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A New Boundary Rigidity Theorem for Holomorphic Self-Mappings of the Unit Ball in $\mathbb{C}^n$

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Abstract: In this paper, we give some new boundary rigidity properties for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$. Compared with the previous related work, our result only requires conditions on the first-order derivatives, thought at n linearly independent different boundary points.
Keywords: Holomorphic mapping, Schwarz lemma, Boundary rigidity, Unit ball.

1. Introduction

Let $\mathbb{C}$ be the complex plane, and let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Denote by $\mathbb{C}^n$ the n-dimensional complex Hilbert space with the inner product and the norm given by

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j\overline{w_j}, \quad \|z\| = (\langle z, z\rangle)^{\frac{1}{2}},$$

where $z,\ w \in \mathbb{C}^n$. Let

$$\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

be the open unit ball in $\mathbb{C}^n$. The unit sphere is defined by $\partial\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$. Let $H(\mathbb{B}^n)$ be the family of all holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$. Throughout this paper, we write a point $z \in \mathbb{C}^n$ as a column vector in

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the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix},$$

and the symbol $'$ stands for the transpose of vectors or matrices. For $f \in H(\mathbb{B}^n)$, we also write it as the $n \times 1$ matrix form $f = (f_1, f_2, \cdots, f_n)'$, where f_j is a holomorphic function from $\mathbb{B}^n$ to $\mathbb{C}$, $j = 1, \cdots, n$. The complex Jacobian matrix of $f \in H(\mathbb{B}^n)$ at a point $a \in \mathbb{B}^n$ is given by

$$J_f(a) = \left(\frac{\partial f_j}{\partial z_k}(a)\right)_{n\times n} = \begin{pmatrix} \frac{\partial f_1}{\partial z_1}(a) & \cdots & \frac{\partial f_1}{\partial z_n}(a) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial z_1}(a) & \cdots & \frac{\partial f_n}{\partial z_n}(a) \end{pmatrix}.$$

It is clear that $J_f(a)$ is a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$. Denote by $Aut(\mathbb{B}^n)$ the set of all holomorphic automorphisms of $\mathbb{B}^n$. In what follows, a domain is a connected open subset in $\mathbb{C}^n$.

The Schwarz lemma is one of the most important results in the classical complex analysis. And almost all results in the holomorphic geometric function theory have the Schwarz lemma lurking in the background. Here we refer the reader to [1-7], as well as, many references therein for discussions related to such studies. From the point of view of applications, it has been a very natural task to obtain various boundary versions of the Schwarz lemma. In the case of one complex variable, the following boundary Schwarz lemma is classical:

Theorem 1.1 ([8]) Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function with $f(0) = 0$ and let f be holomorphic at $z = 1$ with $f(1) = 1$. Then the following two conclusions hold.

(1) $f'(1) \geq 1$.
(2) $f'(1) = 1$ if and only if $f(z) \equiv z$.

Theorem 1.1 has the following extension.

Theorem 1.1$'$ Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic mapping with $f(0) = 0$ and let f be holomorphic at $z = \alpha \in \partial\mathbb{D}$ with $f(\alpha) = \beta \in \partial\mathbb{D}$. Then the following two conclusions hold.

(1) $\overline{\beta} f'(\alpha)\alpha \geq 1$.

(2) $\overline{\beta}f'(\alpha)\alpha = 1$ if and only if $f(z) \equiv e^{i\theta}z$ and $e^{i\theta} = \beta\alpha^{-1}$, where $\theta \in \mathbb{R}$.

When $\alpha = \beta = 1$, Theorem 1.1$'$ is just Theorem 1.1.

Remark 1.1 Theorem 1.1$'$ needs only to assume that f is C^1 up to the boundary of the unit disk near $z = \alpha$.

In [9, 10], Osserman and Chelst studied the Schwarz lemma at the boundary on the unit disk, respectively. In several complex variables, Burns and Krantz obtained a new Schwarz lemma at the boundary, which gives a new boundary rigidity theorem for holomorphic self-mappings of a bounded strongly pseudoconvex domain (see [11]). Huang obtained a sharp boundary version of the Schwarz lemma for holomorphic self-mappings of simply connected strongly pseudo-convex domains which have an interior fixed point in [12]. See [13] for more on these matters. These results are stated as follows:

Theorem 1.2 ([11]) Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. Let $p \in \partial\Omega$, and let $f : \Omega \to \Omega$ be a holomorphic mapping such that

$$f(z) = z + O(\|z - p\|^4)$$

as $z \to p$. Then $f(z) \equiv z$.

