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Bases of q-Schur Module $\mathcal{A}^\lambda$

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Abstract: In this paper, we construct the so-called q-Schur modules as left principal ideals of cyclotomic q-Schur algebras, and prove that they are isomorphic to those cell modules defined in [3] and [9] in any level r. After that, mainly, we prove that these q-Schur modules are free and construct their bases. This result gives new versions of some known results such as standard basis and the branching theorem. With the help of this realization and the new basis, we give a new proof of the Branch rule of Weyl modules which was first discovered by Wada in [13].
Keywords: q-Schur module, cyclotomic q-Schur algebra, branching theorem.

1. Introduction

Weyl modules for a cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ have been investigated recently in the context of cellular algebras (see [3]). These modules are defined as quotient modules of certain *permutation* modules, that is, as *cell modules* via cellular basis.

However, the classical theory [1] and the works [4],[5] in the case when $m = 1, 2$ suggested that a construction as *submodules* without using cellular basis should exist in the case of Iwahori-Hecke algebra. Following Dipper and James' work [2], when the level r equals to one, *basis* and *structure* appearing in Hecke algebras can still be constructed in q-Schur algebras with a totally different way.

This phenomenon needs a great change to stay valid in the case of cyclotomic q-Schur algebras with large level, which is the inspiration of this paper. We can solve the difficulties by constructing a series of principal left ideals. Each single one is generated by a single element of the cyclotomic q-Schur algebras, which we denote by z_λ. The element z_λ we construct is $\varphi^1_{\lambda w} \cdot T_{w_\lambda} \cdot y_{\lambda'}$ by the right Ariki-Koike algebra $\mathcal{H}_{n,r}$-module

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structure, where the element $y_{\lambda'}$ and morphism $\varphi^d_{\lambda w}$ are defined in 2.3 and 2.4 respectively. i.e., *q-Schur module* $\mathcal{A}^\lambda$ is defined as $\mathscr{S}_{n,r}\cdot\varphi^1_{\lambda w}T_{w_\lambda}y_{\lambda'}$ (Definition 2.4). Then in Theorem 3.1, we prove that the $\mathcal{A}^\mu$ as $\mathscr{S}_{n,r}\cdot z_\mu$ is exactly a realization of Weyl modules in the category of modules over cyclotomic q-Schur algebras which is a generalization of Dipper and James' work [2]. After that, we construct and prove a R-basis of the q-Schur module $\mathcal{A}^\mu$ in the main result as follows:

Theorem 3.5. *Suppose that $\lambda\in\Lambda^+_{n,r}(\boldsymbol{m})$. Then the q-Schur module $\mathcal{A}^\lambda$ is free as a R-module and $\{\varphi^{1_A}_{\mu\lambda}\cdot z_\lambda|A\in\mathcal{T}^{ss}_\mu(\lambda)$ and $\mu\in\Lambda_{n,r}(\boldsymbol{m})\}\subseteq\mathcal{A}^\lambda$ is a basis.*

Here μ is any multipartition (defined in Section 2.1) and A is its semi-standard tableau (defined in Remark 3.3). This theorem is something like "the half way" of the semi-standard basis that appeared in [3]. With the help of this basis constructed, we can show a new version of the Branch rule which happens in the category of modules over a cyclotomic q-Schur algebra.

The paper is organised as follows. In Section 3, we construct some left ideals $\{\mathcal{A}^\mu\}$, which are called *q-Schur modules* over the cyclotomic q-Schur algebra ${}_R\mathscr{S}_{n,r}$, and prove that these q-Schur modules are the same as Weyl modules in [3]. After that, we clarify that these ideals are spanned by the natural basis as $\{\varphi^{1_A}_{\mu\lambda}\cdot z_\lambda|\mu\in\Lambda_{n,r}(\mathbf{m})$ and A$\in\mathcal{T}^{ss}_\mu(\lambda)\}$, just as a parallel work of Dipper and James in [4]. In Section 4, by using of these new bases in q-Schur modules, we construct their filtrations, as a new point of view to the Branch rule in Wada's work [13].

2. Prelimilaries

2.1. Some notations about tableaux

First, we state some notations following [11].

A *composition* λ of n is a finite sequence of non-negative integers $(\lambda_1,\lambda_2,\ldots,\lambda_m)$ such that $|\lambda|=\sum_i\lambda_i=n$. There is a partial order $\trianglelefteq$(resp. $\trianglerighteq$) within compositions of n as: we denote $\lambda\trianglelefteq\mu$ (resp. $\lambda\trianglerighteq\mu$) when $\sum_{i=1}^k\lambda_i\leq\sum_{i=1}^k\mu_i$(resp. $\sum_{i=1}^k\lambda_i\geq\sum_{i=1}^k\mu_i$) for all $1\leq k\leq m$. Moreover, if a composition λ satisfies that $\lambda_1\geq\lambda_2\geq\cdots\lambda_m$, it is called a *partition.* For later use, let $\Lambda(n)$ (resp. $\Lambda^+(n)$) denote the set of all compositions (resp. all partitions) of n.

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $1, \ldots, n$ with Coxeter generators $s_i := (i, i+1)$, and $\mathfrak{S}_\lambda$ the Young subgroup corresponding to the composition λ of n, which is denoted by:

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\mathbf{a}} = \mathfrak{S}_{\{1,\ldots,a_1\}} \times \mathfrak{S}_{\{a_1+1,\ldots,a_2\}} \times \cdots \times \mathfrak{S}_{\{a_{m-1}+1,\ldots,a_m\}},$$

where $\mathbf{a} = [a_0, a_1, \ldots, a_m]$ with $a_0 = 0$ and $a_i = \lambda_1 + \cdots + \lambda_i$ for all $i = 1, \ldots, m$. We denote by $\mathscr{D}_\lambda$ the set of distinguished representatives of right $\mathfrak{S}_\lambda$-cosets and write $\mathscr{D}_{\lambda\mu} := \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$, which is the set of distinguished representatives of double cosets $\mathfrak{S}_\lambda \setminus \mathfrak{S}_n / \mathfrak{S}_\mu$.

One can identify a composition λ with *Young diagram* and we say that λ is the *shape* of the corresponding Young diagram. A λ-*tableau* is a filling of the n boxes of the Young diagram of λ of the numbers $1, 2, \ldots, n$. Denote the set of λ-tableaux by $\mathcal{T}(\lambda)$ and usually denote $\mathfrak{t}$ as an element of $\mathcal{T}(\lambda)$.

If $\lambda \in \Lambda(n)$, it is well-known that symmetric group $\mathfrak{S}_n$ has a right group action on $\mathcal{T}(\lambda)$, which is simply interchanging the components of a tableau in $\mathcal{T}(\lambda)$.

