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# Explicit Plancherel Measure for $\mathrm{PGL}_2(\mathbb{F})$

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**Abstract:** In this paper we compute an explicit Plancherel formula for $\mathrm{PGL}_2(\mathbb{F})$ where $\mathbb{F}$ is a non-archimedean local field by a method developed by Busnell, Henniart and Kutzko. Let $G$ be connected reductive group over a non-archimedean local field $\mathbb{F}$. We show that we can obtain types and covers (as defined by Bushnell and Kutzko in "Smooth representations of reductive $p$-adic groups: structure theory via types" Pure Appl. Math, 2009) for $G/Z$ coming from types and covers of $G$ in a very explicit way. We then compute those types and covers for $\mathrm{GL}_2(\mathbb{F})$ which give rise to all types and covers for $\mathrm{PGL}_2(\mathbb{F})$ that are in the principal series. The Bernstein components $\bar{\mathfrak{s}}$ of $\mathrm{PGL}_2(\mathbb{F})$ that correspond to the principal series are of the form $[\bar{\mathbb{T}}, \varphi]_{\bar{G}}$ where $\bar{\mathbb{T}}$ is the diagonal matrices in $\mathrm{GL}_2(\mathbb{F})$ modulo the center and $\varphi$ is a smooth character of $\bar{\mathbb{T}}$. Let $(\bar{J}, \bar{\lambda}_\varphi)$ be an $\bar{\mathfrak{s}}$-type. Then the Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ is a Hilbert algebra and has a measure associated to it called Plancherel measure of $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$. We show that computing the Plancherel measure for $\mathrm{PGL}_2(\mathbb{F})$ essentially reduces to computing the Plancherel measure for $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ for every type $(\bar{J}, \bar{\lambda}_\varphi)$. We get that the Hilbert algebras $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ come in two flavors; they are either $\mathbb{C}[\mathbb{Z}]$ or they are a free algebra in two generators $s_1, s_2$ subject to the relations $s_1^2 = 1$ and $s_2^2 = (q^{-1/2} - q^{-1/2})s_2 + 1$. We denote this latter algebra by $\mathcal{H}(q, 1)$. The Plancherel measure for $\mathbb{C}[\mathbb{Z}]$ as well as the Plancherel measure for $\mathcal{H}(q, 1)$ are known.
**Keywords:** Plancherel Measrue, $\mathrm{PGL}_2$, Types and Covers.

## 1. Introduction

We will start by giving a characterization of the Plancherel measure as given by Dixmier in [Dix77, 18.8.2]. Let $G$ be a locally compact group with Haar measure $\mu_G$ and denote by $\widehat{G}$ the set of irreducible unitary representations of $G$. If $G$ is separable, unimodular and such that the enveloping $C^*$-algebra

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of $G$ is postliminal then there exists a unique measure $\nu$ on the set $\widehat{G}$ (which depends on the Haar measure $\mu_G$) such that for every $u \in L^1(G) \cap L^2(G)$

$$\int_G |u(s)|^2 d\mu_G(s) = \int_{\widehat{G}} tr(\zeta(u)\zeta(u)^*)d\nu(\zeta).$$

The measure $\nu$ is the *Plancherel measure.*

Let us assume in what follows that $G$ is a connected reductive group defined over a non-archimedean local field $\mathbb{F}$. Then $G$ is totally disconnected and we may form the Hecke algebra $\mathcal{H}(G)$ of all locally constant, compactly supported functions. More precisely, let $\mathcal{H}(G)$ to be set of all functions $f : G \longrightarrow \mathbb{C}$ with the properties:

1) there exist a compact open subgroup $K$ of $G$ such that $f(gk) = f(g)$ for all $k \in K$,

2) $f$ has compact support.

For $f_1, f_2 \in \mathcal{H}(G)$ we define the function $f_1 \star f_2(y) = \int_G f_1(x)f_2(x^{-1}y)d\mu_G$. We call the function $f_1 \star f_2$ the convolution of $f_1$ and $f_2$. With the convolution product the set $\mathcal{H}(G)$ obtains the structure of an associative complex algebra.

We continue with the assumption that $G$ is a connected reductive group defined over a non-archimedean local field $\mathbb{F}$, then according to a theorem of Bernstein [Ber74], $G$ is postliminal. Also $G$ may be embedded in some $\mathrm{GL}_N(\mathbb{F})$ for some positive integer $N$ showing that $G$ is separable since $\mathrm{GL}_N(\mathbb{F})$ is separable [Ren10, p. 121]. It is also well known that a connected reductive group defined over a non-archimedean local field $\mathbb{F}$, is unimodular [Ren10, p. 121]; therefore, satisfies all the conditions for the existence of the Plancherel measure.

In the case where $G$ is a connected reductive group defined over a non-archimedean local field we can characterize the Plancherel measure in a slightly different and more convenient way. Indeed,

**Theorem 1.** *The Plancherel measure $\nu$ is the unique measure on $\widehat{G}$ such that*

$$f(1) = \int_{\hat{G}} tr(\zeta(f))d\nu(\zeta), \; f \in \mathcal{H}(G).$$

There has been a considerable amount of research on the topics related to the Plancherel measure for connected reductive groups over a local field. One of the most important expositions is by Waldspurger following the work of

Harish-Chandra [Wal03]. The Plancherel Measure is also used in applications of the celebrated Arthur's Trace formula [Art91]. The main objective of this paper is to compute the Plancherel Measure for $\mathrm{PGL}_2(\mathbb{F})$ where $\mathbb{F}$ is a non–archimedean local field. We will like to mention that the Plancherel measure has been computed by a completely different method by Silberger in [Sil70]. However, Silberger restricts to the case where the characteristic of the residue field is odd while we do not impose any restriction.

## 1.1. The methodology

The method that we use to compute the Plancherel measure relies heavily on the theory of types and covers developed by Kutzko and Bushnell. We therefore make a quick introduction to the theory of types and covers and refer to [BK01] for the proof and definition of the following results.

### 1.1.1. Types and Covers.

**Definition 1.** *A cuspidal data is a pair $(M, (\sigma, W))$ where $M$ is a Levi component of a parabolic subgroup $P$ of $G$ and $(\sigma, W)$ is an irreducible supercuspidal representation of $M$. We should understand by the notation $P = MN$ that $M$ is a Levi component and $N$ is the unipotent part.*

**Definition 2.** *Let $(M_i, (\sigma_i, W_i))$, $i = 1, 2$ be two cuspidal data. We say they define the same* inertial support *if there exist an unramified character $\chi$ of $M_2$ such that*

$$gM_1g^{-1} = M_2 \text{ and } \sigma_2 = \sigma_1^g \otimes \chi.$$

We denote by $[(M, (\sigma, W))]_G$ the class of cuspidal data that define the same inertial support as $(M_1, (\sigma_1, W_1))$ and by $\mathscr{B}(G)$ the set of all these equivalence classes. If $(\pi, V)$ is an irreducible smooth representation, then there exists a parabolic subgroup $P = MN$, a unique element $\mathfrak{s} \in \mathscr{B}(G)$ and a representative $(M, (\sigma, W)) \in \mathfrak{s}$ such that $(\sigma, W)$ is a composition factor of $r_P^G(\pi, V)$. Saying that $(\sigma, W)$ is a composition factor of $r_P^G(\pi, V)$ is equivalent to say $\mathrm{Hom}_M(r_P^G(\pi), \sigma) \neq 0$ or equivalently $\mathrm{Hom}_G(\pi, \iota_P^G(\sigma)) \neq 0$.

**Definition 3.** *If $(\pi, V)$ is an irreducible smooth representation of $G$ and $[(M, (\sigma, W))]_G = \mathfrak{s} \in \mathscr{B}(G)$ satisfies $\mathrm{Hom}_G(\pi, \iota_P^G(\sigma)) \neq 0$. We say then that $(\pi, V)$ has* inertial support *in $\mathfrak{s}$ and denote it by $\mathfrak{I}((\pi, V)) \in \mathfrak{s}$.*

Let $\mathfrak{R}(G)$ be the category of smooth representations of $G$. The Bernstein decomposition that can be found originally in [Ber84] and explained in great detail in [Ren10] gives us that $\mathfrak{R}(G)$ can be decomposed as a product of subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathscr{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

The symbol $\mathfrak{R}^{\mathfrak{s}}(G)$ denotes the full subcategory of $\mathfrak{R}(G)$ where a smooth representation $(\pi, V)$ is an object of $\mathfrak{R}^{\mathfrak{s}}(G)$ if and only if every irreducible subquotient $(\tau, E)$ of $(\pi, V)$ has inertial support in $\mathfrak{s}$.

Let $K$ be a compact open subgroup of $G$ and let $(\rho, W)$ be an irreducible $K$-representation. Let $(\check{\rho}, \check{W})$ be the contragradient representation. Let $\mathcal{H}(G, \rho)$ denote the space of compactly supported functions $f : G \longrightarrow End_{\mathbb{C}}(\check{W})$ such that

$$f(k_1gk_2) = \check{\rho}(k_1)f(g)\check{\rho}(k_2), \text{where } k_1, k_2 \in K,\ g \in G.$$

The operation of convolution

$$f_1 \star f_2(g) = \int_G f_1(x)f_2(x^{-1}g)d\mu_G(x)\ f_1, f_2 \in \mathcal{H}(G, \rho),$$

gives $\mathcal{H}(G, \rho)$ the structure of an associative complex algebra with identity. We denote the identity by $1_{(G,\rho)}$ and is given by

$$1_{(G,\rho)}(x) = \begin{cases} \frac{1}{\mu_G(K)}\check{\rho}(x) & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 4.** *Let $K$ be an open compact subgroup of $G$. Let $(\rho, W)$ be an irreducible representation of $K$. Then denote by*

$$\mathcal{I}_G(\rho) = \left\{x \in G \middle|\, \mathrm{Hom}_{x^{-1}Kx\cap K}(\rho^x, \rho) \neq 0\right\}.$$

*Where $\rho^x(y) = \rho(xyx^{-1})$, for $y \in x^{-1}Kx \cap K$.*

**Lemma 2.** *Using the notation of the preceding definition we have the following list of results.*

i) *If $x \in \mathcal{I}_G(\rho)$ then $KxK \subset \mathcal{I}_G(\rho)$.*

ii) *$x \in \mathcal{I}_G(\rho)$ if and only if there exist a function $f \in \mathcal{H}(G, \rho)$ such that the support of $f$ is $KxK$.*

iii) *$x \in \mathcal{I}_G(\rho)$ if and only if there exist a function $f \in \mathcal{H}(G, \rho)$ such that the support of $f$ contains $KxK$.*

iv) *Let* $\mathcal{H}_x(G,\rho) = \{f \in \mathcal{H}(G,\rho) |\ \mathrm{supp}(f) \subset KxK\}$ *then*

$$\mathcal{H}(G,\rho) = \bigoplus_{x \in K \backslash \mathcal{I}_G(\rho)/K} \mathcal{H}_x(G,\rho).$$

Let $(\pi, V)$ be a smooth representation of $G$ and let $K$ and $(\rho, W)$ be as above; the space $V_\rho = \mathrm{Hom}_K(\rho, V)$ has the structure of a left $\mathcal{H}(G,\rho)$ module with the action given by $(f \cdot T) : w = \int_G \pi(x) T({}^t f(x) w) d\mu_G(x)$, for $f \in \mathcal{H}(G,\rho)$, $T \in V_\rho$ and where $^t$ denotes the transpose operator. Given an $\mathfrak{s} \in \mathscr{B}(G)$ there is a functor $\mathbf{M}_\rho : \mathfrak{R}^{\mathfrak{s}}(G) \to \mathcal{H}(G,\rho)$-Mod given by $\mathbf{M}_\rho(\pi, V) = \mathrm{Hom}_K(\rho, V)$.

**Definition 5.** *The pair* $(K,\rho)$ *is an* $\mathfrak{s}$-type *if an only if the functor* $\mathbf{M}_\rho : \mathfrak{R}^{\mathfrak{s}}(G) \to \mathcal{H}(G,\rho)$-Mod *gives an equivalence of categories.*

**Theorem 3.** *Let* $\mathfrak{s}$ *be an element in* $\mathscr{B}(G)$. *The pair* $(K,\rho)$ *is then an* $\mathfrak{s}$-*type in* $G$ *if and only if, for an irreducible representation* $(\pi, V) \in \mathfrak{R}(G)$, *we have* $\mathfrak{I}(\pi) \in \mathfrak{S}$ *if and only if* $\mathrm{Hom}_K(\rho, \pi) \neq 0$

Let $P_u = MN_u$ be a parabolic subgroup of $G$ with Levi component $M$ and unipotent radical $N_u$. Let us denote by $N_\ell$ the opposite of $N_u$ relative to $M$ and $P_\ell = MN_\ell$ where $N_\ell$ is the unipotent radical for $P_\ell$. Consider the pair $(J,\tau)$ where $J$ is a compact open subgroup and $(\tau, W)$ an irreducible representation of $J$.

**Definition 6.** *The pair* $(J,\tau)$ *is said to be* decomposed with respect to $(M, P_u)$ *if there following conditions hold:*

i) $J = J \cap N_\ell \cdot J \cap M \cdot J \cap N_u$,

ii) *the groups* $J \cap N_\ell$ *and* $J \cap N_u$ *are both contained in the kernel of* $\tau$.

Let $\mathscr{Z}(M)$ denote the center of $M$. Suppose that $(J,\tau)$ is decomposed with respect to $(M, P_u)$. We write $J_\ell$, $J_M$ and $J_u$ for $J \cap N_\ell$, $J \cap M$ and $J \cap N_u$ respectively.

**Definition 7.** *An element* $z \in M$ *is said to be positive if* $zJ_uz^{-1} \subset J_u$ *and* $z^{-1}J_\ell z \subset J_\ell$.

**Definition 8.** *An element* $\zeta \in \mathscr{Z}(M)$ *is said to be strongly* $(P_u, J)$*-positive if*

i) *$\zeta$ is positive,*

ii) *for any compact open subgroups $H_1$,$H_2$ of $N_u$ there exist an integer $m \geq 0$ such that $\zeta^m H_1 \zeta^{-m} \subset H_2$*

iii) *for any compact open subgroups $K_1$,$K_2$ of $N_l$ there exist an integer $m \geq 0$ such that $\zeta^{-m} K_1 \zeta^m \subset K_2$.*

**Definition 9.** *Let $M$ be some Levi subgroup of $G$. Let $J_M$ be a compact open subgroup of $M$ and $(\tau_M, W)$ an irreducible representation of $J_M$. The pair $(J, \tau)$ is a $G$-*cover *of $(J_M, \tau_M)$ if the following conditions hold:*

i) *$(J, \tau)$ is decomposed with respect to $(M, P)$, for any parabolic subgroup $P$ with Levi component $M$,*

ii) *$J \cap M = J_M$ and $\tau|_{J_M} \cong \tau_M$,*

iii) *for every parabolic subgroup $P$ with Levi component $M$ there exists an invertible element in $\mathcal{H}(G, \tau)$ supported in the double coset $Jz_PJ$, where $z_P \in \mathscr{Z}(M)$ is strongly $(P, J)$-positive.*

Let $\mathfrak{s}_M = [L, \sigma]_M \in \mathscr{B}(M)$. We have that $L$ is a Levi component of some parabolic subgroup of $M$ and $\sigma$ a supercuspidal representation of $L$. Then $[L, \sigma]_M$ determines the element $[L, \sigma]_G \in \mathscr{B}(G)$. We introduce the notation $\mathfrak{s}_G = [L, \sigma]_G$.

