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Moduli Spaces of Semistable Sheaves of Dimension 1 on $\mathbb{P}^2$

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Abstract: Let $M(d,\chi)$ be the moduli space of semistable sheaves of rank 0, Euler characteristic χ and first Chern class dH $(d>0)$, with H the hyperplane class in $\mathbb{P}^2$. We give a description of $M(d,\chi)$, viewing each sheaf as a class of matrices with entries in $\bigoplus_{i\geq 0} H^0(\mathcal{O}_{\mathbb{P}^2}(i))$. We show that there is a big open subset of $M(d,1)$ isomorphic to a projective bundle over an open subset of a Hilbert scheme of points on $\mathbb{P}^2$. Finally we compute the classes of $M(4,1)$, $M(5,1)$ and $M(5,2)$ in the Grothendieck ring of varieties, especially we conclude that $M(5,1)$ and $M(5,2)$ are of the same class.
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1. Introduction.

Moduli spaces M of semistable sheaves of dimension 1 on surfaces are very interesting and many people have studied on them. On $K3$ or abelian surfaces, for a large number of M, Yoshioka has given explicitly the deformation classes of them in [9]. Le Potier studied a lot on M for $\mathbb{P}^2$ such as their Picard groups and rationalities in [5]. Drézet and Maican studied sheaves of dimension 1 on $\mathbb{P}^2$ with multiplicity 4,5 and 6, via their locally free resolutions (see [2],[7] and [8]). But except few trivial cases, the classes of M for $\mathbb{P}^2$ in the Grothendieck group of varieties are not known.

Let $M(d,\chi)$ be the moduli space of semistable sheaves of rank 0, first Chern class dH $(d>0)$ and Euler characteristic χ on $\mathbb{P}^2$. $M(d,\chi)\simeq M(d,\chi')$ if $\chi\equiv\pm\chi'$ (mod d). There is a map $\pi: M(d,\chi)\rightarrow |dH|$ sending each sheaf to its support. Fibers of π over integral curves are isomorphic to their (compactified) Jacobians. But fibers of π over non-integral curves are not well understood.

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In this paper we build a 1-1 correspondence between pure sheaves of dimension 1 on $\mathbb{P}^2$ and pairs (E,f) with E direct sums of line bundles on $\mathbb{P}^2$ and $f: E\otimes\mathcal{O}_{\mathbb{P}^2}(-1)\hookrightarrow E$ injective, then after putting a stability condition on these pairs we can view $M(d,\chi)$ as the moduli space of semistable pairs (E,f). From this point of view, we somehow avoid studying fibers of π over non-integral curves. However for a general d, the moduli space is still very complicated. We are only able to describe a big open set of $M(d,\chi)$ with $\chi=1$. We have the following proposition which is a generalization of Proposition 3.3.1 in [2] to all multiplicities.

Proposition 1.1 (Proposition 4.5). *There is an open subset $W^d\subset M(d,1)$ with $M(d,1)-W^d$ of codimension ≥ 2, and $W^d\simeq\mathbb{P}(\mathcal{V}^d)$, where $\mathcal{V}^d$ is a vector bundle of rank $3d$ over $N_0^d := Hilb^{[\frac{(d-1)(d-2)}{2}]}(\mathbb{P}^2)-\Omega_{d-3}^{[\frac{(d-1)(d-2)}{2}]}$ with $Hilb^{[n]}(\mathbb{P}^2)$ the Hilbert scheme of n-points on $\mathbb{P}^2$ and $\Omega_k^{[n]}$ the closed subscheme of $Hilb^{[n]}(\mathbb{P}^2)$ parametrizing n-points lying on a curve of class kH.*

Denote by $[X]$ the class of a variety X in the Grothendieck ring of varieties. For $d\leq 5$ and $g.c.d.(d,\chi)=1$, we compute $[M(d,\chi)]$ and get the following three theorems, with $\mathbb{L}:=[\mathbb{A}^1]$ the class of the affine line.

Theorem 1.2 (Theorem 5.1). *For $d\leq 3$, $M(d,1)=W^d$. Moreover $W^d\simeq|dH|\simeq\mathbb{P}^{3d-1}$ for $d=1,2$; $W^3\simeq\mathcal{C}_3$ with $\mathcal{C}_3$ the universal curve in $\mathbb{P}^2\times|3H|$.*

Theorem 1.3 (Theorem 5.2). $[M(4,1)]=\sum_{i=0}^{17} b_{2i}\mathbb{L}^i$ *and*

$$\begin{aligned}&b_0=b_{34}=1,\ b_2=b_{32}=2,\ b_4=b_{30}=6,\\&b_6=b_{28}=10,\ b_8=b_{26}=14,\ b_{10}=b_{24}=15,\\&b_{12}=b_{14}=b_{16}=b_{18}=b_{20}=b_{22}=16.\end{aligned}$$

In particular the Euler number $e(M(4,1))$ of the moduli space is 192.

Theorem 1.4 (Theorem 6.1). $[M(5,1)]=[M(5,2)]=\sum_{i=0}^{26} b_{2i}\mathbb{L}^i$ *and*

$$\begin{aligned}&b_0=b_{52}=1,\ b_2=b_{50}=2,\ b_4=b_{48}=6,\\&b_6=b_{46}=13,\ b_8=b_{44}=26,\ b_{10}=b_{42}=45,\\&b_{12}=b_{40}=68,\ b_{14}=b_{38}=87,\ b_{16}=b_{36}=100,\\&b_{18}=b_{34}=107,\ b_{20}=b_{32}=111,\ b_{22}=b_{30}=112,\\&b_{24}=b_{26}=b_{28}=113.\end{aligned}$$

In particular the Euler number of both moduli spaces is 1695.

Remark 1.5. *The Euler numbers $e(M(d,\chi))$ have been computed in [4] partially using physics arguments for $M(d,\chi)$ smooth. They have $e(M(d,\chi)) = (-1)^{dim(M(d,\chi))} n_d^0$ with n_d^0 so-called BPS states of weight 0 for the local $\mathbb{P}^2$ (see Equation (4.2) and Table 4 in Section 8.3 in [4]). We see that our result accords with theirs for $d \leq 5$.*

Remark 1.6. *Theorem 1.3 and Theorem 1.4 give the motive decompositions of $M(d,r)$ for $d = 4, 5$, r coprime to d. But according to the result in [11] that these moduli spaces admit affine pavings, we also get cell decompositions of them.*

The structure of the paper is arranged as follows. In Section 2, we construct a 1-1 correspondence between pure sheaves of dimension 1 and pairs (E, f). The stability condition of (E, f) is given in Section 3. In Section 4 we study the big open set W^d in $M(d,1)$ and prove Proposition 1.1. Theorem 1.2 and Theorem 1.3 are proved in Section 5 while Theorem 1.4 is proved in the last section—Section 6. We have Appendix A and B where we prove some technical lemmas used in Section 6.

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2. Pure sheaves of dimension 1 on $\mathbb{P}^2$.

From now on except otherwise stated, a pair (E, f) on $\mathbb{P}^2$ always satisfies the following two conditions:

$$(1) E \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i) \text{ i.e.} E \text{ is a direct sum of line bundles on } \mathbb{P}^2; \tag{2.1}$$

$$(2) f \in Hom(E \otimes \mathcal{O}_{\mathbb{P}^2}(-1), E) \text{ and moreover } f \text{ is injective.} \tag{2.2}$$

Definition 2.1. *We say two pairs (E, f) and (E', f') are isomorphic if $E \simeq E'$ and there exist two isomorphisms φ and ϕ from E to E' such that the following diagram commutes*

$$\begin{tikzcd} E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "f"] \arrow[d, "\varphi\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"'] & E \arrow[d, "\phi"] \\ E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "f'"] & E'. \end{tikzcd} \tag{2.3}$$

Define two sets as follows

$$\mathcal{A} := \{Isomorphism\ classes\ of\ pure\ sheaves\ of\ dimension\ 1\};$$

$$\mathcal{B} := \{Isomorphism\ classes\ of\ pairs\ (E, f)\}.$$

We have a set-map θ from $\mathcal{B}$ to $\mathcal{A}$ sending each pair to its cokernel. We want to prove that θ is bijective. First we have the following lemma.

Lemma 2.2. *Let F be a sheaf of rank 0 and first Chern class dH $(d > 0)$ on $\mathbb{P}^2$, then F is pure of dimension 1 if and only if F lies in the following exact sequence with E_F a direct sum of line bundles.*

$$0 \to E_F \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to E_F \to F \to 0. \tag{2.4}$$

Proof. The "if" part is obvious: F in (2.4) is of rank 0 and has a locally free resolution of length 1, hence F is pure of dimension 1. To show the "only if", it is enough to construct the sequence (2.4) for every pure sheaf F. We first follow the construction given by Le Potier in Proposition 3.10 in [5].

Denote by $Supp(F)$ the support of F. Since F is a torsion sheaf, we can take a point $x \in \mathbb{P}^2 - Supp(F)$. Let $U := \mathbb{P}^2 - \{x\}$, then U is isomorphic to the total space of $\mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$ with a projection $p : U \to \mathbb{P}^1$. F is a sheaf of $\mathcal{O}_U$-modules. F is pure, hence p_*F is pure and hence of form $\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(n_i)$. p_*F has a structure of $p_*\mathcal{O}_U$-module which gives a morphism $f_1 : p_*F \to p_*F \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\widetilde{E_F} := p^*(p_*F)$. Pull f_1 back to U and define the following morphim

$$\tilde{f} := (p^*f_1 - \lambda id_{\widetilde{E_F}}) \otimes p^*id_{\mathcal{O}_{\mathbb{P}^1}(-1)} : \widetilde{E_F} \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-1) \to \widetilde{E_F}, \tag{2.5}$$

where λ is the canonical section of $p^*\mathcal{O}_{\mathbb{P}^1}(1)$. $\tilde{f}$ is injective and the cokernel is the sheaf F.

On the other hand, $\mathbb{P}^2 - U = \{x\}$ is of codimension 2 and $\widetilde{E_F}$ is a direct sum of line bundles on U, hence both $\tilde{f}$ and $\widetilde{E_F}$ can be extended to the whole $\mathbb{P}^2$ and we get a resolution of F on $\mathbb{P}^2$ as in (2.4) and $E_F \simeq j_*\widetilde{E_F} \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i)$ with $j : U \to \mathbb{P}^2$ the open immersion. Hence the lemma. $\square$

Remark 2.3. *The form of E in (E, f) is determined by $h^0(F(n))$ with a finite number of n, where $F = coker(f)$ and $F(n) := F \otimes \mathcal{O}_{\mathbb{P}^2}(n)$. Moreover $E_1 \simeq E_2$ iff $h^0(F_1(n)) = h^0(F_2(n))$ for all n.*

Lemma 2.2 implies that θ is surjective, then we have the injectivity by the following lemma.

Lemma 2.4. *Take any two exact sequences*

$$0 \longrightarrow E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f_1} E_1 \xrightarrow{g_1} F_1 \longrightarrow 0$$

$$0 \longrightarrow E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f_2} E_2 \xrightarrow{g_2} F_2 \longrightarrow 0,$$

with E_i direct sums of line bundles, then $F_1 \simeq F_2$ if and only if $(E_1, f_1) \simeq (E_2, f_2)$.

Proof. We only need to show the "only if". $E_1 \simeq E_2$ if $F_1 \simeq F_2$ by Remark 2.3 . We then want to construct the commutative diagram (2.3). f_i can be represented by square matrices with entries in $\bigoplus_{i\geq 0} H^0(\mathcal{O}_{\mathbb{P}^2}(i))$. After some invertible transformation, we can ask f_i to have the following form

$$f_i = \begin{pmatrix} \mathbb{I}_i & 0 \\ 0 & \mathtt{T}_i \end{pmatrix},$$

with $\mathbb{I}_i$ the identity matrix and $\mathtt{T}_i$ a square matrix with entries in $\bigoplus_{i\geq 1} H^0(\mathcal{O}_{\mathbb{P}^2}(i))$. Hence we can write $E_i \simeq K_i \oplus M_i$ and $E_i \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \simeq K_i \oplus N_i$, such that f_i splits into the direct sum of an identity on K_i and a morphism $t_i : N_i \to M_i$ represented by $\mathtt{T}_i$. We then have the following exact sequence which is a minimal free resolution of F_i (see [3] Page 5 Definition)

$$0 \longrightarrow N_i \xrightarrow{t_i} M_i \xrightarrow{g_i|_{M_i}} F_i \longrightarrow 0.$$

Because of the uniqueness of the minimal free resolution (see [3] Page 6 Theorem 1.6), we have the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r, "t_1"] \arrow[d, "\beta"', "\simeq"] & M_1 \arrow[r, "g_1|_{M_1}"] \arrow[d, "\alpha"', "\simeq"] & F_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & N_2 \arrow[r, "t_2"] & M_2 \arrow[r, "g_2|_{M_2}"] & F_2 \arrow[r] & 0.
\end{tikzcd}
\tag{2.6}
$$

Hence we have $K_1 \simeq K_2$ because $E_1 \simeq E_2$ and $M_1 \simeq M_2$. We define a map $\phi : E_1 \to E_2$ to be $\mathrm{I}_K \oplus \alpha$ with I_K an isomorphism from K_1 to K_2, and similarly we define the other map $\varphi \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)} : E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ to be $\mathrm{I}_K \oplus \beta$. Then we have the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "f_1"] \arrow[d, "\varphi\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"', "\simeq"] & E_1 \arrow[r, "g_1"] \arrow[d, "\phi"', "\simeq"] & F_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "f_2"] & E_2 \arrow[r, "g_2"] & F_2 \arrow[r] & 0.
\end{tikzcd}
\tag{2.7}
$$

This finishes the proof of the lemma. □

We finally get the following proposition.

Proposition 2.5. *There is a 1-1 correspondence between isomorphism classes of pure sheaves of dimension 1 and isomorphism classes of pairs (E, f).*

3. The stability condition.

We put a stability condition on our pairs (E, f), so that the map θ induces a bijection from semistable pairs to semistable sheaves. Given a pair (E, f) and its image F via θ, we write down the exact sequence

$$
0 \longrightarrow E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f} E \xrightarrow{g} F \longrightarrow 0\ . \tag{3.1}
$$

Recall that the slop of a torsion free sheaf E, $\mu(E)$, is defined as follows

$$
\mu(E) := \frac{deg(E)}{rank(E)};
$$

and for a sheaf F of dimension 1 we have

$$
\mu(F) := \frac{\chi(F)}{deg(F)}.
$$

We then have $\mu(E)+1=\mu(F)$ for E,F in the sequence (3.1).