For $\lambda \in \Lambda(n)$, let λ' be the dual partition of λ, i.e., $\lambda'_i := \#\{j; \lambda_j \geq i\}$. There is a unique element $w_\lambda \in \mathfrak{S}_n$ with the *trivial intersection property* in (4.1) of [4]:

$$w_\lambda^{-1} \mathfrak{S}_\lambda w_\lambda \cap \mathfrak{S}_{\lambda'} = \{1\}. \tag{2.1}$$

We can represent w_λ with help of Young diagrams. For example, $\begin{smallmatrix}\square&\square&\square\\ \square&\square&\end{smallmatrix}$ represents $\lambda = (3, 2)$, then $w_\lambda \in \mathfrak{S}_n$ is defined by the equation $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$, where $\mathfrak{t}^\lambda$ (resp. $\mathfrak{t}_\lambda$) is the λ-tableau obtained by putting the number $1, 2, \ldots, n$ in order into the boxes from left to right down successive rows (resp. columns). In the example,

$$\mathfrak{t}^{(3,2)} = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5& \\ \cline{1-2}\end{array}, \qquad \mathfrak{t}_{(3,2)} = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4& \\ \cline{1-2}\end{array}.$$

If we have a λ-tableau $\mathfrak{t}$ here, we also can determine a unique element $d(\mathfrak{t}) \in \mathfrak{S}_n$, such that $\mathfrak{t}^\lambda \cdot d(\mathfrak{t}) = \mathfrak{t}$.

Definition 2.1. *[2] Suppose that $\mathfrak{t}_1$ is a λ-tableau and $\mathfrak{t}_2$ is a μ-tableau, where both λ, $\mu \in \Lambda^+(n)$. Let $\chi(\mathfrak{t}_1, \mathfrak{t}_2)$ be a $n \times n$ matrix whose entry in row i and column j is the cardinality of following set:*

$$\{\textit{entries in the first } i \textit{ rows of } \mathfrak{t}_1\} \cap \{\textit{entries in the first } j \textit{ columns of } \mathfrak{t}_2\}.$$

Remark 2.2. *[2] If* $\mathfrak{t}_1$ *and* $\mathfrak{t}_1'$ *are* λ*-tableaux and* $\mathfrak{t}_2$ *and* $\mathfrak{t}_2'$ *are* μ*-tableaux for* λ *and* $\mu \in \Lambda^+(n)$*, then write* $\chi(\mathfrak{t}_1, \mathfrak{t}_2) \geq \chi(\mathfrak{t}_1', \mathfrak{t}_2')$ *if each entry in* $\chi(\mathfrak{t}_1, \mathfrak{t}_2)$ *is not smaller than corresponding one in* $\chi(\mathfrak{t}_1', \mathfrak{t}_2')$*. Write* $\chi(\mathfrak{t}_1, \mathfrak{t}_2) > \chi(\mathfrak{t}_1', \mathfrak{t}_2')$ *if, in addition,* $\chi(\mathfrak{t}_1, \mathfrak{t}_2) \neq \chi(\mathfrak{t}_1', \mathfrak{t}_2')$*.*

The following properties are immediate from the definitions.

$$\chi(\mathfrak{t}_1 w, \mathfrak{t}_2 w) = \chi(\mathfrak{t}_1, \mathfrak{t}_2) \quad for\ all\ w \in \mathfrak{S}_n. \tag{2.2}$$

$$\chi(\mathfrak{t}_1 w, \mathfrak{t}_2) = \chi(\mathfrak{t}_1, \mathfrak{t}_2) \quad if\ w \in \mathfrak{S}_\lambda. \tag{2.3}$$

$$\chi(\mathfrak{t}_1, \mathfrak{t}_2 w) = \chi(\mathfrak{t}_1, \mathfrak{t}_2) \quad if\ w \in \mathfrak{S}_{\mu'}. \tag{2.4}$$

Let $\mathbf{m} = (m_1, \cdots, m_r) \in \mathbb{Z}^r_{>0}$ be a r-tuple of positive integers. Define a subset of r-compositions of n as:

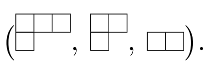
$$\Lambda_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \;\middle|\; \begin{array}{c} \mu^{(k)} = (\mu_1^{(k)}, \cdots, \mu_{m_k}^{(k)}) \in \mathbb{Z}^{m_k}_{\geq 0} \\ \sum_{k=1}^r \sum_{i=1}^{m_k} \mu_i^{(k)} = n \end{array} \right\}.$$

We denote by $|\mu^{(k)}| = \sum_{i=1}^{m_k} \mu_i^{(k)}$ (resp. $|\mu| = \sum_{k=1}^r |\mu^{(k)}|$) the size of $\mu^{(k)}$ (resp. the size of μ). We define the map $\zeta : \Lambda_{n,r}(\mathbf{m}) \to \mathbb{Z}^r_{\geq 0}$ by $\zeta(\mu) = (|\mu^{(1)}|, |\mu^{(2)}|, \cdots, |\mu^{(r)}|)$ for $\mu \in \Lambda_{n,r}(\mathbf{m})$. Put $\Lambda^+_{n,r}(\mathbf{m}) = \{\lambda \in \Lambda_{n,r}(\mathbf{m}) | \lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_{m_k}^{(k)}$ for any $k = 1, \cdots, r\}$.

Let $\lambda' := (\lambda^{(r)\prime}, \ldots, \lambda^{(1)\prime})$ denote the r-composition dual to λ. By concatenating the components of λ, the resulting composition of r will be denoted by

$$\bar{\lambda} := \lambda^{(1)} \vee \cdots \vee \lambda^{(r)}.$$

We can also identify $\lambda \in \Lambda_{n,r}(\mathbf{m})$ with a series of Young diagrams. For example, $\lambda = ((31), (21), (2))$ is identified with



Similarly, we can define two tableaux $\mathfrak{t}^\lambda$ and $\mathfrak{t}_\lambda$ in multi-composition case. Let $\mathfrak{t}^\lambda$ (resp. $\mathfrak{t}_\lambda$) be the λ-tableau obtained by setting the numbers $1, \ldots, r$ in order into the boxes down successive rows (resp. columns) in the first (resp. last) diagram of λ, then in the second (resp. second last) diagram and so on. Due to the example above, we have

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 8 & 9 \\ \hline\end{array}\right).$$

$$\mathfrak{t}_\lambda = \left(\begin{array}{|c|c|c|}\hline 6 & 8 & 9 \\ \hline 7 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\right).$$

Give the element $w_\lambda \in \mathfrak{S}_n$ by $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$ corresponding to a r-partition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of n. More precisely, if $\mathfrak{t}^i$ denote the i-th subtableau of $\mathfrak{t}^\lambda$, then define $w_{(i)}$ by $\mathfrak{t}^i w_{(i)} = \mathfrak{t}_i$.

2.2. Ariki-Koike algebras and cyclotomic q-Schur algebras

Now recall the notion of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ from [3] and the presentations of $\mathscr{S}_{n,r}$ by generators and fundamental relations given in [14].

Let R be a commutative ring, and take parameters $q, Q_1, \cdots, Q_r \in R$ such that q is invertible in R. The Ariki-Koike algebra $\mathcal{H}_{n,r}$ is the associative algebra with 1 over R generated by $T_0, T_1, \ldots, T_{n-1}$ with the following defining relations:

$$\begin{aligned}
&(T_0 - Q_1)(T_0 - Q_2) \cdots (T_0 - Q_r) = 0, && \\
&(T_i - q)(T_i + q^{-1}) = 0 && (1 \le i \le n-1), \\
&T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, && \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} && (1 \le i \le n-2), \\
&T_i T_j = T_j T_i && (|i-j| \ge 2).
\end{aligned}$$

The subalgebra of $\mathcal{H}_{n,r}$ generated by $T_1, \cdots, T_{n-1}$ is isomorphic to the *Iwahori-Hecke algebra* $\mathcal{H}_n$ (sometimes we write it $\mathcal{H}(\mathfrak{S}_n)$) in [11]. For $w \in \mathfrak{S}_n$, denote by $\ell(w)$ the length of w and by T_w the *standard basis* of $\mathcal{H}_n$ corresponding to w.