Theorem 1.3 ([12]) Let $\Omega \subset\subset \mathbb{C}^n (n > 1)$ be a simply connected pseudoconvex domain with C^∞ boundary. Suppose that $p \in \partial\Omega$ is a strongly pseudoconvex point with at least C^3 boundary. If $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(z_0) = z_0$ for some $z_0 \in \Omega$ and

$$f(z) = z + o(\|z - p\|^2)$$

as $z \to p$, then $f(z) \equiv z$.

In [14], Huang studied a semi-rigidity property for holomorphic mappings between unit balls with different dimensions. And Krantz explored versions of the Schwarz lemma at the boundary point of a domain, and reviewed results of several authors (see [15]).

In this paper, we prove a new version of the boundary rigidity theorem for holomorphic self-mappings of the unit ball. With different from those results in [10-13, 15], we need not consider the partial derivatives of order 2 and order 3.

2. Some lemmas

In this section, we exhibit some notations and collect several basic lemmas, which will be used in the subsequent section.

Lemma 2.1 ([16]) For given $a \in \mathbb{B}^n$, let $A = sI_n + \frac{a\overline{a}'}{1+s}$, where $s = \sqrt{1-\|a\|^2}$, I_n is the unit square matrix of order n. Then

$$\varphi_a(z) = A\frac{a-z}{1-\overline{a}'z}$$

is a holomorphic automorphism of $\mathbb{B}^n$ which interchanges 0 and a. Moreover, φ_a is biholomorphic in a neighborhood of $\overline{\mathbb{B}^n}$, and

$$A^2 = s^2 I_n + a\overline{a}', \quad Aa = a, \quad J_{\varphi_a}(z) = A\left[-\frac{I_n}{1-\overline{a}'z} + \frac{(a-z)\overline{a}'}{(1-\overline{a}'z)^2}\right].$$

In the case of the unit disk, φ_a is just the Möbius transformation. That is,

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}, \; z \in \mathbb{D}.$$

In [17], Wu proved what is now called the Carathéodory-Cartan-Kaup-Wu theorem, which generalizes the classical Schwarz lemma for holomorphic mappings to higher dimension. This theorem is stated as follows:

Lemma 2.2 ([17]) Let Ω be a bounded domain in $\mathbb{C}^n$, and let f be a holomorphic self-mapping of Ω which fixes a point $p \in \Omega$. Then
(1) The eigenvalues of $J_f(p)$ all have modulus not exceeding 1;
(2) $|\det J_f(p)| \le 1$;
(3) If $|\det J_f(p)| = 1$, then f is a biholomorphism of Ω.

Lemma 2.3 ([16]) Let $\varphi \in Aut(\mathbb{B}^n)$ and $\varphi(0) = 0$. Then φ is a unitary transformation on $\mathbb{C}^n$. This is, there exists a unique unitary square matrix U such that

$$\varphi(z) = Uz$$

for any $z \in \mathbb{B}^n$.

Lemma 2.4 ([18]) Let $T \in \mathbb{C}^{m\times n}$ be an $m \times n$ matrix and $1 \le m \le n$. Then there are unitary square matrices U of order m and V of order n

respectively such that

$$T = U \begin{pmatrix} \lambda_1 & & 0 & 0\cdots 0 \\ & \ddots & & \cdots \\ 0 & & \lambda_m & 0\cdots 0 \end{pmatrix} V,$$

where $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, and $\lambda_1^2, \cdots, \lambda_m^2$ are the all eigenvalues of $T\overline{T}'$.

3. Main results

In this section, we first generalize Theorem 1.1$'$ to higher dimension. As application, we give a new boundary rigidity theorem for holomorphic self-mappings of the unit ball.

Theorem 3.1 Let $f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ be a holomorphic mapping with $f(0) = 0$. We have the following two conclusions.