**Proposition 4.** *Let $M$ be a Levi subgroup of $G$ and let $\mathfrak{s}_M$ be an element in $\mathscr{B}(M)$. Consider $\mathfrak{s}_G \in \mathscr{B}(M)$ as above. Assume that $(J_M, \tau_M)$ is an $\mathfrak{s}_M$-type and that $(J, \tau)$ is a $G$ cover of $(J_M, \tau_M)$. Then $(J, \tau)$ is an $\mathfrak{s}_G$-type.*

Let $R, S$ be two rings with identity and let $\phi : R \longrightarrow S$ be a ring homomorphism sending the identity in $R$ to the identity in $S$. We can regard $S$ as a $R$-$S$ bimodule where $R$ acts on the left by $ra = \phi(r)a$ and $S$ acts on the right by multiplication. We denote by $\phi_*$ the covariant functor $\phi_* : R\text{-Mod} \longrightarrow S\text{-Mod}$ where $\phi_*(M) = \mathrm{Hom}_R(S, M)$.

**Theorem 5.** *Let $(J_M, \tau_M)$ be an $\mathfrak{s}_M$-type and $(J, \tau)$ a $G$ cover of $(J_M, \tau_M)$. Let $P = MN$ be a parabolic subgroup of $G$. Let us choose Haar measures $\mu_G$ and $\mu_M$ such that $\mu_G(J) = \mu_M(J_M) = 1$ and form the algebras $\mathcal{H}(M, \tau_M)$, $\mathcal{H}(G, \tau)$ with respect to these measures. Then there exists an injective algebra* *homomorphism* $t_P^i : \mathcal{H}(M, \tau_M) \longrightarrow \mathcal{H}(G, \tau)$ *such that the diagram*

$$\begin{array}{ccc} \mathfrak{R}^{\mathfrak{s}_M}(M) & \xrightarrow{\iota_P^G} & \mathfrak{R}^{\mathfrak{s}_G}(G) \\ \downarrow \mathbf{M}_{\tau_M} & & \downarrow \mathbf{M}_{\tau} \\ \mathcal{H}(M, \tau_M)\text{-Mod} & \xrightarrow{(t_P^i)_*} & \mathcal{H}(G, \tau)\text{-Mod} \end{array}$$

*is commutative. Moreover if* $a \in \mathcal{I}_M(\tau_M)$ *is a positive element with respect to* $(P, J)$*. Then for* $h \in \mathcal{H}(M, \tau_M)$ *with support* $J_M a J_M$*,* $t_P^i(h)$ *has support in* $JaJ$ *and* $t_P^i(h)(a) = h(a)\delta_P^{-1}(a)$*.*

We see that theorem 5 gives that the functor of induction $\mathrm{Ind}_P^G$ corresponds a functor $(t_P^i)_*$ given by the algebra homomorphism $t_P^i$. We can obtain a similar result for unitary induction.

**Theorem 6.** *Let* $(J_M, \tau_M)$ *be an* $\mathfrak{s}_M$*-type and* $(J, \tau)$ *a* $G$ *cover of* $(J_M, \tau_M)$*. Let* $P = MN$ *be a parabolic subgroup of* $G$*. Let us choose Haar measures* $\mu_G$ *and* $\mu_M$ *such that* $\mu_G(J) = \mu_M(J_M) = 1$ *and form the algebras* $\mathcal{H}(M, \tau_M)$*,* $\mathcal{H}(G, \tau)$ *with respect to these measures. Then there exists an injective algebra homomorphism* $t_P^u : \mathcal{H}(M, \tau_M) \longrightarrow \mathcal{H}(G, \tau)$ *such that the diagram*

$$\begin{array}{ccc} \mathfrak{R}^{\mathfrak{s}_M}(M) & \xrightarrow{\iota_P^G} & \mathfrak{R}^{\mathfrak{s}_G}(G) \\ \downarrow \mathbf{M}_{\tau_M} & & \downarrow \mathbf{M}_{\tau} \\ \mathcal{H}(M, \tau_M)\text{-Mod} & \xrightarrow{(t_P^u)_*} & \mathcal{H}(G, \tau)\text{-Mod} \end{array}$$

*is commutative. Moreover if* $a \in \mathcal{I}_M(\tau_M)$ *is a positive element with respect to* $(P, J)$*. Then for* $h \in \mathcal{H}(M, \tau_M)$ *with support* $J_M a J_M$*,* $t_P^u(h)$ *has support in* $JaJ$ *and* $t_P^u(h)(a) = h(a)\delta_P^{-1/2}(a)$*.*

**1.1.2. Hecke Algebra of a Cover.** We end this chapter with a result of Bushnell and Kutzko [BK01]. This result is useful to compute the Hecke algebra of a cover. Let

$$\mathfrak{t} = [L, \sigma]_L \in \mathscr{B}(L).$$

The element $\mathfrak{t}$ also defines an element $\mathfrak{s} = [\mathfrak{t}]_G = [L, \sigma]_G \in \mathscr{B}(G)$.

**Assumptions.**

i) There is a open compact-mod-center subgroup $\tilde{J}_L$ of $L$ and and an irreducible smooth representations $\tilde{\tau_L}$ of $\tilde{J}_L$ such that $\sigma \cong c\text{-}\mathrm{Ind}_{\tilde{J}_L}^G(\tilde{\tau_L})$.

ii) The representation $\tau_L = \tilde{\tau_L}|_{J_L}$ is irreducible, where $J_L$ denotes the unique maximal compact subgroup of $\tilde{J}_L$.

iii) An element $x \in L$ intertwines $\tau_L$ if and only if $x \in \tilde{J}_L$.

Let $(J, \tau)$ be a $G$-cover for $(J_L, \tau_L)$. We therefore get by proposition 4 that $(J, \tau)$ is an $\mathfrak{s}$-type. For a parabolic subgroup $P = LN$ of $G$ we get by 5 an injective homomorphism $t^i_P : \mathcal{H}(L, \tau_L) \longrightarrow \mathcal{H}(G, \tau)$. We can identify $\mathcal{H}(L, \tau_L)$ with a subalgebra of $\mathcal{H}(G, \tau)$. Let $K$ be a compact subgroup of $G$ that contains $J$. We can then form the subalgebra of $\mathcal{H}(K, \tau)$ consisting of functions with support contained in $K$. We can let $N_G(L)$, the $G$ normalizer of $L$, act on $\mathscr{B}(L)$ by $x \cdot [L, \gamma]_L = [L, \gamma^x]$. We write $\mathbf{W}_\mathfrak{s}$ for the (finite) $N_G(L)/L$-stabilizer of $\mathfrak{t}$.

**Theorem 7.** *With the notation above*

i) *The map*

$$\mathcal{H}(L, \tau_L) \bigotimes \mathcal{H}(K, \tau) \longrightarrow \mathcal{H}(G, \tau)$$

$$(f \otimes \phi) \longmapsto f \star \phi$$

*is an injective homomorphism of* $\mathcal{H}(L, \tau_L)$-$\mathcal{H}(K, \tau)$-*bimodules.*

ii) *We have* $\dim_{\mathbb{C}}(\mathcal{H}(K, \tau)) \leq |\mathbf{W}_\mathfrak{s}|$.

iii) *If* $\dim_{\mathbb{C}}(\mathcal{H}(K, \tau)) = |\mathbf{W}_\mathfrak{s}|$, *then the map of i) is an isomorphism.*

**1.1.3. Explicit Plancherel Mearsure.** The method that we use to compute the Plancherel measure for $\mathrm{PGL}_2(\mathbb{F})$ follows closely a method developed by Bushnell, Henniart and Kutzko; all the results in this subsection can be found in [BHK11].

**Definition 10.** *A* Hilbert algebra *is a complex algebra $A$ with an involution and carrying a positive Hermitian form* $[\,,]$ *such that:*

i) $[x, y] = [y^*, x^*],\ x, y \in A$;

ii) $[xy, z] = [y, x^* z],\ x, y, z \in A$;

iii) *for every $x \in A$ the mapping $y \mapsto xy$ of $A$ into $A$ is continuous with respect the topology induced by* $[\,,]$;

iv) *the set of elements $xy$ for $x, y \in A$ is dense in $A$.*

A Hilbert algebra is called *normalized* if it has a unit $\mathbf{e}$ and the inner product satisfies $[\mathbf{e}, \mathbf{e}] = \mathbf{1}$.

Let $A$ be a normalized Hilbert algebra, and let $\tilde{A}$ be the Hilbert space obtained by completing $A$ with respect to $[\,,]$. The action of $A$ on itself by left multiplication induces an injection of $A$ into the space $\mathscr{L}(\tilde{A})$ of continuous linear operators. We denote by ${}_rC^*(A)$ the closure of the image of $A$ in $\mathscr{L}(\tilde{A})$ and by $\widehat{{}_rC^*(A)}$ the set of all irreducible unitary representations of the $C^*$-algebra ${}_rC^*(A)$.

**Theorem 8.** *Let $A$ be a normalized Hilbert algebra, with unit element* $\mathbf{e}$*, such that ${}_rC^*(A)$ is liminal. There is then a unique positive measure $\nu_A$ on $\widehat{{}_rC^*(A)}$ such that:*

$$[a, \mathbf{e}] = \int_{\widehat{{}_rC^*(A)}} tr(\pi(a))\nu_A(\pi)$$

*We refer to $\nu_A$ as the* Plancherel measure *for $A$.*

Let $K$ be a compact open subgroup of $G$ and let $(\rho, W)$ be an irreducible representation of $K$. There exists a unique, up to a constant, positive Hermitian form $\langle\,,\rangle$ on $W$ invariant under $K$. The algebra $\mathrm{End}_{\mathbb{C}}(\hat{W})$ carries an involution $a \to a^*$ given by $\langle a^*w, w'\rangle = \langle w, aw'\rangle$, $a \in \mathrm{End}_{\mathbb{C}}(\hat{W})$ $w, w' \in \hat{W}$. This involution induces an involution $h \to h^*$ on the Hecke algebra $\mathcal{H}(G, \rho)$ given by $h^*(x) = h(x^{-1})^*$ for $h \in \mathcal{H}(G, \rho)$, $g \in G$. We regard $\mathcal{H}(G, \rho)$ as a normalized Hilbert algebra with positive definite Hermitian form $[\,,]$ given by

$$[f, h] = \frac{\mu_G(K)}{\dim\rho} tr(f * h^*(1))$$

Let $(\pi, V)$ be an irreducible unitary representation of $G$. Then the subrepresentation of smooth vectors $(\pi^\infty, V^\infty)$ is an irreducible smooth representation of $G$. If we denote by ${}_r\widehat{G}$ the subset of $\widehat{G}$ that supports the Plancherel measure then the Bernstein decomposition induces a disjoint union

$${}_r\widehat{G} = \dot{\bigcup}_{\mathfrak{s} \in \mathscr{B}} {}_r\widehat{G}(\mathfrak{s})$$

where ${}_r\widehat{G}(\mathfrak{s})$ is the set of all $(\pi, V) \in {}_r\hat{G}$ such that $(\pi^\infty, V^\infty) \in \mathfrak{R}^{\mathfrak{s}}(G)$. The sets ${}_r\widehat{G}(\mathfrak{s})$ are open in ${}_r\widehat{G}$; we conclude that a subset $S$ of ${}_r\widehat{G}$ is Borel if and only if ${}_r\widehat{G}(\mathfrak{s}) \cap S$ is Borel for every $\mathfrak{s}$. The functor $\mathbf{M}_\rho$ induces a homeomorphism

$$\widehat{\mathbf{M}}_\rho : {}_r\widehat{G}(\mathfrak{s}) \longrightarrow {}_r\widehat{C^*}(\mathcal{H}(G,\rho))$$

where $\widehat{\mathbf{M}}_\rho((\pi,V)) = \mathbf{M}_\rho((\pi^\infty, V^\infty))$. Moreover, if $S$ is a Borel subset of ${}_r\hat{G}(\mathfrak{s})$ and $\nu$ is the Plancherel measure on $\widehat{G}$ associated to the Haar measure $\mu_G$, then

$$\nu(S) = \frac{\dim\rho}{\mu_G(K)}\nu_{\mathcal{H}(G,\rho)}(\widehat{\mathbf{M}}_\rho(S)). \tag{A}$$

It follows that in order to obtain an explicit Plancherel measure for $G$ it is enough to obtain an explicit Plancherel measure $\nu_{\mathcal{H}(G,\rho)}$ where $(K,\rho)$ is an $\mathfrak{s}$-type for every $\mathfrak{s} \in \mathscr{B}(G)$.

### 1.2. The Affine Hecke Algebras $\mathcal{H}(q_1, q_2)$

The results of this and the following subsection with the exception of corollary 15 are taken from a paper by P. Kutzko and L. Morris [KM09]. In the paper the authors compute the Plancherel measure for the affine Hecke algebras $\mathcal{H}(q_1,q_2)$ in two real parameters $q_1 \geq q_2 \geq 1$. This will be useful for us because we will show later that for certain types $(K,\rho)$ for $\mathrm{PGL}_2(G)$ the Hecke algebra $\mathcal{H}(G,\rho)$ is isomorphic to an affine Hecke algebra $\mathcal{H}(q_1,q_2)$.

**Definition 11.** *Let $q_1 \geq q_2 \geq 1$ be two fixed real numbers and set $\gamma_i = q_i^{1/2}, c_i = \gamma_i - \gamma_i^{-1}$ for $i = 1, 2$. We let $\mathcal{H}(q_1,q_2)$ be the complex algebra with identity $\mathbf{1}$ and two generators $s_i,\ i = 1,2$ subject only to the relations*

$$s_i^2 = c_i s_i + \mathbf{1},\ i = 1,2$$

Viewed as a complex vector space the algebra $\mathcal{H}(q_1,q_2)$ has a basis consisting of elements $w = \Pi_{i=1}^k u_i,\ u_i \in \{s_1, s_2\}$ where $u_i \neq u_{i+1}$ for $1 \leq i \leq k-1$. (We allow for the case $k = 0$ as well; in that case, we set $w = \mathbf{1}$.) We refer to these elements as *words* and denote the set of words by $\mathcal{W}$.

We can give $\mathcal{H}(q_1,q_2)$ the structure of a Hilbert algebra. Let $x \to x^*$ be the involution characterised by the following properties.

i) $s_i^* = s_i,\ i = 1,2$

ii) $x \to x^*$ is multiplication reversing and conjugate linear.

We define the functional $\Lambda : \mathcal{H}(q_1,q_2) \to \mathbb{C}$ by setting $\Lambda(\mathbf{1}) = 1,\ \Lambda(w) = 0$, $w \in \mathcal{W}$, $w \neq \mathbf{1}$. For $x, y \in \mathcal{H}(q_1,q_2)$ we set $[x,y] = \Lambda(xy^*)$.

**Proposition 9.** *$\mathcal{H}(q_1,q_2)$ is a Hilbert algebra with respect to $[\,,]$.*

Let $\rho$ be a 1-dimensional algebra homomorphism from $\mathcal{H}(q_1,q_2)$ into $\mathbb{C}$. Then $\rho(s_i)^2 = c_i\rho(s_i)+1$ and this quadratic equation has two solutions i.e $\rho(s_i)=\gamma_i$ or $\rho(s_i) = -\gamma_i^{-1}$. We therefore have 4 1-dimensional algebra homomorphisms.