For each r-composition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$, define $[\lambda] := [a_0, a_1, \ldots, a_r]$ such that $a_0 := 0$ and $a_i := \sum_{j=1}^{i} |\lambda^{(j)}|$. In the case of Iwahori-Hecke algebras, we can define an element $m_\lambda \in \mathcal{H}_n$ (resp. $n_\lambda \in \mathcal{H}_n$) as $m_\lambda := \sum\limits_{w \in \mathfrak{S}_{\bar{\lambda}}} T_w$ (resp. $n_\lambda := \sum\limits_{w \in \mathfrak{S}_{\bar{\lambda}}} (-q)^{\ell(w)} T_w$) and $w_\lambda \in \mathfrak{S}_n$ is defined in the above subsection.

Definition 2.3. *Let $\mathcal{H}_{n,r}$ be a cyclotomic Hecke algebra with generators $\{T_0, T_1, \ldots, T_{n-1}\}$, and elements $L_1 = T_0$, $L_i = q^{-1} T_{i-1} L_{i-1} T_{i-1}$ for $i = 2, \cdots, n$, and put $\pi_0 = 1$, $\pi_a(x) = \Pi_{j=1}^{a} (L_j - x)$ for any $x \in R$ and any positive integer a. Following [3], we can construct a series of numbers as $\boldsymbol{a} = [\lambda] = [a_0, a_1, \ldots, a_r]$. Define that*

$$u_{\boldsymbol{a}}^+ = \pi_{a_1}(Q_2) \cdots \pi_{a_{r-1}}(Q_r) \quad \textit{and} \quad u_{\boldsymbol{a}}^- = \pi_{a_1}(Q_{r-1}) \cdots \pi_{a_{r-1}}(Q_1),$$

and, for $\lambda \in \Lambda_{n,r}(\boldsymbol{m})$*, define that*

$$x_\lambda := u^+_{[\lambda]} m_{\bar{\lambda}} = m_{\bar{\lambda}} u^+_{[\lambda]} \text{ and } y_\lambda := u^-_{[\lambda]} n_{\bar{\lambda}} = n_{\bar{\lambda}} u^-_{[\lambda]}.$$

Define the right ideal as $M^\lambda := x_\lambda \mathcal{H}_{n,r}$ *which is always called* permutation module.

The cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ associated to $\mathcal{H}_{n,r}$ is defined by

$${}_R\mathscr{S}_{n,r} = {}_R\mathscr{S}_{\Lambda_{n,r}}(\mathbf{m}) = \mathrm{End}_{\mathcal{H}_{n,r}}\Big(\bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} M^\mu\Big).$$

In order to describe a presentation of ${}_R\mathscr{S}_{n,r}$, we prepare some notations.

Put $m = \sum_{k=1}^r m_k$, and define a "*dominant order in multipartitions*". i.e., for $\lambda, \mu \in \Lambda_{n,r}(\mathbf{m})$ and $1 \leq l \leq r$, $1 \leq j \leq m_l$,

$$\lambda \trianglerighteq \mu \quad \Leftrightarrow \quad \sum_{i=1}^{l-1} |\lambda^{(i)}| + \sum_{k=1}^{j} \lambda_k^{(l)} \geq \sum_{i=1}^{l-1} |\mu^{(i)}| + \sum_{k=1}^{j} \mu_k^{(l)}.$$

For $(i,k) \in \Gamma'(\mathbf{m}) := \Gamma(\mathbf{m}) \setminus \{(m_r, r)\}$, we define the elements $E_{(i,k)}$, $F_{(i,k)} \in {}_R\mathscr{S}_{n,r}$ [14] by:

$$E_{(i,k)}(m_\mu \cdot h) = \begin{cases} q^{-\mu_{i+1}^{(k)}+1}\Big(\sum_{x \in X_\mu^{\mu+\alpha_{(i,k)}}} q^{\ell(x)} T_x^*\Big) h^\mu_{+(i,k)} m_\mu \cdot h & \text{if } \mu + \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{if } \mu + \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m}), \end{cases}$$

$$F_{(i,k)}(m_\mu \cdot h) = \begin{cases} q^{-\mu_{i}^{(k)}+1}\Big(\sum_{y \in X_\mu^{\mu-\alpha_{(i,k)}}} q^{\ell(x)} T_y^*\Big) m_\mu \cdot h & \text{if } \mu - \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{if } \mu - \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m}), \end{cases}$$

for any $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $h \in \mathcal{H}_{n,r}$, where $h^\mu_{+(i,k)} = \begin{cases} 1 & (i \neq m_k), \\ L_{N+1} - Q_{k+1} & (i = m_k). \end{cases}$

For $\lambda \in \Lambda_{n,r}(\mathbf{m})$, we define the element $1_\lambda \in {}_R\mathscr{S}_{n,r}$ by

$$1_\lambda(m_\mu \cdot h) = \delta_{\lambda\mu} m_\lambda \cdot h$$

for $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $h \in \mathcal{H}_{n,r}$. In addition, we see that $\{1_\lambda | \lambda \in \Lambda_{n,r}(\mathbf{m})\}$ is a set of pairwise orthogonal idempotents, and then $1 = \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda$.

Definition 2.4. *For any $\mu \in \Lambda_{n,r}(\boldsymbol{m})$ and $m \in \mathbb{N}$, we now define a left principal ideal of cyclotomic q-Schur algebra as in the case $m = 1$ in [2]:*

*$\mathcal{A}^\mu \triangleq \mathcal{S}_{n,r}\varphi^1_{\mu\omega}T_{w_\mu}y_{\mu'}$ with $\varphi^1_{\mu\omega} \in Hom_{\mathcal{H}_{n,r}}(\mathcal{H}_{n,r}, M^\mu)$ is defined as $\varphi^1_{\mu\omega}(h) := x_\mu h$ for any $h \in \mathcal{H}_{n,r}$. Meanwhile, the element $T_{w_\mu}y_{\mu'}$ acts on $\varphi^1_{\mu\omega}$ induced by the right $\mathcal{H}_{n,r}$-module structure of M^μ. From now on, the module $\mathcal{A}^\mu$ is called a q-*Schur module*, and denote the element $\varphi^1_{\mu\omega}T_{w_\mu}y_{\mu'} \in \mathscr{S}_{n,r}$ by z_μ.*

Note that here we needs to put the restriction that $\mu \in \Lambda_{n,r}(\mathbf{m})$ since in the last section, the results only make sense in this restricted situation.

Recall in [6] that the set of all $[\lambda]$ forms a poset $\Lambda[m,r]$ (where $m = \sum_i a_i$) which has the same set $\Lambda(m,r)$ as all compositions of m with at most r parts but with different order. Partial ordering on $\Lambda[m,r]$ is given by $\preceq$: $[a_i] \preceq [b_i]$ if $a_i \leq b_i$ for all $i = 1, \dots, r$, while $\Lambda(m,r)$ has the usual dominance order $\trianglelefteq$.

