-\frac{1}{2}d\iota_{Y_0}\eta_1 \\
&\quad -d\circ\iota_{Y_1}\eta_0-\iota_{Y_1}d\eta_0+\frac{1}{2}d\iota_{Y_1}\eta_0+\iota_{Y_1}\iota_{Y_0}H \\
&= [Y_0^i\frac{\partial}{\partial x^i}, Y_1^j\frac{\partial}{\partial x^j}]+\frac{1}{2}d\circ\iota_{Y_0}\eta_1+\iota_{Y_0}d\eta_1 \\
&\quad -\frac{1}{2}d\circ\iota_{Y_1}\eta_0-\iota_{Y_1}d\eta_0+\iota_{Y_1}\iota_{Y_0}H \\
&= (Y_0^i\frac{\partial Y_1^j}{\partial x^i}-Y_1^i\frac{\partial Y_0^j}{\partial x^i})\frac{\partial}{\partial x^j}+\frac{1}{2}(\frac{\partial Y_0^i}{\partial x^p}\eta_{1i}+Y_0^i\frac{\partial \eta_{1i}}{\partial x^p} \\
&\quad -\frac{\partial Y_1^i}{\partial x^p}\eta_{10}-Y_1^i\frac{\partial\eta_{0i}}{\partial x^p})dx^p+(\frac{\partial\eta_{1p}}{\partial x^j}Y_0^j-\frac{\partial\eta_{1j}}{\partial x^p}Y_0^j \\
&\quad -\frac{\partial\eta_{0p}}{\partial x^j}Y_1^j+\frac{\partial\eta_{0j}}{\partial x^p}Y_1^j)dx^p+Y_0^iY_1^jH_{ijp}dx^p \\
&= (Y_0^i\frac{\partial Y_1^j}{\partial x^i}-Y_1^i\frac{\partial Y_0^j}{\partial x^i})\frac{\partial}{\partial x^j} \\
&\quad +(\frac{1}{2}(\frac{\partial Y_0^i}{\partial x^p}\eta_{1i}-\frac{\partial Y_1^i}{\partial x^p}\eta_{0i})+(\frac{\partial\eta_{1p}}{\partial x^j}Y_{0j}-\frac{\partial\eta_{0p}}{\partial x^j}Y_{1j}) \\
&\quad +\frac{1}{2}(Y_1^i\frac{\partial\eta_{0i}}{\partial x^p}-Y_0^i\frac{\partial\eta_{1i}}{\partial x^p})+Y_0^iY_1^jH_{ijp})dx^p,
\end{aligned}
$$

where x^i can be z^i or $\overline{z^i}$, $H=\frac{1}{3!}H_{ijk}dx^i\wedge dx^j\wedge dx^k$, H_{ijk} is skew-symmetric with respect to i,j,k.

We also have known that H is fixed, M is compact, we can get some proper larger constant c_1 such that $c_1\geq max\{H_{ijk},\frac{\partial^\gamma H_{ijk}}{\partial x^\gamma}:|\gamma|\leq k\}$. So by the definition of Hölder norms, we get the 1st inequality.

(2) By the definition of the d_E, it is sufficient to prove the inequality holds for $A = X + \varphi \in E^*$. Since by direct computation, we have

$$
\begin{aligned}
& (d_E(X+\varphi))(Y_0+\eta_0, Y_1+\eta_1) \\
:= \;& (d_E(X^i \frac{\partial}{\partial x^i} + \varphi_j dx^j))(Y_0^k \frac{\partial}{\partial x^k} + \eta_{0l} dx^l, Y_1^k \frac{\partial}{\partial x^k} + \eta_{1l} dx^l) \\
:= \;& Y_0 < X+\varphi, Y_1+\eta_1 > - Y_1 < X+\varphi, Y_0+\eta_0 > \\
& - < X+\varphi, [Y_0+\eta_0, Y_1+\eta_1]_H > \\
= \;& \frac{1}{2} Y_0^i \frac{\partial}{\partial x^i}(\varphi_j Y_1^j + X^k \eta_{1k}) - \frac{1}{2} Y_1^i \frac{\partial}{\partial x^i}(\varphi_j Y_0^j + X^k \eta_{0k}) \\
& < X^i \frac{\partial}{\partial x^i} + \varphi_j dx^j, (Y_0^i \frac{\partial Y_1^j}{\partial x^i} - Y_1^i \frac{\partial Y_0^j}{\partial x^i}) \frac{\partial}{\partial x^j} \\
& + (\frac{1}{2}(\frac{\partial Y_0^i}{\partial x^p} \eta_{1i} - \frac{\partial Y_1^i}{\partial x^p} \eta_{0i}) + (\frac{\partial \eta_{1p}}{\partial x^j} Y_{0j} - \frac{\partial \eta_{0p}}{\partial x^j} Y_{1j}) \\
& + \frac{1}{2}(Y_1^i \frac{\partial \eta_{0i}}{\partial x^p} - Y_0^i \frac{\partial \eta_{1i}}{\partial x^p}) + Y_0^i Y_1^j H_{ijp}) dx^p, > \\
= \;& \frac{1}{2}(Y_0^i Y_1^j \frac{\partial \varphi_j}{\partial x^i} - Y_1^i Y_0^j \frac{\partial \varphi_j}{\partial x^i} + Y_0^i \varphi_j \frac{\partial Y_1^j}{\partial x^i} - Y_1^i \varphi_j \frac{\partial Y_0^j}{\partial x^i} \\
& + Y_0^i \eta_{1k} \frac{\partial X^k}{\partial x^i} - Y_1^i \eta_{0k} \frac{\partial X^k}{\partial x^i} + X^k Y_0^i \frac{\partial \eta_{1k}}{\partial x^i} - X^k Y_1^i \frac{\partial \eta_{0k}}{\partial x^i}) \\
& + \frac{1}{2}(Y_1^i \varphi_j \frac{\partial Y_0^j}{\partial x^i} - Y_0^i \varphi_j \frac{\partial Y_1^j}{\partial x^i}) \\
& + \frac{1}{2}(X^p Y_1^j \frac{\partial \eta_{0p}}{\partial x^j} - X^p Y_0^j \frac{\partial \eta_{1p}}{\partial x^j}) + \frac{1}{4}(X^p \eta_{0i} \frac{\partial Y_1^i}{\partial x^p} - X^p \eta_{1i} \frac{\partial Y_0^i}{\partial x^p}) \\
& + \frac{1}{4}(X^p Y_0^i \frac{\partial \eta_{1i}}{\partial x^p} - X^p Y_1^i \frac{\partial \eta_{0i}}{\partial x^p}) - \frac{1}{2} X^p Y_0^i Y_1^j H_{ijp} \\
= \;& \frac{1}{2} Y_0^i Y_1^j (\frac{\partial \varphi_j}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^j}) + \frac{1}{2}(Y_0^i \eta_{1k} \frac{\partial X^k}{\partial x^i} - Y_1^i \eta_{0k} \frac{\partial X^k}{\partial x^i}) \\
& + \frac{1}{4} X^p (\eta_{0i} \frac{\partial Y_1^i}{\partial x^p} - \eta_{1i} \frac{\partial Y_0^i}{\partial x^p}) + \frac{1}{4} X^p (Y_0^i \frac{\partial \eta_{1i}}{\partial x^p} - Y_1^i \frac{\partial \eta_{0i}}{\partial x^p}) \\
& - \frac{1}{2} X^p Y_0^i Y_1^j H_{ijp},
\end{aligned}
$$

where $Y_0 + \eta_0, Y_1 + \eta_1 \in E$.

So we can get that

$$\begin{aligned} d_E(X+\varphi) &:= \frac{1}{2}((\frac{\partial\varphi_j}{\partial x^i}-\frac{1}{2}X^pH_{ijp})dx^i\wedge dx^j+\frac{1}{2}\frac{\partial X^k}{\partial x^i}dx^i\wedge\frac{\partial}{\partial x^k} \\ &\quad +\frac{1}{2}\frac{\partial}{\partial x^i}\wedge X^p\frac{\partial}{\partial x^p}dx^i+\frac{1}{2}dx^i\wedge X^p\frac{\partial}{\partial x^p}\frac{\partial}{\partial x^i}) \\ &:= \frac{1}{2}((\frac{\partial\varphi_j}{\partial x^i}-\frac{1}{2}X^pH_{ijp})dx^i\wedge dx^j+dX^k\wedge\frac{\partial}{\partial x^k} \\ &\quad \frac{1}{2}\frac{\partial}{\partial x^i}\wedge Xdx^i+\frac{1}{2}dx^i\wedge X\frac{\partial}{\partial x^i}) \end{aligned}$$

Thus d_E is an operator of order 1.

In the above computation, we just assume that $A=X+\varphi\in E^*\subset T_M\oplus T_M^*$, and we didn't use the condition $[E,E]_H\subset E$. There is another way to show d_E is an operator of order 1. If we choose a basis $E:\{e_i=Y_i+\eta_i\}_{i=1}^{2n}, E^*:\{e^i=X^i+\varphi^i\}_{i=1}^{2n}, [e_j,e_k]_H:=c_{jk}^pe_p, c_{jk}^p\in C^\infty(U)$. Also we let $f_i\in C^\infty(U)$.