We set $d = s_1s_2$ and set $D = \mathbb{C}[d,d^{-1}]$. We then get that $\mathcal{H}(q_1,q_2) = D\bigoplus Ds_1$. We have a functor of induction that we denote by $\mathrm{ind}_D^{\mathcal{H}(q_1,q_2)}$ : $D$-Mod $\to \mathcal{H}(q_1,q_2)$-Mod given by $\mathrm{ind}_D^{\mathcal{H}(q_1,q_2)}N = \mathrm{Hom}_D(\mathcal{H}(q_1,q_2),N)$ for a left $D$ module $N$. We consider a 1-dimensional representation $\chi : D \longrightarrow \mathbb{C}$ and denote by $\mathbb{C}_\chi$ the space where $\chi$ acts. It is clear that the representation $\mathbb{C}_\chi$ only depends on the value of $\chi(d)$. We set $(\sigma_\chi, M_\chi) = \mathrm{ind}_D^{\mathcal{H}(q_1,q_2)}\mathbb{C}_\chi$ and note that $(\sigma_\chi,M_\chi)$ is then a two dimensional representation of $\mathcal{H}(q_1,q_2)$.

**Proposition 10.** *All the irreducible unitary representations of $\mathcal{H}(q_1,q_2)$ have dimension less or equal than 2. Moreover, the two dimensional unitary representations are of the form $(\sigma_\chi,M_\chi)$ where $|\chi(d)|=1$ and $\mathrm{Im}\chi(d)\geq 0$ or $\chi(d)\in(-\frac{\gamma_1}{\gamma_2},1)\cup(1,\gamma_1\gamma_2)$.*

**Proposition 11.** *Let $\widehat{\mathcal{H}}(q_1,q_2)$ denote the set of irreducible unitary representations of $\mathcal{H}(q_1,q_2)$. The map $(\pi,V)\mapsto(\pi|_{\mathcal{H}(q_1,q_2)},V)$ is an injection of ${}_r\widehat{C^*}(\mathcal{H}(q_1,q_2))$ into $\widehat{\mathcal{H}}(q_1,q_2)$.*

We can see that proposition 11 implies that all irreducible representations of ${}_rC^*(\mathcal{H}(q_1,q_2))$ are of dimension less or equal than 2. It follows that ${}_rC^*(\mathcal{H}(q_1,q_2))$ is a liminal (or CCR) $C^*$-algebra. We then have that there exist a Plancherel measure $\nu_{\mathcal{H}(q_1,q_2)}$ for $\mathcal{H}(q_1,q_2)$. We write ${}_r\widehat{C^*}(\mathcal{H}(q_1,q_2)) = \hat{A}_2\bigcup\hat{A}_1$ where $\hat{A}_i$ denotes the set of equivalence classes of irreducible representations of dimension $i=1,2$

**Theorem 12.** *Let $\hat{A}_2$ be as above. Let $Y=\{s\in\mathbb{C} : |s|=1 \text{ and } \mathrm{Im}(s)\geq 0\}$. Then the map $\theta : Y\to\hat{A}_2$ given by $\theta(s)=(\sigma_\chi,M_\chi)$ where $\chi(d)=s$ is a homeomorphism.*

We thus get a positive measure $\nu_0$ in $Y$ given by $\nu_0(S)=\nu_{\mathcal{H}(q_1,q_2)}(\theta(S))$. We have that $s_1$ and $s_2$ are invertible elements in $\mathcal{H}(q_1,q_2)$ therefore $d=s_1s_2$ is an invertible element. Let $z=d+d^{-1}$ and set $f(z)=z^2-c_1c_2z-(c_1^2+c_2^2+4)=(z-(\gamma_1\gamma_2+\frac{1}{\gamma_1\gamma_2}))(z+(\frac{\gamma_1}{\gamma_2}+\frac{\gamma_2}{\gamma_1}))$.

**Theorem 13.** *Let $|dz|$ denote the positive measure on $Y$ given by arclength. Let $\mathtt{h} : Y\to\mathbb{C}^+$ be given by $\mathtt{h}(s)=\frac{\chi(z^2-4)}{\chi(f(z))}$ where $\chi$ is the 1-dimensional* *representation of $D$ such that $\chi(d) = s$. Then $|dz|$ is absolutely continuous with respect to $d\nu_0$ and $d\nu_0 = \frac{1}{2\pi}\mathtt{h}|dz|$.*

We can see that we have a very explicit description of the Plancherel measure in $\hat{A}_2$. We now take our attention to the $\hat{A}_1$. We have already mentioned that $\hat{A}_1$ has at most 4 points.

**Theorem 14.** *$\hat{A}_1$ has at most 2 points and consists of the representations (non necessarily different) $\rho_1$, $\rho_2$ where $\rho_1(d) = -\frac{\gamma_2}{\gamma_1}$ and $\rho_2(d) = \frac{1}{\gamma_2\gamma_1}$. The points have positive measure*

$$\nu_{\mathcal{H}(q_1,q_2)}(\rho_1) = \frac{1}{2}\left(\frac{q_1-1}{q_1+1} + \frac{q_2-1}{q_2+1}\right), \nu_{\mathcal{H}(q_1,q_2)}(\rho_2) = \frac{1}{2}\left(\frac{q_1-1}{q_1+1} - \frac{q_2-1}{q_2+1}\right).$$

### 1.3. Explicit Plancherel measure for $\mathcal{H}(q,1)$

We have completed the goal of giving a description of the Plancherel measure $\nu_{\mathcal{H}(q_1,q_2)}$. We will make use of this result while computing the Plancherel measure for $\mathrm{PGL}_2(\mathbb{F})$. We will show later that the Hecke algebras obtained from the types for $\mathrm{PGL}_2(\mathbb{F})$ are isomorphic to the algebras $\mathcal{H}(q,1)$ where $q > 1$ or the group algebra of the integers $\mathbb{C}[\mathbb{Z}]$.

Let $s \in \mathbb{C}$ and define $L(1,s) = (1 - q^{-s})$. The map $t \longrightarrow q^{it}$ from $[0, \frac{\pi}{\ln q}]$ into $Y$ is a Borel isomorphism that takes the measure $\frac{\ln(q)}{2\pi}dt$ to $\frac{1}{2\pi}|dz|$. Let $P(t) = \mathtt{h}(q^{it}) = \frac{\chi(z^2-4)}{\chi(f(z))}$ where $\chi(d) = q^{it}$. We will like to write the function $P(t)$ in terms of functions of the form $L(1,s), s \in \mathbb{C}$. For that end we first compute

$$\begin{aligned}\chi(z^2-4) &= (q^{it}+q^{it})^2 - 4 = q^{2it} + q^{-2it} - 2\\ &= -(1-q^{2it})(1-q^{2it}) = \frac{-1}{L(1,-2it)L(1,2it)}\end{aligned}$$

We now compute $\chi(f(z)) = \chi((z - (q^{1/2}+q^{-1/2})) \cdot \chi((z + (q^{1/2}+q^{-1/2}))$. We get the following equations:

$$\begin{aligned}(1)\qquad \chi(z-(q^{1/2}+q^{-1/2})) &= q^{it}+q^{-it}-(q^{1/2}+q^{-1/2})\\ &= -q^{1/2}(1-q^{it-1/2})(1-q^{-it-1/2})\end{aligned}$$

$$\begin{aligned}(2)\qquad \chi(z+(q^{1/2}+q^{-1/2})) &= q^{it}+q^{-it}+(q^{1/2}+q^{-1/2})\\ &= q^{1/2}(1+q^{it-1/2})(1+q^{-it-1/2})\end{aligned}$$

Multiplying 1 and 2 we get

$$\chi(f(z)) = -q(1-q^{2it-1})(1-q^{-2it-1}) = \frac{-q}{L(1,1-2it)L(1,1+2it)}$$

We then conclude that

$$P(t) = \mathtt{h}(q^{it}) = \frac{\chi(z^2-4)}{\chi(f(z))} = q^{-1}\frac{L(1,1-2it)L(1,1+2it)}{L(1,-2it)L(1,2it)}$$

We also have that $\mathcal{H}(q,1)$ has two distinct one dimensional representations $\rho_1,\rho_2$ with positive Plancherel measure where $\rho_1(d)=q^{-1/2}$, $\rho_2(d)=-q^{-1/2}$.

**Corollary 15.** *The Plancherel measure $\nu_{\mathcal{H}(q,1)}$ of the Hecke algebra $\mathcal{H}(q,1)$ can be identified with the interval $[0,\frac{\pi}{\ln q}]$ and a measure $\frac{\ln(q)}{2\pi}Pdt$ where $dt$ gives the Lebesgue measure in the interval and*

$$P(t) = q^{-1}\frac{L(1,1-2it)L(1,1+2it)}{L(1,-2it)L(1,2it)}$$

*union two points $\rho_1,\rho_2$ where*

$$\nu_{\mathcal{H}(q,1)}(\rho_1) = \nu_{\mathcal{H}(q,1)}(\rho_2) = \frac{1}{2}\left(\frac{q-1}{q+1}\right).$$

## 2. Types and Covers for $\mathbf{GL_2}(\mathbb{F})$ related to $\mathbf{PGL_2}(\mathbb{F})$

We first introduce some notation. Unless explicitly stated otherwise, by $G$ we mean $\mathrm{GL}_2(\mathbb{F})$. We denote by $Z$ the center of $G$, by $B_u$ the upper triangular matrices and by $B_\ell$ the lower triangular matrices. We denote by $\bar{G}$ the quotient group $G/Z=\mathrm{PGL}_2(\mathbb{F})$. Likewise if $x\in\mathrm{GL}_2(\mathbb{F})$ and $S\subset\mathrm{GL}_2(\mathbb{F})$

we denote by $\bar{x}$ and $\bar{S}$ the image of $x$ and $S$ respectively under the quotient map. If $H$ is a subgroup of $G$ and $\rho$ is a representation of $H$ trivial on $H \cap Z$ we will denote by $\overline{\rho}$ the representation of $\overline{H}$ given by $\overline{\rho}(\bar{x}) = \rho(x)$ for $x \in H$. We denote by $\mathcal{O}$ the ring of integers in $\mathbb{F}$ with maximal prime ideal $\mathfrak{p}$ and by $\varpi$ a prime element in $\mathcal{O}$. We let $q$ be the number of elements in the finite field $\mathcal{O}/\mathfrak{p}$ and we let $v$ be the valuation on $\mathbb{F}^{\times}$ such that $v(\varpi) = 1$. Let $G'$ be a connected reductive algebraic group defined over $\mathbb{F}$ and let $X^*(G')$ be the group of rational characters i.e. morphisms of algebraic groups from $G'$ into $\mathrm{GL}_1$ defined over $\mathbb{F}$. We define ${}^oG' = \bigcap_{\chi \in X^*(G')} \ker |\chi|_{\mathbb{F}}$. A smooth character of $G'$ is then unramified if is trivial on ${}^oG'$.

We have that $G$ has up to conjugacy only two Levi subgroups. The first one is the diagonal matrices which we denote by $\mathbb{T}$ and the second one is the whole group $G$. Similarly $\bar{G}$ has up to conjugacy only two Levi subgroups namely $\bar{\mathbb{T}}$ and $\bar{G}$. Let $\bar{\mathfrak{s}} = [\bar{\mathbb{T}}, \varphi]_G \in \mathscr{B}(\bar{G})$ (see Bernstein decomposition 1.1.1). We have then by definition of $\bar{\mathfrak{s}}$ that $\varphi$ is a character for $\bar{\mathbb{T}} \cong \mathbb{F}^{\times}$. If we consider $\varphi$ as a character of $\mathbb{F}^{\times}$, we can regard $\varphi \otimes \varphi^{-1}$ as a character of $\mathbb{T}$ and hence construct an element $\mathfrak{s} \in \mathscr{B}(G)$ by setting $\mathfrak{s} = [\mathbb{T}, \varphi \otimes \varphi^{-1}]_G$. Later we will show that an $\mathfrak{s}$-type can be related to a $\bar{\mathfrak{s}}$-type.

The goal of this section is to construct an $\mathfrak{s}$-type for $\mathfrak{s}$ of the form $[\mathbb{T}, \varphi \otimes \varphi^{-1}]_G \in \mathscr{B}(G)$. Let $[\mathfrak{s}]_{\mathbb{T}} = [\mathbb{T}, \varphi \otimes \varphi^{-1}]_{\mathbb{T}}$. It is rather trivial that the pair $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$ is a $[\mathfrak{s}]_{\mathbb{T}}$-type. We will give a justification of this fact as a warm up. Let $(\pi, V)$ be an irreducible representation of $\mathbb{T}$. Then $(\pi, V) \in \mathfrak{R}^{[\mathfrak{s}]_{\mathbb{T}}}(\mathbb{T})$ if and only if $\pi = \varphi \otimes \varphi^{-1} \otimes \chi$, where $\chi$ is an unramified character i.e trivial in ${}^\circ\mathbb{T}$. Hence the restriction of $\pi$ to ${}^\circ\mathbb{T}$ is equal to $\varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}}$ showing that $\mathrm{Hom}_{{}^\circ\mathbb{T}}(\varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}}, V) \neq 0$ and therefore by [BK01, (4.2)] we get that $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$ is a $[\mathfrak{s}]_{\mathbb{T}}$-type.

It follows from [BK01, (8.3) Theorem] that in order to construct an $\mathfrak{s}$-type it is enough to construct a cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$ (see [BK01, (8.1)] for the definition of a cover). Define the integer $sw(\varphi)$ to be the smallest positive integer $n$ such that $1 + \mathfrak{p}^n \subset \ker \varphi$. Let $J = J_\varphi$ be the compact subgroup of $G$ given by:

$$J = \{[c_{ij}] \in G |\ c_{11}, c_{22} \in \mathcal{O}^{\times},\ c_{12} \in \mathcal{O},\ c_{21} \in \mathfrak{p}^{sw(\varphi^2)}\}$$

and define the function $\lambda = \lambda_\varphi$ on $J$ by $\lambda([c_{ij}]) = \varphi^2(c_{11})\varphi^{-1}(\det[c_{ij}])$. Let us check that $\lambda$ is a one dimensional representation of $J$.

$$\begin{aligned}
\lambda([a_{ij}] \cdot [b_{ij}]) &= \varphi^2(a_{11}b_{11} + a_{12}b_{21})\varphi^{-1}(\det([a_{ij}] \cdot [b_{ij}])) \\
&= \varphi^2(a_{11}b_{11}(1 + a_{12}b_{21}(a_{11}b_{11})^{-1}))\varphi^{-1}(\det([a_{ij}])\det([b_{ij}])) \\
&= \varphi^2(a_{11}b_{11})\varphi^{-1}(\det([a_{ij}])\det([b_{ij}])) = \lambda([a_{ij}])\lambda([b_{ij}]).
\end{aligned}$$

Only the third equality deserves an explanation and that follows from the fact that $a_{12} \in \mathcal{O}$ and $b_{21} \in \mathfrak{p}^{sw(\varphi^2)}$ then $a_{12}b_{21} \in \mathfrak{p}^{sw(\varphi^2)}$. Since $a_{11}$ and $b_{11}$ are units in $\mathcal{O}$ we get that $1 + a_{12}b_{21}(a_{11}b_{11})^{-1} \in 1 + \mathfrak{p}^{sw(\varphi^2)} \subset \ker \varphi^2$, hence the equality. Note that if $[c_{ij}]$ is a diagonal matrix in $J$ then

$$\begin{aligned}([c_{ij}]) &= \varphi^2(c_{11})\varphi^{-1}(\det[c_{ij}]) = \varphi^2(c_{11})\varphi^{-1}(c_{11}c_{22}) \\ &= \varphi(c_{11}c_{22}^{-1}) = \varphi \otimes \varphi^{-1}([c_{ij}]).\end{aligned}$$

In other words the restriction of $\lambda_\varphi$ to $\mathbb{T} \cap J = {}^\circ\mathbb{T}$ is equal to $\varphi \otimes \varphi^{-1}$.