The following results will be useful in the sequel (see (2.8), (3.1), (3.4) in [6]).

Lemma 2.5. *[6] Let $\boldsymbol{a}, \boldsymbol{b} \in \Lambda[m,r]$, and note $\mathcal{H}(\mathfrak{S}_n)$ as the Iwahori-Hecke algebra associated with $\mathfrak{S}_n$.*

(2.5) $\quad u^+_{\boldsymbol{a}}\mathcal{H}_{n,r}u^-_{\boldsymbol{b}'} = 0$ *unless* $\boldsymbol{a} \preceq \boldsymbol{b}$,

(2.6) $\quad u^+_{\boldsymbol{a}}\mathcal{H}(\mathfrak{S}_n)u^-_{\boldsymbol{a}'} = \mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})v_{\boldsymbol{a}}$,

(2.7) $\quad u^+_{\boldsymbol{a}}\mathcal{H}_{n,r}u^-_{\boldsymbol{a}'} = u^+_{\boldsymbol{a}}\mathcal{H}(\mathfrak{S}_n)u^-_{\boldsymbol{a}'}$,

(2.8) $\quad v_{\boldsymbol{a}}\mathcal{H}_{n,r}$ *is a free R-submodule with basis* $\{v_{\boldsymbol{a}}T_w | w \in \mathfrak{S}_n\}$,

where $v_{\boldsymbol{a}} := u^+_{\boldsymbol{a}}T_{w_{\boldsymbol{a}}}u^-_{\boldsymbol{a}'}$.

Definition 2.6. *[12] For $\lambda \in \Lambda^+_{n,r}(\boldsymbol{m})$ and $\mu \in \Lambda_{n,r}(\boldsymbol{m})$, a λ-tableau of type μ denoted as T is said to be* **semistandard** *if the following hold:*

(i) the entries in each row of each component of $T^{(k)}$ of T are non-decreasing;

(ii) the entries in each column of each component $T^{(k)}$ of T are strictly increasing;

(iii) if $(a,b,c) \in \lambda$, and $T(a,b,c) = (i,s)$ then $s \geq c$.

Let $\mathcal{T}^{ss}_\mu(\lambda)$ be the set of semistandard λ-tableau of type μ and denote $\mathcal{T}^{ss}_\Lambda(\lambda) = \bigcup_{\mu \in \Lambda} \mathcal{T}^{ss}_\mu(\lambda)$. Here we use $\Lambda := \Lambda_{n,r}$ for convenience.

The set

$$\{\Psi_{ST}|S,T\in\mathcal{T}^{ss}_{\Lambda}(\lambda),\ \lambda\in\Lambda^+(n,r)\}, \tag{2.9}$$

which is called the *semi-standard* basis of cyclotomic q-Schur algebras in [3], forms a cellular basis of $\mathscr{S}_{n,r}$ in the sense of [10]. Let $\mathscr{S}^{\rhd\lambda}_{n,r}$ be the two sided ideal of $\mathscr{S}_{n,r}$ spanned by all Ψ_{ST}, where $S,T\in\mathcal{T}^{ss}_{\Lambda}(\mu)$ and $\mu\rhd\lambda$ (i.e., $\mu:=\text{shape}(S)=\text{shape}(T)\rhd\lambda$), where shape$(T)$ means the partition corresponding to tableaux T.

In particular, let $\lambda\in\Lambda^+(n,r)$ be a multipartition and recall that T^λ is the unique semistandard λ-tableau of type λ (see [3] and [11]). From the definition, one sees that $\Psi_{T^\lambda T^\lambda}$ can restrict to the identity map on M_λ, and sometimes we denote it by Ψ_λ .

With above notations, we can define the "cell module" as a submodule of $\mathscr{S}_{n,r}/\mathscr{S}^{\rhd\lambda}_{n,r}$:

$$W^\lambda=\mathscr{S}_{n,r}\bar{\Psi}_\lambda,\qquad \textit{where}\ \bar{\Psi}_\lambda:=(\mathscr{S}^{\rhd\lambda}_{n,r}+\Psi_\lambda)/\mathscr{S}^{\rhd\lambda}_{n,r}. \tag{2.10}$$

The module W^λ is called a *Weyl module* in [3].

3. Main theorem and its proof

We now prove q-Schur module given above is isomorphic to those in [3] as "cell modules" when $\lambda\in\Lambda^+_{n,r}(\mathbf{m})$. Recall the definitions given in 2.6. In order to show the next main theorem, we need demonstrate some notations and definitions which may be used in the procedure of proofs. Most of them can be found in paper [3] and book [11]:

1) Std(λ), for a partition (resp. multipartition) λ:
   It is the set consisting of all standard (semistandard) tableaux.

2) $\lambda(\mathfrak{t})$, for λ and μ are partitions (resp. multipartitions) and $\mathfrak{t}$ is a μ-tableau, which satisfies $|\lambda|=|\mu|$:
   It is a μ-tableau of type λ, which replace the components in $\mathfrak{t}$ with its row number in t^λ.

3) $m_{\mathfrak{st}}$, for $\mathfrak{s}$ and $\mathfrak{t}$ are λ-tableau and λ is a partition (resp. multipartition):
   It is an element of Iwahori (resp. Cyclotomic) Hecke algebra, which is $m_{\mathfrak{st}}:=T_{d(\mathfrak{s})^{-1}}\cdot m_\lambda\cdot T_{d(\mathfrak{t})}$ (resp. $m_{\mathfrak{st}}:=T_{d(\mathfrak{s})^{-1}}\cdot x_\lambda\cdot T_{d(\mathfrak{t})}$).

4) m_{ST}, for $S, T \in \mathcal{T}_\lambda(\mu)$ and $\lambda, \mu \in \Lambda_{n,r}(\mathbf{m})$:
It is an element of Iwahori (resp. Cyclotomic) Hecke algebra, which is $m_{ST} := \sum_{\substack{\mu(\mathfrak{s})=S \\ \mu(\mathfrak{t})=T}} m_{\mathfrak{st}}$.

Theorem 3.1. *For each $\lambda \in \Lambda^+_{n,r}(\boldsymbol{m})$, we have the following $\mathscr{S}_{n,r}$-module isomorphism:*

$$\mathcal{A}^\lambda \cong W^\lambda.$$

Proof. Consider the epimorphism:

$$\begin{aligned}\theta : \mathscr{S}_{n,r}\Psi_\lambda \longrightarrow \mathscr{S}_{n,r}z_\lambda; \quad h\Psi_\lambda \mapsto hz_\lambda &= h\varphi^1_{\lambda\omega}T_{w_\lambda}y_{\lambda'} \\ &= h\varphi^1_{\bar{\lambda}\omega} \cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} \cdot v_{[\mu]}.\end{aligned}$$

Suppose that $T \in \mathcal{T}^{ss}_\lambda(\mu)$ and $S \in \mathcal{T}^{ss}_\nu(\mu)$ with $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $\nu \in \Lambda_{n,r}(\mathbf{m})$. By the definition of Ψ_{ST} in [3] and semistandard basis theorem [3] (**6.6**), we trivially find that the set $\{\Psi_{ST} | T \in \mathcal{T}^{ss}_\lambda(\mu),\ S \in \mathcal{T}^{ss}_\nu(\mu)$ *with* $\mu \trianglerighteq \lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m}), \nu \in \Lambda_{n,r}(\mathbf{m})\}$ is a R-basis of $\mathscr{S}_{n,r}\Psi_\lambda$. More precisely, we can write this basis as

(3.1) $\{\Psi_{TT^\lambda} | T \in \mathcal{T}^{ss}_\nu(\lambda)\} \cup \{\Psi_{ST} | T \in \mathcal{T}^{ss}_\lambda(\mu)$ *and* $S \in \mathcal{T}^{ss}_\nu(\mu)$ *with* $\mu \triangleright \lambda\}$.