Then we have

$$\begin{aligned} (d_E(f_ie^i))(e_j,e_k) &:= a(e_j)<f_ie^i,e_k>-a(e_k)<f_ie^i,e_j> \\ &\quad -<f_ie^i,[e_j,e_k]_H> \\ &= Y_j(f_k)-Y_k(f_j)-<f_ie^i,c_{jk}^pe_p> \\ &= Y_j^p\frac{\partial f_k}{\partial x^p}-Y_k^p\frac{\partial f_j}{\partial x^p}-f_pc_{jk}^p. \end{aligned}$$

Thus, we get

$$d_E(f_ie^i) = (Y_j^p\frac{\partial f_k}{\partial x^p}-Y_k^p\frac{\partial f_j}{\partial x^p}-f_pc_{jk}^p)e^j\wedge e^k.$$

Thus d_E is an operator of order 1 since the coefficients only has f, df's terms and the change of the basis not change the order of d_E.

We have also known that

$$\begin{aligned} d_E^* &:= *d_E*^{-1} \\ &:= A_1\cdots A_n\circ d_E\circ A_n\cdots A_1 \\ &= (X_1+g(X_1)+b(X_1))\cdots(X_n+g(X_n)+b(X_n)) \\ &\quad \circ d_E\circ(X_n+g(X_n)+b(X_n))\cdots(X_1+g(X_1)+b(X_1)). \end{aligned}$$

The 3rd equality holds since every element in C_+ can be represented as $X + g(X) + b(X)$ for some $X \in T_M \times_{\mathbb{R}} \mathbb{C}$, where g, b is definite in Definition 2.7. See [8].

So we can get that d_E^* is also an operator of order 1. Since every X_i, g, b is fixed, only depend on the manifold M itself, and by the definition of the Hölder norms, we can get the 2nd inequality.

(3) Since d_E and d_E^* are both operators of order 1, we get that $\triangle_{d_E} := d_E d_E^* + d_E^* d_E$ is an operator of order 2 just as the harmonic operator in Calabi-Yau case.

Since the Green operator G_{d_E} is a strongly elliptic operator just as the Green operator in Calabi-Yau case, we can use the same proof as Lemma 5.7 at P276 and Theorem 4.3 in P436 in [11] just replace the vector bundle $\wedge^* T_M^*$ by $\wedge^* T_M \oplus \wedge^* T_M^*$ and replace the local representation $A = \frac{1}{3!} A_{ijk} dx^i \wedge dx^j \wedge dx^k$ by $A = \frac{1}{3!}(A_{ijk} dx^i \wedge dx^j \wedge dx^k + A^i_{jk} \frac{\partial}{\partial x^i} \wedge dx^j \wedge dx^k + A^{ij}_k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge dx^k + A^{ijk} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}$ to get the 3rd inequality.

□

The following convergent power series we will use to prove the convergence.

Lemma 3.4 (See Lemma 4.1 in [12]). *Let $x_1 = a$ be a constant, and $x_k = c\sum_{i=1}^{k-1} x_i x_{k-i}$, where c is a constant, then $\sum_{i=1}^{\infty} x_i t^i$ converge on $|t| \leq \frac{1}{4ac}$.*

4. A generalized version of $\partial_H \overline{\partial}_H$-lemma

In this section, we will prove a generalized version of $\partial_H \overline{\partial}_H$-lemma on a compact H-twisted generalized Kähler manifold which is similar to the result in Kähler case in [12]. This lemma will be used in the next section to construct the deformations.

Lemma 4.1. *For any $\rho \in \wedge^* T_M^*$,*

$$\begin{aligned} ||\overline{\partial}_H^* G\rho||^2 &\leq (\rho, G\rho), \\ ||\overline{\partial}_H^* G \partial_H \rho||^2 &\leq ||\rho||^2. \end{aligned}$$

Proof. (1)

$$\begin{aligned}
||\overline{\partial}_H^* G\rho||^2 &:= (\overline{\partial}_H^* G\rho, \overline{\partial}_H^* G\rho) \\
&= (\overline{\partial}_H \overline{\partial}_H^* G\rho, G\rho) \\
&= (\rho - \mathbb{H}\rho - \overline{\partial}_H^* \overline{\partial}_H G\rho, G\rho) \\
&= (\rho, G\rho) - (\overline{\partial}_H^* \overline{\partial}_H G\rho, G\rho) \\
&= (\rho, G\rho) - (\overline{\partial}_H G\rho, \overline{\partial}_H G\rho) \\
&\leq (\rho, G\rho)
\end{aligned}$$

The 4th equality holds since $\mathbb{H}\rho \in H^*_{\overline{\partial}_H}(M, S)) \perp G\rho \in Im\triangle_{\overline{\partial}_H}$ with respect to the inner product $(\cdot, \cdot)$ which is definite in Definition 2.7. And the equality holds if and only if $\overline{\partial}_H G\rho = 0$.

(2)

$$\begin{aligned}
||\overline{\partial}_H^* G\partial_H\rho||^2 &:= (\overline{\partial}_H^* G\partial\rho, \overline{\partial}_H^* G\partial_H\rho) \\
&= (\overline{\partial}_H \overline{\partial}_H^* G\partial_H\rho, G\partial_H\rho) \\
&= (\triangle_{\overline{\partial}_H} G\partial_H\rho, G\partial_H\rho) - (\overline{\partial}_H^* G\overline{\partial}_H\partial\rho, G\partial_H\rho) \\
&= (\rho, \partial_H^* G\partial_H\rho) - (G\overline{\partial}_H\partial_H\rho, G\overline{\partial}_H\partial_H\rho) \\
&= (\rho, \triangle_{\overline{\partial}_H} G\rho) - (\rho, \partial_H\partial_H^* G\rho) - ||G\overline{\partial}_H\partial_H\rho||^2 \\
&= (\rho, \rho - \mathbb{H}\rho - \partial_H\partial_H^* G\rho) - ||G\overline{\partial}_H\partial_H\rho||^2 \\
&= ||\rho||^2 - ||\mathbb{H}\rho||^2 - (\partial_H^*\rho, G\partial_H^*\rho) - ||G\overline{\partial}_H\partial_H\rho||^2 \\
&\leq ||\rho||^2
\end{aligned}$$

The 3rd and 4th equality holds since $\triangle_{\partial_H} = \triangle_{\overline{\partial}_H}$, and $\overline{\partial}_H G = G\overline{\partial}_H$ which we give a simple proof: let $G\overline{\partial}_H\rho := \beta \in Im\Delta_{\overline{\partial}_H}, G\rho := \alpha \in Im\Delta_{\overline{\partial}_H}$, then $\Delta_{\overline{\partial}_H}\alpha = \rho - \mathbb{H}\rho, \overline{\partial}_H\rho = \Delta_{\overline{\partial}_H}\beta$. since $\overline{\partial}_H^*\overline{\partial}_H^2\alpha + \overline{\partial}_H\overline{\partial}_H^*\overline{\partial}_H\alpha = \overline{\partial}_H\overline{\partial}_H^*\overline{\partial}_H\alpha + \overline{\partial}_H^2\overline{\partial}_H^*\alpha$, we have that $\Delta_{\overline{\partial}_H}\overline{\partial}_H\alpha = \overline{\partial}_H\Delta_{\overline{\partial}_H}\alpha = \overline{\partial}_H\rho - \overline{\partial}_H(\mathbb{H}\rho) = \overline{\partial}_H\rho$. So $G\overline{\partial}_H\rho = G\Delta_{\overline{\partial}_H}\overline{\partial}_H\alpha = \overline{\partial}_H\alpha = \overline{\partial}_H G\rho$.(Or see P147, theorem 5.2(d) in [21] and the elliptic proposition see Proposition 6 in Page 35 [4].). The last inequality holds since $G_{\partial_H}|_{Im\triangle_{\partial_H}} = \triangle_{\partial_H}^{-1}$, we assume $G_{\partial_H}\partial_H^*\rho := \alpha$, then $(\partial_H^*\rho, G_{\partial_H}\partial_H^*\rho) = (\triangle_{\partial_H}\alpha, \alpha) \geq 0$, see more details in [7]. And the equality holds if and only if $\rho \in Im\triangle_{\overline{\partial}_H}$, $\overline{\partial}_H\partial_H\rho = 0$, and $\partial_H^*\rho = 0$.

□

Proposition 4.2. *Let (M, J) be a compact H-twisted Generalized Calabi-Yau manifold, then for any $\rho \in \wedge^* T_M^*$,*

$$s = \overline{\partial}_H^* G \partial_H \rho$$

is a solution to the equation $\overline{\partial}_H s = \partial_H \rho$ with condition $\overline{\partial}_H \partial_H \rho = 0$, such that

$$||s||^2 \leq (\partial_H \rho, G \partial_H \rho).$$

Moreover, if $\mathbb{H}s = 0$, and $\overline{\partial}_H^* s = 0$, s is uniquely determined.

Proof.

$$\begin{aligned}
\overline{\partial}_H s &= \overline{\partial}_H \overline{\partial}_H^* G \partial_H \rho \\
&= \partial_H \rho - \mathbb{H} \partial_H \rho - \overline{\partial}_H^* \overline{\partial}_H G \partial_H \rho \\
&= \partial_H \rho - \overline{\partial}_H^* \overline{\partial}_H G \partial_H \rho \\
&= \partial_H \rho
\end{aligned}$$

The 3rd equality holds since $\partial_H \rho \in Im \partial_H \perp H^*_{\overline{\partial}_H}(M, \wedge^* T_M^*)$; the last equality holds since the assumption $\overline{\partial}_H \partial_H \rho = 0$. So s is a solution and

$$||s||^2 \leq (\partial_H \rho, G \partial_H \rho)$$

holds by the Lemma 4.1 above.