The group $J$ is compact and the pair $(J_\varphi, \lambda_\varphi)$ is our candidate to be a $G$-cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$ and therefore an $\mathfrak{s}$-type. We have only two parabolic subgroups with Levi component $\mathbb{T}$, namely the upper triangular matrices and the lower triangular matrices. For a matrix $[c_{ij}] \in G$, if $c_{11} \neq 0$ we have

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c_{21}c_{11}^{-1} & 1 \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ 0 & (c_{11}c_{22} - c_{21}c_{12})c_{11}^{-1} \end{bmatrix} \begin{bmatrix} 1 & c_{12}c_{11}^{-1} \\ 0 & 1 \end{bmatrix} \tag{3}$$

and if $c_{22} \neq 0$ we have

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 & c_{12}c_{22}^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (c_{11}c_{22} - c_{21}c_{12})c_{22}^{-1} & 0 \\ 0 & c_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c_{21}c_{22}^{-1} & 1 \end{bmatrix} \tag{4}$$

It follows at once that for a parabolic subgroup $P$ with Levi component $\mathbb{T}$ we get $(J, \lambda)$ is decomposed with respect to $(\mathbb{T}, P)$ (see [BK01, (6.1)] for the definition of being decomposed). Indeed, if $P = B_u$ the upper triangular matrices then 3 gives us the desired decomposition and if $P = B_\ell$ is the lower triangular matrices then equation 4 will do the job. We have remarked before that the restriction of $\lambda_\varphi$ to $\mathbb{T} \cap J = {}^\circ\mathbb{T}$ is equal to $\varphi \otimes \varphi^{-1}$. Hence in order to show that $(J_\varphi, \lambda_\varphi)$ is a $G$-cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$ we need to prove the existence of an invertible element in $\mathcal{H}(G, \lambda)$ supported on $Jz_PJ$ where $z_P$ is a strongly positive element with respect to $(J, P)$ for every parabolic subgroup $P$ with Levi component $\mathbb{T}$. (see [BK01, (6.14)] if you need to recall the definition of strongly positive).

Consider the set $\mathcal{I}_G(\lambda) = \{x \in G |\, \mathrm{Hom}_{x^{-1}Jx \cap J}(\lambda^x, \lambda) \neq 0\}$. Let

$$\mathcal{H}_x(G, \lambda) = \{f \in \mathcal{H}(G, \lambda) |\, \mathrm{supp}(f) \subset JxJ\}.$$

We see that for $f \in \mathcal{H}_x(G, \lambda)$, $f$ is determined by its value at $x$ and since $\lambda$ is one dimensional, $\mathcal{H}_x(G, \lambda)$ has dimension one for $x \in \mathcal{I}_G(\lambda)$. Let $g_x \in$

$\mathcal{H}_x(G,\lambda)$ for $x \in \mathcal{I}_G(\lambda)$ be given by $g_x(x) = 1$. Likewise we can define $g_{\bar{x}} \in \mathcal{H}_{\bar{x}}(\bar{G},\bar{\lambda})$ for $\bar{x} \in \mathcal{I}_{\bar{G}}(\bar{\lambda})$ be given by $g_{\bar{x}}(\bar{x}) = 1$.

**Lemma 16.** *Let $G$ be a connected reductive group over a non archimedean local field. Let $J$ be a compact subgroup and $\lambda$ a one dimensional representation of $J$. Let $g_x \in \mathcal{H}_x(G,\lambda)$ for $x \in \mathcal{I}_G(\lambda)$ be given by $g_x(x) = 1$. Then $g_x^* = g_{x^{-1}}$.*

*Proof.* Let $y = j_1 x^{-1} j_2$ where $j_1, j_2 \in J$. Then

$$\begin{aligned} g_x^*(y) &= (g_x(y^{-1}))^* = (g_x(j_2^{-1} x j_1^{-1}))^* = (\check{\lambda}(j_2^{-1})\check{\lambda}(j_1^{-1}))^* \\ &= (\check{\lambda}(j_1^{-1}))^*(\check{\lambda}(j_2^{-1}))^* = \check{\lambda}(j_1)\check{\lambda}(j_2) = g_{x^{-1}}(y) \end{aligned}$$

If $y$ is not in $Jx^{-1}J$ then $y^{-1}$ is not in $JxJ$ so $g_x^*(y) = (g_x(y^{-1}))^* = 0 = g_{x^{-1}}(y)$. □

Let $\Pi = \left[\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then for $n \geq 1$, $\Pi^n$ is easily seen to be strongly positive for $(J, B_u)$ and $\Pi^{-n}$ is strongly positive for $(J, B_\ell)$. The elements $\Pi, \Pi^{-1}$ are in $\mathcal{I}_G(\lambda)$ then by [BH06, 11.2 Lemma] we can claim the existence of $g_\Pi$ and $g_{\Pi^{-1}}$.

**Proposition 17.** *If $\varphi^2|_{\mathcal{O}^\times} \neq 1$ then the pair $(J_\varphi, \lambda_\varphi) = (J, \lambda)$ is a $G$-cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$.*

*Proof.* It will be enough to show that $g_\Pi$ and $g_{\Pi^{-1}}$ are invertible to finish the proof of the proposition. We claim that the element $g_\Pi * g_{\Pi^{-1}}$ has support in $J$. We know that $g_\Pi * g_{\Pi^{-1}}$ has support in $(J\Pi J\Pi^{-1}J) \cap \mathcal{I}_G(\lambda)$. Using the decomposition of $J = J_\ell \cdot J_\mathbb{T} \cdot J_u$ with respect to $B_u$, we get that $\Pi J \Pi^{-1} = (\Pi J_\ell \Pi^{-1})(\Pi J_\mathbb{T} \Pi^{-1})(\Pi J_u \Pi^{-1}) = \Pi J_\ell \Pi^{-1} J_\mathbb{T} J_u \subset \Pi J_\ell \Pi^{-1} J$. The last equality is true because $\Pi$ is positive element with respect to $(J, B_u)$, so $\Pi J_u \Pi^{-1} \subset J_u$ and $\Pi$ commutes with every element of $J_\mathbb{T}$. Let $c(y) = \left[\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right]$ and suppose that $c(y) \in (\Pi J_\ell \Pi^{-1} - J)$. We see that $v(y) = sw(\varphi^2) - 1$.

We will prove that $c(y)$ does not intertwine $\lambda$. Since $\varphi^2|_{\mathcal{O}^\times} \neq 1$ there exists $a \in \mathcal{O}^\times$ such that $\varphi^2(a) \neq 1$. Let us first consider the case where $sw(\varphi^2) = 1$. Then $y^{-1} \in \mathcal{O}$. Let

$$x_1 = \begin{bmatrix} a & (a-1)y^{-1} \\ 0 & 1 \end{bmatrix}, \; x_2 = \begin{bmatrix} 1 & (a-1)y^{-1} \\ 0 & a \end{bmatrix}$$

We have $x_1, x_2 \in J$ and $c(y) x_1 c(y)^{-1} = x_2$, but

$$\lambda(x_1) = \varphi^2(a)\varphi^{-1}(a) = \varphi(a) \neq \varphi(a^{-1}) = \lambda(x_2)$$

so $c(y)$ does not intertwine $\lambda$.

If $sw(\varphi^2) \geq 2$ take $z \in \mathfrak{p}^{sw(\varphi^2)-1}$ such that $\varphi^2(1+z) \neq 1$. Then $zy^{-1} \in \mathcal{O}$ and $zy \in \mathfrak{p}^{sw(\varphi^2)}$. Consider the elements

$$x_1 = \begin{bmatrix} 1 & -zy^{-1} \\ -zy & 1 \end{bmatrix}, \ x_2 = \begin{bmatrix} 1+z & -zy^{-1} \\ 0 & 1-z \end{bmatrix}$$

We have $x_1, x_2 \in J$ and $c(y)x_1c(y)^{-1} = x_2$ but $\lambda(x_1) = \varphi^2(1)\varphi^{-1}(1-z^2)$ and $\lambda(x_2) = \varphi^2(1+z)\varphi^{-1}(1-z^2)$. Since $\varphi^2(1+z) \neq 1$ we obtain $\lambda(x_1) \neq \lambda(x_2)$. This finishes the proof of our claim.

We then get that $g_\Pi * g_{\Pi^{-1}}$ is a constant multiple of the identity. If we show that $g_\Pi * g_{\Pi^{-1}}(1) \neq 0$ we will finish the proof of the proposition. Let us compute $g_\Pi * g_{\Pi^{-1}}(1)$

$$\begin{aligned} g_\Pi * g_{\Pi^{-1}}(1) &= \int_G g_\Pi(x) g_{\Pi^{-1}}(x^{-1}) d\mu_G(x) \\ &= \int_{J\Pi J} g_\Pi(x) g_{\Pi^{-1}}(x^{-1}) d\mu_G(x) \\ &= \mu(J\Pi J) = q \neq 0 \end{aligned}$$

The first equality is just the definition, the second equality follows from the fact that the support of $g_\Pi$ is contained in $J\Pi J$ and the third equality is true because if $x \in J\Pi J$ then $x = j_1 \Pi j_2$ for some $j_1, j_2 \in J$ then

$$\begin{aligned} g_\Pi(x) g_{\Pi^{-1}}(x^{-1}) &= g_\Pi(j_1\Pi j_2) g_{\Pi^{-1}}(j_2^{-1}\Pi^{-1}j_1^{-1}) \\ &= \check{\lambda}(j_1)\check{\lambda}(j_2)\check{\lambda}(j_2^{-1})\check{\lambda}(j_1^{-1}) = 1. \end{aligned}$$

This finishes the proof of the proposition. □

We will now consider the case where $\varphi^2|_{\mathcal{O}^\times} = 1$. In this case $J_\varphi$ is the Iwahori subgroup and we denote it by $I$. If we let $K = GL_2(\mathcal{O})$ and $w = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, we have a decomposition $K = GL_2(\mathcal{O}) = I \cup IwI$. The element $w$ intertwines $\lambda$, therefore by [BH06, 11.2 Lemma] the existence of an element $g_w \in \mathcal{H}(G, \lambda)$.

**Proposition 18.** *If $\varphi^2|_{\mathcal{O}^\times} = 1$ then the pair $(J_\varphi, \lambda_\varphi) = (J, \lambda)$ is a $G$-cover for $({}^\circ\mathbb{T}, \varphi \otimes \varphi^{-1}|_{{}^\circ\mathbb{T}})$.*

*Proof.* The support of $g_w * g_w$ is contained in $K = I \cup IwI$, therefore there exists complex numbers $a$ and $b$, such that $g_w * g_w = ag_w + bg_1$.

$$\begin{aligned} g_w * g_w(1) &= \int_G g_w(x) g_w(x^{-1}) d\mu_G(x) \\ &= \int_{IwI} g_w(x) g_w(x^{-1}) d\mu_G(x) = \mu_G(IwI) \neq 0 \end{aligned}$$

We get that $b = g_w * g_w(1) \neq 0$. So $g_w$ satisfies a quadratic equation with non-zero constant term and therefore it is invertible. Let $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right]$. Conjugation by $\alpha$ is an automorphism of $G$ that fixes $I$ and also we have that $\lambda(\alpha x \alpha^{-1}) = \lambda(x)$ for all $x \in I$. Therefore there are elements $g_\alpha$ and $g_{\alpha^{-1}}$. We have that $g_\alpha * g_{\alpha^{-1}}$ has support contained in $I\alpha I \alpha^{-1} I = I$. We see that $g_\alpha * g_{\alpha^{-1}} = \mu_G(I\alpha I) \neq 0$. We then conclude that $g_\alpha$ is invertible and since $g_w$ is invertible we get that $g_w * g_\alpha$ is invertible. The support of $g_w * g_\alpha$ lies in $IwI\alpha I = Iw\alpha I = I\Pi I$. Similarly $g_{\alpha^{-1}} * g_w$ is invertible and has support in $I\alpha^{-1}IwI = I\alpha^{-1}wI = I\Pi^{-1}I$ so we get that the proposition holds true. □

## 3. Types and Covers for $\mathbf{GL_2}(\mathbb{F})$ mod the center

We have computed types and covers for $G$ and we mentioned before that we want to relate them to types and covers for $\bar{G}$. Let $P = LN$ be a parabolic subgroup with Levi component $L$ and unipotent subgroup N. Let $\sigma$ be a supercuspial representation of $L$ trivial on $Z$. Then $\bar{P} = \bar{L}\bar{N}$ is a parabolic subgroup of $\bar{G}$ with Levi component $\bar{L}$ and unipotent subgroup $\bar{N}$. Let $\mathfrak{s} = [L, \sigma]_G$ and denote by $\bar{\mathfrak{s}} = [\bar{L}, \bar{\sigma}]_{\bar{G}}$, where $\bar{\sigma}(\bar{x}) = \sigma(x)$ for $x \in L$. Finally, let $(\rho, W)$ be an irreducible representation of a compact open subgroup $J$ of $G$.