(1) If f is holomorphic at $z = \alpha \in \partial\mathbb{B}^n$ with $f(\alpha) = \beta \in \partial\mathbb{B}^n$, then

$$\overline{\beta}' J_f(\alpha)\alpha \geq 1.$$

(2) If there exist linearly independent $\alpha_1, \cdots, \alpha_n \in \partial\mathbb{B}^n$ such that f is holomorphic at $z = \alpha_k$ with $f(\alpha_k) = \beta_k \in \partial\mathbb{B}^n\,(k = 1, \cdots, n)$, then the following n equalities

$$\overline{\beta_k}' J_f(\alpha_k)\alpha_k = 1\,(k = 1, \cdots, n) \tag{3.1}$$

hold if and only if

$$f(z) \equiv Uz$$

and $U = (\beta_1, \cdots, \beta_n)(\alpha_1, \cdots, \alpha_n)^{-1}$ is a unitary square matrix of order n.

When $n = 1$, Theorem 3.1 is just Theorem 1.1$'$.

Proof. (1) Take

$$g(\zeta) = \overline{\beta}' f(\zeta\alpha), \quad \zeta \in \mathbb{D}.$$

Then $g : \mathbb{D} \longrightarrow \mathbb{D}$ is a holomorphic function with $g(0) = 0$ and g is holomorphic at $\zeta = 1$ with $g(1) = 1$. Hence, by Theorem 1.1, we obtain

$$1 \leq g'(1) = \overline{\beta}' J_f(\alpha)\alpha.$$

The proof of (1) is complete.

(2) Suppose that the equalities hold in (3.1). For any $k = 1, \cdots, n$, take

$$g_k(\zeta) = \overline{\beta_k}' f(\zeta \alpha_k), \quad \zeta \in \mathbb{D}.$$

From $f(\alpha_k) = \beta_k \in \partial \mathbb{B}^n$ and (3.1), we know that $g_k : \mathbb{D} \longrightarrow \mathbb{D}$ is a holomorphic function with $g_k(0) = 0$, g_k is holomorphic at $\zeta = 1$ with $g_k(1) = 1$ and $g_k'(1) = \overline{\beta_k}' J_f(\alpha_k)\alpha_k = 1\,(k = 1, \cdots, n)$. Hence, by Theorem 1.1, we obtain $g_k(\zeta) \equiv \zeta$ for each $k = 1, \cdots, n$. It follows that

$$1 = g_k'(0) = \overline{\beta_k}' J_f(0)\alpha_k, \quad k = 1, \cdots, n. \tag{3.2}$$

Assume that

$$f(z) = J_f(0)z + P_2(z) + P_3(z) + \cdots, \; z \in \mathbb{B}^n,$$

where $P_j(z)$ is a homogenous polynomial mapping of order j, $j = 2, 3, \cdots$. Then

$$f(e^{i\theta} z) = J_f(0) z e^{i\theta} + P_2(z) e^{i2\theta} + P_3(z) e^{i3\theta} + \cdots, \; z \in \mathbb{B}^n, \quad \theta \in \mathbb{R}.$$

This implies

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z) e^{-i\theta} d\theta = J_f(0)z.$$

It follows that for any $z \in \mathbb{B}^n$,

$$\|J_f(0)z\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta} z)\| d\theta < 1. \tag{3.3}$$

That means

$$\|J_f(0)\alpha_k\| \leq 1, \quad k = 1, \cdots, n. \tag{3.4}$$

Thus, (3.2) and (3.4) yield

$$\|J_f(0)\alpha_k\| = 1.$$

This, together with $\|\beta_k\| = 1$ and (3.2), gives

$$\beta_k = J_f(0)\alpha_k, \; k = 1, \cdots, n.$$

Hence,

$$J_f(0) = (\beta_1, \cdots, \beta_n)(\alpha_1, \cdots, \alpha_n)^{-1}.$$

On the other hand, by Lemma 2.4, we set

$$J_f(0) = U \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} V,$$

where U and V are both unitary square matrices of order n, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Then

$$\overline{U}'\beta_k = \overline{U}' J_f(0)\alpha_k = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} (V\alpha_k) \in \partial\mathbb{B}^n, \quad k = 1, \cdots, n. \quad (3.5)$$

This shows that $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ maps these linearly independent unit vectors $V\alpha_1, \cdots, V\alpha_n$ into the unit vectors.