We now prove the uniqueness of the solution under the assumptions given.
If there exists another solution s', we have $\overline{\partial}_H (s - s') = 0$. Also we have $\mathbb{H}(s - s') = 0$, $\overline{\partial}_H^* (s - s') = 0$ by assumption, then $s - s' = 0$. □

5. An application on a compact *H*-twisted generalized Calabi-Yau manifold

In [14], it has been proved that a sufficiently small deformations of a compact H-twisted generalized Calabi-Yau manifold are unobstructed. In this section, we shall re-state that the deformation can be L^2 convergence in a fixed neighbourhood by using another convergent power series which is parallel to that in [12].

Theorem 5.1. *Let (M, J) be a compact H-twisted generalized Calabi-Yau manifold, then there exists a globally L^2 convergent power series which determines the deformations in $t < \frac{1}{4ac}$,*

$$\epsilon(t)\rho_0 = \sum_{i=1}^{N} \epsilon_i \rho_0 t^i + \sum_{k\geq 2} \sum_{k_1+\cdots+k_N=k, k_i\geq 0} \epsilon_{k_1\cdots k_N}\rho_0 (t^1)^{k_1}\cdots(t^N)^{k_N},$$

which satisfies:
(1)$\overline{\partial}_H \epsilon(t)\rho_0 + \frac{1}{2}[\epsilon(t), \epsilon(t)]_H \rho_0 = 0$;
(2)$\epsilon_{k_1\cdots k_N}\rho_0$ is $\overline{\partial}_H^*$-closed, and ∂_H-exact for any $k_1 + \cdots + k_N = k \geq 2$;
(3)$\epsilon(t)\rho_0$ is L^2 convergence in $t < \frac{1}{4ac}$.

Proof. In step 1, we recall the construction of the deformation $\epsilon(t)$, in step 2, we prove that $\epsilon(t)$ is L^2 convergence in $t < \frac{1}{4ac}$.

Step 1:

The construction of a power series solution $\epsilon(t)$ to the integrable condition has been given in [14], here we just recall it in order to make the proof more clearly.

The integrable condition which is called Maurer-Cartan equation is:

$$d_E\epsilon(t) + \frac{1}{2}[\epsilon(t), \epsilon(t)]_H = 0.$$

By Lemma 3.1, we can rewrite it:

$$\overline{\partial}_H(\epsilon(t)\rho_0) + \frac{1}{2}[\epsilon(t), \epsilon(t)]_H\rho_0 = 0,$$

where $\rho_0 \in U_0$ is the canonical bundle.

Choose a basis of $H^2_{d_E}(M, \wedge^* E^*) : \{\epsilon_i\}_{i=1}^N$. We now recall the construction of a power series solution $\epsilon(t)$ to the integrable condition in [14]. We write $\epsilon(t)$ in the form of

$$\epsilon(t) := \epsilon_i t^i + \epsilon_{ij} t^i t^j + \cdots + \epsilon_{i_1\cdots i_N}(t^1)^{i_1}\cdots(t^N)^{i_N} + \cdots,$$

such that $\epsilon_{i_1\cdots i_k}\rho_0$ are $\overline{\partial}_H^*$-closed and ∂_H-exact for all $k \geq 2$.

For simplicity the notation, let us assume that $dim_{\mathbb{C}} H^2_{d_E}(M, \wedge^* E^*) = 1$ and a basis of $H^2_{d_E}(M, \wedge^* E^*) : \{\epsilon_1\}$. So we shall discuss $\epsilon(t)$ in the form of

$$\epsilon(t) := \epsilon_1 t + \epsilon_2 t^2 + \cdots ,$$

such that $\epsilon_k \rho_0$ be $\overline{\partial}^*_H$-closed and ∂_H-exact for all $k \geq 2$.
To compare the coefficients of t in the Maurer-Cartan equation, we get

$$\begin{aligned} k = 1, \overline{\partial}_H(\epsilon_1 \rho_0) &= 0, \\ k = 2, \overline{\partial}_H(\epsilon_2 \rho_0) &= -\frac{1}{2}[\epsilon_1, \epsilon_1]_H \rho_0, \\ &\cdots \\ k, \overline{\partial}_H(\epsilon_k \rho_0) &= -\frac{1}{2}\sum_{i=1}^{k-1}[\epsilon_i, \epsilon_{k-i}]_H \rho_0. \end{aligned}$$

Let

$$\psi_k := -\frac{1}{2}\sum_{i=1}^{k-1}[\epsilon_i, \epsilon_{k-i}]_H \rho_0.$$

We show that $\overline{\partial}_H \psi_k = 0$ by induction on k.
For $k = 2$, we have

$$\begin{aligned} \overline{\partial}_H \psi_2 &:= \overline{\partial}_H(-\frac{1}{2}[\epsilon_1, \epsilon_1]_H \rho_0) \\ &= -\frac{1}{2}(d_E[\epsilon_1, \epsilon_1]_H)\rho_0 \\ &= -\frac{1}{2}([d_E\epsilon_1, \epsilon_1]_H - [\epsilon_1, d_E\epsilon_1]_H)\rho_0 \\ &= 0 \end{aligned}$$

The last equality holds since $\epsilon_1 \in H^2_{d_E}(M, \wedge^* E^*)$, hence $d_E \epsilon_1 = 0$.

If we suppose that i from 2 to $k-1$, $\overline{\partial}_H\psi_i = 0$ holds. For $i = k$, we have

$$\begin{aligned}
& -2\overline{\partial}_H\psi_k \\
:= \;& \overline{\partial}_H(\sum_{i=1}^{k-1}[\epsilon_i, \epsilon_{k-i}]_H\rho_0) \\
= \;& \sum_{i=1}^{k-1}(d_E[\epsilon_i, \epsilon_{k-i}]_H)\rho_0 \\
= \;& \sum_{i=1}^{k-1}([d_E\epsilon_i, \epsilon_{k-i}]_H - [\epsilon_i, d_E\epsilon_{k-i}]_H)\rho_0 \\
= \;& -\frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}([[\epsilon_j, \epsilon_{i-j}]_H, \epsilon_{k-i}]_H)\rho_0 + \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{i-j-1}([\epsilon_i, [\epsilon_j, \epsilon_{n-i-j}]_H)\rho_0 \\
= \;& \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}([\epsilon_{k-i}, [\epsilon_j, \epsilon_{i-j}]_H]_H)\rho_0 + \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{i-j-1}([\epsilon_i, [\epsilon_j, \epsilon_{n-i-j}]_H)\rho_0 \\
= \;& 2\sum_{i+j+h=k, 1\leq i<j<h}([\epsilon_i, [\epsilon_j, \epsilon_h]_H]_H + [\epsilon_j, [\epsilon_h, \epsilon_i]_H]_H + [\epsilon_h, [\epsilon_i, \epsilon_j]_H]_H)\rho_0 \\
& + \sum_{2i+j=k, 1\leq i\neq j}(2[\epsilon_i, [\epsilon_i, \epsilon_j]_H]_H + [\epsilon_j, [\epsilon_i, \epsilon_i]_H]_H)\rho_0 \\
& + \sum_{3i=k, 1\leq i}[\epsilon_i, [\epsilon_i, \epsilon_i]_H]_H\rho_0 \\
= \;& 0
\end{aligned}$$

The 2nd equality holds by the definition of $[\cdot,\cdot]_H$ and d_E; the 3rd equality is the induction; and the last equality holds since the Jacobian identity. See P21 in [2].

Next, we show that ψ_k is ∂_H-exact by induction on k, and there exists an unique $\epsilon_k\rho_0$, such that $\epsilon_k\rho_0$ is ∂_H-exact, $\overline{\partial}_H^*$-closed for any $k \geq 2$.

Since $\epsilon_1 \in H^2_{d_E}(M, \wedge^* E^*)$, in Calabi-Yau case, we have $\epsilon_1\rho_0 \in H^2_{\overline{\partial}_H}(M, \wedge^* T^*_M)$. Hence $\triangle_{\overline{\partial}_H}(\epsilon_1\rho_0) = \triangle_{\partial_H}(\epsilon_1\rho_0) = 0$ since $\triangle_{\partial_H} = \triangle_{\overline{\partial}_H}$. So we get $\partial_H(\epsilon_1\rho_0) = 0$(We need this since it is the condition of the formula (3.1) in Lemma 3.2.).

Therefore, for $k = 2$, by using the formula (3.1) in Lemma 3.2, we have

$$\begin{aligned}
\psi_2 \;&:=\; -\frac{1}{2}[\epsilon_1, \epsilon_1]_H \\
&=\; \frac{1}{2}\partial_H(\epsilon_1\epsilon_1\rho_0),
\end{aligned}$$

i.e. ψ_2 is ∂_H-exact.

Since

$$\begin{aligned}\frac{1}{2}\overline{\partial}_H\partial_H(\epsilon_1\epsilon_1\rho_0) &:= \overline{\partial}_H\psi_2\\ &= 0,\end{aligned}$$

by Proposition 4.2, we can get that the equation

$$\overline{\partial}_H\epsilon_2\rho_0 = \frac{1}{2}\partial_H\epsilon_1\epsilon_1\rho_0$$

has the unique solution

$$\epsilon_2\rho_0 = \frac{1}{2}\overline{\partial}_H^*G\partial_H(\epsilon_1\epsilon_1\rho_0).$$

which satisfying that $\epsilon_2\rho_0$ is $\overline{\partial}_H^*$-closed and ∂_H-exact thus ∂_H-closed.