**Proposition 19.** *With the notation above, if $(J, \rho)$ is an $\mathfrak{s}$-type then $\rho(z) = 1$ for all $z \in Z \cap J$ and $(\bar{J}, \bar{\rho})$ is an $\bar{\mathfrak{s}}$-type, where $\bar{\rho}(\bar{x}) = \rho(x)$ for $x \in J$.*

*Proof.* Let $f \in \mathrm{Ind}_P^G \sigma$, then $z \cdot f(x) = \sigma(z) f(x) = f(x)$ for all $z \in Z$, therefore the representation $\mathrm{Ind}_P^G \sigma$ has a trivial central character. Since $(\rho, W)$ is an $\mathfrak{s}$-type $\mathrm{Hom}_J(\rho, \mathrm{Ind}_P^G \sigma) \neq 0$ so $\rho$ has also a trivial central character. So it makes sense to talk about $\bar{\rho}$ as a representation of $\bar{J} \cong J/(J \cap Z)$. Now let $(\bar{\pi}, V)$ be an irreducible representation of $\bar{G}$ such that $\mathfrak{I}((\bar{\pi}, V)) \in \bar{\mathfrak{s}}$. Consider $(\pi, V)$ the representation of $G$ such that $\pi(x) = \bar{\pi}(\bar{x})$. Since $\mathfrak{I}((\bar{\pi}, V)) \in \bar{\mathfrak{s}}$ then there exist an unramified character $\bar{\psi}$ of $\bar{L}$ such that $\mathrm{Hom}_{\bar{G}}(\bar{\pi}, \mathrm{Ind}_{\bar{P}}^{\bar{G}} \bar{\sigma}^{\bar{g}} \otimes \bar{\psi}) \neq 0$. Let $\psi$ be the representation of $L$ such that $\psi(x) = \bar{\psi}(\bar{x})$ for $x \in L$ and let $g \in \bar{g}$. For $f \in \underline{\mathrm{Ind}_P^G \sigma^g \otimes \psi}$ we get $z \cdot f(x) = \sigma(gzg^{-1})\psi(z) f(x) = f(x)$. We can thus consider $\overline{\mathrm{Ind}_P^G \sigma^g \otimes \psi}$ to be the $\overline{G}$-representation acting on the same vector space than $\mathrm{Ind}_P^G \sigma^g \otimes \psi$ and where $\bar{x}f = f(x)$ for all $\bar{x} \in \overline{G}$, $f \in \mathrm{Ind}_P^G \sigma^g \otimes \psi$. We conclude that the map $f \mapsto \bar{f}$ where $\bar{f}(\bar{x}) = f(x)$ is a $\bar{G}$-isomorphism from $\overline{\mathrm{Ind}_P^G \sigma^g \otimes \psi}$ to $\mathrm{Ind}_{\bar{P}}^{\bar{G}} \bar{\sigma}^{\bar{g}} \otimes \bar{\psi}$. Then

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma^g \otimes \psi) = \mathrm{Hom}_{\bar{G}}(\bar{\pi}, \overline{\mathrm{Ind}_P^G \sigma^g \otimes \psi}) \cong \mathrm{Hom}_{\bar{G}}(\bar{\pi}, \mathrm{Ind}_{\bar{P}}^{\bar{G}} \bar{\sigma}^{\bar{g}} \otimes \bar{\psi})$$

Since $\mathrm{Hom}_{\bar{G}}(\bar{\pi}, \mathrm{Ind}_{\bar{P}}^{\bar{G}} \bar{\sigma}^{\bar{g}} \otimes \bar{\psi}) \neq 0$, we get $\mathfrak{I}((\pi, V)) \in \mathfrak{s}$. Using the fact that $(\rho, W)$ is an $\mathfrak{s}$-type we get that $V_\rho \neq 0$. Note that

$$V_\rho = \mathrm{Hom}_J(\rho, \pi) = \mathrm{Hom}_{\bar{J}}(\bar{\rho}, \bar{\pi}) = V_{\bar{\rho}} \neq 0.$$

We can see that all the arguments can be reversed i.e. we can start with the assumption that $V_{\bar{\rho}} \neq 0$ and conclude that $\mathfrak{I}((\bar{\pi}, V)) \in \bar{\mathfrak{s}}$. We make use of theorem 3 to conclude that $(\bar{J}, \bar{\rho})$ is an $\bar{\mathfrak{s}}$-type. □

**Remark 1.** *The proof above does not use the fact that $G$ is $\mathrm{GL}_2(\mathbb{F})$. The same proof works for a general connected reductive group $G$ over a non-archimedean local field.*

We then have by proposition 19 a way to relate types in $G$ to types in $\bar{G}$. We would like to have a similar result for covers. Let $P_u = MN_u$ be a parabolic subgroup of $G$. Let us denote by $N_\ell$ the opposite of $N_u$ relative to $M$. Consider the pair $(J, \tau)$ where $J$ is a compact open subgroup and $(\tau, W)$ an irreducible representation of $J$. If the pair $(J, \tau)$ is decomposed with respect to $(M, P_u)$, we have by definition of decomposition that $J = J \cap N_\ell \cdot J \cap M \cdot J \cap N_u$. Then we get $\overline{J} = \overline{J \cap N_\ell} \cdot \overline{J \cap M} \cdot \overline{J \cap N_u}$. We claim that $\overline{J \cap N_\ell} = \overline{J} \cap \overline{N_\ell}$, $\overline{J \cap N_u} = \overline{J} \cap \overline{N_u}$ and $\overline{J \cap M} = \overline{J} \cap \overline{M}$. Let us first show that $\overline{J \cap N_u} = \bar{J} \cap \overline{N_u}$. We certainly have $\overline{J \cap N_u} \subset \bar{J} \cap \bar{N_u}$, for the reverse containment take $\overline{x}_u \in \overline{J} \cap \overline{N}_u$. Consider then $j \in J$ and $x_u \in N_u$ such that $jz = x_u$ for some $z \in M$. We can then decompose $j = j_\ell j_M j_u$ where $j_\ell \in J \cap N_\ell$, $j_M \in J \cap M$ and $j_u \in J \cap N_u$. We then have that $x_u = jz = j_\ell (z j_M) j_u$. Since the map from $N_\ell \times M \times N_u \to G$ given by $(x_\ell, x_M, x_u) \to x_\ell x_M x_u$ is an injective map we get from the equality $x_u = j_\ell(z j_M) j_u$ that $x_u = j_u \in J$ so $x_u \in J \cap N_u$ and thus $\overline{x}_u \in \overline{J \cap N_u}$. Similarly we can show that $\overline{J \cap N_\ell} = \overline{J} \cap \overline{N}_\ell$. It remains to show that $\overline{J \cap M} = \bar{J} \cap \overline{M}$. We again have $\overline{J \cap M} \subset \bar{J} \cap \bar{M}$, for the reverse containment we take $\bar{x} \in \bar{J} \cap \bar{M}$ then there is $j \in J$ and $m \in M$ such that $\bar{j} = \bar{m} = \bar{x}$. Therefore $m^{-1} j \in Z \subset M$ so $j \in M$ we conclude that $j \in J \cap M$ and thus $\bar{x} = \bar{j} \in \overline{J \cap M}$. We then have that $\overline{J \cap M} = \bar{J} \cap \overline{M}$. Since the definition of $\bar{\tau}$ in $\bar{J}$ is given by $\bar{\tau}(\bar{x}) = \tau(x)$ for

$x \in J$. We see that if $\tau(x) = 1$ then $\bar{\tau}(\bar{x}) = 1$. We can assure that $\bar{J} \cap \bar{N}_\ell$ and $\bar{J} \cap \bar{N}_u$ are contained in $\ker \bar{\tau}$.

**Proposition 20.** *Let $J$ be a compact open subgroup of $G$ and $\rho$ and irreducible representation of $J$ such that $\rho(z) = 1$ for all $z \in Z \cap J$. The map from $\mathcal{H}(G, \rho)$ into $\mathcal{H}(\bar{G}, \bar{\rho})$ given by $f \to \bar{f}$ where $\bar{f}(\bar{x}) = \int_Z f(zx)\mu_Z(z)$ is a homomorphism of involutive algebras such that $supp(\bar{f}) \subset \overline{supp(f)}$.*

*Proof.* Let $f \in \mathcal{H}(G, \rho)$ if $\bar{x} \notin \overline{supp(f)}$ we get $f(zx) = 0$ for all $z \in Z$. Therefore

$$\overline{f}(\bar{x}) = \int_Z f(zx) d\mu_Z(z) = 0$$

we conclude that $\bar{x} \notin supp(\bar{f})$ and thus the containment $supp(\bar{f}) \subset \overline{supp(f)}$.

Let $f_1, f_2 \in \mathcal{H}(G, \rho)$ then

$$\begin{aligned}
\overline{f_1 * f_2}(\bar{y}) &= \int_Z f_1 * f_2(zy) d\mu_Z(z) = \int_Z \int_G f_1(x) f_2(x^{-1}zy) d\mu_G(x) d\mu_Z(z) \\
&= \int_G \int_Z f_1(x) f_2(x^{-1}zy) d\mu_Z(z) d\mu_G(x) \\
&= \int_{Z\backslash G} \int_Z \int_Z f_1(ux) f_2(u^{-1}zx^{-1}y) d\mu_Z(z) d\mu_Z(u) d\mu_{Z\backslash G}(x) \\
&= \int_{Z\backslash G} \int_Z f_1(ux) \bar{f}_2(\bar{x}^{-1}\bar{y}) d\mu_Z(u) d\mu_{Z\backslash G}(x) \\
&= \int_{Z\backslash G} \bar{f}_1(\bar{x}) \bar{f}_2(\bar{x}^{-1}\bar{y}) d\mu_{Z\backslash G}(x) = \bar{f}_1 * \bar{f}_2(\bar{y})
\end{aligned}$$

The first two equalities follow from the definitions. The third equality is an application of Fubini's theorem. The fourth equality is derived from a well known fact (See for instance [BH06, 10a.2.]). The rest of the equalities are just mere use of the definitions. Let $f \in \mathcal{H}(G, \rho)$ we check that the map $f \mapsto \bar{f}$ commutes with involution.

$$\begin{aligned}
\left(\bar{f}\right)^* (\bar{y}) &= (\overline{f}(\bar{y}^{-1}))^* = \left(\int_Z f(zy^{-1}) d\mu_Z(z)\right)^* \\
&= \int_Z (f(zy))^* d\mu_Z(z) = \int_Z (f^*(z^{-1}y)) d\mu_Z(z) = \overline{f^*}(\bar{y}).
\end{aligned}$$

We now need to check that the identity $1_{(G,\rho)}$ of $\mathcal{H}(G, \rho)$ goes to the identity $1_{(\bar{G},\bar{\rho})}$ of $\mathcal{H}(\bar{G}, \bar{\rho})$. We have that $\bar{1}_{(G,\rho)}$ has support on $\bar{J}$. Take $\bar{k} \in \bar{J}$, we might assume $k \in J$ then

$$\begin{aligned}\bar{1}_{(G,\rho)}(\bar{k}) &= \int_Z 1_{(G,\rho)}(zk)d\mu_Z(z) = \int_{J\cap Z} 1_{(G,\rho)}(zk)d\mu_Z(z)\\ &= \frac{\check{\rho}(k)}{\mu_G(J)}\mu_Z(J\cap Z) = \frac{\check{\rho}(k)}{\mu_{Z\backslash G}(J)} = 1_{(\bar{G},\bar{\rho})}.\end{aligned}$$

This finishes the proof of the proposition. □

We have then that if $f \in \mathcal{H}(G,\rho)$ is invertible then $\bar{f}$ is invertible. If $H$ is a subgroup of $G$ the condition $xHx^{-1} \subset H$ implies $\bar{x}\bar{H}\bar{x}^{-1} \subset \bar{H}$. We see then that if $x$ is a strongly positive element with respect to $(J,P)$ then $\bar{x}$ is strongly positive with respect to $(\bar{J},\bar{P})$. We then have the following corollary.

**Corollary 21.** *Let $L \subset M$ where $L$ and $M$ are Levi components of parabolic subgroups of $G$. Let $\mathfrak{s}_M \in \mathscr{B}(M)$. Suppose there is a pair $(L,\sigma) \in \mathfrak{s}_M$ such that $\sigma$ is trivial in $Z$. If $(J_M,\tau_M)$ is a $\mathfrak{s}_M$-type and $(J,\tau)$ is a $G$ cover for $(J_M,\tau_M)$ then $(\bar{J},\bar{\tau})$ is a $\bar{G}$ cover for $(\bar{J}_M,\bar{\tau}_M)$ and $(\bar{J},\bar{\tau})$ is an $[\bar{\mathfrak{s}}]_{\bar{G}}$-type.*

## 4. Hecke Algebras $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$

Let us choose the Haar measure $\mu_G$ such that $\mu_G(I) = 1$ and suppose that $\phi^2|_{\mathcal{O}^\times} = 1$. Recall the element $w = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ and the existence of the element $g_w$ in the case $\varphi^2|_{\mathcal{O}^\times} = 1$. We have already seen in the proof of proposition 18 that $g_w * g_w = ag_w + b1_{(G,\lambda_\varphi)}$, and $b = \mu_G(IwI) = [I : wIw^{-1} \cap I] = q$. Let $x_0, x_1, x_2 \dots x_{q-1} \in \mathcal{O}$, be different coset representatives of $\mathcal{O}/\mathfrak{p}$ where $x_0 = 0$ and for each $x_i$, $0 \le i \le q-1$, set $[x_i] = \left[\begin{smallmatrix}1&x_i\\0&1\end{smallmatrix}\right]$. The matrices $[x_i]$ are a full set of coset representatives of $(wIw^{-1} \cap I)\backslash I$. Let us now compute the complex number $a$.

(5)

$$\begin{aligned}a = g_w * g_w(w) &= \int_G g_w(x)g_w(x^{-1}w)d\mu_G(x) = \int_{IwI} g_w(x)g_w(x^{-1}w)d\mu_G(x)\\ &= \sum_{i=0}^{q-1}\int_{[x_i]wI} g_w(x)g_w(x^{-1}w)d\mu_G(x) = \varphi(-1)(q-1).\end{aligned}$$

Only the last two equalities deserve an explanation. The first one follows from the fact that $IwI$ is the disjoint union of the left cosets $[x_i]wI$ for $0 \le i \le q-1$. For the last equality we see that $[x_0] = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. If $x \in wI$, say

$x = wy$ for $y \in I$, then $x^{-1}w = y^{-1} \in I$ so $g_w(x^{-1}w) = 0$. We get

$$\int_{[x_0]wI} g_w(x)g_w(x^{-1}w)d\mu_G(x) = 0.$$

Let us now consider the case $x \in [x_i]wI$, for $1 \leq i \leq q-1$. Let $y \in I$ be such that $x = [x_i]wy$, then

$$\begin{aligned} g_w(x)g_w(x^{-1}w) &= g_w([x_i]wy)g_w(y^{-1}w[x_i]^{-1}w) \\ &= \check{\lambda}_\varphi([x_i])\check{\lambda}_\varphi(y)\check{\lambda}_\varphi(y^{-1})g_w(w[x_i]^{-1}w) \\ &= g_w(w[x_i]^{-1}w) = \varphi(-1). \end{aligned}$$

The last equality follows easily from the decomposition

$$w[x_i]^{-1}w = \begin{bmatrix} x_i^{-1} & 1 \\ 0 & -x_i \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -x_i \\ 0 & 1 \end{bmatrix};$$

$$\begin{aligned} g_w(w[x_i]^{-1}w) &= \check{\lambda}_\varphi(\begin{bmatrix} x_i^{-1} & 1 \\ 0 & -x_i \end{bmatrix})\check{\lambda}_\varphi(\begin{bmatrix} 1 & -x_i \\ 0 & 1 \end{bmatrix}) \\ &= \varphi^2(x_i^{-1})\varphi^{-1}(-1) = \varphi(-1) \end{aligned}$$

We make a summary of this in the following lemma.