Definition 3.1. *We say a pair (E,f) is (semi)stable if for any subsheaf $E'\subsetneq E$ and E' a direct sum of line bundles such that $f^{-1}(E')\simeq E'\otimes\mathcal{O}_{\mathbb{P}^2}(-1)$, we have $\mu(E')(\leq)<\mu(E)$.*

Lemma 3.2. *θ induces a bijection from semistable pairs to semistable sheaves.*

Proof. Look at the sequence (3.1). To prove the lemma, we only need to prove that $\forall F'\subsetneq F$, $\exists E'$ a direct sum of line bundles and $f^{-1}(E')\simeq E'\otimes\mathcal{O}_{\mathbb{P}^2}(-1)$, such that $F\simeq coker(f|_{f^{-1}(E')})$. Keep notations the same as in the proof of Lemma 2.2, and we see that p_*F' is a direct sum of line bundles on $\mathbb{P}^1$ and $p_*F'\subset p_*F$. By following the construction in the proof of Lemma 2.2, we get $E'\simeq j_*p^*p_*F'$ a direct sum of line bundles and moreover $f^{-1}(E')\simeq E'\otimes\mathcal{O}_{\mathbb{P}^2}(-1)$, hence the lemma. □

Remark 3.3. *By Remark 2.3, we see that E_1 might not isomorphic to E_2 while F_1 is only S-equivalent to F_2.*

From now on let H be the hyperplane class in $\mathbb{P}^2$ and $u_{d,\chi}$ the class in the Grothendieck group of coherent sheaves on $\mathbb{P}^2$, which is of rank 0, first Chern class dH and Euler characteristic χ. Denote by $M(d,\chi)$ the moduli space parametrizing semistable sheaves of class $u_{d,\chi}$. Then $M(d,\chi)$ is irreducible (see [5] Theorem 3.1) and the stable locus $M(d,\chi)^s$ is smooth. $M(d,\chi)\simeq M(d,\chi')$ if $\chi\equiv\pm\chi'\pmod d$. $M(d,\chi)=M(d,\chi)^s$ if and only if $g.c.d.(d,\chi)=1$. Hence $M(d,1)=M(d,1)^s$ and the moduli space is smooth of dimension d^2+1 for all $d\geq 1$. Moreover there is a universal sheaf on $M(d,1)\times\mathbb{P}^2$ by Theorem 3.19 in [5].

Let $g.c.d(d,\chi)=1$, then we can assign every point F in $M(d,\chi)$ uniquely to a pair (E,f) such that $rank(E)=d$ and $c_1(E)=(\chi-d)H$. We view every point in $M(d,\chi)$ as a pair (E,f) and stratify $M(d,\chi)$ by the form of E, then every stratum is a constructible set in $M(d,\chi)$.

We write down the following lemma for future use.

Lemma 3.4. *Let (E,f) be a semistable pair. Let D',D'' be two direct summands of E such that $D'\simeq D''$ and $f(D'\otimes\mathcal{O}_{\mathbb{P}^2}(-1))\subset D''$. Then we have $\mu(D')\leq\mu(E)$. In particular E must have the form $\oplus_{i=1}^k\mathcal{O}_{\mathbb{P}^2}(n+i)^{\oplus a_i}$ with n some integer and $a_i>0$ for all $1\leq i\leq k$.*

Proof. Since $D' \simeq D''$ and they are both direct summands of E, $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'' \Leftrightarrow f^{-1}(D'') = D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \simeq D'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$. Hence the first statement.

Write $E \simeq \oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ with $a_i > 0$ and $n_1 > n_2 > \ldots > n_k$, then we want to show that $n_i - n_{i+1} = 1$ for all $1 \leq i \leq k-1$. Assume $\exists i_0$ such that $n_{i_0} - n_{i_0+1} \geq 2$, then $f((\oplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \oplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ and $\mu(\oplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}) > \mu(E)$, which is a contradiction. Hence the statement. □

4. A big open subset in $M(d, 1)$.

We want to give a concrete description of an open subset in $M(d, 1)$, where the pairs (E, f) satisfy that $E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$. We first have the following lemma.

Lemma 4.1. *If $E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$, then (E, f) is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. The lemma is equivalent to Claim 4.2 in [6]. We also prove it here. Because of Lemma 3.4, we only need to prove the "if". By direct observation we see that if $E' \subset E$ is a direct sum of line bundles such that $\mu(E') > \mu(E)$, then E' is a direct summand of E. Hence the lemma. □

Define $\widetilde{W}^d := \{(E, f) | E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}\} \subset M(d, 1)$, we then have the following lemma.

Lemma 4.2. *$M(d, 1) - \widetilde{W}^d$ is of codimension ≥ 2 in $M(d, 1)$.*

Proof. For any point $x \in \mathbb{P}^2$, denote Y_x to be the open subset of $M(d, 1)$ where the pair (E, f) satisfies that $x \notin Supp(coker(f))$. $M(d, 1)$ can be covered by finitely many Y_x. According to Proposition 3.14 in [5], $Y_x \cap (M(d, 1) - \widetilde{W}^d)$ is of codimension ≥ 2 in Y_x, hence the lemma. □

Now we look at a pair (E, f) in $\widetilde{W}^d$. $f(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \not\subset \mathcal{O}_{\mathbb{P}^2}$ by Lemma 4.1, hence the restriction $f_{restr} : \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$ is nonzero. Therefore we can ask f to identify $\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with a summand $\mathcal{O}_{\mathbb{P}^2}(-1)$

and then f can be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ A & 0 & B \end{pmatrix},$$

where A is a $(d-1)\times 1$ matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and B a $(d-1)\times(d-2)$ matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. B provides a morphism $f_B : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d-2}$. By Lemma 4.1 the stability condition is equivalent to the following condition

Condition 4.3. *For any* $1 < d' < d$, $f_B(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d'-1}) \nsubseteq \mathcal{O}_{\mathbb{P}^2}^{\oplus d'-2}$.

Let $f_{B^t} : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1}$ be a morphism represented by the transform of B, B^t. Then Condition 4.3 is equivalent to the following condition

Condition 4.4. *For any* $1 < d'' \leq d-2$, $f_{B^t}^{-1}(\mathcal{O}_{\mathbb{P}^2}^{\oplus d''}) \not\simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d''}$.

We have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \arrow[r, "f_{B^t}"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1} \arrow[r, "f_q"] & Q_f \arrow[r] & 0 . \\ & & \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[u, "f_{A^t}"] \arrow[ur, "\sigma_f := f_q \circ f_{A^t}"'] & & \end{tikzcd} \tag{4.1}$$

The injectivity of f_{B^t} is because of the injectivity of f. Let $F := coker(f)$ and $F^\vee := \mathcal{E}xt^1(F, \mathcal{O}_{\mathbb{P}^2})$, then $F^{\vee\vee} \simeq F$ and moreover F and $F^\vee$ are determined by each other (see [10] Lemma A.0.13). We write down a commutative diagram as follows

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \arrow[r] \arrow[d, "\simeq"] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r] \arrow[d, "f_{B^t} \oplus f_{A^t}"] & \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r] \arrow[d, "\sigma_f"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \arrow[r, "f_{B^t}"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1} \arrow[r, "f_q"] \arrow[d] & Q_f \arrow[r] \arrow[d, "\delta"] & 0 \\ & & F^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "\simeq"] \arrow[d] & F^\vee \otimes \mathcal{O}_{p^2}(-2) \arrow[d] & \\ & & 0 & 0 & \end{tikzcd} \tag{4.2}$$

We see that the isomorphism class of $F^\vee$ is determined by the pair (Q_f, σ_f), hence so is the isomorphism class of F.

Define $W^d := \{[(E,f)] \in \widetilde{W}^d | Q_f \textit{ is torsion free}\}$. Then we have the following proposition.

Proposition 4.5. *$W^d \simeq \mathbb{P}(\mathcal{V}^d)$, where $\mathcal{V}^d$ is a vector bundle of rank $3d$ over $N_0^d := Hilb^{[\frac{(d-1)(d-2)}{2}]}(\mathbb{P}^2) - \Omega_{d-3}^{[\frac{(d-1)(d-2)}{2}]}$ with $Hilb^{[n]}(\mathbb{P}^2)$ the Hilbert scheme of n-points on $\mathbb{P}^2$ and $\Omega_k^{[n]}$ the closed subscheme of $Hilb^{[n]}(\mathbb{P}^2)$ parametrizing n-points lying on a curve of class kH.*

Proof. Condition 4.4 is satisfied automatically for Q_f torsion free. Hence W^d consists of all the pairs (Q_f, σ_f) in diagram (4.1) with Q_f torsion free.

Define $\bar{d} := \frac{(d-1)(d-2)}{2}$. If Q_f in diagram (4.1) is torsion free, then by direct calculation we know that $Q_f \simeq I_{\bar{d}} \otimes \mathcal{O}_{\mathbb{P}^2}(d-2)$, with I_n the ideal sheaf of a 0-dimensional subscheme of length n on $\mathbb{P}^2$.

The following lemma shows that N_0^d parametrizes all the torsion free Q_f in diagram (4.1).

Lemma 4.6. *$I_{\bar{d}}(d-2) := I_{\bar{d}} \otimes \mathcal{O}_{\mathbb{P}^2}(d-2)$ has the following resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1} \longrightarrow I_{\bar{d}}(d-2) \longrightarrow 0\,. \tag{4.3}$$

if and only if $H^0(I_{\bar{d}}(d-3)) = 0$.

Proof. The lemma is equivalent to Proposition 4.5 in [1] and it is also a straightforward consequence of Corollary 3.9 in [3] Page 38 and Proposition 3.1 in [3] Page 32. □

Up to scalars, σ_f can be viewed as an element in $\mathbb{P}H^0(Q_f(2))$ which is exactly $det(f)$.

Because N_0^d is open in $Hilb^{[\bar{d}]}(\mathbb{P}^2)$, on $\mathbb{P}^2 \times N_0^d$ we have a universal sheaf $\mathcal{I}_{\bar{d}}$ which restricted to the fiber over each point $[I_{\bar{d}}] \in N_0^d$ is the ideal sheaf $I_{\bar{d}}$. We have the diagram

$$\begin{tikzcd} \mathcal{I}_{\bar{d}} \arrow[r] & \mathbb{P}^2 \times N_0^d \arrow[dl, "q"'] \arrow[d, "p"] \\ \mathbb{P}^2 & N_0^d. \end{tikzcd} \tag{4.4}$$

$R^i p_*(\mathcal{I}_{\bar{d}} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(d)) = 0$ for $i \geq 1$ and $p_*(\mathcal{I}_{\bar{d}} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(d))$ is locally free of rank $3d$. Define $\mathcal{V}^d := p_*(\mathcal{I}_{\bar{d}} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(d))$. There is a 1-1 correspondence between points in $\mathbb{P}(\mathcal{V}^d)$ and isomorphism classes of (Q_f, σ_f) with Q_f torsion free. To prove the proposition, it is enough to construct a family $\mathcal{F}$ of stable sheaves of class $u_{d,1}$ over $\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d)$.

We have the following commutative diagram

$$\begin{tikzcd} \mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d) \arrow[r, "id_{\mathbb{P}^2}\times\pi"] \arrow[d, "\tilde{p}"] & \mathbb{P}^2 \times N_0^d \arrow[d, "p"] \arrow[dr, "q"] & \\ \mathbb{P}(\mathcal{V}^d) \arrow[r, "\pi"] & N_0^d & \mathbb{P}^2 \end{tikzcd} \tag{4.5}$$

Denote by $\mathcal{O}_\pi(1)$ the relative polarization on $\mathbb{P}(\mathcal{V}^d)$ over N_0^d. We have the following exact sequence on $\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d)$

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}(\mathcal{V}^d)} \to \tilde{p}^*\mathcal{O}_\pi(1) \otimes (id_{\mathbb{P}^2} \times \pi)^*(\mathcal{I}_{\bar{d}} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(d)) \to \mathcal{F}^\vee \to 0. \tag{4.6}$$

We see that fiberwise (4.6) is the first vertical exact sequence from the right hand side in (4.2) tensored by $\mathcal{O}_{\mathbb{P}^2}(2)$. Hence $\mathcal{F}^\vee$ is a family of stable sheaves of class $u_{d,1}^\vee$. We get $\mathcal{F}$ by taking the dual. Hence the proposition. □

We now have a concrete description of W^d.

Proposition 4.7. *$M(d,1) - W^d$ is of codimension ≥ 2 in $M(d,1)$.*

Proof. By Lemma 4.2, we only need to show that $\widetilde{W}^d - W^d$ is of codimension at least 2 in $M(d,1)$. Denote by $|dH|$ the linear system of divisors of class dH, then non-integral curves form a closed subset of codimension ≥ 2 in $|dH|$. Therefore by Proposition 2.8 and Lemma 3.2 in [5], we know that stable sheaves with non-integral supports form a closed subset of codimension ≥ 2 in $M(d,1)$. We then want to show that if Q_f in (4.1) is not torsion free, then $Supp(F) = Supp(coker(f))$ is non-integral.

Denote by T_f the torsion of Q_f. Since Q_f has a free resolution of length 1, T_f must be a pure sheaf supported on a curve in $|d'H|$. Look back to the diagram (4.2), the map δ restricted to T_f gives a nonzero element in $\mathrm{Hom}(T_f, F^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-2))$. If $d' < d$, then $Supp(F^\vee) = Supp(F)$ can not be integral. Now assume $d' = d$ and we look at the following exact sequence

$$0 \to T_f \to Q_f \to Q_f^{tf} \to 0.$$

The torsion free sheaf Q_f^{tf} has the form $I_n(m) := I_n \otimes \mathcal{O}_{\mathbb{P}^2}(m)$ such that $m + d' = deg(Q_f) = d - 2$. On the other hand, the surjective morphism f_q

induces a surjective morphism from $\mathcal{O}_{\mathbb{P}^2}^{\oplus d-1}$ to Q_f^{tf}, hence $m \geq 1$ and thus $d' < d-2 < d$ which is a contradiction. This finishes the proof. $\square$

5. $M(d,1)$ with $d \leq 4$.

In this section we study $M(d,1)$ with $d \leq 4$. Notice that for $d \leq 4$, up to isomorphism $M(d,1)$ is the only moduli space such that there is no strictly semistable locus, since $M(d,\chi) \simeq M(d,\chi')$ if $\chi \equiv \pm\chi' \pmod{d}$.

For $d \leq 3$, $M(d,1)$ is very easy to understand and the following theorem is already known by Theorem 3.5 and Theorem 5.1 in [5]. But however using our new description we give another proof. Recall that we have defined a big open subset $W^d \subset M(d,1)$ in the previous section.

Theorem 5.1. *For $d \leq 3$, $M(d,1) = W^d$. Moreover $W^d \simeq |dH| \simeq \mathbb{P}^{3d-1}$ for $d = 1,2$; $W^3 \simeq \mathcal{C}_3$ with $\mathcal{C}_3$ the universal curve in $\mathbb{P}^2 \times |3H|$.*

Proof. By Lemma 3.4 we see that for $d \leq 3$, the sheaf E in a stable pair (E,f) can only have the form $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$. From the proof of Proposition 4.7 we see that the torsion of Q_f can only be supported on a curve of degree no bigger than $d-3$, hence Q_f is always torsion free for $d \leq 3$. Hence the first statement. By direct observation, we get the form of W^d for $d \leq 3$. This finishes the proof. $\square$

Denote by $[X]$ the class of a variety X in the Grothendieck ring of varieties. Define $\mathbb{L} := [\mathbb{A}^1]$ with $\mathbb{A}^1$ the affine line. We have the following theorem.

Theorem 5.2. $[M(4,1)] = \sum_{i=0}^{17} b_{2i}\mathbb{L}^i$ *and*

$$\begin{aligned}
&b_0 = b_{34} = 1;\ b_2 = b_{32} = 2;\ b_4 = b_{30} = 6;\\
&b_6 = b_{28} = 10;\ b_8 = b_{26} = 14;\ b_{10} = b_{24} = 15;\\
&b_{12} = b_{14} = b_{16} = b_{18} = b_{20} = b_{22} = 16.
\end{aligned}$$

In particular the Euler number $e(M(4,1))$ of the moduli space is 192.

To prove Theorem 5.2, we first define two strata as follows.

$$\begin{aligned}
M_1 :=\ & \{[(E,f)] \in M(4,1) | E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}\};\\
M_2 :=\ & \{[(E,f)] \in M(4,1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}.
\end{aligned}$$

According to Lemma 3.4, $M(4,1) = M_1 \sqcup M_2$.