If we suppose that for $i = 2$ to $k-1$, $\epsilon_i\rho_0$ exists uniquely, such that $\epsilon_i\rho_0$ is ∂_H-exact, $\overline{\partial}_H^*$-closed hold. For $i = k$, we have

$$\psi_k := -\frac{1}{2}\sum_{i=1}^{k-1}[\epsilon_i, \epsilon_{k-i}]_H\rho_0 \quad = \quad \frac{1}{2}\partial_H(\sum_{i=1}^{k-1}\epsilon_i\epsilon_{k-i}\rho_0)$$

is also ∂_H-exact.

Since

$$\begin{aligned}\frac{1}{2}\overline{\partial}_H\partial_H(\sum_{i=1}^{k-1}\epsilon_i\epsilon_{k-i}\rho_0) &:= \overline{\partial}_H\psi_k\\ &= 0,\end{aligned}$$

by Proposition 4.2, we can get that the equation

$$\overline{\partial}_H\epsilon_k\rho_0 = \frac{1}{2}\partial_H\sum_{i=1}^{k-1}\epsilon_i\epsilon_{k-i}\rho_0$$

has the solution

$$\epsilon_k\rho_0 = \frac{1}{2}\overline{\partial}_H^*G\partial_H(\sum_{i=1}^{k-1}\epsilon_i\epsilon_{k-i}\rho_0).$$

(Since $U_i := \wedge^i E^*\cdot U_0$ and U_0 is a complex line bundle whose generator is ρ_0, any $\alpha\in U_i$ can be represented as $A\rho_0$, where $A\in\wedge^i E^*$.)

Hence by Proposition 4.2, we can get that $\mathbb{H}(\epsilon_k\rho_0)=0$ and $\epsilon_k\rho$ is ∂_H-exact, $\overline{\partial}_H^*$-closed, and

$$||\epsilon_k\rho_0||^2 \leq \frac{1}{4}(\partial_H(\sum_{i=1}^{k-1}\epsilon_i\epsilon_{k-i}\rho), G\partial_H(\sum_{i=1}^{k-1}\epsilon_i\epsilon_{k-i}\rho_0)).$$

For the general case of $t=(t^1,\cdots,t^N)$, we can also give the same discussion.

Comparing the coefficients of $t=(t^1,\cdots,t^N)$ in the Maurer-Cartan equation, we get

$$\begin{aligned}
k=1, \overline{\partial}_H(\epsilon_i\rho_0) &= 0,\\
k=2, \overline{\partial}_H(\epsilon_{ij}\rho_0) &= -\frac{1}{2}([\epsilon_i,\epsilon_j]_H+[\epsilon_j,\epsilon_i]_H)\rho_0,\\
&\cdots\\
k, \overline{\partial}_H(\epsilon_{i_1\cdots i_N}\rho_0) &= -\frac{1}{2}\sum_{j_\alpha+h_\alpha=i_\alpha, j_\alpha,h_\alpha\geq 0, \alpha=1\cdots N}[\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H\rho_0.\\
&\quad (i_1+\cdots+i_N=k)
\end{aligned}$$

Let

$$\psi_{i_1\cdots i_N} := -\frac{1}{2}\sum_{j_\alpha+h_\alpha=i_\alpha, j_\alpha,h_\alpha\geq 0, \alpha=1\cdots N}[\epsilon_{j_1\ldots j_N},\epsilon_{h_1\cdots h_N}]_H\rho_0. \quad (i_1+\cdots+i_N=k)$$

We show that $\overline{\partial}_H\psi_{i_1\ldots i_N}=0$ by induction on $i_1+\ldots+i_N=k$.
For $k=2$, we have

$$\begin{aligned}
\overline{\partial}_H\psi_{ij} &:= \overline{\partial}_H(-\frac{1}{2}([\epsilon_i,\epsilon_j]_H+[\epsilon_j,\epsilon_i]_H)\rho_0)\\
&= -\frac{1}{2}(d_E([\epsilon_i,\epsilon_j]_H+[\epsilon_j,\epsilon_i]_H))\rho_0\\
&= -\frac{1}{2}([d_E\epsilon_i,\epsilon_j]_H-[\epsilon_i,d_E\epsilon_j]_H)+[d_E\epsilon_j,\epsilon_i]_H\\
&\quad -[\epsilon_j,d_E\epsilon_i]_H)\rho_0\\
&= 0.
\end{aligned}$$

The last equality holds since $\epsilon_i\in H^2(M,\wedge^*E^*)$, hence $d_E\epsilon_i=0$.

If we suppose that from $i_1+\cdots+i_N=2$ to $k-1$, $\overline{\partial}_H\psi_{i_1\cdots i_N}=0$ holds. For $i_1+\cdots+i_N=k$, we have

$$\begin{aligned}
& -2\overline{\partial}_H\psi_{i_1\cdots i_N}\\
:= & \overline{\partial}_H(\sum_{j_\alpha+h_\alpha=i_\alpha, j_\alpha,h_\alpha\geq 0,\alpha=1\cdots N}[\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H\rho_0)\\
= & \sum_{j_\alpha+h_\alpha=i_\alpha, j_\alpha,h_\alpha\geq 0,\alpha=1\cdots N}(d_E([\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H))\rho_0\\
= & \sum_{j_\alpha+h_\alpha=i_\alpha, j_\alpha,h_\alpha\geq 0,\alpha=1\cdots N}([d_E\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H)-[\epsilon_{j_1\cdots j_N},d_E\epsilon_{h_1\cdots h_N}]_H))\rho_0\\
= & 2\sum_{j_\alpha+h_\alpha+p_\alpha=i_\alpha}([\epsilon_{j_1\cdots j_N},[\epsilon_{h_1\cdots h_N},\epsilon_{p_1\cdots p_N}]_H]_H+\\
& [\epsilon_{h_1\cdots h_N},[\epsilon_{p_1\cdots p_N},\epsilon_{j_1\cdots j_N}]_H]_H+[\epsilon_{p_1\cdots p_N},[\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H]_H)\rho_0\\
& +\sum_{2j_\alpha+h_\alpha=i_\alpha}(2[\epsilon_{j_1\cdots j_N},[\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H]_H\\
& +[\epsilon_{h_1\cdots h_N},[\epsilon_{j_1\cdots j_N},\epsilon_{j_1\cdots j_N}]_H]_H)\rho_0+\sum_{3j_\alpha=i_\alpha}[\epsilon_{j_1\cdots j_N},[\epsilon_{j_1\cdots j_N},\epsilon_{j_1\cdots j_N}]_H]_H\rho_0\\
= & 0
\end{aligned}$$

The 2nd equality holds by the definition of $[\cdot,\cdot]_H$ and d_E; the 3rd equality is the induction; and the last equality holds since the Jacobian identity.

Next, we show that $\psi_{i_1\cdots i_N}$ is ∂_H-exact by induction on $i_1+\cdots+i_N=k$, and there exists an unique $\epsilon_{i_1\ldots i_N}\rho_0$, such that $\epsilon_{i_1\ldots i_N}\rho_0$ is ∂_H-exact, $\overline{\partial}_H^*$-closed for any $i_1+\cdots+i_N=k\geq 2$.

Since $\epsilon_i\in H^2_{d_E}(M,\wedge^*E^*)$, in Calabi-Yau case, we have $\epsilon_i\rho_0\in H^2_{\overline{\partial}_H}(M,\wedge^*T^*_M)$. Hence $\triangle_{\overline{\partial}_H}(\epsilon_i\rho_0)=\triangle_{\partial_H}(\epsilon_i\rho_0)=0$ since $\triangle_{\partial_H}=\triangle_{\overline{\partial}_H}$. So we get $\partial_H(\epsilon_i\rho_0)=0$.

Therefore, for $k=2$, by using the formula (3.1) in Lemma 3.2, we have

$$\begin{aligned}
\psi_{ij} &:= -\frac{1}{2}([\epsilon_i,\epsilon_j]_H+[\epsilon_i,\epsilon_j]_H)\\
&= \frac{1}{2}\partial_H((\epsilon_i\epsilon_j+\epsilon_j\epsilon_i)\rho_0),
\end{aligned}$$

i.e. ψ_{ij} is ∂_H-exact.

Since

$$\begin{aligned}
\frac{1}{2}\overline{\partial}_H\partial_H(\epsilon_i\epsilon_j\rho_0+\epsilon_j\epsilon_i\rho_0) &:= \overline{\partial}_H\psi_{ij}\\
&= 0
\end{aligned}$$

by Proposition 4.2, we can get that the equation

$$\overline{\partial}_H \epsilon_{ij}\rho_0 = \frac{1}{2}\partial_H(\epsilon_i\epsilon_j + \epsilon_j\epsilon_i)\rho_0$$

has the unique solution

$$\epsilon_{ij}\rho_0 = \frac{1}{2}\overline{\partial}^*_H G\partial_H((\epsilon_i\epsilon_j + \epsilon_j\epsilon_i)\rho_0).$$

which satisfying that $\epsilon_{ij}\rho_0$ is $\overline{\partial}^*_H$-closed and∂_H-exact.