Now, we claim that $\lambda_1 = \cdots = \lambda_n = 1$. It follows from (3.3) that

$$0 \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq 1.$$

So without loss of generality, we assume that

$$0 \leq \lambda_n < 1.$$

Write

$$A_1 = (V\alpha_1, \cdots, V\alpha_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Since $\alpha_1, \cdots, \alpha_n$ are linearly independent, we know that A_1 is a nonsingular square matrix of order n, and

$$(a_{1j}, a_{2j}, \cdots, a_{nj})' \in \partial\mathbb{B}^n, \quad j = 1, \cdots, n. \quad (3.6)$$

By (3.5) and (3.6), we have

$$\lambda_1^2|a_{1j}|^2 + \cdots + \lambda_n^2|a_{nj}|^2 = 1 = |a_{1j}|^2 + \cdots + |a_{nj}|^2, \quad j = 1, \cdots, n.$$

It follows from this and $0 \leq \lambda_n < 1$ that

$$a_{nj} = 0, \quad j = 1, \cdots, n.$$

This contradicts the nonsingularity of A_1. Thus,

$$\lambda_1 = \cdots = \lambda_n = 1.$$

Hence, $J_f(0)$ is a unitary square matrix and $|\det J_f(0)| = 1$. By Lemma 2.2 and Lemma 2.3, we get

$$f(z) \equiv J_f(0)z$$

and $J_f(0) = (\beta_1, \cdots, \beta_n)(\alpha_1, \cdots, \alpha_n)^{-1}$ is a unitary square matrix of order n.

Conversely, suppose that

$$f(z) \equiv Uz$$

and $U = (\beta_1, \cdots, \beta_n)(\alpha_1, \cdots, \alpha_n)^{-1}$ is a unitary square matrix of order n. Then

$$\overline{\beta_k}' J_f(\alpha_k)\alpha_k = \overline{\alpha_k}'\overline{U}' U\alpha_k = 1, \quad k = 1, \cdots, n.$$

The proof of (2) is complete.

In particular, Theorem 3.1 shows that if $f(\alpha_k) = \alpha_k \, (k = 1, \cdots, n)$, then the equalities hold in (3.1) if and only if

$$f(z) \equiv z.$$

Thus, we obtain the following boundary rigidity theorem for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$.

Corollary 3.1 (Boundary Rigidity Theorem) Let $f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ be a holomorphic mapping with $f(0) = 0$ and there exist linearly independent $\alpha_1, \cdots, \alpha_n \in \partial\mathbb{B}^n$ such that f is holomorphic at $z = \alpha_k$ with $f(\alpha_k) = \alpha_k \in \partial\mathbb{B}^n \, (k = 1, \cdots, n)$. Then the following n equalities

$$\overline{\alpha_k}' J_f(\alpha_k)\alpha_k = 1 \, (k = 1, \cdots, n) \tag{3.7}$$

hold if and only if

$$f(z) \equiv z.$$

When $n = 1$, Corollary 3.1 is just (2) of Theorem 1.1.

Remark 3.1 In Corollary 3.1, one of the conditions is that it requires n fixed points which are linearly independent on $\partial\mathbb{B}^n$. If we remove even one fixed point, then $f(z) \equiv z$ does not hold. In fact, write

$$e_j = (0, \cdots, 0, \overbrace{1}^{\text{j-th}}, 0, \cdots, 0)' \in \partial \mathbb{B}^n, \quad j = 1, \cdots, n.$$

Take

$$f(z) = (z_1, \cdots, z_{n-1}, z_n(z_1^2 + \cdots + z_{n-1}^2))', \quad z \in \mathbb{B}^n.$$

Then for each $z \in \mathbb{B}^n$, we obtain

$$\begin{aligned}
&\|f(z)\|^2 \\
&= |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^2 |z_1^2 + \cdots + z_{n-1}^2|^2 \\
&\leq |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^2 (|z_1|^2 + \cdots + |z_{n-1}|^2)^2 \\
&< 1 - |z_n|^2 + |z_n|^2 (1 - |z_n|^2) \\
&\leq 1.
\end{aligned}$$

Hence, $f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ is a holomorphic mapping and $f(0) = 0$. It is obvious that f is holomorphic at e_j,

$$f(e_j) = e_j \quad \text{and} \quad J_f(e_j) = I_n, \quad j = 1, \cdots, n-1.$$

It is clear that the equality holds in (3.7). However,

$$f(z) = (z_1, \cdots, z_{n-1}, z_n(z_1^2 + \cdots + z_{n-1}^2))' \neq z.$$

Therefore, we must make use of at least n fixed points on $\partial \mathbb{B}^n$ in Corollary 3.1, which can imply the conclusion.