Lemma 5.3. *A pair (E, f) with $rank(E) = 4$ and $deg(E) = -3$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. We only need to prove the lemma for $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. We want to show that if $\exists E' \subset E$, E' is a direct sum of line bundles with $\mu(E') > \mu(E)$ and $f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then $\exists D, D' \subset E$ two direct summands with $D \simeq D'$ and $\mu(D) > \mu(E)$, such that $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'$. With no loss of generality, we assume that E' has the form $\bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ with $a_i > 0$ and $n_i - n_{i+1} = 1$.

Let $E' \simeq E'' \subset E$ and E'' is not a direct summand of E. Then E'' has to be one of the following two cases:

(1) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(2) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

Let E'' be in case (1). If E' is a direct summand of E, then $f^{-1}(E') \neq E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ because by Nakayama's lemma $E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \subset f^{-1}(E') \Rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \subset f^{-1}(E')$ which contradicts the injectivity.

If $E' = E''$ and $f^{-1}(E') = E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then again by Nakayama's lemma we have $f(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$. Hence we get $D = D' = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$.

If $E' = E''$ and $f^{-1}(E')$ is a direct summand of $E \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ isomorphic to $E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then we have $D = \mathcal{O}_{\mathbb{P}^2}^{2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) = D'$ and $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'$, since there is no nonzero morphism from $\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ to $\mathcal{O}_{\mathbb{P}^2}(-2)$. Hence case (1) is done.

Case (2) is analogous and this finishes the proof of the lemma. $\square$

For a pair $(E, f) \in M_2$, f can be represented by the following matrix

$$\begin{pmatrix} b_1 & b_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. The injectivity of f implies that $det(f) = b_1a_2 - b_2a_1 \neq 0$. Moreover by Lemma 5.3 (E, f) is stable if and only if $kb_1 \neq k'b_2$ for any $(k, k') \in \mathbb{C}^2 - \{0\}$.

Lemma 5.4. $[M_2] = [\mathbb{P}^2 \times \mathbb{P}^{13}]$.

Proof. We have the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "(b_1,b_2)"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \arrow[r, "f_r"] & R_f \arrow[r] & 0\,. \\
& & \mathcal{O}_{\mathbb{P}^2}(-3) \arrow[u, "(a_1,a_2)"] \arrow[ur, "\omega_f:=f_r\circ(a_1,a_2)"'] & &
\end{tikzcd} \tag{5.2}
$$

Since $kb_1 \neq k'b_2$ for any $(k,k') \in \mathbb{C}^2 - \{0\}$, we have $R_f \simeq I_1(1)$ and every $I_1(1)$ can be put in sequence (5.2). Hence M_2 consists of all the isomorphism classes of pairs (R_f, ω_f) with $R_f \simeq I_1(1)$. Hence M_2 is isomorphic to a projective bundle over $Hilb^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_1(4))) \simeq \mathbb{P}^{13}$. Hence the lemma. $\square$

The big open subset W^4 defined in the previous section is contained in M_1. We have $[W^4] = [(Hilb^{[3]}(\mathbb{P}^2) - \Omega_1^{[3]}) \times \mathbb{P}^{11}]$ by Proposition 4.5.

Lemma 5.5. *$\Omega_1^{[3]} \simeq \mathcal{C}_1^{[3]}$ with $\mathcal{C}_1^{[3]}$ the relative Hilbert scheme of 3-points on the universal family $\mathcal{C}_1 \subset \mathbb{P}^2 \times |H|$, and hence $[\Omega_1^{[3]}] = [\mathbb{P}^2 \times \mathbb{P}^3]$.*

Proof. We have a natural map $\xi : \mathcal{C}_1^{[3]} \to \Omega_1^{[3]}$. ξ is an isomorphism because there is at most one curve in $|H|$ passing through any 3 points. $\mathcal{C}_1 \to |H|$ is a $\mathbb{P}^1$-bundle, hence the map $p : \mathcal{C}_1^{[3]} \to |H|$ is a projective bundle with fibers isomorphic to $(\mathbb{P}^1)^{[3]} \simeq \mathbb{P}^3$, therefore $[\Omega_1^{[3]}] = [\mathcal{C}_1^{[3]}] = [\mathbb{P}^2 \times \mathbb{P}^3]$. $\square$

Now we want to compute $[M_1 - W^4]$. Look back to diagram (4.1), we want to see what Q_f will be if it is not torsion free for $d = 4$. We know that the torsion of Q_f can only be supported on a curve of degree no bigger than $d - 3 = 1$ (See the proof of Proposition 4.7). We write down the following exact sequence

$$0 \to T_f \to Q_f \to Q_f^{tf} \to 0,$$

with T_f the torsion of Q_f and Q_f^{tf} a torsion free sheaf of rank 1.

Since T_f is supported on a curve in $|H|$ and $h^0(T_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \leq h^0(Q_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$, $T_f \simeq \mathcal{O}_H(t) \simeq \mathcal{O}_{\mathbb{P}^1}(t)$ with $t \leq 0$. Let $Q_f^{tf} \simeq I_n(m)$ with $m > 0, n \geq 0$. Then we have $m = 1$ and $n - t = 1$ by direct calculation.

If $t = 0, n = 1$, then we have the following commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & \mathcal{O}_H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r] \arrow[d] & Q_f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \arrow[r] \arrow[d] & I_1(1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{5.3}
$$

which contradicts Condition 4.4. Hence we have $t = -1, n = 0$ and Q_f lies in the following exact sequence

$$0 \to \mathcal{O}_H(-1) \to Q_f \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0. \tag{5.4}$$

$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H(-1)) \simeq H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{C}$, so for a fixed projective line $\mathbb{P}^1$ of class H, if (5.4) does not split, Q_f is unique up to isomorphism.

Lemma 5.6. *Q_f in (5.4) also lies in the following exact sequence (5.5) if and only if (5.4) does not split.*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to Q_f \to 0. \tag{5.5}$$

Proof. If $Q_f \simeq \mathcal{O}_H(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, it certainly can not lie in (5.5). If the sequence (5.4) does not split, then Q_f is unique, so we only need to construct the sequence (5.5) with Q_f contains $\mathcal{O}_H(-1)$ as its torsion. Write $\mathbb{P}^2 = \mathtt{Proj}\mathbb{C}[x_0, x_1, x_2]$. With no loss of generality we assume that $\mathcal{O}_H(-1)$ is supported on $\{x_0 = 0\}$, then the following matrix represents a morphism $f_{B^t} : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ such that $coker(f_{B^t})$ contains $\mathcal{O}_{\{x_0=0\}}(-1)$ as its torsion.

$$f_{B^t} := \begin{pmatrix} x_0 & 0 \\ x_1 & x_2 \\ 0 & x_0 \end{pmatrix}. \tag{5.6}$$

This finishes the proof. □

Remark 5.7. *f_{B^t} defined in (5.6) also satisfies the stability condition i.e. Condition 4.4.*

Lemma 5.8. *Decompose $|H|$ into cells and write $|H| = \cup_{i=0}^{2}\mathbb{A}^i$. Then $\mathbb{A}^i$ parametrizes isomorphism classes of Q_f such that there are pairs $(Q_f, \sigma_f) \in M_1 - W^4$ and T_f are supported on curves in $\mathbb{A}^i \subset |H|$.*

Proof. Lemma 5.6 implies that there is a 1-1 correspondence between isomorphism classes of Q_f and points in $|H|$. We need to decompose $|H|$ into cells so that we have a universal family over $\mathbb{P}^2 \times \mathbb{A}^i$ for each i.

We have the following diagram

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r] & \mathbb{P}^2 \times |H| , \arrow[dl, "q"'] \arrow[d, "p"] \\ \mathbb{P}^2 & |H| \end{tikzcd} \tag{5.7}$$

with $\mathcal{C}_1$ the universal curve of degree 1.

$\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H(-1)) = 0$ for all $i \neq 1$ and $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H(-1)) \simeq \mathbb{C}$, therefore $L := \mathcal{E}xt^1_p(q^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1))$ is a line bundle on $|H|$. Moreover $ch(L) = -ch(R^\bullet p_* \circ R^\bullet \mathcal{H}om(q^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)))$, so by Grothendieck-Hirzbruch-Riemann-Roch Theorem we can compute and get that $c_1(L) = c_1(\mathcal{O}_{|H|}(1))$ and hence $L \simeq \mathcal{O}_{|H|}(1)$.

L has a nowhere vanishing global section on each $\mathbb{A}^i$, in other words, we have an exact sequence on $\mathbb{P}^2 \times \mathbb{A}^i$

$$0 \to \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)|_{\mathbb{P}^2\times\mathbb{A}^i} \to \mathcal{Q}^i \to q^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^2\times\mathbb{A}^i} \to 0, \tag{5.8}$$

such that restricted on the fiber over any point $y \in \mathbb{A}^i$ it does not split. Hence $\mathcal{Q}^i$ is the family we want and hence the lemma. □

We rewrite diagram (4.1) for $d = 4$ as the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \arrow[r, "f_{B^t}"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r, "f_q"] & Q_f \arrow[r] & 0 . \\ & & \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[u, "f_{A^t}"] \arrow[ur, "\sigma_f = f_q \circ f_{A^t}"'] & & \end{tikzcd} \tag{5.9}$$

Denote $\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$ to be the union of the projective bundles $\mathbb{P}(p_*(\mathcal{Q}^i \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)|_{\mathbb{P}^2\times\mathbb{A}^i}))$ over $\mathbb{A}^i$ with fibers isomorphic to $\mathbb{P}H^0(Q_f(2)) \simeq \mathbb{P}^{11}$. Analogously, isomorphism classes of pairs (Q_f, σ_f) can be parametrized by $\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$. However $\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2))) \not\subset M(4, 1)$. Look back to diagram (4.2), it is easy to see that

$$\mathbb{P}(p_*(\mathcal{Q}\otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))\cap M(4,1)=\{[(Q_f,\sigma_f)]|Im(\sigma_f)\not\subset T_f\}, \tag{5.10}$$

with $Im(\sigma_f)$ the image of σ_f and T_f the torsion of Q_f.

The complement of (5.10) in $\mathbb{P}(p_*(\mathcal{Q}\otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$ is the union of the projective bundles $\mathbb{P}(p_*(\mathcal{O}_{\mathcal{C}_1}\otimes q^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^2\times\mathbb{A}^i}))$ over $\mathbb{A}^i$ with fibers isomorphic to $\mathbb{P}H^0(\mathcal{O}_H(1))\simeq\mathbb{P}^1$. The following lemma is a straightforward consequence.

Lemma 5.9. $[M_1-W^4]=[\mathbb{P}^2\times\mathbb{P}^{11}-\mathbb{P}^2\times\mathbb{P}^1]$.

Proof of Theorem 5.2. By Lemma 5.4, Lemma 5.5 and Lemma 5.9, we have

$$[M(4,1)]=[(Hilb^{[3]}(\mathbb{P}^2)-\mathbb{P}^2\times\mathbb{P}^3)\times\mathbb{P}^{11}+\mathbb{P}^2\times(\mathbb{P}^{11}-\mathbb{P}^1)+\mathbb{P}^2\times\mathbb{P}^{13}],$$

which leads to the theorem by direct calculation. □

6. $M(5,1)$ and $M(5,2)$.

Up to isomorphism $M(5,1)$ and $M(5,2)$ are the only two moduli spaces with $d=5$ such that there is no strictly semistable locus. In this section we prove the following theorem.

Theorem 6.1. $[M(5,1)]=[M(5,2)]=\sum_{i=0}^{26}b_{2i}\mathbb{L}^i$ *and*

$$\begin{aligned}&b_0=b_{52}=1,\ b_2=b_{50}=2,\ b_4=b_{48}=6,\\&b_6=b_{46}=13,\ b_8=b_{44}=26,\ b_{10}=b_{42}=45,\\&b_{12}=b_{40}=68,\ b_{14}=b_{38}=87,\ b_{16}=b_{36}=100,\\&b_{18}=b_{34}=107,\ b_{20}=b_{32}=111,\ b_{22}=b_{30}=112,\\&b_{24}=b_{26}=b_{28}=113.\end{aligned}$$

In particular the Euler number of both moduli spaces is 1695.

◊ **Computation for** $[M(5,1)]$

According to Lemma 3.4 we first stratify $M(5,1)$ into three strata defined as follows.

$$\begin{aligned}M_1&:=\{[(E,f)]\in M(5,1)|E\simeq\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus4}\};\\M_2&:=\{[(E,f)]\in M(5,1)|E\simeq\mathcal{O}_{\mathbb{P}^2}^{\oplus2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus2}\oplus\mathcal{O}_{\mathbb{P}^2}(-2)\};\\M_3&:=\{[(E,f)]\in M(5,1)|E\simeq\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus2}\}.\end{aligned}$$

Lemma 6.2. *A pair (E, f) with $rank(E) = 5$ and $deg(E) = -4$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. See Appendix A. □

For a pair $(E, f) \in M_3$, f can be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_1 & 0 & 0 & 0 & b_1 \\ a_2 & 0 & 0 & 0 & b_2 \end{pmatrix}, \tag{6.1}$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(4))$. The injectivity of f implies that $det(f) = b_2a_1 - b_1a_2 \neq 0$. Moreover by Lemma 6.2 (E, f) is stable if and only if $kb_1 \neq k'b_2$ for any $(k, k') \in \mathbb{C}^2 - \{0\}$.

Lemma 6.3. $[M_3] = [\mathbb{P}^2 \times \mathbb{P}^{19}]$.

Proof. The proof is analogous to that of Lemma 5.4. M_3 is isomorphic to a projective bundle over $Hilb^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_1(5))) \simeq \mathbb{P}^{19}$. Hence the lemma. □

We stratify M_2 into two strata as follows.

$$\begin{aligned} M_2^s &:= \{[(E, f)] \in M_2 | f|_{\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)} \textit{is surjective onto } \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\}; \\ M_2^c &:= M_2 - M_2^s. \end{aligned}$$

For a pair $(E, f) \in M_2^s$, f can be represented by the following matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ b_1 & b_2 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 \end{pmatrix}, \tag{6.2}$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. The injectivity of f implies that $det(f) = b_1a_2 - b_2a_1 \neq 0$. Moreover by Lemma 6.2 (E, f) is stable if and only if $kb_1 \neq k'b_2$ for any $(k, k') \in \mathbb{C}^2 - \{0\}$.

Lemma 6.4. $[M_2^s] = [Gr(2, 6) \times \mathbb{P}^{16} - \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2]$ *with* $Gr(2, 6)$ *the Grassmannian parametrizing 2-dimensional linear subspaces of* $\mathbb{C}^6$.

Proof. We have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "{(b_1,b_2)}"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \arrow[r, "f_r"] & R_f \arrow[r] & 0\,. \\
 & & \mathcal{O}_{\mathbb{P}^2}(-3) \arrow[u, "{(a_1,a_2)}"] \arrow[ur, "{\omega_f := f_r \circ (a_1,a_2)}"'] & &
\end{tikzcd} \tag{6.3}$$

Since $kb_1 \neq k'b_2, \forall (k,k') \in \mathbb{C}^2 - \{0\}$, the isomorphism classes of R_f 1-1 correspond to points in $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(2))) = Gr(2,6)$. Denote by $\mathcal{G}$ the tautological bundle on $Gr(2,6)$. Then on $Gr(2,6) \times \mathbb{P}^2$ we have the following exact sequence.

$$0 \to q^*\mathcal{O}_{\mathbb{P}^2}(-2) \to p^*\mathcal{G}^\vee \to \mathcal{R} \to 0, \tag{6.4}$$

with p and q the projections to $Gr(2,6)$ and $\mathbb{P}^2$ respectively. $\mathcal{R}$ restricted to the fiber over $[(b_1,b_2)] \in Gr(2,6)$ is R_f. Hence isomorphism classes of (R_f, ω_f) are parametrized by the projective bundle $\mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3)))$ over $Gr(2,6)$ with fibers isomorphic to $\mathbb{P}^{16}$.