If we suppose that for $i_1 + \cdots + i_N = 2$ to $k-1$, $\epsilon_{i_1\cdots i_N}\rho_0$ exists uniquely, such that $\epsilon_{i_1\cdots i_N}\rho_0$ is ∂_H-exact, $\overline{\partial}^*_H$-closed hold. For $i_1 + \cdots + i_N = k$, we have

$$\begin{aligned}\psi_{i_1\cdots i_N} &:= -\frac{1}{2}\sum_{j_\alpha + h_\alpha = i_\alpha}[\epsilon_{j_1\cdots j_N}, \epsilon_{h_1\cdots h_N}]_H\rho_0 \\ &= \frac{1}{2}\partial_H(\sum_{j_\alpha + h_\alpha = i_\alpha}\epsilon_{j_1\cdots j_N}\epsilon_{h_1\cdots h_N}\rho_0)\end{aligned}$$

is also ∂_H-exact.

Since

$$\begin{aligned}\frac{1}{2}\overline{\partial}_H\partial_H(\sum_{j_\alpha + h_\alpha = i_\alpha}\epsilon_{j_1\cdots j_N}\epsilon_{h_1\cdots h_N}\rho_0) &:= \overline{\partial}_H\psi_{i_1\cdots i_N} \\ &= 0,\end{aligned}$$

by Proposition 4.2, we can get that the equation

$$\overline{\partial}_H\epsilon_{i_1\cdots i_N}\rho_0 = \frac{1}{2}\partial_H\sum_{j_\alpha + h_\alpha = i_\alpha}\epsilon_{j_1\cdots j_N}\epsilon_{h_1\cdots h_N}\rho_0$$

has the unique solution

$$\epsilon_{i_1\cdots i_N}\rho_0 = \frac{1}{2}\overline{\partial}^*_H G\partial_H(\sum_{j_\alpha + h_\alpha = i_\alpha}\epsilon_{j_1\cdots j_N}\epsilon_{h_1\cdots h_N}\rho_0).$$

And by Proposition 4.2, we can get that $\mathbb{H}(\epsilon_{i_1\cdots i_N}\rho_0) = 0$, $\epsilon_{i_1\cdots i_N}\rho$ is ∂_H-exact, $\overline{\partial}^*_H$-closed, and $||\epsilon_{i_1\cdots i_N}\rho_0||^2 \leq \frac{1}{4}(\partial_H(\sum_{j_\alpha + h_\alpha = i_\alpha}\epsilon_{j_1\cdots j_N}\epsilon_{h_1\cdots h_N}\rho_0), G\partial_H(\sum_{j_\alpha + h_\alpha = i_\alpha}\epsilon_{j_1\cdots j_N}\epsilon_{h_1\cdots h_N}\rho_0))$.

Step 2: We show $\epsilon(t)$ L^2 converges at $|t| \leq \frac{1}{4ac}$ by using the same convergent power series as in the [12].

Firstly, we have

$$
\begin{aligned}
||\epsilon_{k_1\cdots k_N}\rho_0|| &:= \frac{1}{2}||\overline{\partial}_H G\partial_H^*(\sum_{i_\alpha+j_\alpha=k_\alpha}\epsilon_{i_1\cdots i_N}\epsilon_{j_1\dots j_N}\rho_0)|| \\
&\leq \sum_{i_\alpha+j_\alpha=k_\alpha}||\epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}\rho_0|| \\
&\leq \sum_{i_\alpha+j_\alpha=k_\alpha}||\epsilon_{i_1\cdots i_N}\rho_0||_{s+\alpha}\cdot||\epsilon_{j_1\cdots j_N}\rho_0||, \qquad (5.1)
\end{aligned}
$$

where $k\geq 2\in\mathbb{N}, 0\leq\alpha\leq 1$. The 1st inequality holds since Lemma 4.1 and the last inequality holds by the definition of the Hölder norms.

Secondly, we set $\{x_i\}_{i=1}^{\infty}$, such that $x_1=a, x_k=c\sum_{i=1}^{k-1}x_ix_{k-i}$ just as in Lemma 3.4 and c is a constant definite in inequality (3.2) after Lemma 3.3. We set $c:=max\{1,c\}$ which we still denote it as c. We will show that

$$
x_k \geq \sum_{k_1+\cdots+k_N=k}||\epsilon_{k_1\cdots k_N}\rho_0||_{s+\alpha}
$$

by induction on k.

Since $\rho_0,\epsilon_i,s\geq 2,\alpha$ fixed, we can assume that $a=max\{\sum_{i=1}^N||\epsilon_i\rho_0||,\ \sum_{i=1}^N||\epsilon_i\rho_0||_{s+\alpha},\ ||\rho_0||\}$ is a constant.

For $k=2$, we have

$$
\begin{aligned}
x_2 &:= cx_1x_1 \\
&= c(\sum_{i=1}^N||\epsilon_i\rho_0||_{s+\alpha})(\sum_{i=1}^N||\epsilon_i\rho_0||_{s+\alpha}) \\
&\geq \frac{1}{2}\sum_{i,j=1}^N||\overline{\partial}_H^*G\partial_H(\epsilon_i\epsilon_j\rho_0)||_{s+\alpha} \\
&\geq \sum_{i,j=1}^N||\epsilon_{ij}\rho_0||_{s+\alpha}.
\end{aligned}
$$

The 1st inequality holds since Lemma 3.3, and the last inequality holds since the construction of $\epsilon(t)$

If for $\leq k-1$, the inequality above holds, for k, we have

$$
\begin{aligned}
x_k &:= c\sum_{i+j=k} x_i x_j \\
&\geq c\sum_{i+j=k}\Big(\sum_{i_1+\cdots+i_N=i} ||\epsilon_{i_1\cdots i_N}\rho_0||_{s+\alpha}\Big)\Big(\sum_{j_1+\cdots+j_N=j} ||\epsilon_{j_1\cdots j_N}\rho_0||_{s+\alpha}\Big) \\
&\geq c\sum_{i+j=k}\Big(\sum_{i_1+\cdots+i_N=i} ||\epsilon_{i_1\cdots i_N}||_{s+\alpha}\Big)\Big(\sum_{j_1+\cdots+j_N=j} ||\epsilon_{j_1\cdots j_N}\rho_0||_{s+\alpha}\Big) \\
&= \frac{1}{2}\sum_{i_1+\cdots+i_N+j_1+\cdots+j_N=k} ||\overline{\partial}_H^* G\partial_H(\epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}\rho_0)||_{s+\alpha} \\
&\geq \sum_{k_1+\cdots+k_N=k} ||\epsilon_{k_1\cdots k_N}\rho_0||_{s+\alpha}. \qquad (5.2)
\end{aligned}
$$

The 1st inequality holds since the induction, the 2nd equality holds since ρ_0 is fixed and we consider $||\epsilon_{i_1\cdots i_N}\rho_0||_{s+\alpha}$ equivalent to $||\epsilon_{i_1\cdots i_N}||_{s+\alpha}$, and the constant c' is different, for simplity, we consider $max\{c, c'\}$ and still denote as c.

We also need to show

$$
x_k \geq \sum_{k_1+\cdots+k_N=k} ||\epsilon_{k_1\cdots k_N}\rho_0||
$$

by induction on k.

For $k = 2$, we have

$$
\begin{aligned}
x_2 &:= c x_1 x_1 \\
&= c\Big(\sum_{i=1}^{N} ||\epsilon_i\rho_0||_{s+\alpha}\Big)\Big(\sum_{i=1}^{N} ||\epsilon_i\rho_0||\Big) \\
&\geq \sum_{i,j=1}^{N} ||\epsilon_{ij}\rho_0||.
\end{aligned}
$$

If for $\leq k-1$, the inequality above holds, for k, we have

$$
\begin{aligned}
x_k &:= c \sum_{i+j=k} x_i x_j \\
&\geq c \sum_{i+j=k} \Big(\sum_{i_1+\cdots+i_N=i} ||\epsilon_{i_1\cdots i_N}\rho_0||_{s+\alpha} \Big) \Big(\sum_{j_1+\cdots+j_N=j} ||\epsilon_{j_1\cdots j_N}\rho_0|| \Big) \\
&\geq \frac{1}{2} \sum_{i_1+\cdots+i_N+j_1+\cdots+j_N=k} ||\overline{\partial}_H^* G \partial_H \epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}\rho_0|| \\
&\geq \sum_{k_1+\cdots+k_N=k} ||\epsilon_{k_1\cdots k_N}\rho_0||
\end{aligned}
$$

by formula (5.1).

Hence, we get

$$
\begin{aligned}
& ||\epsilon(t)\rho_0|| \\
:= \ & ||\sum_{i=1}^{N} \epsilon_i \rho_0 t^i + \sum_{k\geq 2, k_1+\cdots+k_N=k} \epsilon_{k_1\cdots k_N}\rho_0 (t^1)^{k_1}\cdots(t^N)^{k_N}|| \\
:= \ & ||\sum_{i=1}^{N} \epsilon_i \rho_0|| \cdot |t^i| + \frac{1}{2} \sum_{k\geq 2, k_1+\cdots+k_N=k} ||\overline{\partial}_H^* G \partial_H \sum_{i_\alpha+j_\alpha=k_\alpha} \epsilon_{i_1\cdots i_N} \\
& \epsilon_{j_1\cdots j_N}\rho_0 (t^1)^{k_1}\cdots(t^N)^{k_N}|| \\
\leq \ & \sum_{i=1}^{N} ||\epsilon_i\rho_0|| \cdot |t^i| + \frac{1}{2} \sum_{k\geq 2, k_1+\cdots+k_N=k} || \sum_{i_\alpha+j_\alpha=k_\alpha} \epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}\rho_0|| \\
& \cdot |t^1|^{k_1}\cdots|t^N|^{k_N} \\
\leq \ & a|t| + c \sum_{k\geq 2, i_1+\cdots+i_N+j_1+\cdots+j_N=k} ||\epsilon_{i_1\cdots i_N}\rho_0||_{s+\alpha} \cdot ||\epsilon_{j_1\cdots j_N}\rho_0|| \cdot |t|^k \\
& (t = max_i\{t^i\}) \\
\leq \ & x_1|t| + \sum_{k\geq 2} x_k |t|^k \\
< \ & +\infty
\end{aligned}
$$

at $|t| \leq \frac{1}{4ac}$.