Remark 3.2 In Corollary 3.1, it follows from $f(0) = 0$ and $\alpha_1, \cdots, \alpha_n$ are linearly independent that there are $2C_n^0$ conditions. And for any $k = 1, \cdots, n$, $f(\alpha_k) = \alpha_k \in \partial \mathbb{B}^n$ and $\overline{\alpha_k}' J_f(\alpha_k)\alpha_k = 1$ show that there are $2C_n^1$ conditions. Thus, Corollary 3.1 uses

$$2C_n^0 + 2C_n^1 = 2(n+1)$$

conditions to get $f(z) \equiv z$.

In Theorem 1.2, $f(z) = z + O(\|z-p\|^4)$ if and only if

$$f(p) = p, \quad J_f(p) = I_n,$$

$$\frac{\partial^2 f}{\partial z_j \partial z_k}(p) = 0, \quad j,k = 1,\cdots,n, \tag{3.8}$$

and

$$\frac{\partial^3 f}{\partial z_j \partial z_k \partial z_l}(p) = 0, \quad j,k,l = 1,\cdots,n. \tag{3.9}$$

On the other hand, $J_f(p) = I_n$ if and only if $\frac{\partial f}{\partial z_k}(p) = e_k$, $k = 1,\cdots,n$, which means that there are C_n^1 conditions. Meanwhile, (3.8) and (3.9) show that there are C_{n+1}^2 and C_{n+2}^3 conditions, respectively. From this and $f(p) = p$, Theorem 1.2 uses

$$C_n^0 + C_n^1 + C_{n+1}^2 + C_{n+2}^3 = 1 + n + \frac{n(n+1)}{2} + \frac{n(n+1)(n+2)}{6}$$

conditions to get $f(z) \equiv z$.

In Theorem 1.3, $f(z) = z + o(\|z-p\|^2)$ if and only if $f(p) = p$, $J_f(p) = I_n$ and $\frac{\partial^2 f}{\partial z_j \partial z_k}(p) = 0$ for $j,k = 1,\cdots,n$. This, together with $f(z_0) = z_0$, means that Theorem 1.3 uses

$$2C_n^0 + C_n^1 + C_{n+1}^2 = 2 + n + \frac{n(n+1)}{2}$$

conditions to get $f(z) \equiv z$.

Compared with Theorem 1.2 and Theorem 1.3, Corollary 3.1 uses fewer conditions. Of course, we focus on the unit ball in $\mathbb{C}^n$, which is a special strongly pseudoconvex domain or strongly convex domain.

Next, we present a version of the boundary Schwarz lemma on the unit disk, which is an extension of Theorem 1.1$'$ as well.

Theorem 3.2 Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic mapping with $f(0) = a$ and let f be holomorphic at $z = \alpha \in \partial\mathbb{D}$ with $f(\alpha) = \beta \in \partial\mathbb{D}$. Then the following two conclusions hold.

(1) $\overline{\beta} f'(\alpha)\alpha \geq \frac{|1-\overline{a}\beta|^2}{1-|a|^2}$.

(2) $\overline{\beta} f'(\alpha)\alpha = \frac{|1-\overline{a}\beta|^2}{1-|a|^2}$ (3.10)

if and only if $f(z) \equiv \varphi_a(e^{i\theta} z)$ and $e^{i\theta} = \varphi_a(\beta)\alpha^{-1}$, where $\theta \in \mathbb{R}$.

When $a = 0$, Theorem 3.2 is just Theorem 1.1$'$.

Proof. Take

$$g(z) = \varphi_a \circ f(z), \quad z \in \mathbb{D}.$$

Then by Lemma 2.1, $g : \mathbb{D} \longrightarrow \mathbb{D}$ is a holomorphic function with $g(0) = 0$ and g is holomorphic at $z = \alpha$ with $g(\alpha) = \varphi_a(\beta) \in \partial\mathbb{D}$. It follows from Theorem 1.1$'$ that

$$\begin{aligned}
1 &\le \overline{\varphi_a(\beta)} g'(\alpha)\alpha \\
&= \overline{\varphi_a(\beta)} \varphi_a'(\beta) f'(\alpha)\alpha \\
&= \overline{\varphi_a(\beta)} \left[\frac{-(1-|a|^2)}{(1-\overline{a}\beta)^2} \right] f'(\alpha)\alpha \\
&= \frac{\overline{a} - \overline{\beta}}{1 - a\overline{\beta}} \left[\frac{-(1-|a|^2)}{(1-\overline{a}\beta)^2} \right] f'(\alpha)\alpha \\
&= \frac{\overline{\beta} - \overline{a}}{(1-\overline{a}\beta)\overline{\beta}} \left(\frac{1-|a|^2}{|1-\overline{a}\beta|^2} \right) \overline{\beta} f'(\alpha)\alpha \\
&= \overline{\beta} f'(\alpha)\alpha \left(\frac{1-|a|^2}{|1-\overline{a}\beta|^2} \right).
\end{aligned}$$