Then, obviously, we have that

$$W^\lambda \cong \mathscr{S}_{n,r}\Psi_\lambda / (\mathscr{S}_{n,r}\Psi_\lambda \cap \mathscr{S}^{\triangleright\lambda}_{n,r}).$$

We claim that, with $\mu \trianglerighteq \lambda$ and $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, $\nu \in \Lambda_{n,r}(\mathbf{m})$, if $\theta(\Psi_{ST}) = \theta(\Psi_{ST}\Psi_{T^\lambda T^\lambda}) = \Psi_{ST}\varphi^1_{\lambda\omega}T_{w_\lambda}y_{\lambda'} \neq 0$,then $\mu = \lambda$.

Consider the action on the unit of $\mathcal{H}_{n,r}$:

$$\begin{aligned}\Psi_{ST}\varphi^1_{\lambda\omega}T_{w_\lambda}y_{\lambda'}(1) &= m_{ST}T_{w_\lambda}y_{\lambda'} \\ &= \sum_{\substack{\mathfrak{t}\in\mathrm{Std}(\mu)\\ \lambda(\mathfrak{t})=T}} m_{S\mathfrak{t}}T_{w_\lambda}y_{\lambda'} = \sum_{\substack{\mathfrak{t}\in\mathrm{Std}(\mu)\\ \lambda(\mathfrak{t})=T}} \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\mu)\\ \nu(\mathfrak{s})=S}} m_{\mathfrak{st}}T_{w_\lambda}y_{\lambda'} \\ &= \sum_{\mathfrak{s},\mathfrak{t}} T_{d(\mathfrak{s})}x_\mu T_{d(\mathfrak{t})}T_{w_\lambda}y_{\lambda'} \\ &= \sum_{\mathfrak{s},\mathfrak{t}} T_{d(\mathfrak{s})}m_{\bar{\mu}}u^+_{[\mu]}T_{d(\mathfrak{t})}T_{w_\lambda}u^-_{[\lambda']}n_{\bar{\lambda}'} \\ &= (*).\end{aligned}$$

Recall that by Lemma 2.5, $u^+_{\mathbf{a}}\mathcal{H}_{n,r}u^-_{\mathbf{b}'} = 0$ unless $\mathbf{a} \preceq \mathbf{b}$. $\Psi_{ST}\varphi^1_{\lambda\omega}T_{w_\lambda}y_{\lambda'} \neq 0$ implies that for some $\mathfrak{s}$ and $\mathfrak{t}$ above, that $T_{d(\mathfrak{s})}x_{\bar{\mu}}u^+_{[\mu]}T_{d(\mathfrak{t})}T_{w_\lambda}u^-_{[\lambda']}y_{\bar{\lambda}'} \neq 0$. Thus, this condition shows that $[\mu] \preceq [\lambda]$. On the other hand, with the assumption in above claim, i.e., $\mu \trianglerighteq \lambda$, it is obvious that $[\mu] \succeq [\lambda]$ by the definition of $[\mu]$, $[\lambda]$ and $\trianglerighteq$, $\succeq$. So $[\mu] = [\lambda]$. Then we find

$$
\begin{aligned}
(*) \quad &= \sum_{\substack{\mathfrak{s},\mathfrak{t} \\ [\mu]=[\lambda]}} T_{d(\mathfrak{s})}m_{\bar{\mu}}u^+_{[\mu]}T_{d(\mathfrak{t})}T_{w_\lambda}u^-_{[\mu]'}n_{\bar{\lambda}'} \\
&= \sum_{\substack{\mathfrak{s},\mathfrak{t} \\ [\mu]=[\lambda] \\ h'\in\mathfrak{S}_{[\mu]}}} T_{d(\mathfrak{s})}m_{\bar{\mu}}h'v_{[\mu]}n_{\bar{\lambda}'} \qquad \text{by (2.6) and (2.7) in Lemma 2.5} \\
&= \sum_{\substack{\mathfrak{s},\mathfrak{t} \\ [\mu]=[\lambda] \\ h'_i\in\mathfrak{S}_{\{|\lambda_{i-1}|+1,\cdots,|\lambda_i|\}}}} T_{d(\mathfrak{s})}m_{\mu^{(1)}\vee\cdots\vee\mu^{(r)}}h'_1\cdots h'_m n_{\lambda^{(1)\prime}\vee\cdots\vee\lambda^{(r)\prime}}v_{[\mu]} \quad \text{by [8]} \\
&= \sum_{\substack{\mathfrak{s},\mathfrak{t} \\ [\mu]=[\lambda] \\ h'_i\in\mathfrak{S}_{\{|\lambda_{i-1}|+1,\cdots,|\lambda_i|\}}}} T_{d(\mathfrak{s})}(m_{\mu^{(1)}}h'_1n_{\lambda^{(1)\prime}})\cdots(m_{\mu^{(r)}}h'_m n_{\lambda^{(r)\prime}})v_{[\mu]} \quad .
\end{aligned}
$$

Since $[\lambda] = [\mu]$, the fact that this is non-zero implies, by [4] (**4.1**), $\lambda^{(i)} \trianglerighteq \mu^{(i)}$ for all $i = 1, \ldots, r$. On the other hand, by [8] (**1.6**), $\mu \trianglerighteq \lambda$ and $[\mu] = [\lambda]$ imply $\mu^{(i)} \trianglerighteq \lambda^{(i)}$, with $1 \leq i \leq r$. Hence $\mu^{(i)} = \lambda^{(i)}$ for all i, and therefore, $\mu = \lambda$. This completes the proof of above claim.

By the claim and the display in (3.1), one sees that

$$\ker\theta = \{\Psi_{ST} \mid T \in \mathcal{T}^{ss}_\lambda(\mu) \text{ and } S \in \mathcal{T}^{ss}_\nu(\mu) \text{ with } \mu \triangleright \lambda\} = \mathscr{S}_{n,r}\Psi_\lambda \cap \mathscr{S}^{\triangleright\lambda}_{n,r}.$$

Therefore, $\mathcal{A}^\lambda \cong W^\lambda$. □

Definition 3.2. *[4] For $w \in \mathfrak{S}_n$ and $S \in \mathcal{T}_\lambda(\mu)$ with $\lambda, \mu \in \Lambda(n, r)$, define a map*

$$\mathfrak{S}_n \times \mathcal{T}_\lambda(\mu) \longrightarrow \mathscr{D}_\lambda \qquad (w, S) \longmapsto w_S \tag{3.2}$$

where the element w_S is defined by the row-standard λ-tableau $\mathfrak{t}^\lambda w_S$ for which i belongs to the row a if the place occupied by i in $\mathfrak{t}^\mu w$ is occupied by a in S.