**Lemma 22.** *Let $w = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$, and let $\varphi^2|_{\mathcal{O}^\times} = 1$. Then $g_w$ is invertible and satisfies the equation $g_w^2 = (q-1)\varphi(-1) \cdot g_w + q \cdot 1_{(G,\lambda_\varphi)}$.*

**Proposition 23.** *Let $\varphi^2|_{\mathcal{O}^\times} = 1$ and fix a Haar measure $\mu_{\bar{G}}$ such that $\mu_{\bar{G}}(\bar{I})$. There is a homomorphism of the involutive algebra $\mathcal{H}(q,1)$ into $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$.*

*Proof.* We know that $\alpha = [\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}]$ intertwines $\lambda_\varphi$. Then $\bar{\alpha}$ intertwines $\bar{\lambda}_\varphi$, hence the existence of the element $g_{\bar{\alpha}} \in \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$. Note that $\alpha^2 \in Z$. We then have that $g_{\bar{\alpha}}^2$ has support on $\bar{I}$. We also have by an easy calculation that $g_{\bar{\alpha}}^2(1) = \mu_{\bar{G}}(\bar{I}\bar{\alpha}) = 1$. We conclude that $g_{\bar{\alpha}}^2 = 1_{(\bar{G},\bar{\lambda}_\varphi)}$. We also have by lemma 16 that $g_{\bar{\alpha}}^* = g_{\bar{\alpha}^{-1}} = g_{\bar{\alpha}}$. By the discussion above we get that $g_w^2 = \varphi(-1)(q-1)g_w + q1_{(G,\lambda_\varphi)}$. If we let $u = \frac{\varphi(-1)}{q^{1/2}} g_w$, we see that $u$ satisfies the equation $u^2 = (q^{1/2} - q^{-1/2})u + 1_{(G,\lambda_\varphi)}$. Since $w = w^{-1}$ we get again by lemma 16 that $g_w^* = g_w$ and because $\frac{\varphi(-1)}{q^{1/2}}$ is a real number we get $u^* = u$. We then have that $\bar{u}^2 = (q^{1/2} - q^{-1/2})\bar{u} + 1_{(\bar{G},\bar{\lambda}_\varphi)}$ and $\bar{u}^* = \bar{u}$. Therefore we have an involutive algebra homorphism from $\mathcal{H}(q,1)$ into $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)$ by extending the map $s_1 \mapsto \bar{u}$ and $s_2 \mapsto g_{\bar{\alpha}}$. □

**Corollary 24.** *Let $\varphi^2|_{\mathcal{O}^\times} = 1$. Consider the element $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right]$ and $g_\alpha \in \mathcal{H}(G,\lambda_\varphi)$. Then $\bar{g}_\alpha = \mu_Z(Z\cap I) g_{\bar{\alpha}}$. Let $\Pi = \left[\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then $\bar{g}_\Pi = \mu_Z(Z\cap I) g_{\bar{\Pi}}$.*

*Proof.* We can see that the support of $\bar{g}_\alpha$ is contained in $\bar{I}\bar{\alpha}\bar{I} = \bar{I}\bar{\alpha}$.

$$\begin{aligned} \bar{g}_\alpha(\bar{\alpha}) &= \int_Z g_\alpha(z\alpha) d\mu_Z(z) = \int_{(Z\cap I)} g_\alpha(z\alpha) d\mu_Z(z) \\ &= \int_{(Z\cap I)} \lambda_\varphi(z) d\mu_Z(z) = \int_{(Z\cap I)} d\mu_Z(z) \\ &= \mu_Z(Z\cap I) \end{aligned}$$

We also have that that the support of $\bar{g}_\Pi$ is contained in $\bar{I}\bar{\Pi}\bar{I}$. We have that $z\Pi \in I\Pi I$ implies that $\det(z) \in \mathcal{O}^\times$ so $z \in I$. We thus get that

$$\begin{aligned} \bar{g}_\Pi(\bar{\Pi}) &= \int_Z g_\Pi(z\Pi) d\mu_Z(z) = \int_{(Z\cap I)} g_\Pi(z\Pi) d\mu_Z(z) \\ &= \mu_Z(Z\cap I) \end{aligned}$$

□

**Lemma 25.** *Let $\varphi^2|_{\mathcal{O}^\times} \neq 1$. We let $g = q^{-1/2} g_{\bar{\Pi}}$ we then have $g^* = g^{-1}$.*

*Proof.* In the proof of proposition 17 we showed that $g_\Pi * g_{\Pi^{-1}}$ is a non-zero multiple of the identity. We also have $\overline{g_\Pi * g_{\Pi^{-1}}} = a \cdot g_{\bar{\Pi}} * g_{\bar{\Pi}^{-1}}$ where $a$ is a non-zero constant. We then get $g_{\bar{\Pi}} * g_{\bar{\Pi}^{-1}}$ is a multiple of the identity. We have that

$$\begin{aligned} g_{\bar{\Pi}} * g_{\bar{\Pi}^{-1}}(1) &= \int_{\bar{G}} g_{\bar{\Pi}}(\bar{x}) g_{\bar{\Pi}^{-1}}(\bar{x}^{-1}) d\mu_{\bar{G}}(\bar{x}) \\ &= \int_{\bar{J}\bar{\Pi}\bar{J}} g_{\bar{\Pi}}(\bar{x}) g_{\bar{\Pi}^{-1}}(\bar{x}^{-1}) d\mu_{\bar{G}}(\bar{x}) \\ &= [\bar{J} : \bar{J} \cap \bar{\Pi}\bar{J}\bar{\Pi}^{-1}] = q \end{aligned}$$

We thus get that if $g = q^{-1/2} g_{\bar{\Pi}}$ then $g^{-1} = q^{-1/2} g_{\bar{\Pi}^{-1}} = g^*$. □

**Theorem 26.** *Let $\varphi$ be a character of $\mathcal{O}^\times$. Let us fix a Haar measure $\mu_{\bar{\mathbb{T}}}$ such that $\mu_{\bar{\mathbb{T}}}({}^\circ\bar{\mathbb{T}}) = 1$. Then $\mathcal{H}(\bar{\mathbb{T}},\varphi) \cong \mathbb{C}[\mathbb{Z}]$ as Hilbert algebras.*

*Proof.* We have that $\mathcal{H}(\bar{\mathbb{T}},\varphi)=\bigoplus_{n\in\mathbb{Z}}\mathcal{H}_{\bar{\Pi}^n}(\bar{\mathbb{T}},\varphi)$. Let $h_{\bar{\Pi}}\in\mathcal{H}(\bar{\mathbb{T}},\varphi)$ be supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}$ and such that $h_{\bar{\Pi}}(\bar{\Pi})=1$. Then for a positive integer $n$, $h_{\bar{\Pi}}^n$ is supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}^n$. Let $h_{\bar{\Pi}^{-1}}\in\mathcal{H}(\bar{\mathbb{T}},\varphi)$ be supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}^{-1}$ and such that $h_{\bar{\Pi}^{-1}}(\bar{\Pi}^{-1})=1$. The element $h_{\bar{\Pi}}*h_{\bar{\Pi}^{-1}}$ has support in $^\circ\bar{\mathbb{T}}$.

$$\begin{aligned} h_{\bar{\Pi}}*h_{\bar{\Pi}^{-1}}(1) &= \int_{^\circ\bar{\mathbb{T}}\bar{\Pi}} h_{\bar{\Pi}}(x)h_{\bar{\Pi}^{-1}}(x^{-1})d\mu_{\bar{\mathbb{T}}}(x) \\ &= \mu_{\bar{\mathbb{T}}}(^\circ\bar{\mathbb{T}}\bar{\Pi})=1. \end{aligned}$$

Let $h=h_{\bar{\Pi}}$. We then see that $h^{-1}=h_{\bar{\Pi}^{-1}}$. Making use of lemma 16 we get $h^*=h^{-1}$. We have that $h^{-1}$ has support in $^\circ\bar{\mathbb{T}}\bar{\Pi}^{-1}$ and therefore for a positive integer $n$, $h^{-n}$ has support in $^\circ\bar{\mathbb{T}}\bar{\Pi}^{-n}$. Then for $n$ an arbitrary integer we get that $h^n$ spans the space $\mathcal{H}_{\bar{\Pi}^n}(\bar{\mathbb{T}},\varphi)$. We then get that the set $\{h^n\}_{n\in\mathbb{Z}}$ is a basis for $\mathcal{H}(\bar{\mathbb{T}},\varphi)$, therefore $\mathcal{H}(\bar{\mathbb{T}},\varphi)=\mathbb{C}[h,h^{-1}]$. Since $h^*=h^{-1}$ we get that $(h^n)^*=(h^*)^n=h^{-n}$. Then for integers $n,m$ we get $[h^n,h^m]=h^n*(h^m)^*(1)=h^{n-m}(1)$. So if $n\neq m$ we get $[h^n,h^m]=0$ and if $n=m$ we get $[h^n,h^n]=1$. Hence $\{h^n\}_{n\in\mathbb{Z}}$ is a orthonormal basis. Let $f_n:\mathbb{Z}\longrightarrow\mathbb{C}$ be defined by $f_n(n)=1$ and $f_n(m)=0$ for $m\neq n$. It is well known that $f_1^n=f_n$ for all $n\in\mathbb{Z}$ and that $\{f_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $\mathbb{C}[\mathbb{Z}]$. We deduce that the map that sends $h\mapsto f_1$ induces an algebra homomorphism $F:\mathcal{H}(\bar{\mathbb{T}},\varphi)\longrightarrow\mathbb{C}[\mathbb{Z}]$ that sends $h^n\mapsto f_1^n=f_n$. Then $F$ sends an orthonormal basis to an orthonormal basis, so is an isomorphism that preserves the inner-product. In order to finish the proof we need to show that $F$ commutes with the involution. We see that $f_n^*(m)=\overline{f_n(-m)}=0$ for $m\neq -n$, and $f_n^*(-n)=\overline{f_n(n)}=1$, so $f_n^*=f_{-n}$. Therefore $F((h^n)^*)=F(h^{-n})=f_{-n}=f_n^*=F(h^n)^*$ for all $n\in\mathbb{Z}$, this implies that $F(x^*)=F(x)^*$ and this finishes the proof of the theorem. □

**Theorem 27.** *Let $\varphi$ be a character of $\mathcal{O}^\times$.Choose the Haar measure $\mu_{\bar{G}}$ to be the one such that $\mu_{\bar{G}}(\bar{J}_\varphi)=1$. Then the Hecke algebra $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)\cong\mathbb{C}[\mathbb{Z}]$ if $\varphi^2|_{\mathcal{O}^\times}\neq 1$ and $\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)\cong\mathcal{H}(q,1)$ if $\varphi^2|_{\mathcal{O}^\times}=1$. Where the isomorphism are isomorphism of normalized Hilbert algebras.*

*Proof.* Let $K=GL_2(\mathcal{O})$. We claim that the map

$$\begin{gathered} \mathcal{H}(\bar{\mathbb{T}},\varphi)\otimes_{\mathbb{C}}\mathcal{H}(\bar{K},\bar{\lambda}_\varphi)\longrightarrow\mathcal{H}(\bar{G},\bar{\lambda}_\varphi) \\ f\otimes g\longrightarrow f*g \end{gathered}$$

is an isomorphism of vector spaces. Let $\bar{\mathfrak{s}}=[\bar{\mathbb{T}},\overline{\varphi\otimes\varphi^{-1}}]_{\bar{G}}$. Then by theorem 7 we have the inequality $\dim_{\mathbb{C}}\mathcal{H}(\bar{K},\bar{\lambda})\leq|\mathbf{W}_{\bar{\mathfrak{s}}}|$ and that the claim holds true if we have equality. We take an element $\bar{x} \in \bar{G}$ and $\bar{t} \in \bar{\mathbb{T}}$. We see that the only way that $\bar{x}\bar{t}\bar{x}^{-1} \in \bar{\mathbb{T}}$, is if $\bar{x} \in \bar{\mathbb{T}} \cup \bar{w}\bar{\mathbb{T}}$. Then $|\mathbf{W}_{\bar{\mathfrak{s}}}| \leq |N_{\bar{G}}(\bar{\mathbb{T}})/\bar{\mathbb{T}}| \leq 2$. In the case $\varphi^2|_{\mathcal{O}^\times} \neq 1$, take $a \in \mathcal{O}^\times$ such that $\varphi(a) \neq \varphi(a)^{-1}$. Let

$$x_1 = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } x_2 = \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

Then for an unramified character $\chi$ of $\bar{\mathbb{T}}$ and an element $\bar{s} \in \bar{\mathbb{T}}$, we get

$$\begin{aligned}(\overline{\varphi \otimes \varphi^{-1}}^{\bar{s}} \otimes \chi)(\bar{x}_1) = \overline{\varphi \otimes \varphi^{-1}}(\bar{x}_1) = \varphi(a) \neq \varphi(a^{-1}) = \overline{\varphi \otimes \varphi^{-1}}(\bar{x}_2) \\ = \overline{\varphi \otimes \varphi^{-1}}(\bar{w}\bar{x}_1\bar{w}^{-1}) = \overline{\varphi \otimes \varphi^{-1}}^{\bar{w}}(\bar{x}_1).\end{aligned}$$

Therefore $\bar{w}$ does not stabilize $\bar{\mathfrak{s}}$, so $|\mathbf{W}_{\bar{\mathfrak{s}}}| = 1$. We conclude that

$$\dim_{\mathbb{C}} \mathcal{H}(\bar{K}, \bar{\lambda}) = 1.$$

We have thus proved the claim in the case $\varphi^2|_{\mathcal{O}^\times} \neq 1$. If $\varphi^2|_{\mathcal{O}^\times} = 1$ then $\bar{w}$ intertwines $\overline{\varphi \otimes \varphi^{-1}}$ and $\mathcal{I}_{\bar{K}}(\bar{\lambda}_\varphi) = \bar{I} \cup \bar{I}\bar{w}\bar{I}$. We then obtain that $g_{\bar{w}}$ and $1_{(\bar{G},\bar{\lambda}_\varphi)}$ are a basis for $\mathcal{H}(\bar{K}, \bar{\lambda})$ and so $\dim_{\mathbb{C}} \mathcal{H}(\bar{K}, \bar{\lambda}) = 2 = |\mathbf{W}_{\bar{\mathfrak{s}}}|$. This finishes the proof of the claim.

As a consequence of our claim for $\varphi^2|_{\mathcal{O}^\times} \neq 1$ we get that for the upper triangular matrices $B$ the map

$$t^i_{\bar{B}} : \mathcal{H}(\bar{\mathbb{T}}, \bar{\varphi}) \longrightarrow \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$$

from theorem 5 is an algebra isomorphism. We then conclude that the map

$$t^u_{\bar{B}} : \mathcal{H}(\bar{\mathbb{T}}, \bar{\varphi}) \longrightarrow \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$$

from theorem 6 is also an isomorphism. We claim that the map $t^u_{\bar{B}}$ preserves the inner product and involution and is therefore an isomorphism of Hilbert algebras.

Let $h_{\bar{\Pi}} \in \mathcal{H}(\bar{\mathbb{T}}, \varphi)$ be supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}$ and such that $h_{\bar{\Pi}}(\bar{\Pi}) = 1$. Then for a positive integer $n$, $h^n_{\bar{\Pi}}$ is supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}^n$. Let $h_{\bar{\Pi}^{-1}} \in \mathcal{H}(\bar{\mathbb{T}}, \varphi)$ be supported in $^\circ\bar{\mathbb{T}}\bar{\Pi}^{-1}$ and such that $h_{\bar{\Pi}^{-1}}(\bar{\Pi}^{-1}) = 1$. We have shown in the proof of theorem 26. That for $h = h_{\bar{\Pi}}$, $h^* = h^{-1}$. The way that $t^u_{\bar{B}}$ was constructed gives us that $t^u_{\bar{B}}(h) = \delta^{-1/2}(\Pi) g_{\bar{\Pi}} = q^{-1/2} g_{\bar{\Pi}}$. Let us set $g = q^{-1/2} g_{\bar{\Pi}}$ as in lemma 25. We then have

$$t^u_{\bar{B}}((h^n)^*) = t^u_{\bar{B}}(h^{-n}) = g^{-n} = (g^n)^* = t^u_{\bar{B}}(h^n)^*.$$

Since $\{h^n : n \in \mathbb{Z}\}$ is a basis for $\mathcal{H}(\bar{\mathbb{T}}, \bar{\varphi})$ we get that for all $f \in \mathcal{H}(\bar{\mathbb{T}}, \bar{\varphi})$, $t^u_{\bar{B}}(f^*) = t^u_{\bar{B}}(f)^*$.