This gives

$$\overline{\beta} f'(\alpha)\alpha \ge \frac{|1-\overline{a}\beta|^2}{1-|a|^2}.$$

The proof of (1) is complete.

Suppose that the equality holds in (3.10). Then $\overline{\varphi_a(\beta)} g'(\alpha)\alpha = 1$, which implies $g(z) \equiv e^{i\theta} z$ and $e^{i\theta} = \varphi_a(\beta)\alpha^{-1}$ by Theorem 1.1$'$, where $\theta \in \mathbb{R}$. That is $f(z) \equiv \varphi_a(e^{i\theta} z)$.

Conversely, suppose that $f(z) \equiv \varphi_a(e^{i\theta} z)$ and $e^{i\theta} = \varphi_a(\beta)\alpha^{-1}$. Then a simple calculation shows that the equality holds in (3.10). The proof of (2) is complete.

Lastly, we establish the high-dimensional version of Theorem 3.2 as follows, which also extends Theorem 3.1.

Theorem 3.3 Let $f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ be a holomorphic mapping with $f(0) = a$. We have the following two conclusions.

(1) If f is holomorphic at $z = \alpha \in \partial\mathbb{B}^n$ with $f(\alpha) = \beta \in \partial\mathbb{B}^n$, then

$$\overline{\beta}' J_f(\alpha)\alpha \ge \frac{|1-\overline{a}'\beta|^2}{1-\|a\|^2}.$$

(2) If there exist linearly independent $\alpha_1, \cdots, \alpha_n \in \partial\mathbb{B}^n$ such that f is holomorphic at $z = \alpha_k$ with $f(\alpha_k) = \beta_k \in \partial\mathbb{B}^n$ $(k = 1, \cdots, n)$, then the following n equalities

$$\overline{\beta_k}' J_f(\alpha_k)\alpha_k = \frac{|1 - \overline{a}'\beta_k|^2}{1 - \|a\|^2} \ (k = 1, \cdots, n) \tag{3.11}$$

hold if and only if

$$f(z) \equiv \varphi_a(Uz)$$

and $U = (\varphi_a(\beta_1), \cdots, \varphi_a(\beta_n))(\alpha_1, \cdots, \alpha_n)^{-1}$ is a unitary square matrix of order n.

When $n = 1$, Theorem 3.3 is just Theorem 3.2.

Proof. It is clear that

$$g = \varphi_a \circ f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$$

is a holomorphic mapping with $g(0) = 0$. Moreover, by Lemma 2.1, we know that g is holomorphic at α with $g(\alpha) = \varphi_a(\beta) \in \partial\mathbb{B}^n$. It follows from Theorem 3.1 that

$$1 \le \overline{g(\alpha)}' J_g(\alpha)\alpha = \overline{\varphi_a(\beta)}' J_{\varphi_a}(\beta) J_f(\alpha)\alpha. \tag{3.12}$$