However $M_2^s \subsetneq \mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3)))$ and the complement of M_2^s in $\mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3)))$ consists of all (R_f, ω_f) such that the images of ω_f are contained in the torsions of R_f.

If b_1 is prime to b_2, then R_f is torsion free. If b_1 is not prime to b_2, then R_f lies in the following exact sequence.

$$0 \to \mathcal{O}_H(-1) \to R_f \to I_1(1) \to 0, \tag{6.5}$$

with H a hyperplane in $\mathbb{P}^2$. The closed subset $|H| \times Hilb^{[1]}(\mathbb{P}^2) \hookrightarrow Gr(2,6)$ parametrizes all the R_f that are not torsion free.

We write down the following diagram.

$$\begin{tikzcd}
\mathbb{P}^2 & {|H| \times Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2} \arrow[l, "q"'] \arrow[d, "p_1"'] \arrow[dr, "p"] & \\
 & \mathcal{C}_1 \hookrightarrow |H| \times \mathbb{P}^2 & {|H| \times Hilb^{[1]}(\mathbb{P}^2)}
\end{tikzcd} .$$

Those (R_f, ω_f) not in M_2^s are parametrized by the projective bundle $\mathbb{P}(p_*(p_1^*\mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$ over $|H| \times Hilb^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(\mathcal{O}_H(2))) \simeq \mathbb{P}^2$.

Hence we have $M_2^s \simeq \mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3))) - \mathbb{P}(p_*(p_1^*\mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$, hence the lemma. $\square$

For a pair $(E,f)\in M_2^c$, f can be represented by the following matrix

$$\left(\begin{array}{ccccc} b_1 & b_2 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ a_1 & a_2 & 0 & b_3 & 0\\ 0 & 0 & 0 & 0 & 1\\ e_1 & e_2 & 0 & a_3 & 0 \end{array}\right), \tag{6.6}$$

where $b_i\in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a_i\in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and $e_i\in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. $det(f)\neq 0$. Lemma 6.2 implies that (E,f) is stable if and only if $kb_1\neq k'b_2, \forall (k,k')\in \mathbb{C}^2-\{0\}$ and $k''a_3\neq b\cdot b_3, \forall (k'',b)\in\mathbb{C}\times H^0(\mathcal{O}_{\mathbb{P}^2}(1))-\{(0,0)\}$.

Lemma 6.5. $[M_2^c]=[Hilb^{[1]}(\mathbb{P}^2)\times Hilb^{[2]}(\mathbb{P}^2)\times\mathbb{P}^{17}-Hilb^{[1]}(\mathbb{P}^2)\times\mathbb{P}^1\times\mathbb{P}^1]$.

Proof. We first write down the following two exact sequences.

$$0\longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)\xrightarrow{(b_1,b_2)}\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\xrightarrow{f_r} R_f\longrightarrow 0 \tag{6.7}$$

$$0\longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)\xrightarrow{(a_3,b_3)}\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\xrightarrow{f_s} S_f\longrightarrow 0 \tag{6.8}$$

Because of the stability condition, we see that both R_f and S_f are torsion free and hence $R_f\simeq I_1(1)$ and $S_f\simeq I_2(2)$. On the other hand, any $I_1(1)$ ($I_2(2)$) can be put in the sequence (6.7) ((6.8)).

We write down a commutative diagram as follows.

(6.9)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "(a_3,b_3)\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"] \arrow[d, "(b_1,b_2)\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"'] & \mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "f_s\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"] \arrow[d, "(b_1,b_2)\otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)}\oplus(b_1,b_2)"] & S_f(-1) \arrow[r] \arrow[d, "id_{S_f}\otimes(b_1,b_2)"] & 0\\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \arrow[r, "(a_3,b_3)^{\oplus 2}"'] \arrow[d, "f_r\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"'] & (\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2})^{\oplus 2} \arrow[r, "f_s^{\oplus 2}"'] \arrow[d, "f_r\otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)}\oplus f_r"] & S_f^{\oplus 2} \arrow[r] \arrow[d, "id_{S_f}\otimes f_r"] & 0\\
0 \arrow[r] & R_f\otimes\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "id_{R_f}\otimes(a_3,b_3)"'] \arrow[d] & R_f\oplus R_f(1) \arrow[r, "id_{R_f}\otimes f_s"'] \arrow[d] & R_f\otimes S_f \arrow[r] \arrow[d] & 0\\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We have another commutative diagram

$$(6.10)\qquad \begin{tikzcd}
\mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "(e_1,a_1)\oplus(e_2,a_2)"] & (\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2})^{\oplus 2} \arrow[r, "f_s^{\oplus 2}"] \arrow[d, "f_r\otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)}\oplus f_r"'] & S_f^{\oplus 2} \arrow[d, "id_{S_f}\otimes f_r"] \\
& R_f\oplus R_f(1) \arrow[r, "id_{R_f}\otimes f_s"'] & R_f\otimes S_f.
\end{tikzcd}$$

Isomorphism classes of $(E,f)\in M_2^c$ are parametrized by (R_f,S_f,ω_f) with $\omega_f:\mathcal{O}_{\mathbb{P}^2}(-2)\to R_f\otimes S_f$ the composed map in (6.10). We write down the following diagram.

$$\begin{tikzcd}
& \mathbb{P}^2 & Hilb^{[1]}(\mathbb{P}^2)\times Hilb^{[2]}(\mathbb{P}^2)\times\mathbb{P}^2 \arrow[l, "q"'] \arrow[dl, "p_1"'] \arrow[d, "p"] \arrow[dr, "p_2"] & \\
& Hilb^{[1]}(\mathbb{P}^2)\times\mathbb{P}^2 & Hilb^{[1]}(\mathbb{P}^2)\times Hilb^{[2]}(\mathbb{P}^2) & Hilb^{[2]}(\mathbb{P}^2)\times\mathbb{P}^2
\end{tikzcd}.$$

Denote by $\mathcal{I}_1$ ($\mathcal{I}_2$) the universal family of ideal sheaves on $Hilb^{[1]}(\mathbb{P}^2)\times\mathbb{P}^2$ ($Hilb^{[2]}(\mathbb{P}^2)\times\mathbb{P}^2$). Isomorphism classes of (R_f,S_f,ω_f) are parametrized by the projective bundle $\mathbb{P}(p_*(p_1^*\mathcal{I}_1\otimes p_2^*\mathcal{I}_2\otimes q^*\mathcal{O}_{\mathbb{P}^2}(5)))$ over $Hilb^{[1]}(\mathbb{P}^2)\times Hilb^{[2]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_1\otimes I_2\otimes\mathcal{O}_{\mathbb{P}^2}(5)))\simeq\mathbb{P}^{17}$.

There are still points in $\mathbb{P}(p_*(p_1^*\mathcal{I}_1\otimes p_2^*\mathcal{I}_2\otimes q^*\mathcal{O}_{\mathbb{P}^2}(5)))$ that we must exclude. They are points (R_f,S_f,ω_f) such that the images of ω_f are contained in the torsions of $R_f\otimes S_f$.

We write down the following exact sequence.

$$(6.11)\qquad 0\to I_1\to\mathcal{O}_{\mathbb{P}^2}\to\mathcal{O}_x\to 0,$$

with $\mathcal{O}_x$ the skyscraper sheaf supported at a single point x. Tensor (6.11) by I_2, and we get

$$0\to Tor^1(\mathcal{O}_x,I_2)\to I_1\otimes I_2\to I_2\to I_2\otimes\mathcal{O}_x\to 0.$$

We see that the torsion of $I_1\otimes I_2$ is isomorphic to $Tor^1(\mathcal{O}_x,I_2)$. Tensor (6.8) by $\mathcal{O}_x$ and we get

$$0\longrightarrow Tor^1(\mathcal{O}_x,I_2(2))\longrightarrow\mathcal{O}_x\xrightarrow{(a_3,b_3)}\mathcal{O}_x^{\oplus 2}\longrightarrow I_2(2)\otimes\mathcal{O}_x\longrightarrow 0.$$

Hence we see that the torsion of $R_f\otimes S_f$ is either zero or isomorphic to $\mathcal{O}_x$. The later implies that $R_f\simeq I_{\{x\}}(1)$ and $Tor^1(\mathcal{O}_x,I_2)\neq 0\Leftrightarrow(a_3,b_3)|_x=0$.

We then want to parametrize all (I_1, I_2) such that $I_1 \otimes I_2$ contain torsion. We first write down the following diagram

$$
(6.12)\qquad
\begin{tikzcd}
\mathbb{P}^2 & & \mathbb{P}^2 \times Hilb^{[1]}(\mathbb{P}^2) \times |H| \arrow[ll, "q"'] \arrow[dl, "q_1"'] \arrow[d, "p_3"] \\
\mathcal{C}_1 \hookrightarrow \mathbb{P}^2 \times |H| & & Hilb^{[1]}(\mathbb{P}^2) \times |H| \hookleftarrow \mathbb{P}(\mathcal{V}^1),
\end{tikzcd}
$$

where $\mathcal{V}^1$ is the rank 2 vector bundle on $Hilb^{[1]}(\mathbb{P}^2)$ defined as $s_*(\mathcal{I}_1 \otimes t^*\mathcal{O}_{\mathbb{P}^2}(1))$ with s and t the projection from $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ to $Hilb^{[1]}(\mathbb{P}^2)$ and $\mathbb{P}^2$ respectively. Let $\mathcal{Z}$ be defined by the following Cartesian diagram

$$
(6.13)\qquad
\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] & \mathbb{P}(p_{3*}(q_1^*(\mathcal{O}_{\mathcal{C}_1}) \otimes q^*\mathcal{O}_{\mathbb{P}^2}(1))) \arrow[d] \\
\mathbb{P}(\mathcal{V}^1) \arrow[r] & Hilb^{[1]}(\mathbb{P}^2) \times |H|.
\end{tikzcd}
$$

$\mathbb{P}(\mathcal{V}^1)$ parametrizes all the pairs $([x], [C]) \in Hilb^{[1]}(\mathbb{P}^2) \times |H|$ such that $x \in C$, and $\mathcal{Z}$ parametrizes $([x_1], [(C, x_2)]) \in Hilb^{[1]}(\mathbb{P}^2) \times \mathcal{C}_1$ such that $x_1 \in C$. Define $\imath : \mathcal{Z} \to Hilb^{[1]}(\mathbb{P}^2) \times Hilb^{[2]}(\mathbb{P}^2)$ such that $\imath(([x_1], [(C, x_2)])) = ([x_1], [x_1, x_2, C])$. It is easy to see that $\imath$ is an embedding with its image exactly the set of points (I_1, I_2) such that $I_1 \otimes I_2$ have torsion.

$p_*(p_1^*\mathcal{I}_1 \otimes p_1^*\mathcal{I}_2 \otimes p^*\mathcal{O}_{\mathcal{Z}}) \simeq p_*(p_1^*\mathcal{I}_1 \otimes p_1^*\mathcal{I}_2) \otimes \mathcal{O}_{\mathcal{Z}}$ by the flatness of p. $p_1^*\mathcal{I}_1 \otimes p_1^*\mathcal{I}_2 \otimes p^*\mathcal{O}_{\mathcal{Z}}$ contains $p_1^*\mathcal{O}_{\mathcal{Z}_1}$ as its torsion where $\mathcal{Z}_1$ is the universal subscheme in $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$. Hence we can embed $\mathcal{Z}$ into $\mathbb{P}(p_*(p_1^*\mathcal{I}_1 \otimes p_2^*\mathcal{I}_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(5)))$ by taking the non-zero constant section of $p_1^*\mathcal{O}_{\mathcal{Z}_1}$.

Hence we have $M_2^c \simeq \mathbb{P}(p_*(p_1^*\mathcal{I}_1 \otimes p_2^*\mathcal{I}_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(5))) - \mathcal{Z}$. The lemma follows since $[\mathcal{Z}] = [Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times \mathbb{P}^1]$. □

Finally let $(E, f) \in M_1 - W^5$. Rewrite (4.1) for $d = 5$ as follows

$$
(6.14)\qquad
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \arrow[r, "f_{B^t}"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \arrow[r, "f_q"] & Q_f \arrow[r] & 0\,. \\
& & \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[u, "f_{A^t}"] \arrow[ur, "\sigma_f = f_q \circ f_{A^t}"'] & &
\end{tikzcd}
$$

Notice that the torsion T_f of Q_f contains neither $\mathcal{O}_H$ nor $\mathcal{O}_{2H}(x)$ as a subsheaf with x a single point on the curve, otherwise we will have a diagram similar to diagram (5.3) which contradicts Condition 4.4. Also $H^0(T_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$. Hence we see that if $c_1(T_f) = 2H$, then $\chi(T_f) = 1$ and $Q_f^{tf} \simeq$

$\mathcal{O}_{\mathbb{P}^2}(1)$. Moreover since T_f does not contain $\mathcal{O}_H(n)$ with $n \geq 0$, T_f is stable and hence by Theorem 5.1 there is only one stable sheaf for each curve in $|2H|$. Hence $T_f \simeq \mathcal{O}_{2H}$.

We have the following commutative diagram.

(6.15)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \arrow[r, "\imath"] \arrow[d, "\simeq"] & K \arrow[r] \arrow[d, "j"'] & T_f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \arrow[r, "f_{B^t}"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \arrow[r, "f_q"] \arrow[d, "f_{tq}"'] & Q_f \arrow[r] \arrow[d] & 0 \\
 & & Q_f^{tf} \arrow[r, "\simeq"] \arrow[d] & Q_f^{tf} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

We stratify $M_1 - W^5$ into three strata as follows.

$$\begin{aligned}
\Pi_1 :=& \quad \{[(E,f)] \in M_1 - W^5 | T_f \simeq \mathcal{O}_{2H}, Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(1)\}; \\
\Pi_2 :=& \quad \{[(E,f)] \in M_1 - W^5 | T_f \simeq \mathcal{O}_H(-1), Q_f^{tf} \simeq I_2(2)\}; \\
\Pi_3 :=& \quad \{[(E,f)] \in M_1 - W^5 | T_f \simeq \mathcal{O}_H(-2), Q_f^{tf} \simeq I_1(2)\}.
\end{aligned}$$

A priori there is the fourth possibility that $T_f \simeq \mathcal{O}_H(-3), Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$, we will explain why this case is excluded later in the computation for $[\Pi_3]$.

Lemma 6.6. $[\Pi_1] = [\mathbb{P}^5 \times \mathbb{P}^{14} - \mathbb{P}^5 \times \mathbb{P}^4]$.

Proof. Notice that $\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{2H}) = 0$ for all $i \neq 1$ and $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{2H}) \simeq \mathbb{C}$, and the proof is analogous to that of Lemma 5.9. $\square$

Let $(E,f) \in \Pi_2$. Since $Q_f^{tf} \simeq I_2(2)$, we have the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(a,b)} \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{g} I_2(2) \to 0, \tag{6.16}$$

with $b \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and b prime to a. Take $\mathrm{Hom}(-, \mathcal{O}_H(-1))$ on (6.16) and we get

(6.17)
$$0 \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1)) \to \mathrm{Ext}^1(I_2(2), \mathcal{O}_H(-1)) \xrightarrow{\tilde{g}} \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_H(-1)) \to 0,$$

Lemma 6.7. *Q_f with $T_f \simeq \mathcal{O}_H(-1)$ and $Q_f^{tf} \simeq I_2(2)$ lies in diagram (6.15) if and only if the image of the following exact sequence via $\tilde{g}$ is not zero*

$$0 \to \mathcal{O}_H(-1) \to Q_f \to I_2(2) \to 0, \tag{6.18}$$

i.e. (6.18) is not contained in the image of $Hom(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1))$.