For example, $S = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 1&2\\\cline{1-2}\end{array}$ and $\mathfrak{t}^\mu w = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$ with $\mu = (3, 2)$ and $\lambda = (2, 2, 1)$, then $\mathfrak{t}^\lambda w_S = \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$.

Remark 3.3. *Let $\mathcal{T}^{ss}_{\lambda}(\mu)$ be the set of all semi-standard μ-tableaux of type λ, with λ and $\mu \in \Lambda_{n,r}(\boldsymbol{m})$. For any $S \in \mathcal{T}^{ss}_{\lambda}(\mu)$, we define $1_S := 1_{\bar{S}}$. Since S is a semi-standard μ-tableau of type λ, it implies that $\bar{S}$ is a row-standard $\bar{\mu}$-tableau of type $\bar{\lambda}$, as in [7].*

We compare the definition of semi-standard tableaux which appears in [3] with that in [7]. Note that every entry in S is written as the symbol (i,j) and is replaced by $i + \sum_{k=1}^{j-1} m_k$, for $1 \le i \le m_j$, $1 \le j \le n$.

Then, by the definition above, we obtain the following consequence:

Lemma 3.4. *Suppose that $u \in \mathfrak{S}_r$ and $w \in \mathfrak{S}_{\mu^{(1)'} \vee \cdots \vee \mu^{(r)'}}$, with $\lambda, \mu \in \Lambda_{n,r}(\boldsymbol{m})$. Then $\varphi^{1}_{\bar{\lambda}\omega} T_u T_w$ is a linear combination of terms $\varphi^{d}_{\bar{\lambda}\omega}$ $(d \in \mathscr{D}_{\bar{\lambda}})$ for which $\chi(t^{\bar{\lambda}}d, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)}) = \chi(t^{\bar{\lambda}}u, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)})$.*

Proof. The conclusion is ture when $w = 1$ since $\varphi^{1}_{\bar{\lambda}\omega} T_u = \varphi^{u}_{\bar{\lambda}\omega}$ for some $u \in \mathfrak{S}_n$. Below we assume that $w \neq 1$.

For some $w' \in \mathfrak{S}_n$ and some $a = (i, i+1) \in \mathfrak{S}_{\mu^{(1)'} \vee \cdots \vee \mu^{(r)'}}$, we have that $w = w'a$, and without lose generality, we can set $(i, i+1) \in \mathfrak{S}_{\mu^{(1)'}}$ satisfying:

$$w' = w'_1 \cdots w'_r,\ w = w_1 \cdots w_r \quad \text{with} \quad \begin{aligned} w'_1(i, i+1) &= w_1, \\ w_i &= w'_i \quad for\ i = 2, \cdots, r. \end{aligned}$$

By induction on length $\ell(w)$, we have $\varphi^{1}_{\bar{\lambda}\omega} T_u T_{w'}$ as a linear combination of terms $\varphi^{d}_{\bar{\lambda}\omega}$ $(d \in \mathscr{D}_{\bar{\lambda}})$ for which $\chi(t^{\bar{\lambda}}d, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)}) = \chi(t^{\bar{\lambda}}u, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)})$.

Consider

$$\varphi^{1}_{\bar{\lambda}\omega} T_u T_w = \varphi^{1}_{\bar{\lambda}\omega} T_u T_w T_a = \sum_{\chi(t^{\bar{\lambda}}d, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)}) = \chi(t^{\bar{\lambda}}u, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)})} C_d \varphi^{d}_{\bar{\lambda}\omega} T_a.$$

By [2] or [4], we have

(3.3)

$$\varphi^{d}_{\bar{\lambda}\omega} T_a = \begin{cases} q\varphi^{d}_{\bar{\lambda}\omega} & \text{if } i, i+1 \text{ belong to the same row of } t^{\bar{\lambda}}d, \\ \varphi^{da}_{\bar{\lambda}\omega} & \text{if the row index of } i \text{ in } t^{\bar{\lambda}} \text{ is less than that of } i+1, \\ q\varphi^{da}_{\bar{\lambda}\omega} + (q-1)\varphi_{\bar{\lambda}}\varphi^{d}_{\bar{\lambda}\omega} & \text{otherwise.} \end{cases}$$

Then the proof is completed through checking the formula above case by case. □

By the definition in Remark 3.3, we can show the following theorem on basis, which is the main result in this paper.

Theorem 3.5. *Suppose that $\lambda \in \Lambda^+_{n,r}(\boldsymbol{m})$. Then the q-Schur module $\mathcal{A}^\lambda$ is free as an R-module and $\{\varphi^{1_A}_{\mu\lambda} \cdot z_\lambda | A \in \mathcal{T}^{ss}_\mu(\lambda)$ and $\mu \in \Lambda_{n,r}(\boldsymbol{m})\} \subseteq \mathcal{A}^\lambda$ is a basis.*

Proof. With the help of Theorem 3.1, it is enough to show that $\{\varphi^{1_A}_{\mu\lambda} z_\lambda | A \in \mathcal{T}^{ss}_\mu(\lambda)$ *and* $\mu \in \Lambda_{n,r}(\mathbf{m})\} \subseteq \mathcal{A}^\lambda$ is R-linearly independent. We calculate the action of the element $\varphi^{1_A}_{\lambda\mu} \cdot z_\mu$ on the unit of $\mathcal{H}_{n,r}$,

$$\begin{aligned}
\varphi^{1_A}_{\lambda\mu} \cdot z_\mu(1) &= \varphi^{1_A}_{\lambda\mu} \varphi^{1}_{\mu\omega} T_{w_\mu} y_{\mu'}(1) \\
&= \varphi^{1_A}_{\lambda\mu}(x_\mu) T_{w_\mu} y_{\mu'} \\
&= \Big(\sum_{d \in \mathfrak{S}_{\bar\lambda} 1_A \mathfrak{S}_{\bar\mu}} T_d\Big) \cdot u^+_{[\mu]} T_{w_\mu} n_{\bar\mu'} u^-_{[\mu']} && \text{by [7]} \\
&= \Big(\sum_{d \in \mathfrak{S}_{\bar\lambda} 1_A \mathfrak{S}_{\bar\mu}} T_d\Big) \cdot T_{w_{(1)} \cdots w_{(r)}} u^+_{[\mu]} T_{w_{[\mu]}} u^-_{[\mu']} n_{\bar\mu'} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu}(x_{\bar\mu}) \cdot T_{w_{(1)} \cdots w_{(r)}} v_{[\mu]} n_{\mu^{(r)\prime} \vee \cdots \vee \mu^{(1)\prime}} && \text{by Lemma 2.5} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu}(x_{\bar\mu}) \cdot T_{w_{(1)} \cdots w_{(r)}} \cdot n_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \cdot v_{[\mu]} && \text{by [6]} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu}(x_{\mu^{(1)} \vee \cdots \vee \mu^{(r)}} T_{w_{(1)} \cdots w_{(r)}} n_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}}) \cdot v_{[\mu]} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu} \varphi^{1}_{\bar\mu\omega} \cdot T_{w_{(1)} \cdots w_{(r)}} n_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}}(1) \cdot v_{[\mu]}
\end{aligned}$$