□

Remark 5.2. *(1)Since*

$$\begin{aligned}\overline{\partial}_H(\epsilon_{i_1\cdots i_N}\rho_0) &= -\frac{1}{2}\sum_{j_\alpha+h_\alpha=i_\alpha, j_\alpha,h_\alpha\geq 0,\alpha=1\cdots N}[\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H\rho_0, \\ & \quad (i_1+\cdots+i_N=k) \\ \overline{\partial}^*_H(\epsilon_{i_1\cdots i_N}\rho_0) &= 0,\end{aligned}$$

we can get

$$\begin{aligned}\triangle_{\overline{\partial}_H}(\epsilon_{i_1\cdots i_N}\rho_0) &= \overline{\partial}^*_H\overline{\partial}_H(\epsilon_{i_1\cdots i_N}\rho_0) \\ &= -\frac{1}{2}\sum_{j_\alpha+h_\alpha=i_\alpha, j_\alpha,h_\alpha\geq 0,\alpha=1\cdots N}\overline{\partial}^*_H([\epsilon_{j_1\cdots j_N},\epsilon_{h_1\cdots h_N}]_H\rho_0).\end{aligned}$$

So by the assumption that $\epsilon_i\rho_0$ is smooth on M, and $\triangle_{\overline{\partial}_H}$ is a strongly elliptic operator, we can get that $\epsilon_{i_1\cdots i_N}\rho_0$ is smooth on M by the elliptic regularity theorem in [17], and the induction on $i_1+\cdots+i_N=k$.

(2)From P281 in [11], we have known that $\epsilon(t)\rho_0$ is smooth on $M\times\triangle_\delta$ for a sufficiently small neighbourhood of $t\in\triangle_\delta$.

6. Global canonical family

In this section, if we assume that the deformation is smooth in a fixed neighbourhood, we shall get the global canonical sections for deformations of compact H-twisted Generalized Kähler manifold by using the parallel method in [12] and [22] if we assume the existence of a global canonical family of deformation. There are already many works on [18, 19].

First, we have the following equivalence to criterion the holomorphism with respect to the generalized complex structure.

Proposition 6.1. *[20] On compact generalized Kähler manifold M. For any $\rho\in U_{0,M}$, $e^{-\epsilon(t)}\rho$ is holomorphic with respect to the generalized complex structure $J_{\epsilon(t)}$ induced by $\epsilon(t)$ on M_t if and only if*

$$\overline{\partial}_H\rho-\partial_H(\epsilon(t)\rho)=0.$$

Proof. Since if $A\in E$,

$$((1+\epsilon)A)\rho \quad = \quad e^{-\epsilon}Ae^{\epsilon}\rho,$$

(See P16, Proposition 7 in [14].) we have

$$\begin{aligned}((1+\epsilon)A)e^{-\epsilon}\rho_0 &= e^{-\epsilon}Ae^{\epsilon}e^{-\epsilon}\rho \\ &= e^{-\epsilon}A\rho \\ &= 0.\end{aligned}$$

The last equality is by the definition of U_0. Hence we get $e^{-\epsilon(t)}\rho \in U_{0,M_t}$.

Since

$$e^{\epsilon}d_H e^{-\epsilon}\rho = \overline{\partial}_H\rho - \partial_H(\epsilon\rho),$$

we have

$$d_H e^{-\epsilon}\rho = 0 \Leftrightarrow \overline{\partial}_H\rho - \partial_H(\epsilon\rho) = 0.$$

Since $e^{-\epsilon}\rho \in U_{0,M_t}$, we have

$$\partial_{H,t}e^{-\epsilon}\rho = 0,$$

where $d_H = \partial_{H,t} + \overline{\partial}_{H,t}$ on M_t. So we have

$$\overline{\partial}_{H,t}e^{-\epsilon}\rho = 0 \Leftrightarrow \overline{\partial}_H\rho - \partial_H(\epsilon\rho) = 0.$$

□

In compact generalized Calabi-Yau case, we have

Proposition 6.2 (See P16, Proposition 7 in [14].). *If we assume that $\epsilon(t)$ smooth with fixed convergence radius exists, then*

$$\epsilon(t)\rho_0 \in H^2_{\overline{\partial}_H}(M, \wedge^* T^*_M),$$

where $H^2_{\overline{\partial}_H}(M, \wedge^* T^*_M)$ is the cohomology group of the complex $(\wedge^* T^*_M, \overline{\partial}_H)$, and

$$\rho^c_t := e^{-\epsilon(t)}\rho_0$$

holomorphic with respect to $J_{\epsilon(t)}$ with fixed convergence radius.

Proof. For any test form η on M,

$$\begin{aligned}(\partial_H(\epsilon(t)\rho_0),\eta) &= (\epsilon(t)\rho_0,\partial_H^*\eta)\\ &= lim_{k\to\infty}\sum_{\|I\|\le k}(\epsilon_I t^I\rho_0,\partial_H^*\eta)\\ &= \sum_{i=1}^N(\partial_H\epsilon_i t^i\rho_0,\eta)+lim_{k\to\infty}\sum_{2\le\|I\|\le k}(\partial_H\epsilon_I t^I\rho_0,\eta)\\ &= 0.\end{aligned}$$

The last equality holds since $\epsilon_i\rho_0$ are harmonic, and $\epsilon_I\rho_0$ are ∂_H-exact for $\|I\|] \ge 2$.
So we have $\partial_H(\epsilon(t)\rho_0) = 0$ in the distribution sense.
Also, $\overline{\partial}_H\rho_0 = 0$, and $\epsilon(t)$ is smooth by assumption, we have $\overline{\partial}_H\rho - \partial_H(\epsilon(t)\rho) = 0$ and by Propositon 6.1, we get the conclusion. □

Proposition 6.3. *Let $(M_t, J_{\epsilon(t)})$ be the deformation of a compact H-twisted generalized Calabi-Yau manifold M, then we have*

$$[\rho_t^c] = [\rho_0] + \sum_{i=1}^N[-\epsilon_i\rho_0]t + O(|t|^2),$$

where $[\rho_0]$ means a representation in $H^0_{\overline{\partial}_H}(M,\wedge^*T^*_M)$, and $O(|t|^2)$ denotes the terms in
$H^4_{\overline{\partial}_H}(M,\wedge^*T^*_M)\bigoplus\cdots\bigoplus H^{2n}_{\overline{\partial}_H}(M,\wedge^*T^*_M)$ of order at least 2 in t.

Proof. On convergence radius, we have

$$\begin{aligned}[][\rho_t^c] &:= [\rho_0]+\sum_{i=1}^N[\mathbb{H}(-\epsilon_i\rho_0)]t+\sum_{I,|I|\ge 2}[\mathbb{H}((-1)^{|I|}\epsilon_I\rho_0)]t^{|I|}+\cdots\\ &= [\rho_0]+\sum_{i=1}^N[-\epsilon_i\rho_0]t+O(|t|^2),\end{aligned}$$

where $t^I := (t_1)^{i_1}...(t_N)^{i_N},\ \epsilon_I := \epsilon_{i_1...t_N}$.
The last equality holds since $\epsilon_i\rho_0$ are harmonic, and $\epsilon_I\rho_0$ are ∂_H-exact in Calabi-Yau case.

□

Next, like in [12], as an extension of the case of the generalized Calabi-Yau manifolds, we assume that on compact generalized Kähler M, there exists

$$\epsilon(t) \quad := \quad \epsilon_i t^i + \epsilon_{ij} t^i t^j + \cdots + \epsilon_{i_1 \cdots i_N} (t^1)^{i_1} \cdots (t^N)^{i_N} + \cdots ,$$

satisfying
(1) ϵ_i form a basis of $H^2_{d_E}(M, \wedge^* E^*)$;
(2) the integral condition: $d_E \epsilon(t) + \frac{1}{2}[\epsilon(t), \epsilon(t)]_H = 0$;
(3)$||\epsilon(t)\rho_0|| \leq \sum_{\|I\| \geq 1} ||\epsilon_I|| ||\rho_0|| |t^{\|I\|} < +\infty$ is convergence in $t < \frac{1}{4ac}$, where for any $\rho_0 \neq 0 \in H^0_{\overline{\partial}_H}(M, \wedge^* T^*_M)$, $x_k \geq \sum_{k_1 + \cdots + k_N = k} ||\epsilon_{k_1 \cdots k_N}||_{s+\alpha}$, x_k is in Lemma 3.4, and $\epsilon(t)$ has the convergence as the same as that in Theorem 5.1. As an analogue, we can get that

Proposition 6.4. *If we assume that $\epsilon(t)$ smooth with convergence radius $\frac{1}{4ac}$ exists, and $(M_t, J_{-\epsilon(t)})$ be the deformation, then for any $\rho_0 \neq 0 \in H^0_{\overline{\partial}_H}(M, \wedge^* T^*_M)$, we can construct a smooth power series*

$$\rho_t \quad := \quad \rho_0 + \sum_{K, |K| \geq 1} \rho_K t^K \in U_0,$$

such that
(1) $\rho_t^c := e^{-\epsilon(t)} \rho_t$ holomorphic with respect to $J_{\epsilon(t)}$,
(2) ρ_K is ∂_H-exact and $\overline{\partial}^*_H$-closed, ($|K| \geq 1$)
(3) ρ_t converges with radius $\frac{1}{16a^2c^2}$.