Proof. The map $\tilde{g}$ gives the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "\simeq"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[d, "(a,b)"] & \\
0 \arrow[r] & \mathcal{O}_H(-1) \arrow[r] \arrow[d, "\simeq"] & \widetilde{Q_f} \arrow[r] \arrow[d, "\delta"] & \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d, "g"] & 0 \ (*) \\
0 \arrow[r] & \mathcal{O}_H(-1) \arrow[r] & Q_f \arrow[r] \arrow[d] & I_2(2) \arrow[r] \arrow[d] & 0 \ (**) \\
 & & 0 & 0 &
\end{tikzcd} \tag{6.19}$$

where the sequence $(*)$ is the image of the sequence $(**)$ via $\tilde{g}$ and $\widetilde{Q_f}$ is the Cartesian product of Q_f and $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ over $I_2(2)$.

From (6.19) we see that $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \widetilde{Q_f}) \simeq \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$ and $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, Q_f) \simeq \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, I_2(2))$. Moreover the map f_{tq} in (6.15) factors through a surjective map $s : \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ since $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}) \twoheadrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}, I_2(2))$, and s lifts to a map $\tilde{s} : \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \to \widetilde{Q_f}$ such that $f_q = \delta \circ \tilde{s}$.

If the sequence $(*)$ in (6.19) splits, then $\widetilde{Q_f} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_H(-1)$ and $\delta \circ \tilde{s}$ can not be surjective. Hence $(*)$ does not split.

On the other hand, $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_H(-1)) \simeq \mathbb{C}$, hence $\widetilde{Q_f}$ is unique up to isomorphism if $(*)$ does not split. We see that in this case $\widetilde{Q_f} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus Q_f^1$ with Q_f^1 lying in the following non-splitting sequence

$$0 \to \mathcal{O}_H(-1) \to Q_f^1 \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0.$$

By Lemma 5.6 Q_f^1 lies in the following sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{f_{q_1}} Q_f^1 \to 0.$$

We then have the following commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \arrow[r, "\simeq"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \arrow[d, "(a,b)"] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \arrow[r, "f_q"] \arrow[d, "f_{q_1} \oplus id_{\mathcal{O}_{\mathbb{P}^2}}"] & Q_f \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r] \arrow[d] & Q_f^1 \oplus \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & Q_f \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{6.20}
$$

Hence Q_f lies in (6.19) and hence the lemma. $\square$

Lemma 6.8. $[\Pi_2] = [Hilb^{[2]}(\mathbb{P}^2) \times |H| \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^{14} - \mathbb{P}^1)]$.

Proof. Lemma 6.7 implies that for fixed $\mathcal{O}_H(-1)$ and $I_2(2)$, isomorphism classes of Q_f are parametrized by $\mathbb{P}(\mathrm{Ext}^1(I_2(2), \mathcal{O}_H(-1))) - \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1)))$. Hence isomorphism classes of all Q_f are parametrized by the following scheme

$$\mathbb{P}(\mathcal{E}xt_p^1(\mathcal{I}_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1))) - Hilb^{[2]}(\mathbb{P}^2) \times |H|;$$

where p and q are projections from $\mathbb{P}^2 \times Hilb^{[2]}(\mathbb{P}^2) \times |H|$ to $Hilb^{[2]}(\mathbb{P}^2) \times |H|$ and $\mathbb{P}^2$ respectively, $\mathcal{I}_2$ and $\mathcal{C}_1$ are the pull back of the universal ideal sheaf and the universal curve to $\mathbb{P}^2 \times Hilb^{[2]}(\mathbb{P}^2) \times |H|$ from $\mathbb{P}^2 \times Hilb^{[2]}(\mathbb{P}^2)$ and $\mathbb{P}^2 \times |H|$ respectively. Notice that we embed $Hilb^{[2]}(\mathbb{P}^2) \times |H|$ into $\mathbb{P}(\mathcal{E}xt_p^1(\mathcal{I}_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)))$ by taking the nonzero constant section of the line bundle $\mathcal{H}om_p(q^*\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)) \simeq p_*\mathcal{O}_{\mathcal{C}_1} \simeq \mathcal{O}_{Hilb^{[2]}(\mathbb{P}^2) \times |H|}$.

Analogously the space parametrizing (Q_f, σ_f) is the difference of two projective bundles with fibers isomorphic to $\mathbb{P}(H^0(Q_f(2))) \simeq \mathbb{P}^{14}$ and $\mathbb{P}(H^0(\mathcal{O}_H(1))) \simeq \mathbb{P}^1$ respectively over the space parametrizing Q_f. Hence the lemma. $\square$

Now we do the computation for $[\Pi_3]$ and we will also explain why the case that $T_f \simeq \mathcal{O}_H(-3), Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ is not included. Notice that the map f_{tq} in (6.15) is not surjective on global sections. We first write down the following commutative diagram.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r, "j"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \arrow[r, "f_{tq}"] \arrow[d] & Q_f^{tf} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d, "\tau"'] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4+n} \arrow[r, "g"] \arrow[d, "\tilde{\tau}"] & Q_f^{tf} \arrow[r] & 0 \\
 & \mathcal{O}_{\mathbb{P}^2}^{\oplus n} \arrow[r, "\simeq"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}^{\oplus n} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{6.21}
$$

where $n = 1$ if $Q_f^{tf} \simeq I_1(2)$ and $n = 2$ if $Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$.

From (6.21) we see that $H^i(G(1-i)) = 0$ for $i > 0$, hence by Castelnuovo-Mumford regularity $G(1)$ is globally generated. Therefore the map $\tau \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)} : G(1) \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus n}$ must be surjective on global sections, since otherwise τ is not surjective. Hence $h^0(K(1)) = h^0(G(1)) - nh^0(\mathcal{O}_{\mathbb{P}^2}(1))$. So if $n = 2, Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$, then we have $h^0(K(1)) = 2$ which implies that K can not contain $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$ as a subsheaf. Hence we only have $n = 1$ and $Q_f^{tf} \simeq I_1(2)$.

$T_f \simeq \mathcal{O}_H(-2)$ hence $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), T_f) = 0$, the inclusion $\imath$ in (6.15) is unique up to isomorphisms of $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$ for a fixed K. Hence f_{B^t} is determined by the inclusion j and hence is determined by f_{tq}. Parametrizing f_{B^t} is equivalent to parametrizing the surjective map f_{tq}, hence equivalent to parametrizing $\tilde{\tau}$. We first assume $Q_f^{tf} \simeq I_{[0,0,1]}(2)$, then g can be represented by a 1×5 matrix $(x_0^2, x_0x_1, x_0x_2, x_1x_2, x_1^2)$. $\tilde{\tau}$ can be represented by $\underline{h} := (h_0, h_1, h_2, h_3, h_4)$ with $h_i \in \mathbb{C}$. We want to parametrize the class of $\underline{h}$ modulo scalars.

The sheaf G can be generated by 6 generators $< \epsilon_0, \epsilon_1, \epsilon_2, \eta_0, \eta_1, \eta_2 >$ in $H^0(G(1))$ with two syzygies $(x_0\epsilon_0 + x_1\epsilon_1 - x_2\epsilon_2 = 0, x_0\eta_0 + x_1\eta_1 - x_2\eta_2 = 0)$.

The map τ is determined by $\tau^0 := h^0(\tau \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)}) : H^0(G(1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, and also τ is induced by $\tilde{\tau}$. Hence τ^0 is determined by $\underline{h}$ and we can write down explicitly the images of ϵ_i and η_i as follows

$$
\begin{aligned}
&\tau^0(\epsilon_0) = h_2x_1 - h_1x_2;\ \tau^0(\epsilon_1) = h_0x_2 - h_2x_0;\ \tau^0(\epsilon_2) = h_0x_1 - h_1x_0\\
&\tau^0(\eta_0) = h_3x_1 - h_4x_2;\ \tau^0(\eta_1) = h_1x_2 - h_3x_0;\ \tau^0(\eta_2) = h_1x_1 - h_4x_0.
\end{aligned}
$$

We can get (6.21) if and only if τ^0 is surjective. In other words, the following 3×6 matrix has rank 3.

$$Mat_\tau := \begin{pmatrix} 0 & -h_2 & -h_1 & 0 & -h_3 & -h_4 \\ h_2 & 0 & h_0 & h_3 & 0 & h_1 \\ -h_1 & h_0 & 0 & -h_4 & h_1 & 0 \end{pmatrix}$$

By direct computation we see that

$$rank(Mat_\tau) < 3 \Leftrightarrow h_1h_2 - h_0h_3 = h_1^2 - h_0h_4 = h_1h_3 - h_2h_1 = 0.$$

Hence we know that f_{tq} are parametrized by $P_\tau := \mathbb{P}^4 - \{h_1h_2 - h_0h_3 = h_1^2 - h_0h_4 = h_1h_3 - h_2h_1 = 0\}$.

One can easily compute that $[P_\tau] = [\mathbb{P}^4 - (\mathbb{P}^1 + \mathbb{A}^2 + \mathbb{A}^1)]$. Moreover we can cover $Hilb^{[1]}(\mathbb{P}^2)$ by finitely many Zariski open subsets U_i such that $(I_1(2), \tilde{\tau})$ with $[I_1] \in U_i$ are parametrized by $U_i \times P_\tau$. For example, we can take U_i such that $p_*(\mathcal{I}_1 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2))|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus 5}$, where p and q are the projections from $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ to $Hilb^{[1]}(\mathbb{P}^2)$ and $\mathbb{P}^2$ respectively and $\mathcal{I}_1$ the universal ideal sheaf.

$(I_1(2), \tilde{\tau})$ determines Q_f and analogously we know that (Q_f, σ_f) are parametrized by a difference of two projective bundles over the space parametrizing Q_f. Hence we have the following lemma as a direct consequence.

Lemma 6.9. $[\Pi_3] = [Hilb^{[1]}(\mathbb{P}^2) \times (\mathbb{P}^4 - (\mathbb{P}^1 + \mathbb{A}^2 + \mathbb{A}^1)) \times (\mathbb{P}^{14} - 1)]$.

We have already known that $[W^5] = \mathbb{P}^{14} \times [Hilb^{[6]}(\mathbb{P}^2) - \Omega_2^{[6]}]$. The proof of the following lemma is postponed to the appendix.

Lemma 6.10. $[\Omega_2^{[6]}] = \mathbb{L}^{11} + 3\mathbb{L}^{10} + 8\mathbb{L}^9 + 18\mathbb{L}^8 + 30\mathbb{L}^7 + 39\mathbb{L}^6 + 38\mathbb{L}^5 + 28\mathbb{L}^4 + 15\mathbb{L}^3 + 6\mathbb{L}^2 + 2\mathbb{L}^1 + 1$.

Proof. See Appendix B. □

Proof of Theorem 6.1 for $M(5,1)$. We have

$$[M(5,1)] = [M_3] + [M_2^s] + [M_2^c] + \sum_{i=1}^{3} [\Pi_i] + ([Hilb^{[6]}(\mathbb{P}^2)] - [\Omega_2^{[6]}]) \times [\mathbb{P}^{14}].$$

Combine Lemma 6.3, Lemma 6.4, Lemma 6.5, Lemma 6.6, Lemma 6.8, Lemma 6.9 and Lemma 6.10, and we get the result by direct computation. □

$\Diamond$ **Computation for** $[M(5,2)]$

We stratify $M(5,2)$ into three strata defined as follows.

$$\begin{aligned} M_2 :=& \quad \{[(E,f)] \in M(5,1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}\}; \\ M_3 :=& \quad \{[(E,f)] \in M(5,1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}; \\ M_3' :=& \quad \{[(E,f)] \in M(5,1) | E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}. \end{aligned}$$

Here we use notation M_3' instead of M_4 because we want to specify the lower index of the subspace to be $h^0(F)$ with F any sheaf in it.

Lemma 6.11. *A pair (E,f) with $rank(E) = 5$ and $deg(E) = -3$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. See Appendix A. $\square$

For a pair $(E,f) \in M_3'$, f can be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ d & 0 & 0 & b & 0 \\ c & 0 & 0 & a & 0 \end{pmatrix}, \tag{6.22}$$

where $b \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$, $c \in H^0(\mathcal{O}_{\mathbb{P}^2}(4))$ and $d \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. $det(f) = ad - bc \neq 0$ and by Lemma 6.2 (E,f) is stable if and only if b is prime to a.

Lemma 6.12. $[M_3'] = [Hilb^{[2]}(\mathbb{P}^2) \times \mathbb{P}^{18}]$.

Proof. We have the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(a,b)} \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \to I_2(2) \to 0, \tag{6.23}$$

and for every $[I_2] \in Hilb^{[2]}(\mathbb{P}^2)$, $I_2(2)$ lies in (6.23). Hence analogous to Lemma 5.4, M_3' is isomorphic to a projective bundle over $Hilb^{[2]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_2(5))) \simeq \mathbb{P}^{18}$. Hence the lemma. $\square$

For a pair $(E,f)\in M_3$, f can be represented by the following matrix

$$\begin{pmatrix} B & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ A & 0 & 0\end{pmatrix},$$

where A is a 1×3 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ and B a 2×3 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. The parametrizing space of B is of class $[Hilb^{[3]}(\mathbb{P}^2)-|H|\times\mathbb{P}^3+|H|]$ by Lemma 5.5 and Lemma 5.8. We have the following lemma.

Lemma 6.13. $[M_3]=[(Hilb^{[3]}(\mathbb{P}^2)-|H|\times\mathbb{P}^3+|H|)\times\mathbb{P}^{17}-|H|\times\mathbb{P}^2]$.

Proof. M_3 is the union of a projective bundle over $Hilb^{[3]}(\mathbb{P}^2)-\Omega_1^{[3]}$ with fiber isomorphic to $\mathbb{P}(H^0(I_3(5)))\simeq\mathbb{P}^{17}$ and a difference of two projective bundles over $|H|$ with fibers isomorphic to $\mathbb{P}^{17}$ and $\mathbb{P}(H^0(\mathcal{O}_H(2)))\simeq\mathbb{P}^2$ respectively. Hence the lemma. □

We stratify M_2 into two strata as follows.

$$M_2^s:=\left\{[(E,f)]\in M_2\;\middle|\;\begin{array}{c} f_{rs}:\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\xrightarrow{f|_{\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}}}E\twoheadrightarrow\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}\\ \textit{is injective.}\end{array}\right\};$$

$$M_2^c:=M_2-M_2^s.$$

For a pair $(E,f)\in M_2^s$, f can be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ A & 0 & 0 & B\end{pmatrix},$$

where A is a 3×2 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and B a 3×1 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$.