Then, following from the calculation in [2], for $A, B \in \mathcal{T}_{\bar\lambda}(\bar\mu)$, we write $A \sim B$ if A and B are *row equivalent* (which has been defined in [3], i.e., if one tableau A can be changed to B by a sequence of elementary row permutations.). Thus, $\mathfrak{S}_{\bar\lambda} 1_A \mathfrak{S}_{\bar\mu} = \bigcup_{B \sim A} \mathfrak{S}_{\bar\lambda} 1_B$. In addition, if $w \in \mathfrak{S}_n$, we denote by $\overline{w}$ the unique element of $\mathfrak{S}_\lambda w \cap \mathscr{D}_\lambda$ for some $\lambda \in \Lambda(n,r)$, i.e., the shortest element in $\mathfrak{S}_\lambda w$.

$$\begin{aligned}
& \varphi^{1_A}_{\bar\lambda\bar\mu} \varphi^{1}_{\bar\mu\omega} \cdot T_{w_{(1)} \cdots w_{(r)}} n_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \\
= \; & \Big(\sum_{B \sim A} \varphi^{1_B}_{\bar\lambda\omega} T_{w_{(1)} \cdots w_{(r)}}\Big) n_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \\
= \; & \Big(\sum_{B \sim A} \varphi^{1}_{\bar\lambda\omega} T_{1_B} T_{w_{(1)} \cdots w_{(r)}}\Big) n_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \\
= \; & \Big(\sum_{B \sim A} q^{K_B} \varphi^{1}_{\bar\lambda\omega} T_{\overline{1_B w_{(1)} \cdots w_{(r)}}} + s_B\Big) \cdot n_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} && \text{by [2]}
\end{aligned}$$

where K_B is an integer and s_B is a linear combination of terms $\varphi^d_{\bar\lambda\omega}$ for which

$$\chi(t^{\bar\lambda}1_B, t^{\bar\mu}) > \chi(t^{\bar\lambda}d, t^{\bar\mu}w_{(1)}\cdots w_{(r)}).$$

Moreover, $\chi(t^{\bar\lambda}1_A, t^{\bar\mu}) > \chi(t^{\bar\lambda}1_B, t^{\bar\mu}) = \chi(t^{\bar\lambda}\overline{1_B w_{(1)}\cdots w_{(r)}}, t^{\bar\mu}1_B w_{(1)}\cdots w_{(r)})$ if $B \sim A$ but $B \neq A$. Hence

(3.4)
$$\varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^1_{\bar\mu\omega}\cdot T_{w_{(1)}\cdots w_{(r)}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = (q^K\varphi^1_{\bar\lambda\omega}T_{\overline{1_A w_{(1)}\cdots w_{(r)}}} + s)\cdot n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$$

where K is an integer and s is a linear combination of terms $\varphi^d_{\bar\lambda\omega}$ with

$$\chi(t^{\bar\lambda}1_A, t^{\bar\mu}) > \chi(t^{\bar\lambda}d, t^{\bar\mu}w_{(1)}\cdots w_{(r)}).$$

Now suppose that $\sum_A c_A\varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^1_{\bar\mu\omega}\cdot T_{w_{(1)}\cdots w_{(r)}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = 0$, where $c_A \in R$ and the sum is over $A \in \mathcal{T}^{ss}_{\bar\lambda}(\mu)$. Choose $D \in \mathcal{T}^{ss}_{\bar\lambda}(\mu)$ such that $c_A = 0$ for all A with $\chi(t^{\bar\lambda}1_A, t^{\bar\mu}) > \chi(t^{\bar\lambda}1_D, t^{\bar\mu})$. If we can prove that $c_D = 0$, it will follow that every coefficient $c_A = 0$, and then the proof is completed.

By (3.4), there exists an integer K and $s \in M^\lambda$ such that

$$\sum_A c_A\varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^1_{\bar\mu\omega}\cdot T_{w_{(1)}\cdots w_{(r)}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$$
$$= c_D q^K\varphi^1_{\bar\lambda\omega}T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} + s n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$$

where s is a linear combination of terms $\varphi^d_{\bar\lambda\omega}(d \in \mathscr{D}_{\bar\lambda})$ for which

$$\chi(t^{\bar\lambda}d, t^{\bar\mu}w_{(1)}\cdots w_{(r)}) \not\geq \chi(t^{\bar\lambda}1_D, t^{\bar\mu}). \tag{3.5}$$

Now, suppose

$$c_D q^K\varphi^1_{\bar\lambda\omega}T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} + s n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = 0$$

and by Lemma 3.4, $\varphi^1_{\bar\lambda\omega}T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$ is the linear combination of terms $\varphi^d_{\bar\lambda\omega}$ $(d \in \mathscr{D}_{\bar\lambda})$ for which $\chi(t^{\bar\lambda}d, t^{\bar\mu}w_{(1)}\cdots w_{(r)}) = \chi(t^{\bar\lambda}\overline{1_D w_{(1)}\cdots w_{(r)}}, t^{\bar\mu}w_{(1)}\cdots w_{(r)}) = \chi(t^{\bar\lambda}1_D, t^{\bar\mu})$, while $sy_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$ is a linear combination of terms $\varphi^1_{\bar\lambda\omega}(d \in \mathscr{D}_\lambda)$ for which $\chi(t^{\bar\lambda}, t^{\bar\mu}) \neq \chi(t^{\bar\lambda}1_D, t^{\bar\mu})$ by (3.5). Therefore,

$$c_D q^K\varphi^1_{\bar\lambda\omega}T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = 0.$$

But $\varphi^1_{\bar\lambda\omega}T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} \neq 0$, since the numbers strictly increase down the columns for every component of D. Therefore, $c_D = 0$, as we claimed.

Now, we have already known that the elements $\varphi^{1_A}_{\bar{\lambda}\bar{\mu}}\varphi^1_{\bar{\mu}\omega}\cdot T_{w_{(1)}\cdots w_{(r)}}n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$ is linearly independent. It implies that $\varphi^{1_A}_{\lambda\mu}\varphi^1_{\mu\omega}T_{w_\mu}y_{\mu'} = \varphi^{1_A}_{\bar{\lambda}\bar{\mu}}\varphi^1_{\bar{\mu}\omega}\cdot T_{w_{(1)}\cdots w_{(r)}}n_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}\cdot v_{[\mu]}$ are R-linearly independent, since by Lemma 2.5 it is trivial that $a\cdot v_{[\mu]}=0$ if and only if $a=0$ for any $a\in\mathcal{H}(\mathfrak{S}_r)$. □

It should be necessary to make a comparison of the basis of $\mathcal{A}^\lambda$ constructed here with that of W^λ. Via the isomorphism given in Theorem 3.1, we know indeed θ assigns the cellular basis element $\Psi_{ST}=\Psi_{ST}\Psi_{T^\lambda T^\lambda}$ to the basis element $\Psi_{ST}\cdot z_\lambda = \Psi_{ST}\cdot\varphi^1_{\lambda\omega}T_{w_\lambda}y_{\lambda'}$, where $T\in\mathcal{T}^{ss}_\lambda(\lambda)=\{T^\lambda\}$, $S\in\mathcal{T}^{ss}_\nu(\lambda), \nu\in\Lambda_{n,r}(\mathbf{m})$ and $\lambda\in\Lambda_{n,r}(\mathbf{m})^+$. Following from the notations of (5.8), (5.9) in [7], one shows that $\varphi^{1_S}_{\mu\lambda}=\Psi_{ST^\lambda}$. Therefore,

$$\theta : \Psi_{ST^\lambda} + \mathscr{S}_{n,r} \mapsto \Psi_{ST^\lambda}z_\lambda = \varphi^{1_S}_{\nu\lambda}\cdot z_\lambda.$$

4. Application to the Branch rule

In this section, by using this embedding and restriction functors introduced in [13], we give a new proof of the Branch rule in a cyclotomic q-Schur algebra of rank n to one of rank $n+1$.