Proof. In step 1, we construct ρ_t, in step 2, we prove ρ_t is L^2 convergent in $t < \frac{1}{16a^2c^2}$, and in step 3, we prove ρ_t is smooth.

Step 1: We shall construct ρ_t.

If $\rho_t^c := e^{-\epsilon(t)} \rho_t$ is holomorphic with respect to $J_{\epsilon(t)}$, by Proposition 6.1, we have

$$\overline{\partial}_H \rho_t = \partial_H (\epsilon(t) \rho_t).$$

Let

$$\rho_t \quad := \quad \rho_0 + \sum_{k \geq 1} \sum_{i_1 + \cdots + i_N = k} \rho_{i_1 \cdots i_N} (t^1)^{i_1} \cdots (t^N)^{i_N} \in U_{0,M},$$
$$\epsilon(t) \quad := \quad \sum_{k \geq 1} \sum_{i_1 + \cdots + i_N = k} \epsilon_{i_1 \cdots i_N} (t^1)^{i_1} \cdots (t^N)^{i_N} \in \wedge^2 E^*.$$

Comparing the coefficients of $t = (t^1, \cdots, t^N)$, we have

$$\begin{aligned} k &= 0,\ \overline{\partial}_H \rho_0 = 0, \\ k &\geq 1,\ \overline{\partial}_H \rho_{i_1 \cdots i_N} = \partial_H \Big(\sum_{j_\alpha + h_\alpha = i_\alpha, j_\alpha, h_\alpha \geq 0} \epsilon_{j_1 \cdots j_N} \rho_{h_1 \cdots h_N} \Big).\ (i_1 + \cdots + i_N = k) \end{aligned}$$

Let

$$\eta_{i_1 \cdots i_N} := \partial_H \Big(\sum_{j_\alpha + h_\alpha = i_\alpha, j_\alpha, h_\alpha \geq 0} \epsilon_{j_1 \cdots j_N} \rho_{h_1 \cdots h_N} \Big).$$

We shall prove that $\overline{\partial}_H \eta_{i_1 \cdots i_N} = 0$ by induction on $i_1 + \cdots + i_N = k$ and construct $\rho_{i_1 \cdots i_N}$.

For $k = 1$, we have

$$\begin{aligned} \overline{\partial}_H \eta_i &:= \overline{\partial}_H \partial_H (\epsilon_i \rho_0) \\ &= -\partial_H \overline{\partial}_H (\epsilon_i \rho_0) \\ &= -\partial_H (d_E \epsilon_i) \rho_0 \\ &= 0. \end{aligned}$$

So by Proposition 4.2, the equation

$$\overline{\partial}_H \rho_i = \partial_H (\epsilon_i \rho_0)$$

has the unique solution

$$\rho_i = \overline{\partial}_H^* G \partial_H (\epsilon_i \rho_0).$$

which satisfying that ρ_i is $\overline{\partial}_H^*$-closed, ∂_H-exact, $\mathbb{H}\rho_i = 0$, and

$$||\rho_i||^2 \leq (\partial_H (\epsilon_i \rho_0), G \partial_H (\epsilon_i \rho_0)).$$

If we suppose that for $i_1+\cdots+i_N\leq k-1$, $\overline{\partial}_H\eta_{i_1\cdots i_N}=0$ holds and $\rho_{i_1\cdots i_N}$ is constucted. For $i_1+\cdots+i_N=k$, we have

$$\begin{aligned}
\overline{\partial}_H\eta_{i_1\cdots i_N} &:= \overline{\partial}_H\partial_H(\sum_{j_\alpha+h_\alpha=i_\alpha,j_\alpha,h_\alpha\geq 0}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N})\\
&= -\partial_H\overline{\partial}_H(\sum_{j_\alpha+h_\alpha=i_\alpha,j_\alpha,h_\alpha\geq 0}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N})\\
&= -\partial_H\sum_{j_\alpha+h_\alpha=i_\alpha}((d_E\epsilon_{j_1\cdots j_N})\rho_{h_1\cdots h_N}+\epsilon_{j_1\cdots j_N}(\overline{\partial}_H\rho_{h_1\cdots h_N}))\\
&= -\partial_H\sum_{j_\alpha+h_\alpha=i_\alpha}(-\sum_{l_\alpha+p_\alpha=j_\alpha}\frac{1}{2}[\epsilon_{l_1\cdots l_N},\epsilon_{p_1\cdots p_N}]_H\rho_{h_1\cdots h_N}\\
&\quad+\sum_{l_\alpha+p_\alpha=h_\alpha}\epsilon_{j_1\cdots j_N}\partial_H(\epsilon_{l_1\cdots l_N}\rho_{p_1\cdots p_N}))\\
&= \frac{1}{2}\partial_H\sum_{l_\alpha+p_\alpha+h_\alpha=i_\alpha}(-\partial_H(\epsilon_{l_1\cdots l_N}\epsilon_{p_1\cdots p_N}\rho_{h_1\cdots h_N})\\
&\quad+\epsilon_{l_1\cdots l_N}\partial_H(\epsilon_{p_1\cdots p_N}\rho_{h_1\cdots h_N})+\epsilon_{p_1\cdots p_N}\partial_H(\epsilon_{l_1\cdots l_N}\rho_{h_1\cdots h_N})\\
&\quad-\partial_H(\sum_{l_\alpha+p_\alpha=h_\alpha}\epsilon_{j_1\cdots j_N}\partial_H(\epsilon_{l_1\cdots l_N}\rho_{p_1\cdots p_N}))\\
&= \partial_H\sum_{l_\alpha+p_\alpha+h_\alpha=i_\alpha}+\epsilon_{l_1\cdots l_N}\partial_H(\epsilon_{p_1\cdots p_N}\rho_{h_1\cdots h_N})\\
&\quad-\partial_H(\sum_{l_\alpha+p_\alpha=h_\alpha}\epsilon_{j_1\cdots j_N}\partial_H(\epsilon_{l_1\cdots l_N}\rho_{p_1\cdots p_N}))\\
&= 0.
\end{aligned}$$

The 3rd equality holds since the definition of Lie derivation d_E; the 4th equality is by induction and using the integral condition of $\epsilon(t)$; the 5th equality holds since Lemma 3.2 and each $\epsilon_{i_1\cdots i_N}\in\wedge^2E^*$; and the 6th equality holds since $\partial_H^2=0$.

From the discussion above, we know that the conditions of Proposition 4.2 hold. So we have the equation

$$\overline{\partial}_H\rho_{i_1\cdots i_N}=\partial_H(\sum_{j_\alpha+h_\alpha=i_\alpha,j_\alpha,h_\alpha\geq 0}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N})$$

has the unique solution

$$\rho_{i_1\cdots i_N}=\overline{\partial}_H^*G\partial_H(\sum_{j_\alpha+h_\alpha=i_\alpha,j_\alpha,h_\alpha\geq 0}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N})\in U_{0,M}.$$

which satisfying that $\rho_{i_1\cdots i_N}$ is $\overline{\partial}_H^*$-closed, ∂_H-exact, $\mathbb{H}\rho_{i_1\cdots i_N}=0$, and

$$||\rho_{i_1\cdots i_N}||^2 \leq (\partial_H(\sum_{j_\alpha+h_\alpha=i_\alpha}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N}), G\partial_H(\sum_{j_\alpha+h_\alpha=i_\alpha}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N})).$$

Step 2: We show ρ_t is L^2 converges at $t<\frac{1}{16a^2c^2}$.
We have already known that

$$\begin{aligned}
&\sum_{k_1+\cdot+k_N=k}||\rho_{k_1\cdots k_N}|| \\
:= &\sum_{k_1+\cdot+k_N=k}||\overline{\partial}_H^* G\partial_H(\sum_{j_\alpha+h_\alpha=k_\alpha, j_\alpha, h_\alpha\geq 0}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N})|| \\
\leq &(2c)\sum_{k_1+\cdot+k_N=k}||\sum_{j_\alpha+h_\alpha=k_\alpha, j_\alpha, h_\alpha\geq 0}\epsilon_{j_1\cdots j_N}\rho_{h_1\cdots h_N})|| \\
\leq &\sum_{k_1+\cdot+k_N=k}\sum_{j_\alpha+h_\alpha=k_\alpha, j_\alpha, h_\alpha\geq 0}(2c)||\epsilon_{j_1\cdots j_N}||_{s+\alpha}\cdot||\rho_{h_1\cdots h_N}||. \qquad (6.1)
\end{aligned}$$

The 1st inequality holds since Lemma 4.1, and the 2nd inequality holds by the definition of the Hölder norms.