We now check it preserves the inner product on the basis elements. Let $n$ be a positive integer, since $\bar{\Pi}^n$ is a positive element and $h^n$ has support in ${}^\circ\bar{\mathbb{T}}\bar{\Pi}^n$ then $t^u_{\bar{B}}(h^n) = g^n$ has support on $\bar{J}\bar{\Pi}^n\bar{J}$, so $g^n(1) = 0$. Also we deduce that $g^n = cg_{\bar{\Pi}^n}$ for a complex number $c$. Then by lemma 25, $g^{-1} = g^*$ then $g^{-n} = (g^n)^* = (cg_{\bar{\Pi}^n})^* = c^* g_{\bar{\Pi}^{-n}}$ where $c^*$ is the complex conjugate of $c$. Then the support of $g^{-n}$ is in $\bar{J}\bar{\Pi}^{-n}\bar{J}$ hence the equality $g^{-n}(1) = 0$. We let $n, m$ be any two integers then

$$\begin{aligned} [t^u_{\bar{B}}(h^n), t^u_{\bar{B}}(h^m)] &= [g^n, g^m] \\ &= g^n * (g^m)^*(1) = g^{n-m}(1) \end{aligned}$$

We deduce that $[t^u_{\bar{B}}(h^n), t^u_{\bar{B}}(h^m)] = 1$ for $m = n$, and zero otherwise. In any case $[t^u_{\bar{B}}(h^n), t^u_{\bar{B}}(h^m)] = [h^n, h^m]$. Therefore $t^u_{\bar{B}}$ is an isomorphism of Hilbert algebras showing that that $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi) \cong \mathbb{C}[\mathbb{Z}]$.

Let us now consider the case where $\varphi^2|_{\mathcal{O}^\times} = 1$. By the first claim we have that $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ has a basis given by $\{g^n_{\bar{\Pi}}, g^n_{\bar{\Pi}} * g_{\bar{w}}\}$. By proposition 23 we get a homomorphism of involutive algebras from

$$F : \mathcal{H}(q, 1) \longrightarrow \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$$

such that $F(s_1) = \bar{u} = \frac{\varphi(-1)}{q^{1/2}}\bar{g}_w$ and $F(s_2) = g_{\bar{\alpha}}$. We already had mentioned in the proof of proposition 18 that $g_w * g_\alpha$ is invertible and has support in $I\Pi I$. Hence $g_w * g_\alpha$ is a non-zero multiple of $g_\Pi$. Since $g_w * g_\alpha$ is invertible we have that $\overline{g_w * g_\alpha}$ is invertible and thus a non-zero multiple of $g_{\bar{\Pi}}$. We have that

$$\begin{aligned} F(s_1 s_2) = \frac{\varphi(-1)}{q^{1/2}}\bar{g}_w * g_{\bar{\alpha}} &= \frac{\varphi(-1)}{q^{1/2}}\bar{g}_w * \bar{g}_\alpha \\ = \frac{\varphi(-1)}{q^{1/2}}\overline{g_w * g_\alpha} &= \frac{\varphi(-1)}{q^{1/2}}\bar{g}_\Pi = \frac{\varphi(-1)}{q^{1/2}}g_{\bar{\Pi}} \end{aligned}$$

and hence $F(s_1 s_2)$ is a non-zero multiple of $g_{\bar{\Pi}}$. We can say then that for an integer $n$, $F((s_1 s_2)^n) = c_n g^n_{\bar{\Pi}}$ where $c_n$ is a non-zero constant, and from here we can therefore also say that $F((s_1 s_2)^n s_1) = \frac{\varphi(-1)}{q^{1/2}} c_n g^n_{\bar{\Pi}} * \bar{g}_w$. Recall from subsection 1.2 that if we set $d = s_1s_2$ and set $D = \mathbb{C}[d, d^{-1}]$ we get that $\mathcal{H}(q, 1) = D \bigoplus Ds_1$. Therefore the set $\{d^n\} \cup \{d^n s_1\}$ where $n$ ranges over the integers is a basis for $\mathcal{H}(q, 1)$. We conclude that our map $F$ sends a basis of $\mathcal{H}(q, 1)$ to a basis of $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ therefore $F$ is bijective.

It only remains in order to finish the proof to check that $F$ preserves inner products. We see that the set

$$\mathcal{W}' = \{(s_1s_2)^n\}_{n\geq 1} \cup \{(s_2s_1)^n\}_{n\geq 1} \cup \{(s_1s_2)^n s_1\}_{n\geq 0} \cup \{s_2(s_1s_2)^n\}_{n\geq 0}$$

of all words in the letters $s_1, s_2$, together with the identity forms a basis for $\mathcal{H}(q, 1)$. Thus

$$F(\mathcal{W}') = \{g^n_{\bar{\Pi}}\}_{n\geq 1} \cup \{(g^n_{\bar{\Pi}})^*\}_{n\geq 1} \cup \{g^n_{\bar{\Pi}} * \bar{u}\}_{n\geq 0} \cup \{g_{\bar{\alpha}} * g^n_{\bar{\Pi}}\}_{n\geq 0}.$$

We contend that for $f \in F(\mathcal{W}')$, $f(\bar{1}) = 0$. Since $\bar{\Pi}$ is strongly positive and $g_{\bar{\Pi}}$ is in the image of the map $t^u_{\bar{B}}$, the support of $g^n_{\bar{\Pi}}$ is in $\bar{I}\bar{\Pi}^n\bar{I}$, for $n \geq 1$. We have that $z\Pi^n \notin I$ for all $z \in Z$ which implies $\bar{\Pi}^n \notin \bar{I}$, so $\bar{1} \notin \bar{I}\bar{\Pi}^n\bar{I}$, hence $g^n_{\bar{\Pi}}(\bar{1}) = 0$. We also obtain that $(g^n_{\bar{\Pi}})^*$ has support in $\bar{I}\bar{\Pi}^{-n}\bar{I}$, and $\bar{1} \notin \bar{I}\bar{\Pi}^{-n}\bar{I}$. Therefore $(g^n_{\bar{\Pi}})^*(1) = 0$.

The support of $g^n_{\bar{\Pi}} * \bar{u}$ for $n \geq 0$ is contained in $\bar{I}\bar{\Pi}^n\bar{I}\bar{w}\bar{I}$. We get that $\bar{1} \in \bar{I}\bar{\Pi}^n\bar{I}\bar{w}\bar{I}$ if and only if there $\exists\ z \in Z$ , $x_1, x_2, x_3 \in I$ such that

$$x_1 z\Pi^n x_2 w x_3 = 1.$$

Then $z\Pi^n \in \mathrm{GL}_2(\mathcal{O})$ so $n = 0$; then $\bar{I}\bar{\Pi}^n\bar{I}\bar{w}\bar{I} = \bar{I}\bar{w}\bar{I}$, but $\bar{1} \notin \bar{I}\bar{w}\bar{I}$.

The support of $g_{\bar{\alpha}} * g^n_{\bar{\Pi}}$, for $n \geq 0$, is contained in $\bar{\alpha}\bar{I}\bar{\Pi}^n\bar{I} = \bar{I}\bar{\alpha}\bar{\Pi}^n\bar{I}$. So $\bar{1} \in \bar{I}\bar{\alpha}\bar{\Pi}^n\bar{I}$ if and only if $\exists\ z \in Z$ such that $z\alpha\Pi^n \in I$. We have that

$$\alpha\Pi^n = \begin{bmatrix} 0 & 1 \\ \varpi^{n+1} & 0 \end{bmatrix}$$

so if $z = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$, we get

$$z\alpha\Pi^n = \begin{bmatrix} 0 & a \\ a\varpi^{n+1} & 0 \end{bmatrix}.$$

This last matrix is not in $I$ because the entry in the first row and first column is zero and thus not a unit.

We have that the functional $[\ , \mathbf{1}]$ i.e the inner product with $\mathbf{1}$, is the unique linear functional that is zero in the space spanned by $\mathcal{W}$ and is 1 at $\mathbf{1}$. Therefore the functional $x \mapsto F(x)(1)$ for $x \in \mathcal{H}(q, 1)$ is equal to $[\ , \mathbf{1}]$. We then get that for $x, y \in \mathcal{H}(q, 1)$,

$$\begin{aligned}[x, y] &= [y^* x, \mathbf{1}] = F(y^* x)(1) = F(y)^* F(x)(1) \\ &= [F(y)^*, F(x)^*] = [F(x), F(y)].\end{aligned}$$

This finishes the proof of the theorem. $\square$

## 5. Description of the Plancherel Measure

We see that if $(\pi, V)$ is a supercuspidal representation of $\bar{G}$, then the matrix coefficients are compactly supported and thus $(\pi, V)$ is square integrable therefore $(\pi, V)$ has positive Plancherel measure. We have by a proposition in Dixmier's book [Dix77, 18.8.5] that the Plancherel measure for $(\pi, V)$ is equal to the formal dimension. We hence pay the rest of our attention to the case where the representations are not supercuspidal.

In the proof of theorem 27 and using the notation in there we showed that if $\varphi^2|_{\mathcal{O}^\times} \neq 1$ then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi) & \xrightarrow{g \mapsto f_1} & \mathbb{C}[\mathbb{Z}] \\ {\scriptstyle t^u_{\bar{B}}} \uparrow & & \uparrow {\scriptstyle \mathrm{Id}} \\ \mathcal{H}(\bar{\mathbb{T}}, \varphi) & \xrightarrow{h \mapsto f_1} & \mathbb{C}[\mathbb{Z}] \end{array}$$

Let $\varphi_t$ for $t \in \mathbb{C}$ be the representation of $\bar{\mathbb{T}}$ given by $\varphi_t(\bar{\Pi}) = q^t$ and $\varphi_t(a) = \varphi(a)$ for all $a \in {}^\circ\bar{\mathbb{T}}$. We denote by $\mathbb{C}_{\varphi_t}$ the vector space where $\varphi_t$ acts. Let $(\rho_t, V_t) = i_{\bar{B}}^{\bar{G}}(\varphi_t)$. Suppose that $(\rho_t, V_t)$ is a pre-unitary representation of $G$. We will then denote by $[(\rho_t, V_t)]$ the unitary representation obtained from $(\rho_t, V_t)$ by completion. Let us denote by $\mathbb{C}_t$ the $\mathbb{C}[\mathbb{Z}]$ module given by $f_1 \cdot a = q^t a$ for $a \in \mathbb{C}_t$. Using the theory of types we get that for a representation $(\varphi_t, \mathbb{C}_{\varphi_t})$ corresponds the $\mathcal{H}(\bar{\mathbb{T}}, \varphi)$ module $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We can then regard $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ as a $\mathbb{C}[\mathbb{Z}]$-module by demanding $f_1 \cdot \Phi = h \cdot \Phi$ for all $\Phi \in \mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We contend that $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ is isomorphic to $\mathbb{C}_t$ as $\mathbb{C}[\mathbb{Z}]$-modules. Indeed, we have that $\Phi$ is a linear map from $\mathbb{C}$ to $\mathbb{C}$ and thus is given by multiplication of the complex number $\Phi(1)$. We get that for $s \in \mathbb{C}$

$$\begin{aligned} h \cdot \Phi(s) &= \int_{\bar{\mathbb{T}}} \varphi_t(x) \Phi({}^t h(x) s) d\mu_{\bar{\mathbb{T}}}(x) = \Phi(1) s \int_{\bar{\mathbb{T}}} \varphi_t(x) {}^t h(x) d\mu_{\bar{\mathbb{T}}}(x) \\ &= \Phi(1) s \int_{{}^\circ\bar{\mathbb{T}}\bar{\Pi}} \varphi_t(x) h(x) d\mu_{\bar{\mathbb{T}}}(x) = q^t \Phi(s) \end{aligned}$$

The first equality follows by the way $\mathcal{H}(\bar{\mathbb{T}},\varphi)$ acts on $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi,\varphi_t)$. The second equality we already mentioned that follows because $\Phi$ acts by multiplication by the complex number $\Phi(1)$. The third equality is true because the support of $h$ is contained in $^\circ\bar{\mathbb{T}}\bar{\Pi}$ and the transpose of a complex number does nothing to it. Finally for the last equality we have that for $x \in {^\circ\bar{\mathbb{T}}\bar{\Pi}}$ we can write $x = a\bar{\Pi}$ for $a \in {^\circ\bar{\mathbb{T}}}$ then $h(x) = h(a\bar{\Pi}) = \varphi(a)^{-1}$ so $\varphi_t(x)h(x) = q^t\varphi(a)\varphi(a)^{-1} = q^t$. Therefore the isomorphism between the $\mathbb{C}[\mathbb{Z}]$-modules $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi,\varphi_t)$ and $\mathbb{C}_t$. We conclude that the representation $(\rho_t, V_t)$ corresponds to the $\mathbb{C}[\mathbb{Z}]$-module $\mathbb{C}_t$.

**Theorem 28.** *Let us fix a Haar measure such that $\mu_{\bar{G}}(\bar{K}) = 1$. Let $\varphi$ be a character of $\mathbb{F}^\times$ such that $\varphi^2|_{\mathcal{O}^\times} \neq 1$ and let $\bar{\mathfrak{s}} = [\bar{\mathbb{T}}, \overline{\varphi\otimes\varphi^{-1}}]_{\bar{G}}$. Then the map*

$$t \mapsto [(\rho_{it}, V_{it})]$$

*gives a Borel isomorphism between the interval $[-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$ and ${}_r\widehat{G}(\bar{\mathfrak{s}})$. Moreover, if $\nu$ is the Plancherel measure with respect to $\mu_G$ and $dt$ is the Lebesgue measure we get for $t \in [-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$*

$$d\nu([(\rho_{it}, V_{it})]) = \frac{(q+1)(q^{sw(\varphi^2)-1})\ln q}{2\pi}dt$$

*Proof.* It is well known that the spectrum of the reduced $C^*$-algebra of $\mathbb{C}[\mathbb{Z}]$ can be realized as the unit circle with measure given by arc length and where the total measure is 1, see for example [Rud87, p. 88–92]. We then can parametrize the unit circle by sending $t \in [-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$ to $q^{it}$. We also have by the discussion preceding the statement of this theorem that to the point $q^{it}$ we associate the unitary representation $\mathbb{C}_{it}$ of $\mathbb{C}[\mathbb{Z}]$ that corresponds to the unitary representation $[(\rho_{it}, V_{it})]$ in ${}_r\widehat{G}(\bar{\mathfrak{s}})$. We get by equation A in subsection 1.1.3 that

$$d\nu([(\rho_{it}, V_{it})]) = \frac{\dim\lambda_\varphi}{\mu_G(\bar{J}_\varphi)}\frac{\ln q}{2\pi}dt = \frac{\ln q}{\mu_G(\bar{J}_\varphi)2\pi}dt$$

It remains to calculate the number $\mu_G(\bar{J}_\varphi)^{-1}$. We see that $\mu_G(\bar{J}_\varphi)^{-1} = [\bar{K} : \bar{J}_\varphi] = [K : J_\varphi] = (q+1)(q^{sw(\varphi^2)-1})$. This finishes the proof of the theorem. □

We now want to give a description of the Plancherel measure restricted to $\widehat{G}(\bar{\mathfrak{s}})$ when $\bar{\mathfrak{s}} = [\bar{G}, \overline{\varphi\otimes\varphi^{-1}}]_{\bar{G}}$ and $\varphi^2|_{\mathcal{O}^\times} = 1$. We first introduce the following result easily deduced from a result stated by Bushnell and Henniart in [BH06, p. 68].