By Lemma 2.1, we obtain

$$\begin{aligned}
&\overline{\varphi_a(\beta)}' J_{\varphi_a}(\beta) \\
&= \frac{\overline{a}' - \overline{\beta}'}{1 - \overline{\beta}' a} A^2 \left[-\frac{I_n}{1 - \overline{a}'\beta} + \frac{(a - \beta)\overline{a}'}{(1 - \overline{a}'\beta)^2} \right] \\
&= \frac{1}{|1 - \overline{a}'\beta|^2} (\overline{a}' - \overline{\beta}')(s^2 I_n + a\overline{a}') \left[-I_n + \frac{(a - \beta)\overline{a}'}{1 - \overline{a}'\beta} \right] \\
&= \frac{1}{|1 - \overline{a}'\beta|^2} [(1 - \overline{\beta}' a)\overline{a}' - s^2 \overline{\beta}'] \left[-I_n + \frac{(a - \beta)\overline{a}'}{1 - \overline{a}'\beta} \right] \\
&= \frac{1}{|1 - \overline{a}'\beta|^2} \left[s^2 \overline{\beta}' - (1 - \overline{\beta}' a)\overline{a}' + (1 - \overline{\beta}' a)\frac{(\|a\|^2 - \overline{a}'\beta)\overline{a}'}{1 - \overline{a}'\beta} + \frac{s^2(1 - \overline{\beta}' a)\overline{a}'}{1 - \overline{a}'\beta} \right] \\
&= \frac{1}{|1 - \overline{a}'\beta|^2} \left[s^2 \overline{\beta}' - (1 - \overline{\beta}' a) \left(1 - \frac{\|a\|^2 - \overline{a}'\beta}{1 - \overline{a}'\beta} - \frac{1 - \|a\|^2}{1 - \overline{a}'\beta} \right) \overline{a}' \right] \\
&= \frac{1 - \|a\|^2}{|1 - \overline{a}'\beta|^2} \overline{\beta}'.
\end{aligned} \tag{3.13}$$

This, together with (3.12), implies

$$\overline{\beta}' J_f(\alpha)\alpha \geq \frac{|1-\overline{a}'\beta|^2}{1-\|a\|^2}.$$

The proof of (1) is complete.

Suppose that the equalities hold in (3.11). Then by (3.12) and (3.13), we have

$$\overline{g(\alpha_k)}' J_g(\alpha_k)\alpha_k = 1, \quad k = 1, \cdots, n.$$

Hence, by Theorem 3.1, we get

$$g(z) \equiv Uz$$

and

$$U = (g(\alpha_1), \cdots, g(\alpha_n))(\alpha_1, \cdots, \alpha_n)^{-1}$$

is a unitary square matrix of order n. This, together with $g(z) = \varphi_a(f(z))$ and $\varphi_a = \varphi_a^{-1}$, gives

$$f(z) \equiv \varphi_a(Uz)$$

and $U = (\varphi_a(\beta_1), \cdots, \varphi_a(\beta_n))(\alpha_1, \cdots, \alpha_n)^{-1}$ is a unitary square matrix of order n.

Conversely, suppose that $f(z) \equiv \varphi_a(Uz)$ and

$$U = (\varphi_a(\beta_1), \cdots, \varphi_a(\beta_n))(\alpha_1, \cdots, \alpha_n)^{-1}$$

is a unitary square matrix of order n. Then similar to the proof in (3.13), for any $k = 1, \cdots, n$, we obtain

$$\begin{aligned} & \overline{\beta_k}' J_f(\alpha_k)\alpha_k \\ & = \overline{\varphi_a(U\alpha_k)}' J_{\varphi_a}(U\alpha_k)U\alpha_k \\ & = \frac{1-\|a\|^2}{|1-\overline{a}'U\alpha_k|^2}\overline{U\alpha_k}' U\alpha_k \\ & = \frac{1-\|a\|^2}{|1-\overline{a}'\varphi_a(\beta_k)|^2}. \end{aligned} \tag{3.14}$$

By Lemma 2.1, we have

$$\begin{aligned}
&1-\overline{a}'\varphi_a(\beta_k)\\
&=1-\overline{a}'A\frac{a-\beta_k}{1-\overline{a}'\beta_k}\\
&=1-\overline{a}'\frac{a-\beta_k}{1-\overline{a}'\beta_k}\\
&=1-\frac{\|a\|^2-\overline{a}'\beta_k}{1-\overline{a}'\beta_k}\\
&=\frac{1-\|a\|^2}{1-\overline{a}'\beta_k},\quad k=1,\cdots,n.
\end{aligned}\tag{3.15}$$

Combine (3.14) and (3.15) to yield

$$\overline{\beta_k}'J_f(\alpha_k)\alpha_k=\frac{|1-\overline{a}'\beta_k|^2}{1-\|a\|^2},\quad k=1,\cdots,n.$$

The proof of (2) is complete.

Remark 3.3 From the proof of Theorem 3.1, Theorem 3.2 and Theorem 3.3, it is clear that we need only to assume that f is C^1 up to the boundary of the unit ball or the unit disk near $z=\alpha_1,\cdots,\alpha_n$ and $z=\alpha$, respectively.

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