We stratify M_2^s into two strata as follows.

$$\Xi_1:=\{[(E,f)]\in M_2^s|B\simeq(x_0,x_1,x_2)^t\};\quad \Xi_2:=M_2^s-\Xi_1.$$

If $B \simeq (x_0, x_1, x_2)^t$, then (E, f) always satisfies the stability condition. We have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "(x_0,x_1,x_2)"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r, "f_0"] & E_0 \arrow[r] & 0 \\ & & \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \arrow[u, "f_{A^t}"] \arrow[ur, "\xi_f := f_0 \circ f_{A^t}"'] & & \end{tikzcd} \tag{6.24}$$

with E_0 a rank 2 bundle which is the dual of the kernel of the surjective map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{(x_0,x_1,x_2)^t} \mathcal{O}_{\mathbb{P}^2}(1)$. Isomorphism classes of ξ_f are parametrized by $Gr(2, 15)$ since $h^0(E_0(2)) = 15$. Moreover $det(f) \neq 0 \Leftrightarrow$ the image of ξ_f is a rank two subsheaf of $E_0 \Leftrightarrow \mathrm{Im}(\xi_f)$ is not contained in a rank one subsheaf of E_0.

Assume $\mathrm{Im}(\xi_f)$ is contained in a rank one subsheaf $E_1 \subsetneq E_0$. Since E_0 is locally free, we ask E_1 to be a line bundle. Hence either $E_1 \simeq \mathcal{O}_{\mathbb{P}^2}$ or $E_1 \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. Notice that for any n a map $\mathcal{O}_{\mathbb{P}^2}(n) \to E_0$ always factors through map f_0 in (6.24).

Since $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0) \simeq \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$, all inclusions $i : \mathcal{O}_{\mathbb{P}^2} \hookrightarrow E_0$ are parametrized by $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) - \{0\}$. Moreover $\forall i, i' \in H^0(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) - \{0\}, i \not\equiv i'$, $\mathrm{Im}(i) \cap \mathrm{Im}(i') = \emptyset$. Hence all ξ_f such that $\mathrm{Im}(\xi_f)$ are contained in $\mathcal{O}_{\mathbb{P}^2} \simeq E_1 \subsetneq E_0$ are parametrized by $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(2))) \times \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3})) \simeq Gr(2, 6) \times \mathbb{P}^2$.

Let $\imath \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}) - \{0\}$. All inclusions $j : \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow E_0$ that do not factor through $\imath : \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^2}$ are parametrized by $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0) - \tilde{\imath}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}))$, where $\tilde{\imath} :$ $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0) \hookrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0)$ is the map induced by $\imath$. Moreover $\forall j, j' \in H^0(\mathcal{O}_{\mathbb{P}^2}(-1), E_0) - \{0\}, j \not\equiv j'$, $\mathrm{Im}(j) \cap \mathrm{Im}(j') = \emptyset$, and $\forall \imath, \imath' \in H^0(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}) - \{0\}, \imath \not\equiv \imath'$, $\mathrm{Im}(\imath) \cap \mathrm{Im}(\imath') = \emptyset$. Hence all ξ_f such that $\mathrm{Im}(\xi_f)$ are contained in $\mathcal{O}_{\mathbb{P}^2}(-1)$ but not $\mathcal{O}_{\mathbb{P}^2}$ in E_0 are parametrized by $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times (\mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0)) - \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0)) \times \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}))) \simeq \mathbb{P}^2 \times (\mathbb{P}^7 - \mathbb{P}^2 \times \mathbb{P}^2)$.

We have the following lemma as a direct consequence.

Lemma 6.14. $[\Xi_1] = [Gr(2, 15) - Gr(2, 6) \times \mathbb{P}^2 - \mathbb{P}^2 \times (\mathbb{P}^7 - \mathbb{P}^2 \times \mathbb{P}^2)]$.

For a pair $(E, f) \in \Xi_2$, f can be represented by the following matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & b_1 \\ a_5 & a_6 & 0 & 0 & b_2 \end{pmatrix}, \tag{6.25}$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$. By Lemma 6.2 (E, f) is stable if and only if $kb_1 \neq k'b_2, ka_1 \neq k'a_2, \forall (k, k') \in \mathbb{C} - \{0\}$.

We write down the following two exact sequences.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(b_1,b_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{f_r} R_f \longrightarrow 0 \tag{6.26}$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{(a_1,a_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{f_s} S_f \longrightarrow 0 \tag{6.27}$$

$R_f \simeq I_1(1)$. Either $S_f \simeq I_4(2)$ or S_f lies in the following exact sequence.

$$0 \to \mathcal{O}_H(-1) \to S_f \to I_1(1) \to 0. \tag{6.28}$$

Isomorphism classes of (R_f, S_f) are parametrized by $Hilb^{[1]}(\mathbb{P}^2) \times Gr(2, 6)$.

We have two commutative diagrams as follows.

(6.29)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-3) \arrow[r, "(a_1,a_2)\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"] \arrow[d, "(b_1,b_2)\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-2)}"'] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \arrow[r, "f_s\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"] \arrow[d, "(b_1,b_2)^{\oplus 2}"] & S_f(-1) \arrow[r] \arrow[d, "id_{S_f}\otimes(b_1,b_2)"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \arrow[r, "(a_1,a_2)^{\oplus 2}"] \arrow[d, "f_r\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-2)}"'] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \arrow[r, "f_s^{\oplus 2}"] \arrow[d, "f_r^{\oplus 2}"] & S_f^{\oplus 2} \arrow[r] \arrow[d, "id_{S_f}\otimes f_r"] & 0 \\
0 \arrow[r] & R_f \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "id_{R_f}\otimes(a_1,a_2)"] \arrow[d] & R_f^{\oplus 2} \arrow[r, "id_{R_f}\otimes f_s"] \arrow[d] & R_f \otimes S_f \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "(a_3,a_4)\oplus(a_5,a_6)"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \arrow[r, "f_s^{\oplus 2}"] \arrow[d, "f_r^{\oplus 2}"'] & S_f^{\oplus 2} \arrow[d, "id_{S_f}\otimes f_r"] \\
& R_f^{\oplus 2} \arrow[r, "id_{R_f}\otimes f_s"'] & R_f \otimes S_f.
\end{tikzcd}
\tag{6.30}
$$

Isomorphism classes of $(E,f)\in\Xi_2$ are parametrized by (R_f,S_f,ω_f) with $\omega_f:\mathcal{O}_{\mathbb{P}^2}(-2)\to R_f\otimes S_f$ the composed map in (6.30). Hence firstly we have a projective bundle over $Hilb^{[1]}(\mathbb{P}^2)\times Gr(2,6)$ with fibers isomorphic to $\mathbb{P}(H^0(R_f\otimes S_f(2)))\simeq\mathbb{P}^{15}$, which contains Ξ_2 as an open subset. The complement of Ξ_2 in that projective bundle is the set of all (R_f,S_f,ω_f) such that $\mathrm{Im}(\omega_f)$ are contained in the torsions of $R_f\otimes S_f$.

Torsion free S_f are parametrized by $Gr(2,6)-\mathbb{P}^2\times\mathbb{P}^2$. For S_f torsion free, $R_f\otimes S_f$ has torsion if and only if $(a_1,a_2)|_x=0$ with $R_f\simeq I_x(1)$, and the nonzero torsion must be isomorphic to $\mathcal{O}_x$. Define $\mathcal{V}_1^i:=p_*(\mathcal{I}_1\otimes q^*\mathcal{O}_{\mathbb{P}^2}(i))$ with $\mathcal{I}_1,p,q$ the same as before. $\mathcal{V}_1^1$ and $\mathcal{V}_1^2$ are two vector bundles of rank 2 and 5 respectively over $Hilb^{[1]}(\mathbb{P}^2)$. Hence (R_f,S_f,ω_f) with S_f torsion free and $\mathrm{Im}(\omega_f)$ contained in the torsion of $R_f\otimes S_f$ are parametrized by $Gr(2,\mathcal{V}_1^2)-Gr(2,\mathcal{V}_1^1)\times\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))\cup Gr(2,h^0(\mathcal{O}_{\mathbb{P}^2}(1)))\times\mathbb{P}(\mathcal{V}_1^1)$, where $Gr(2,\mathcal{V}_1^i)$ is the relative Grassmannian of the vector bundle $\mathcal{V}_1^i$. And

$$
\begin{aligned}
& [Gr(2,\mathcal{V}_1^1)\times\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))\cup Gr(2,h^0(\mathcal{O}_{\mathbb{P}^2}(1)))\times\mathbb{P}(\mathcal{V}_1^1)]\\
= & [Gr(2,\mathcal{V}_1^1)\times\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))]+[Gr(2,h^0(\mathcal{O}_{\mathbb{P}^2}(1)))\times\mathbb{P}(\mathcal{V}_1^1)]-[\mathbb{P}(\mathcal{V}_1^1)]\\
= & [Hilb^{[1]}(\mathbb{P}^2)\times(\mathbb{P}^2+\mathbb{P}^2\times\mathbb{P}^1-\mathbb{P}^1)].
\end{aligned}
$$

Now let S_f lie in (6.28). Write $R_f\simeq I_x(1)$ and $I_y(1)$ the quotient of S_f in (6.28). If $x\neq y$, then $R_f\otimes I_y(1)$ is torsion free and in this case (R_f,S_f,ω_f) with $\mathrm{Im}(\omega_f)$ contained in the torsions of $R_f\otimes S_f$ are parametrized by a projective bundle over $Hilb^{[1]}(\mathbb{P}^2)\times\mathbb{P}^2\times\mathbb{P}^2-Gr(2,\mathcal{V}_1^1)\times\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))$ with fibers isomorphic to $\mathbb{P}(H^0(I_1(1)\otimes\mathcal{O}_H(1)))\simeq\mathbb{P}^2$.

Finally we have a projective bundle over $Gr(2,\mathcal{V}_1^1)\times\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))$ with fibers isomorphic to $\mathbb{P}(H^0(\mathcal{O}_x)\oplus H^0(I_1(1)\otimes\mathcal{O}_H(1)))\simeq\mathbb{P}^3$ parametrizing (R_f,S_f,ω_f) such that $x=y$ and $\mathrm{Im}(\omega_f)$ are contained in the torsions of $R_f\otimes S_f$. Hence we have the following lemma.

Lemma 6.15. $[\Xi_2]=[Hilb^{[1]}(\mathbb{P}^2)\times(Gr(2,6)\times\mathbb{P}^{15}-Gr(2,5)-\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2-\mathbb{P}^2\times\mathbb{P}^3-\mathbb{P}^1+\mathbb{P}^2\times\mathbb{P}^2+\mathbb{P}^1\times\mathbb{P}^2+\mathbb{P}^2)]$.

For a pair $(E, f) \in M_2^c$, f can be represented by the following matrix

$$\begin{pmatrix} b_1 & b_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A_1 & A_2 & 0 & B \end{pmatrix},$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, A_i is a 3×1 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and B a 3×2 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. (E, f) is stable, hence $kb_1 \neq k'b_2, \forall (k, k') \in \mathbb{C}^2 - \{0\}$ and the parametrizing space M_B of B is of class $[Hilb^{[3]}(\mathbb{P}^2) - |H| \times \mathbb{P}^3 + |H|]$ by Lemma 5.5 and Lemma 5.8. We write down the following two exact sequences.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow{f_{B^t}} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{f_r} R_f \longrightarrow 0 \tag{6.31}$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(b_1, b_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{f_s} S_f \longrightarrow 0 \tag{6.32}$$

$S_f \simeq I_1(1)$. Either $R_f \simeq I_3(2)$ or R_f lies in the following exact sequence.

$$0 \to \mathcal{O}_H(-1) \to R_f \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0. \tag{6.33}$$

Isomorphism classes of (R_f, S_f) are parametrized by $M_B \times Hilb^{[1]}(\mathbb{P}^2)$.

We have two commutative diagrams as follows.

(6.34)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \arrow[r, "((b_1,b_2)\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)})^{\oplus 2}"] \arrow[d, "f_{B^t}\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"'] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 4} \arrow[r, "(f_s\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)})^{\oplus 2}"] \arrow[d, "f_{B^t}^{\oplus 2}"] & S_f(-1)^{\oplus 2} \arrow[r] \arrow[d, "id_{S_f}\otimes(b_1,b_2)"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \arrow[r, "(b_1,b_2)^{\oplus 3}"] \arrow[d, "f_r\otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}"'] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 6} \arrow[r, "f_s^{\oplus 3}"] \arrow[d, "f_r^{\oplus 2}"] & S_f^{\oplus 3} \arrow[r] \arrow[d, "id_{S_f}\otimes f_r"] & 0 \\
0 \arrow[r] & R_f\otimes\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "id_{R_f}\otimes(b_1,b_2)"] \arrow[d] & R_f^{\oplus 2} \arrow[r, "id_{R_f}\otimes f_s"] \arrow[d] & R_f\otimes S_f \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "A_1^t\oplus A_2^t"] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 6} \arrow[r, "f_s^{\oplus 2}"] \arrow[d, "f_r^{\oplus 2}"'] & S_f^{\oplus 3} \arrow[d, "id_{S_f}\otimes f_r"] \\ & R_f^{\oplus 2} \arrow[r, "id_{R_f}\otimes f_s"'] & R_f\otimes S_f. \end{tikzcd} \tag{6.35}$$

Isomorphism classes of $(E, f) \in M_2^c$ are parametrized by (R_f, S_f, ω_f) with $\omega_f : \mathcal{O}_{\mathbb{P}^2}(-2) \to R_f \otimes S_f$ the composed map in (6.35). Hence we have a projective bundle over $M_B \times Hilb^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(R_f \otimes S_f(2))) \simeq \mathbb{P}^{16}$, which contains M_2^c as an open subset.

We need to exclude all the points (R_f, S_f, ω_f) that $\mathrm{Im}(\omega_f)$ are contained in the torsions of $R_f \otimes S_f$. Firstly let R_f lie in (6.33), then the torsion of $R_f \otimes S_f$ is isomorphic to $\mathcal{O}_H(-1) \otimes I_1(1)$. These (R_f, S_f, ω_f) are parametrized by a projective bundle over $(\cup_{i=0}^2 \mathbb{A}^i) \times Hilb^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(\mathcal{O}_H(1) \otimes I_1(1))) \simeq \mathbb{P}^2$.

Let $R_f \simeq I_3(2)$. Denote $S_f \simeq I_x(1)$. The torsion of $R_f \otimes S_f$ is a linear subspace of $\mathcal{O}_x^{\oplus 2} \simeq \mathbb{C}^2$ which is the kernel of $B^t|_x$. If $x \notin Supp(\mathcal{O}_{\mathbb{P}^2}(2)/R_f)$, $R_f \otimes S_f$ is torsion free. If $Supp(\mathcal{O}_{\mathbb{P}^2}(2)/R_f) = \{x, y, z\}$ with $y, z \neq x$, for simplicity we let $x = [0, 0, 1]$, $y = [0, 1, 0]$ and $z = [1, 0, 0]$ and the matrix B have the following form.

$$\begin{pmatrix} x_1 & 0 \\ x_0 & x_0 \\ 0 & x_2 \end{pmatrix}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$.

Let $R_f \simeq I_{\{x,2y\}}(2)$, then B can be

$$\begin{pmatrix} x_0 & 0 \\ x_2 & x_0 \\ 0 & x_1 \end{pmatrix}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$.

Let $R_f \simeq I_{\{2x,y\}}(2)$, then B can be

$$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_2 \end{pmatrix}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$.

Let $R_f \simeq I_{\{3x\}}(2)$, then B can be

$$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ kx_2 & x_1 \end{pmatrix}, for\ any\ k \in \mathbb{C}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$ if $k \neq 0$, $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x^{\oplus 2}$ if $k = 0$.