**Lemma 29.** *Let $\phi$ be a character of $F^\times$ then there exists an unique irreducible subrepresentation of $i_B^G(\delta_B^{-1/2})$ trivial on the center that we denote by $St_G$ and an exact sequence of representations of $G$.*

$$0 \to \phi \circ \det \cdot St_G \longrightarrow i_B^G(\phi \otimes \phi \cdot \delta_B^{-1/2}) \longrightarrow \phi \circ \det \to 0$$

We see that a necessary and sufficient condition for $\phi \otimes \phi$ and $\phi \circ \det$ to be trivial in $Z$ is that $\phi^2 = 1$. We then have that if $\phi^2 = 1$ we get an exact sequence of representations of $\bar{G}$.

$$0 \to \overline{\phi \circ \det} \cdot \overline{St_G} \longrightarrow i_{\bar{B}}^{\bar{G}}(\phi \cdot \delta_{\bar{B}}^{-1/2}) \longrightarrow \overline{\phi \circ \det} \to 0$$

Let us denote the representation $\overline{\phi \circ \det} \cdot \overline{St_G}$ by $St_{(\bar{G},\phi)}$. We see that the representation $St_{(\bar{G},\phi)}$ is the unique subrepresentation of $i_{\bar{B}}^{\bar{G}}(\phi \cdot \delta_{\bar{B}}^{-1/2})$.

Recall that $d = s_1 s_2 \in \mathcal{H}(q,1)$ and that $D = \mathbb{C}[d, d^{-1}]$. In the proof of Theorem 27 we showed that if $\varphi^2|_{\mathcal{O}^\times} = 1$ then we have a commutative diagram

(6)
$$\begin{array}{ccc} \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi) & \longrightarrow & \mathcal{H}(q,1) \\ {\scriptstyle t_{\bar{B}}^u}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{H}(\bar{\mathbb{T}}, \varphi) & \longrightarrow & D \end{array}$$

Where the horizontal map at the top of the diagram is given by $F$ and at the bottom by $h \mapsto \varphi(-1)d$ and where $i$ means inclusion. The commutative diagram 6 induces a commutative diagram at the level of modules.

(7)
$$\begin{array}{ccc} \mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)\text{-Mod} & \longrightarrow & \mathcal{H}(q,1)\text{-Mod} \\ {\scriptstyle \iota_{\bar{B}}^{\bar{G}}}\uparrow & & \uparrow{\scriptstyle i_*} \\ \mathcal{H}(\bar{\mathbb{T}}, \varphi)\text{-Mod} & \longrightarrow & D\text{-Mod} \end{array}$$

Let $\chi_u$ be the unique unramified character of $\bar{\mathbb{T}}$ such that $\chi_u(\bar{\Pi}) = \varphi(-1)\varphi(\bar{\Pi})^{-1}$ and let $\chi_t$ be the unramified character of $\bar{\mathbb{T}}$ given by $\chi_t(\bar{\Pi}) = q^t \chi_u(\bar{\Pi})$. We let $\varphi_t = \chi_t \otimes \varphi$ seen as a representation of $\bar{\mathbb{T}}$ and we denote by $\mathbb{C}_{\varphi_t}$ the vector space where $\varphi_t$ acts. Let $(\rho_t, V_t) = i_{\bar{B}}^{\bar{G}}(\varphi_t)$. Suppose that $(\rho_t, V_t)$ is a pre-unitary representation of $G$. We will then denote by $[(\rho_t, V_t)]$ the unitary representation obtained from $(\rho_t, V_t)$ by completion. Let us denote by $\mathbb{C}_t$ the $D$ module given by $d \cdot a = q^t a$ for $a \in \mathbb{C}_t$. Using the theory of types we get that to a representation $(\varphi_t, \mathbb{C}_{\varphi_t})$ corresponds the $\mathcal{H}(\bar{\mathbb{T}}, \varphi)$ module $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We can then regard $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ as a $D$-module by demanding $d \cdot \Phi = \varphi(-1)h \cdot \Phi$ for all $\Phi \in \mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$. We contend that $\mathrm{Hom}_{\mathcal{O}^\times}(\varphi, \varphi_t)$ is isomorphic to $\mathbb{C}_t$ as $D$-modules. Indeed, we have that $\Phi$ is a linear map from $\mathbb{C}$ to $\mathbb{C}$ thus is given by multiplication of the complex number $\Phi(1)$. We have that for $s \in \mathbb{C}$

$$\begin{aligned} d \cdot \Phi &= \varphi(-1)h \cdot \Phi(s) = \varphi(-1) \int_{\bar{\mathbb{T}}} \varphi_t(x) \Phi({}^t h(x) s) d\mu_{\bar{\mathbb{T}}}(x) \\ &= \varphi(-1)\Phi(1) s \int_{\bar{\mathbb{T}}} \varphi_t(x) {}^t h(x) d\mu_{\bar{\mathbb{T}}}(x) \\ &= \varphi(-1)\Phi(1) s \int_{{}^\circ\bar{\mathbb{T}}\bar{\Pi}} \varphi_t(x) h(x) d\mu_{\bar{\mathbb{T}}}(x) \\ &= \varphi^2(-1) q^t \Phi(s) = q^t \Phi(s) \end{aligned}$$

We can then see that representation $(\rho_t, V_t)$ corresponds to the $\mathcal{H}(q,1)$-module ind $\mathbb{C}_t$.

We also have two 1–dimensional unitary representations $\rho_1, \rho_2$ of $\mathcal{H}(q,1)$ that contribute to the Plancherel measure. These representations are characterized by $\rho_1(d) = q^{-1/2}$ and $\rho_2(d) = -q^{-1/2}$. We will like to know what are the representations of $\bar{G}$ that correspond to $\rho_1$ and $\rho_2$. Let $(\pi_1, V_1)$ be the $\bar{G}$ representation that correspond to $\rho_1$. Since $\rho_1(d) = q^{-1/2}$ then the restriction of $\rho_1$ to $D$ is equal to $\mathbb{C}_{-1/2}$. It follows that $\mathrm{Hom}_D(\rho_1, \mathbb{C}_{-1/2}) \neq 0$ so $\mathrm{Hom}_{\mathcal{H}(q,1)}(\rho_1, \mathrm{ind}\, \mathbb{C}_{-1/2}) \neq 0$. We then have that that

$$\mathrm{Hom}_{\bar{G}}((\pi_1, V_1), i_B^{\bar{G}}(\varphi_{-1/2})) \neq 0.$$

We have that $\varphi^2$ is an unramified character, and the way that $\chi_u$ was defined give us that $\varphi^2 \chi_u^2$ is an unramified character trivial on $\bar{\Pi}$. We therefore have that $\varphi^2 \chi_u^2 = 1$. It follows that the representation $(\pi_1, V_1)$ is the unique subrepresentation $St_{(\bar{G}, \varphi\chi_u)}$. We can deduce by the same method that the representation that corresponds to $\rho_2$ is $St_{(\bar{G}, \varphi\chi_{-u})}$ where $\chi_{-u}$ is the unique unramified character of $\bar{\mathbb{T}}$ such that $\chi_{-u}(\bar{\Pi}) = -\varphi(-1)\varphi(\bar{\Pi})^{-1}$.

**Theorem 30.** *Let us fix a Haar measure such that $\mu_{\bar{G}}(\bar{K}) = 1$. Let $\varphi$ be a character of $\mathbb{F}^\times$ such that $\varphi^2|_{\mathcal{O}^\times} = 1$ and let $\bar{\mathfrak{s}} = [\overline{\mathbb{T}, \varphi \otimes \varphi^{-1}}]_{\bar{G}}$. Then there is a Borel isomorphism from $[0, \frac{\pi}{\ln q}] \cup \{p_1, p_2\}$ into ${}_r\widehat{G}(\bar{\mathfrak{s}})$ given by $t \mapsto [(\rho_{it}, V_{it})]$ for $t \in [0, \frac{\pi}{\ln q}]$ and $p_1 \mapsto [St_{(\bar{G}, \varphi\chi_u)}]$ and $p_2 \mapsto [St_{(\bar{G}, \varphi\chi_{-u})}]$. Moreover, if $\nu$ is the Plancherel measure with respect to $\mu_G$ and $dt$ is the*

*Lebesgue measure. We get for* $t \in [0, \frac{\pi}{\ln q}]$

$$d\nu([(\rho_{it}, V_{it})]) = \frac{\ln(q)(q+1)}{2\pi q}\frac{L(1, 1-2it)L(1, 1+2it)}{L(1, -2it)L(1, 2it)}dt$$

*and* $\nu([St_{(\bar{G},\varphi\chi_u)}]) = \nu([St_{(\bar{G},\varphi\chi_{-u})}]) = \frac{q-1}{2}$.

*Proof.* The proof of this theorem is just a restatement of facts that have been proven before. Let us work first with the Haar measure $\mu'_{\bar{G}}$ where $\mu'_{\bar{G}}(\bar{I}) = 1$. Then in Corollary 15 we have that the Plancherel measure $\nu_{\mathcal{H}(q,1)}$ of the Hecke algebra $\mathcal{H}(q, 1)$ can be identified with the interval $[0, \frac{\pi}{\ln q}]$ and a measure $\frac{\ln(q)}{2\pi}Pdt$ where $dt$ gives the Lebesgue measure in the interval and

$$P(t) = q^{-1}\frac{L(1, 1-2it)L(1, 1+2it)}{L(1, -2it)L(1, 2it)}$$

union two points $\rho_1, \rho_2$ where

$$\nu_{\mathcal{H}(q,1)}(\rho_1) = \nu_{\mathcal{H}(q,1)}(\rho_2) = \frac{1}{2}\left(\frac{q-1}{q+1}\right).$$

We have by theorem 27 that $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi) \cong \mathcal{H}(q, 1)$ as Hilbert algebras and the isomorphism is given explicitly in the proof. We then conclude that the Plancherel measure $\nu_{\mathcal{H}(q,1)}$ of the Hilbert algebra $\mathcal{H}(q, 1)$ can be identified with the Plancherel measure $\nu_{\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)}$ of the Hilbert algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$. We have by equation A in subsection 1.1.3 the Plancherel measure $\nu_{\mathcal{H}(\bar{G},\bar{\lambda}_\varphi)}$ can be identified with the Plancherel measure $\nu$ of $G$ restricted to ${}_r\widehat{G}(\mathfrak{s})$. We can therefore finally identify the interval $[0, \frac{\pi}{\ln q}]$ and a measure $\frac{\ln(q)}{2\pi}Pdt$ where $dt$ gives the Lebesgue measure in the interval and

$$P(t) = q^{-1}\frac{L(1, 1-2it)L(1, 1+2it)}{L(1, -2it)L(1, 2it)}$$

union two points $\rho_1, \rho_2$ where

$$\nu_{\mathcal{H}(q,1)}(\rho_1) = \nu_{\mathcal{H}(q,1)}(\rho_2) = \frac{1}{2}\left(\frac{q-1}{q+1}\right).$$

with the Plancherel measure $\nu$ of $G$ restricted to ${}_r\widehat{G}(\mathfrak{s})$. Using the corresponding modules from the diagram 7 we get the map from $[0, \frac{\pi}{\ln q}] \cup \{p_1, p_2\}$ into ${}_r\widehat{G}(\mathfrak{s})$ given by $t \mapsto [(\rho_{it}, V_{it})]$ for $t \in [0, \frac{\pi}{\ln q}]$ and $p_1 \mapsto [St_{(\bar{G},\varphi\chi_u)}]$ and $p_2 \mapsto [St_{(\bar{G},\varphi\chi_{-u})}]$ gives the final identification. The theorem then follows except for the fact that we have to change the Haar measure to be the one where $\mu_{\bar{G}}(\bar{K}) = 1$. We then have to rescale the Plancherel measure $\nu$, by the number $[\bar{K} : \bar{I}] = [K : I] = q + 1$. The reason for the rescaling is that the isomorphism of Hilbert algebras $\mathcal{H}(q, 1)$ and $\mathcal{H}(\bar{G}, \bar{\lambda}_\varphi)$ is given under the assumption that Haar measure $\mu'_{\bar{G}}(\bar{I}) = 1$ we then have $\mu_{\bar{G}}(\bar{I}) = \frac{1}{[\bar{K}:\bar{I}]}\mu'_{\bar{G}}(\bar{I})$. Since changing the Haar measure by a factor of $\frac{1}{[\bar{K}:\bar{I}]}$ changes the Plancherel measure $\nu$ by a factor of $[\bar{K} : \bar{I}]$ our rescaling is justified. □

**Remark 2.** *If we use the notation and the results on section 23 of the book by Bushnell and Henniart [BH06, p. 143], we see that for a character* $\psi$ *of* $\mathbb{F}$ *trivial on* $\mathfrak{p}$ *but not on* $\mathcal{O}$ *we have the equality* $\gamma(1, s, \psi) = q^{(s-1/2)} \frac{L(1,1-s)}{L(1,s)}$. *We also have that the complex conjugate of* $\gamma(1, s, \psi)$ *is*

$$\gamma(1, -s, \psi) = q^{(-s-1/2)} \frac{L(1, 1 + s)}{L(1, -s)}.$$

*Letting* $s = 2it$ *and multiplying* $\gamma(1, s, \psi)$ *with its conjugate we get the equality*

$$|\gamma(1, s, \psi)|^2 = \frac{1}{q} \frac{L(1, 1 - 2it)L(1, 1 + 2it)}{L(1, -2it)L(1, 2it)}.$$

*We may replace then*

$$\frac{1}{q} \frac{L(1, 1 - 2it)L(1, 1 + 2it)}{L(1, -2it)L(1, 2it)}$$

*in the above theorem for the function* $|\gamma(1, 2it, \psi)|^2$.

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## References

[Art91] James Arthur, *A local trace formula*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 73, 5–96.

[Ber74] J. N. Bernstein, *All reductive p-adic groups are tame*, Functional Analysis and its Applications (1974), 3–6.

[Ber84] J. N. Bernstein, *Le "centre" de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32.

[BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006.

[BHK11] Colin J. Bushnell, Guy Henniart, and Philip C. Kutzko, *Types and explicit Plancherel formulæ for reductive $p$-adic groups*, On certain $L$-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 55–80.

[BK01] Colin J. Bushnell and Philip C. Kutzko, *Types in reductive $p$-adic groups: the Hecke algebra of a cover*, Proc. Amer. Math. Soc. **129** (2001), no. 2, 601–607.

[Dix77] Jacques Dixmier, *$C^*$-algebras*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[KM09] Philip Kutzko and Lawrence Morris, *Explicit Plancherel theorems for $\mathcal{H}(q_1, q_2)$ and $\mathbb{SL}_2(F)$*, Pure Appl. Math. Q. **5** (2009), no. 1, 435–467.

[Ren10] David Renard, *Représentations des groupes réductifs $p$-adiques*, Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010.

[Rud87] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[Sil70] Allan J. Silberger, $\mathrm{PGL}_2$ *over the $p$-adics: its representations, spherical functions, and Fourier analysis*, Lecture Notes in Mathematics, Vol. 166, Springer-Verlag, Berlin-New York, 1970.

[Wal03] J.-L. Waldspurger, *La formule de Plancherel pour les groupes $p$-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), no. 2, 235–333.

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