Also we set $\{x_i\}_{i=1}^\infty$, such that $x_1=2a^2c, x_k=(2c)\sum_{i=1}^{k-1}x_i x_{k-i}$ just as in Lemma 3.4. As we assume the existence of $\epsilon(t)$ which is constructed in Theorem 5.1, and we also assume that $x_k\geq\sum_{k_1+\cdots+k_N=k}||\epsilon_{k_1\cdots k_N}\rho_0||_{s+\alpha}$. Next, we will show

$$x_k \geq \sum_{k_1+\cdots+k_N=k}||\rho_{k_1\cdots k_N}||$$

by induction on k.

For $k=1$, we have

$$\begin{aligned}
x_1 &\geq (2c)(\sum_{i=1}^N||\epsilon_i||_{s+\alpha})||\rho_0|| \\
&\geq \sum_{i=1}^N||\rho_i||.
\end{aligned}$$

If for $\leq k-1$, the inequality above holds, for k, we have

$$\begin{aligned}
x_k &:= (2c)\sum_{i+j=k} x_i x_j \\
&\geq (2c)\sum_{i+j=k}\Big(\sum_{i_1+\cdots+i_N=i} ||\epsilon_{i_1\cdots i_N}||_{s+\alpha}\Big)\Big(\sum_{j_1+\cdots+j_N=j} ||\rho_{j_1\cdots j_N}||\Big) \\
&\geq (2c)\sum_{i_1+\cdots+i_N+j_1+\cdots+j_N=k} ||\epsilon_{i_1\cdots i_N}||_{s+\alpha}\cdot||\rho_{j_1\cdots j_N}|| \\
&\geq \sum_{k_1+\cdots+k_N=k} ||\rho_{k_1\cdots k_N}||.
\end{aligned}$$

The 1st inequality is the induction and the formula (5.2), the in the 2nd inequality we need that $\epsilon(t)$ has the convergence in 5.1, and the last inequality holds since the formula (6.1).

Hence, we get

$$\begin{aligned}
||\rho_t|| &:= ||\rho_0 + \sum_{k\geq 1}\sum_{k_1+\cdots+k_N=k} \rho_{k_1\cdots k_N}(t^1)^{k_1}\cdots(t^N)^{k_N}|| \\
&\leq ||\rho_0|| + \sum_{k\geq 1}\sum_{k_1+\cdots+k_N=k} ||\rho_{k_1\cdots k_N}(t^1)^{k_1}\cdots(t^N)^{k_N}|| \\
&\leq a + (2c)\sum_{k\geq 1, i_1+\cdots+i_N+j_1+\cdots+j_N=k} ||\epsilon_{i_1\cdots i_N}||_{s+\alpha}\cdot||\rho_{j_1\cdots j_N}||\cdot|t|^k \\
&\quad (t = max_i\{t^i\}) \\
&\leq a + 2a^2c|t| + (2c)\sum_{k\geq 2, i_1+\cdots+i_N+j_1+\cdots+j_N=k} ||\epsilon_{i_1\cdots i_N}||_{s+\alpha}\cdot||\rho_{j_1\cdots j_N}|| \\
&\quad \cdot|t|^k \\
&\leq a + x_1|t| + \sum_{k\geq 2} x_k|t|^k \\
&< +\infty
\end{aligned}$$

at $|t| \leq \frac{1}{16a^2c^2}$.

Step 3:

Since $\rho_t^c := e^{-\epsilon(t)}\rho_t$ is holomorphic with respect to $J_{\epsilon(t)}$, we can get that

$$\overline{\partial}_{H,t}\rho_t^c = 0.$$

Since $\rho_t^c := e^{-\epsilon(t)}\rho_t \in U_{0,t}$ is the canonical bundle in M_t, where

$$\wedge^* T^*_{M_t} = U_{0,t}\oplus U_{1,t}\oplus ...\oplus U_{2n,t}$$

is the decomposition in Definition 2.3, we can get that

$$\overline{\partial}^*_{H,t}\rho^c_t = 0$$

by $\overline{\partial}^*_{H,t}\rho^c_t \in U_{-1,t} = 0$.

So we get

$$\triangle_{\overline{\partial}^*_{H,t}}\rho^c_t = 0.$$

Thus, by the fact that $\triangle_{\overline{\partial}^*_{H,t}}$ is a strongly elliptic operator and the elliptic regularity theorem, we get that ρ^c_t is smooth.

□

Proposition 6.5. *We have*

$$[\rho^c_t] = [\rho_0] + \sum_{i=1}^{N}[\mathbb{H}(-\epsilon_i\rho_0)]t + O(|t|^2),$$

where $[\rho_0]$ *means a representation in* $H^0_{\overline{\partial}_H}(M, \wedge^* T^*_M)$, *and* $O(|t|^2)$ *denotes the terms in* $H^4_{\overline{\partial}_H}(M, \wedge^* T^*_M) \bigoplus \cdots \bigoplus H^{2n}_{\overline{\partial}_H}(M, \wedge^* T^*_M)$ *of order at least* 2 *in* t.

Proof. On convergence radius, we have

$$\begin{aligned}[\rho^c_t] &:= [\rho_0] + \sum_{i=1}^{N}[\mathbb{H}(-\epsilon_i\rho_0)]t + \sum_{I,|I|\geq 2}[\mathbb{H}((-1)^{|I|}\epsilon_I\rho_0)]t^{|I|} \\ &\quad + \sum_{i,J,|J|\geq 1}[\mathbb{H}(-\epsilon_i\rho_J)]t^{1+|J|} + \sum_{I,J,|I|\geq 2,|J|\geq 1}[\mathbb{H}((-1)^{|I|}\epsilon_I\rho_J)]t^{|I|+|J|} \\ &= [\rho_0] + \sum_{i=1}^{N}[\mathbb{H}(-\epsilon_i\rho_0)]t + O(|t|^2),\end{aligned}$$

where $t^I := (t_1)^{i_1}...(t_N)^{i_N},\ \epsilon_I := \epsilon_{i_1...t_N}$.

□

Remark 6.6. *Here* $-\epsilon_i\rho_0$ *not necessary in* $H^2_{\overline{\partial}_H}(M, \wedge^* T^*_M)$. *In Calabi-Yau case,* $-\epsilon_i\rho_0 \in H^2_{\overline{\partial}_H}(M, \wedge^* T^*_M)$ *and thus* $\partial_H(-\epsilon_i\rho_0) = 0$, *all* $\rho_K = 0$ *just as in Proposition 6.2.*

References

[1] D. Baraglia, *Variation of Hodge structure for generalized complex manifolds*, (2012) arXiv:math.DG/1205.0240v1.

[2] M. Gualtieri, *Generalized complex geometry*, D. Phil. thesis, Oxford University, (2003) arXiv:math.DG/0401221.

[3] M. Gualtieri, *Generalized geometry and the Hodge decomposition*, (2004) arXiv:math.DG/0409093v1.

[4] M. Gualtieri, *18.969 Topics in geometry* (2006) http://www.math.toronto.edu/mgualt/gcg/geometrynotes.pdf.

[5] M. Gualtieri, *Generalized Kähler geometry* (2010) arXiv:1007.3485.

[6] M. Gualtieri, *Generalized complex geometry* (2007) arXiv:math/0703298.

[7] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiiey-interscience, New York (1978).

[8] F. Guan, *A proof of generalized Kähler identities(in Chinese)*, (2008).

[9] R. Goto, *On deformation of generalized Calabi-Yau, hyperKähler,*G_2 *and* $Spin(7)$ *structures I*, (2005) aiXiv:math/0512211v1.

[10] N. Hitchin, *Gerneralized Calabi-Yau manifolds*, Q.J. Math. 54(3) (2003) 281-308.

[11] K. Kodaira, *Complex manifolds and deformation of complex structure*, Springer-Verlag (1981).

[12] K. Liu, S. Rao, and X. Yang, *Quasi-isometry and deformations of CalabiCYau manifolds*,Inventiones Mathematicae, (2014) DOI 10.1007/s00222-014-0516-1, arXiv:math.DG/1207.1182v1.

[13] K. Liu and S. Rao, *Remarks on the Cartan formula and its applications*, Asian Journal of Mathematics, (2012) 157-170, arXiv:1104.1240v1.

[14] Y. Li, *On deformation of generalized complex structures:the generalized Calabi-Yau case*,(2005). arXiv:hep-th/0508030v2.

[15] J. Morrow and K. Kodaira, *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London,(1971).

[16] P. Ševera and A. Weinstein, *Poisson geometry with a 3-form background*, Prog. Theor. Phys. Suppl. 144, (2001) 145-154.

[17] N. Shimakura, *Partial differential operators of elliptic type,* Translated and revised from the (1978) Japanese original by the author, Translations of Mathematical Monographs, 99, American Mathematical Society, Providence, RI, (1992).

[18] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric,* Mathematical aspects of string theory(San Diego, California, 1986). Advanced Series in Mathematical Physics, vol. 1, 629-646. World Scientific Publishing, Singapore(1987).

[19] A. Todorov, *The Weil-Petersson geometry of the moduli space of* $\mathbb{SU}$*(n≥3)(Calabi-Yau) manifolds I.* Commun. Math. Phys. 126(2), 325-346(1989).

[20] K. Wei, *Global canonical family of deformations of compact H-twisted Generalized Calabi-Yau manifolds.*

[21] R. Wells, *Differential analysis on complex manifolds,* Springer-Verlag, 2nd edition (1980).

[22] Q. Zhao and S. Rao, *Applications of deformation formula of holomorphic one-forms,* Pacific Journal of Mathematics Vol. 266, No. 1, (2013)221-256..

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