The projective bundle $\mathbb{P}(\mathcal{V}_1^1)$ as defined before over $Hilb^{[1]}(\mathbb{P}^2)$ parametrizes all (x, C) with x a single point and C a curve of degree 1 passing through x. Hence we have the universal family $\overline{\mathcal{C}}_1 \subset \mathbb{P}^2 \times \mathbb{P}(\mathcal{V}_1^1)$. Denote $\mathcal{Z}_1$ to be the universal family of subschemes in $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ and $\pi : \mathbb{P}(\mathcal{V}_1^1) \to Hilb^{[1]}(\mathbb{P}^2)$ the projection. Define $\overline{\mathcal{C}}_1^0 := \overline{\mathcal{C}}_1 - (\pi \times id_{\mathbb{P}^2})^* \mathcal{Z}_1$. Denote $\mathbb{P}(\mathcal{V}_1^1)^{[2]}$ the relative Hilbert scheme of 2-points on $\mathbb{P}(\mathcal{V}_1^1)$ over $Hilb^{[1]}(\mathbb{P}^2)$. There is a natural embedding $\imath : \mathbb{P}(\mathcal{V}_1^1) \hookrightarrow \mathbb{P}(\mathcal{V}_1^1)^{[2]}$ sending every point to the double-point supported at it. We have the following diagram

$$\begin{tikzcd}
\overline{\mathcal{C}}_1^0 \times_{\mathbb{P}(\mathcal{V}_1^1) \times_\pi \mathbb{P}(\mathcal{V}_1^1)} \overline{\mathcal{C}}_1^0 - p^*\Delta(\mathbb{P}(\mathcal{V}_1^1)) \arrow[r, "\delta'"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "p'"] \\
\mathbb{P}(\mathcal{V}_1^1) \times_\pi \mathbb{P}(\mathcal{V}_1^1) - \Delta(\mathbb{P}(\mathcal{V}_1^1)) \arrow[r, "\delta"'] & \mathbb{P}(\mathcal{V}_1^1)^{[2]} - \imath(\mathbb{P}(\mathcal{V}_1^1)),
\end{tikzcd} \tag{6.36}$$

with Δ the diagonal embedding and $\mathcal{X}$ defined to make (6.36) a Cartesian diagram. Notice that a priori $\mathcal{X}$ may not exist, but if it exists, it parametrizes isomorphism classes of (R_f, S_f, ω_f) with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{x,y,z\}}(2)$ for $\{x, y, z\} \in N_4^0$ i.e. $H^0(I_{\{x,y,z\}}(1)) = 0$, and $\mathrm{Im}(\omega) \subset Tor(R_f \otimes S_f)$.

Lemma 6.16. *$\mathcal{X}$ exists and* $[\mathcal{X}] = [Hilb^{[1]}(\mathbb{P}^2) \times (\mathbb{P}^2 - \mathbb{P}^1) \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)]$.

Proof. Take an affine cover of $Hilb^{[1]}(\mathbb{P}^2) = \cup_i U_i$ with $U_i \simeq \mathbb{A}^2$. It is enough to prove the lemma with $Hilb^{[1]}(\mathbb{P}^2)$ replaced by U_i. Denote by $\mathcal{Z}_1|_{U_i}$, $\mathbb{P}(\mathcal{V}_1^1)|_{U_i}$, $\mathbb{P}(\mathcal{V}_1^1)^{[2]}|_{U_i}$, $\overline{\mathcal{C}}_1|_{U_i}$ and $\overline{\mathcal{C}}_1^0|_{U_i}$ the pull back of these schemes via the open embedding $U_i \hookrightarrow Hilb^{[1]}(\mathbb{P}^2)$. Then we have that $\mathcal{Z}_1|_{U_i} \simeq U_i$, $\mathbb{P}(\mathcal{V}_1^1)|_{U_i} \simeq U_i \times \mathbb{P}^1$, $\mathbb{P}(\mathcal{V}_1^1)^{[2]}|_{U_i} \simeq U_i \times \mathbb{P}^2$, $\overline{\mathcal{C}}_1|_{U_i} \simeq U_i \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\overline{\mathcal{C}}_1^0|_{U_i} \simeq$

$U_i \times \mathbb{P}^1 \times \mathbb{A}^1$. Hence (6.36) becomes the following commutative diagram.

$$\begin{tikzcd} U_i \times (\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)) \times \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "\delta_i'"] \arrow[d, "p_i"'] & \mathcal{X}_i \arrow[d, "p_i'"] \\ U_i \times (\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)) \arrow[r, "\delta_i"'] & U_i \times (\mathbb{P}^2 - \imath(\mathbb{P}^1)), \end{tikzcd} \tag{6.37}$$

with $\mathcal{X}_i \simeq U_i \times (\mathbb{P}^2 - \imath(\mathbb{P}^1)) \times \mathbb{A}^1 \times \mathbb{A}^1$. Hence the lemma. $\square$

Isomorphism classes of (R_f, S_f) with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{2x,y\}}(2)$ for $H^0(I_{\{2x,y\}}(1)) = 0$ and $\mathrm{Im}(\omega) \subset Tor(R_f \otimes S_f)$ are parametrized by $(\mathbb{P}(\mathcal{V}_1^1) \times_{Hilb^{[1]}(\mathbb{P}^2)} \overline{\mathcal{C}}_1^0) - \Delta$, where Δ is defined by the following Cartesian diagram

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d] & \overline{\mathcal{C}}_1^0 \arrow[d] & \\ \mathbb{P}(\mathcal{V}_1^1) \arrow[r, "id"] & \mathbb{P}(\mathcal{V}_1^1) \arrow[r, "\pi"] & Hilb^{[1]}(\mathbb{P}^2) \end{tikzcd}$$

Lemma 6.17. $[(\mathbb{P}(\mathcal{V}_1^1) \times_{Hilb^{[1]}(\mathbb{P}^2)} \overline{\mathcal{C}}_1^0) - \Delta] = [Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)]$.

Proof. Take the affine cover $Hilb^{[1]}(\mathbb{P}^2) = \cup_i U_i$ with $U_i \simeq \mathbb{A}^2$. Replace $Hilb^{[1]}(\mathbb{P}^2)$ by U_i and the lemma follows immediately. $\square$

The normal sheaf of $\overline{\mathcal{C}}_1$ in $\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}_1^1)$ is locally free over $\overline{\mathcal{C}}_1^0$. We denote by $\mathcal{N}_{\mathcal{C}}^0$ the total space of the normal bundle over $\overline{\mathcal{C}}_1^0$. Then isomorphism classes of (R_f, S_f, ω_f) with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{x,2y\}}(2)$ for $H^0(I_{\{x,2y\}}(1)) = 0$ and $\mathrm{Im}(\omega) \subset Tor(R_f \otimes S_f)$ are parametrized by $\mathcal{N}_{\mathcal{C}}^0$.

Lemma 6.18. $[\mathcal{N}_{\mathcal{C}}^0] = [Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)]$.

Proof. $[\mathcal{N}_{\mathcal{C}}^0] = [\mathbb{A}^1 \times (\overline{\mathcal{C}}_1 - (\pi \times id_{\mathbb{P}^2})^* \mathcal{Z}_1)]$. Also we see $[(\pi \times id_{\mathbb{P}^2})^* \mathcal{Z}_1] = [\mathbb{P}^1 \times Hilb^{[1]}(\mathbb{P}^2)]$ and $[\overline{\mathcal{C}}_1] = [Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times \mathbb{P}^1]$. Hence the lemma. $\square$

If $R_f \simeq I_{\{3x\}}(2)$ with $H^0(R_f(-1)) = 0$ and $Tor(R_f \otimes I_x) \simeq \mathcal{O}_x$, then R_f, viewed as an ideal of $\widehat{\mathcal{O}}_{\mathbb{P}^2,x} \simeq \mathbb{C}[[x_0, x_1]]$, is generated by $(kx_0 - x_1^2, \mathfrak{m}^3)$ with $\mathfrak{m}$ the maximal ideal in $\mathbb{C}[[x_0, x_1]]$ and $k \in \mathbb{C}^*$. Hence such R_f are parametrized by (x_0, k) for any fixed $x \in \mathbb{P}^2$. Hence isomorphism classes of these (R_f, S_f, ω_f) such that $\mathrm{Im}(\omega) \subset Tor(R_f \otimes S_f)$ are parametrized by $\mathbb{P}(\mathcal{V}_1^1) \times (\mathbb{A}^1 - \{0\})$.

Finally, let $S_f \simeq I_x(1)$, $R_f \simeq I_{\{3x\}}(2)$ with $H^0(R_f(-1)) = 0$ and $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x^{\oplus 2}$, then R_f is determined by x since $R_f(-2) \simeq I_x^2$. Hence isomorphism classes of these (R_f, S_f, ω_f) such that $\mathrm{Im}(\omega) \subset Tor(R_f \otimes S_f)$ are parametrized by $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}(H^0(\mathcal{O}_x^{\oplus 2})) \simeq Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1$.

Lemma 6.19. *$[M_2^c] = [M_B \times Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^{16} - \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 - \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) - \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2 - \mathbb{P}^1)]$, with $[M_B] = [Hilb^{[3]}(\mathbb{P}^2) - \mathbb{P}^2 \times \mathbb{P}^3 + |H|]$.*

Proof. $[M_2^c] = [M_B \times Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^{16} - \mathbb{P}^2 \times |H| \times Hilb^{[1]}(\mathbb{P}^2) - \mathbb{P}(\mathcal{V}_1^1) \times (\mathbb{A}^1 - 1) - Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 - \mathcal{X} - \mathbb{P}(\mathcal{V}_1^1) \times_{Hilb^{[1]}(\mathbb{P}^2)} \overline{\mathcal{C}}_1^0 + \Delta - \mathcal{N}_{\mathcal{C}}^0]$.

By Lemma 6.16, Lemma 6.17 and Lemma 6.18, we get the lemma by direct computation. □

Proof of Theorem 6.1 for $M(5,2)$. We have

$$[M(5,2)] = [M_3'] + [M_3] + [\Xi_1] + [\Xi_2] + [M_2^c].$$

Combine Lemma 6.12, Lemma 6.13, Lemma 6.14, Lemma 6.15 and Lemma 6.19, we get the result by direct computation. □

Appendix

Appendix A. Proofs of Lemma 6.2 and Lemma 6.11.

Lemma A.1 (Lemma 6.2). *A pair (E, f) with $rank(E) = 5$ and $deg(E) = -4$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. We first prove the lemma for $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. We want to show that if $\exists E' \subset E$ a direct sum of line bundles with $\mu(E') > \mu(E)$ and $f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then $\exists D, D' \subset E$ two direct summands with $D \simeq D'$ and $\mu(D) > \mu(E)$, such that $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'$. With no loss of generality, we assume that E' has the form $\bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ with $a_i > 0$ and $n_i - n_{i+1} = 1$.

Let $E' \simeq E'' \subset E$ with E'' not a direct summand of E. Then E'' has to be one of the following three cases:

(1) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;
(2) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$;
(3) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

By Nakayama's lemma, we know that $E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ can't be the preimage of any direct summand of E and also $f^{-1}(E'') = E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \Rightarrow f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D$ with D the smallest direct summand of E containing E''.

So we assume that $f^{-1}(E'') = E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with E' a direct summand of E isomorphic to E''.

Let E'' be in case (1). By the assumption we have $f(E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$. On the other hand, write $E = \mathcal{O}_{\mathbb{P}^2} \oplus E' \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, so for the other direct summand $\mathcal{O}_{\mathbb{P}^2}$ we have $f(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. Hence $f((\mathcal{O}_{\mathbb{P}^2} \oplus E') \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, and hence we get $D = D' = E' \oplus \mathcal{O}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.

Case (2) is analogous to case (1).

Let E'' be in case (3). By the assumption we have $f(E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Write $E = E' \oplus L$ with $L \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. We can ask f to identify $L \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with the summand $\mathcal{O}_{\mathbb{P}^2}(-2)$ in E.

Denote by $f_o : E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to E''$ the restriction of f. If $f_o((\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, then we have $D = D' = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \subset E'$.

If $f_o((\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \not\subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, f_o induces an isomorphism from the direct summand $\mathcal{O}_{\mathbb{P}^2}(-1) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ of $E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ to the direct summand $\mathcal{O}_{\mathbb{P}^2}(-2)$ of E''. Hence we can ask f_o to identify these two direct summands. Write $E' = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus L'$ with $L' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, then we have $f_o(L' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. Moreover because f identifies $L \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with the summand $\mathcal{O}_{\mathbb{P}^2}(-2)$ in E, $f((L \oplus L') \otimes \mathcal{O}_{\mathbb{P}^2}(-1))$ is contained in the direct summand $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ of E, hence we have $D = L \oplus L'$ and D' is the direct summand $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ containing $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1))$.

This finishes the proof for $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

Let $E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}$. We have the following six possibilities for E''.

(4) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}$;

(5) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$;

(6) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(7) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(8) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(9) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

Analogously $E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ can not be the preimage of any direct summand of E and also $f^{-1}(E'') = E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \Rightarrow f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D$ with D the smallest direct summand of E containing E''. Let E''_3 and E''_4 be the bundles in case (6) and (7) respectively. $f^{-1}(E''_3) = E''_4 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \Rightarrow$

$f((\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1))\otimes\mathcal{O}_{\mathbb{P}^2}(-1))\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$, and $f^{-1}(E''_4)=E''_3\otimes\mathcal{O}_{\mathbb{P}^2}(-1)\Rightarrow f(\mathcal{O}_{\mathbb{P}^2}(1)\otimes\mathcal{O}_{\mathbb{P}^2}(-1))\subset\mathcal{O}_{\mathbb{P}^2}(1)$.

Hence we then assume E' a direct summand of E isomorphic to E'' and $f^{-1}(E'')=E'\otimes\mathcal{O}_{\mathbb{P}^2}(-1)$.

For case (4), by assumption we have $f(\mathcal{O}_{\mathbb{P}^2}\otimes\mathcal{O}_{\mathbb{P}^2}(-1))\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}$ hence $D=D'=\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}$.

Bundles in case (5), case (8) and case (9) can not be direct summands of E, hence these three cases are done.

For case (6), by assumption we have $f((\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1))\otimes\mathcal{O}_{\mathbb{P}^2}(-1))\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$ hence $D=D'=\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$.

For case (7), by assumption we have $f(\mathcal{O}_{\mathbb{P}^2}\otimes\mathcal{O}_{\mathbb{P}^2}(-1))\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$ hence $D=D'=\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$.

This finishes the proof for the whole lemma. □

Lemma A.2 (Lemma 6.11). *A pair (E,f) with $rank(E)=5$ and $deg(E)=-3$ is stable if and only if for any two direct summands D',D'' of E such that $D'\simeq D''$ and $f(D'\otimes\mathcal{O}_{\mathbb{P}^2}(-1))\subset D''$, we have $\mu(D')<\mu(E)$.*

Proof. We use the same notations as in the proof of Lemma A.1, we list out all the possibilities of E'' as follows.

Let $E\simeq\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$.

(1) $E''\subset\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$;

Let $E\simeq\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\mathcal{O}_{\mathbb{P}^2}(-2)$.

(2) $E''\subset\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$;

(3) $E''\subset\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$;

(4) $E''\subset\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$;

Let $E\simeq\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^2}(-2)$.

(5) $E''\subset\mathcal{O}_{\mathbb{P}^2}(1)$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}$;

(6) $E''\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$;

(7) $E''\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$;

(8) $E''\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$;

(9) $E''\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)$;

(10) $E''\subset\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ and $E''\simeq\mathcal{O}_{\mathbb{P}^2}(1)\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\mathcal{O}_{\mathbb{P}^2}(-2)$.

Cases (1) (5) (6) (7) (8) (9) are the same as cases (1) (4) (5) (6) (7) (8) in Lemma A.1 respectively. Case (10) is analogous to case (3) in Lemma A.1. Cases (2) (3) (4) are analogous to case (1). Hence the lemma. □

Appendix B. Proof of Lemma 6.10.

Lemma B.1 (Lemma 6.10). $[\Omega_2^{[6]}] = \mathbb{L}^{11} + 3\mathbb{L}^{10} + 8\mathbb{L}^9 + 18\mathbb{L}^8 + 30\mathbb{L}^7 + 39\mathbb{L}^6 + 38\mathbb{L}^5 + 28\mathbb{L}^4 + 15\mathbb{L}^3 + 6\mathbb{L}^2 + 2\mathbb{L}^1 + 1.$

Proof. Denote by $\mathcal{C}_2$ the universal curve in $\mathbb{P}^2 \times |2H|$ and $\mathcal{C}_2^{[6]}$ the relative Hilbert scheme of 6-points on $\mathcal{C}_2$ over $|2H|$. We have a surjective map $\xi : \mathcal{C}_2^{[6]} \to \Omega_2^{[6]}$. The fiber of ξ over $I_{\{x_1,\dots,x_6\}}$ consists of all curves passing through $x_1,\dots,x_6$ and hence isomorphic to $\mathbb{P}(H^0(I_{\{x_1,\dots,x_6\}}(2)))$. Let $\mathcal{S}_n := \{I_6 \in \Omega_2^{[6]} | h^0(I_6(2)) = n+1\}$, then $\Omega_2^{[6]} = \coprod_{n=0}^{2} \mathcal{S}_n$. Define $\mathcal{R}_n := \xi^{-1}(\mathcal{S}_n)$, then $\mathcal{R}_n \simeq \mathbb{P}(p_*(\mathcal{I}(2)|_{\mathbb{P}^2 \times \mathcal{S}_n}))$ is a projective bundle over $\mathcal{S}_n$ with fibers isomorphic to $\mathbb{P}^n$.

Denote by $\mathcal{C}_2^o$ the family of integral curves in $|2H|$. $\mathcal{C}_2^o$ is open in $\mathcal{C}_2$.

Lemma B.2. $[\mathcal{C}_2^{o[6]}] = [(|2H| - Sym^2(|H|)) \times \mathbb{P}^6]$ *with* $Sym^2(|H|)$ *the symmetric power of order 2 of* $|H|$.

Proof. The subspace $|2H|^o$ in $|2H|$ parametrizing integral curves is $|2H| - Sym^2(|H|)$. $\mathcal{C}_2^{o[6]}$ is a projective bundle over $|2H|^o$ with fibers isomorphic to $\mathbb{P}^6$. Hence the lemma. $\square$

Denote by $\mathcal{C}_2^R \twoheadrightarrow (Sym^2(|H|) - |H|)$ and $\mathcal{C}_2^N \twoheadrightarrow |H|$ the families of reducible curves and non-reduced curves in $|2H|$ respectively. Let C^R be a reducible curve in $|2H|$ and C^N a non-reduced curve. Denote R_n^R $(\mathcal{R}_n^R) = \mathcal{R}_n \cap Hilb^{[6]}(C^R)$ $(\mathcal{C}_2^{R[6]})$ and R_n^N $(\mathcal{R}_n^N) = \mathcal{R}_n \cap Hilb^{[6]}(C^N)$ $(\mathcal{C}_2^{N[6]})$. Then we have the following lemma

Lemma B.3. $[\mathcal{R}_n^N] = [R_n^N \times |H|]$, *for* $n = 0, 1, 2$.

Proof. We can take an affine cover of $|H|$, write $|H| = \cup_j V_j$ such that $\mathcal{C}_2^N|_{V_j} \simeq V_j \times C^N$. Hence the lemma. $\square$

Denote $\mathfrak{m} =< x, y >$ the maximal ideal of $\mathtt{S} := \mathbb{C}[[x,y]]$. To study R_n^N for $n = 0, 1, 2$, we write down a table for ideals in $\mathtt{S}/(x^2)$ as Table I.

Table I	Ideals I of $\mathtt{S}$ containing (x^2)	
Co-length of I	Ideal I	$I\cap(\mathfrak{m}^2-\mathfrak{m}^3)$
1	$\mathfrak{m}$	$\mathbb{C}x^2\oplus\mathbb{C}xy\oplus\mathbb{C}y^2$
2	$\mathfrak{m}^2+(kx+k'y)\mathtt{S},(k,k')\neq 0$	$\mathbb{C}x^2\oplus\mathbb{C}xy\oplus\mathbb{C}y^2$
3	$\mathfrak{m}^2$	$\mathbb{C}x^2\oplus\mathbb{C}xy\oplus\mathbb{C}y^2$
	$\mathfrak{m}^3+(x+ky^2)\mathtt{S}$	$\mathbb{C}x^2\oplus\mathbb{C}xy$
4	$x^2\mathtt{S}+(ky^2+k'xy)\mathtt{S}+\mathfrak{m}^3,(k,k')\neq 0$	$\mathbb{C}x^2\oplus\mathbb{C}(ky^2+k'xy)$
	$(x+ky^2+k'y^3)\mathtt{S}+\mathfrak{m}^4,k\neq 0$	$\mathbb{C}x^2$
	$(x+k'y^3)\mathtt{S}+\mathfrak{m}^4$	$\mathbb{C}x^2\oplus\mathbb{C}xy$
5	$x^2\mathtt{S}+\mathfrak{m}^3$	$\mathbb{C}x^2$
	$x^2\mathtt{S}+(xy+ky^2+k'y^3)\mathtt{S}+\mathfrak{m}^4,k\neq 0$	$\mathbb{C}x^2$
	$x^2\mathtt{S}+(xy+k'y^3)\mathtt{S}+\mathfrak{m}^4$	$\mathbb{C}x^2\oplus\mathbb{C}xy$
	$(x+ky^3+k'y^4)\mathtt{S}+\mathfrak{m}^5,k\neq 0$	$\mathbb{C}x^2$
	$(x+k'y^4)\mathtt{S}+\mathfrak{m}^5$	$\mathbb{C}x^2\oplus\mathbb{C}xy$
6	$x^2\mathtt{S}+(kxy^2+k'y^3)\mathtt{S}+\mathfrak{m}^4,(k,k')\neq 0$	$\mathbb{C}x^2$
	$x^2\mathtt{S}+(xy+ky^3+k'y^4)\mathtt{S}+\mathfrak{m}^5,k\neq 0$	$\mathbb{C}x^2$
	$x^2\mathtt{S}+(xy+k'y^4)\mathtt{S}+\mathfrak{m}^5$	$\mathbb{C}x^2\oplus\mathbb{C}xy$
	$(x+ky^3+k'y^4+k''y^5)\mathtt{S}+\mathfrak{m}^6,(k,k'')\neq 0$	$\mathbb{C}x^2$
	$(x+k''y^5)\mathtt{S}+\mathfrak{m}^6$	$\mathbb{C}x^2\oplus\mathbb{C}xy$

Let C_r^N be the reduced curve supported on C^N, then $C_r^N\simeq\mathbb{P}^1$. Denote by S^i the subset in $Hilb^{[6]}(C^N)$ consisting of $[I_i^r\cap I_{6-i}]$ with $[I_i^r]\in Hilb^{[i]}(C_r^N)$ and i maximal for this expression. $S_n^i:=S^i\cap R_n^N$. We then have

Lemma B.4. *S_n^i are empty except the following 9 terms:*
$[S_2^6]=[\mathbb{P}^6]$; $[S_1^4]=[\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{A}^1]$; $[S_1^3]=[\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{A}^4]$;
$[S_0^2]=[\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{A}^5]$, $[S_2^2]=[\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{A}^3]$;
$[S_0^1]=[\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{A}^4]$, $[S_2^1]=[\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{A}^3]$;
$[S_0^0]=[\mathbb{P}^2\times\mathbb{P}^1\times(\mathbb{A}^4+2\mathbb{A}^3+\mathbb{A}^2)+\mathbb{P}^2\times\mathbb{A}^4+\mathbb{P}^2(\mathbb{P}^3-\mathbb{P}^1\times\mathbb{P}^1)\times\mathbb{A}^3]$,
$[S_1^0]=[\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{A}^1]$.

We omit the proof of Lemma B.4 since it can be done by elementary analysis and computation case by case. Lemma B.4 together with Lemma B.6 gives $[\mathcal{R}_n^N]$ for $n=0,1,2$.

To compute $[\mathcal{R}_n^R]$, we first define $\widetilde{\mathcal{C}_2^R}$ by the following Cartesian diagram.

$$\begin{tikzcd}
\widetilde{\mathcal{C}_2^R} \arrow[r, "\pi_1"] \arrow[d] & \mathcal{C}_2^R \arrow[d] \\
\mathbb{P}^2\times\mathbb{P}^2-\Delta(\mathbb{P}^2) \arrow[r, "\pi"] & Sym^2(\mathbb{P}^2)-\mathbb{P}^2,
\end{tikzcd}$$

where π is the quotient of the free action of the order two permutation group σ_2. The action of σ_2 lifts to $\widetilde{\mathcal{C}_2^R}^{[6]}$ with $\mathcal{C}_2^{R[6]}$ the quotient. Recall that R_n^R ($\mathcal{R}_n^R$) $= \mathcal{R}_n \cap Hilb^{[6]}(C^R)$ ($\mathcal{C}_2^{R[2]}$). Let $\widetilde{\mathcal{R}_n^R} := \pi_2^{-1}(\mathcal{R}_n^R)$ with π_2 the lift of π. $\mathcal{R}_n^R$ is the quotient of $\widetilde{\mathcal{R}_n^R}$ by the action of σ_2.

Lemma B.5. $[\widetilde{\mathcal{R}_n^R}] = [(\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)) \times R_n^R]$ *for* $n = 0, 1, 2$.

Proof. Analogous to Lemma B.3, we can take an affine cover of $\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)$ which trivializes $\widetilde{\mathcal{C}_2^R}$. $\square$

Denote by 0 the only singular point in C^R. $C^R - \{0\} = \mathbb{A}^1 \sqcup \mathbb{A}^1$. $\widehat{\mathcal{O}}_{C^R,0} \simeq \mathtt{S}/(xy) = \mathbb{C}[[x, y]]/(xy)$. We make a table for ideals of $\mathtt{S}/(xy)$ as Table II.

Table II	Ideals I of $\mathtt{S}$ containing (xy)	
Co-length of I	Ideal I	$I \cap (\mathfrak{m}^2 - \mathfrak{m}^3)$
1	$\mathfrak{m}$	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
2	$\mathfrak{m}^2 + (kx + k'y)\mathtt{S}, (k, k') \neq 0$	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
3	$\mathfrak{m}^2$	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
	$\mathfrak{m}^3 + (x + ky^2)\mathtt{S}$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$\mathfrak{m}^3 + (y + kx^2)\mathtt{S}$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
4	$xy\mathtt{S} + (kx^2 + k'y^2)\mathtt{S} + \mathfrak{m}^3, (k, k') \neq 0$	$\mathbb{C}xy \oplus \mathbb{C}(kx^2 + k'y^2)$
	$(x + ky^3)\mathtt{S} + \mathfrak{m}^4$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(y + kx^3)\mathtt{S} + \mathfrak{m}^4$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
5	$xy\mathtt{S} + \mathfrak{m}^3$	$\mathbb{C}xy$
	$xy\mathtt{S} + (x^2 + ky^3)\mathtt{S} + \mathfrak{m}^4, k \neq 0$	$\mathbb{C}xy$
	$xy\mathtt{S} + x^2\mathtt{S} + \mathfrak{m}^4$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$xy\mathtt{S} + (y^2 + kx^3)\mathtt{S} + \mathfrak{m}^4, k \neq 0$	$\mathbb{C}xy$
	$xy\mathtt{S} + y^2\mathtt{S} + \mathfrak{m}^4$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
	$(x + ky^4)\mathtt{S} + \mathfrak{m}^5$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(y + kx^4)\mathtt{S} + \mathfrak{m}^5$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
6	$xy\mathtt{S} + (kx^3 + k'y^3)\mathtt{S} + \mathfrak{m}^4, (k, k') \neq 0$	$\mathbb{C}xy$
	$xy\mathtt{S} + (x^2 + ky^4)\mathtt{S} + \mathfrak{m}^5, k \neq 0$	$\mathbb{C}xy$
	$xy\mathtt{S} + x^2\mathtt{S} + \mathfrak{m}^5$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$xy\mathtt{S} + (y^2 + kx^4)\mathtt{S} + \mathfrak{m}^5, k \neq 0$	$\mathbb{C}xy$
	$xy\mathtt{S} + y^2\mathtt{S} + \mathfrak{m}^5$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
	$(x + ky^5)\mathtt{S} + \mathfrak{m}^6$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(y + kx^5)\mathtt{S} + \mathfrak{m}^6$	$\mathbb{C}xy \oplus \mathbb{C}y^2$

σ_2 acts on $Hilb^{[6]}(C^R)$ by exchanging the two irreducible components of C^R. Write $Hilb^{[6]}(C^R) = H^x \sqcup H^s \sqcup H^y$ such that $\sigma_2(H^x) = H^y$ and $\sigma_2(H^s) = H^s$. Let $H_n^x = \mathcal{R}_n \cap H^x$ and analogously we have H_n^y and H_n^s. Then $\sigma_2(H_n^x) = H_n^y, \sigma_2(H_n^s) = H_n^s$.

Lemma B.6. 1) $[H_0^x] = [\mathbb{A}^6 + 2\mathbb{A}^5 + 3\mathbb{A}^4 + 3\mathbb{A}^3 + 2\mathbb{A}^2 - 1]$;

2) $[H_1^x] = [\mathbb{A}^6 + 2\mathbb{A}^5 + 2\mathbb{A}^4 + 2\mathbb{A}^3 + 2\mathbb{A}^2 + 2\mathbb{A}^1]$;

3) $[H_2^x] = [\mathbb{P}^6]$;

4) $[H_0^s] = [\mathbb{A}^6 + \mathbb{A}^4 \times \mathbb{P}^1 + \mathbb{A}^2 \times \mathbb{P}^1 + \mathbb{P}^1]$;

5) $H_1^s = H_2^s = \emptyset$.

Proof. $\forall I_6 \in Hilb^{[6]}(C^R)$, $\exists\ 0 \leq i \leq 6$, such that $I_6 = I_i^0 \cap I_{6-i}$ with $[I_i^0] \in Hilb^i(\{0\})$ and $[I_{6-i}] \in Hilb^{[6-i]}(C^R - \{0\}) = Hilb^{[6-i]}(\mathbb{A}^1 \sqcup \mathbb{A}^1)$. Then the lemma can be proved by elementary analysis and computation case by case. $\square$

Lemma B.7. $[\mathcal{R}_n^R] = [H_n^x \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))] + [H_n^s \times Sym^2(\mathbb{P}^2) - \mathbb{P}^2]$.

Proof. $\sigma_2(H_n^x \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)) = H_n^y \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))$ and $\sigma_2(H_n^s \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)) = H_n^s \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))$. Hence the lemma. $\square$

Now combine Lemma B.2, Lemma B.3, Lemma B.4, Lemma B.5, Lemma B.6 and Lemma B.7, we get $[\mathcal{R}_n]$ for $n = 0, 1, 2$. Since $[\Omega_2^{[6]}] = \sum_{n=0}^2 [\mathcal{S}_n]$ and $[\mathcal{S}_n \times \mathbb{P}^n] = [\mathcal{R}_n]$, Lemma 6.10 follows after direct computation. $\square$

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