From now on, throughout this paper, we argue under the following setting, most of them are from [13]:

$$\begin{aligned}
&\mathbf{m} = (m_1,\cdots,m_r) \text{ such that } m_k \geq n+1 \text{ for all } k=1,\cdots,r,\\
&\mathbf{m}' = (m_1,\cdots,m_{r-1},m_r-1),\\
&\mathscr{S}_{n+1,r} = {}_R\mathscr{S}_{n+1,r}(\Lambda_{n+1,r}(\mathbf{m})),\\
&\mathscr{S}_{n,r} = {}_R\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}')).
\end{aligned}$$

We will omit the subscript R if there is no risk to cause confusion.

Define the injective map

$$\begin{gathered}
\gamma : \Lambda_{n,r}(\mathbf{m}') \to \Lambda_{n+1,r}(\mathbf{m}),\\
(\lambda^{(1)},\cdots,\lambda^{(r-1)},\lambda^{(r)}) \mapsto (\lambda^{(1)},\cdots,\lambda^{(r-1)},\widehat{\lambda}^{(r)}),
\end{gathered}$$

where $\widehat{\lambda}^{(r)} = (\lambda^{(1)}_1,\cdots,\lambda^{(r)}_{m_r-1},1)$. Put $\Lambda^\gamma_{n+1,r}(\mathbf{m}) = \mathrm{Im}\gamma$, we have

$$\Lambda^\gamma_{n+1,r}(\mathbf{m}) = \{\mu = (\mu^{(1)},\cdots,\mu^{(r)}) \in \Lambda_{n+1,r}(\mathbf{m}) | \mu^{(r)}_{m_r} = 1\},$$

where it is defined that $\mu^{(i)} = (\mu^{(i)}_1,\cdots,\mu^{(r)}_{m_i}) \in \mathbb{Z}^{m_i}_{>0}$ for $1\leq i\leq r$.

For $\lambda \in \Lambda^+_{n+1,r}$, and $T \in \mathcal{T}^{ss}_{\Lambda}(\lambda)$, let $T \setminus (n+1)$ be the standard tableau obtained by removing the node x such that $T(x) = n+1$, and denote the shape of $T \setminus (n+1)$ by Shape$(T \setminus (n+1))$. Note that x here is a removable node of λ, and that Shape$(T \setminus (n+1)) = \lambda \setminus x$.

Proposition 4.1. *[13](*Wada inclusion*) There exists an algebra homomorphism $\iota : \mathscr{S}_{n,r} \to \mathscr{S}_{n+1,r}$ such that*

$$E^{(l)}_{(i,k)} \mapsto E^{(l)}_{(i,k)}\xi, \qquad F^{(l)}_{(i,k)} \mapsto F^{(l)}_{(i,k)}\xi, \qquad 1_\lambda \mapsto 1_{\gamma(\lambda)} \tag{4.1}$$

for $(i,k) \in \Gamma'(\boldsymbol{m}')$, $l \geq 1$, $\lambda \in \Lambda_{n,r}(\boldsymbol{m}')$, where $\xi = \sum_{\lambda \in \Lambda^{\gamma}_{n+1,r}(\boldsymbol{m})} 1_\lambda$ is an idempotent of $\mathscr{S}_{n+1,r}$. In particular, we have that $\iota(1_{\mathscr{S}_{n,r}}) = \xi$, and that $\iota(\mathscr{S}_{n,r}) \subsetneq \xi\mathscr{S}_{n+1,r}\xi$, where $1_{\mathscr{S}_{n,r}}$ is the unit element of $\mathscr{S}_{n,r}$. Moreover, ι is injective.

Define a restriction functor $\mathrm{Res}^{n+1}_n : \mathscr{S}_{n+1,r}$-mod$\to \mathscr{S}_{n,r}$-mod by

$$\mathrm{Res}^{n+1}_n = \mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}\xi, -) \cong \xi\mathscr{S}_{n+1,r} \otimes_{\mathscr{S}_{n+1,r}} -.$$

Recall that, for $\lambda \in \Lambda^+_{n+1,r}$, the q-Schur module $\mathcal{A}^\lambda$ of $\mathscr{S}_{n+1,r}$ has an free R-basis $\{\varphi^{1_A}_{\mu\lambda} z_\lambda | A \in \mathcal{T}^{ss}_\mu(\lambda), \mu \in \Lambda_{n+1,r}(\mathbf{m})\}$. From the definition, we have that

$$\mathrm{Res}^{n+1}_n(\mathcal{A}^\lambda) = \xi\mathcal{A}^\lambda.$$

Thus, $\mathrm{Res}^{n+1}_n(\mathcal{A}^\lambda)$ has an free R-basis $\{\varphi^{1_A}_{\mu\lambda} z_\lambda | A \in \mathcal{T}^{ss}_\mu(\lambda), \mu \in \Lambda^{\gamma}_{n+1,r}(\mathbf{m})\}$.

The following notations are from [8].

For a partition $\lambda = (\lambda_1, \cdots, \lambda_m)$ of n, we identify the boxes in the Young diagram $\mathcal{N}(\lambda)$ with its position coordinates. Thus,

$$\mathcal{N}(\lambda) = \{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | j \leq \lambda_i\}.$$

The elements of $\mathcal{N}(\lambda)$ will be called *nodes*. A node of the form (i, λ_i) (resp. $(i, \lambda_i + 1)$) is called *removable* (resp. *addable*) if $i = m$ or $\lambda_i > \lambda_{i+1}$ for $i \neq m$ (resp. $(i, \lambda_i) = (0,1)$ for $\lambda_1 = \cdots = \lambda_m = 1$ or $i = 1$ or $\lambda_{i-1} > \lambda_i$ if $i \neq 1$).

Let $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)})$ be an r-partition. Then its $\mathcal{N}(\lambda)$ is the union of $\mathcal{N}(\lambda^{(k)})$, $1 \leq k \leq r$. i.e., a set of nodes

$$\mathcal{N}(\lambda) = \{(i,j,k) | i,j \in \mathbb{Z}^+, j \leq \lambda^{(k)}_i,\ 1 \leq k \leq r\}.$$

A node of $\mathcal{N}(\lambda)$ is said to be *removable* (resp. *addable*) if it is a removable (resp. addable) node of $\mathcal{N}(\lambda^{(k)})$ for some k. Denote by $\mathcal{R}_\lambda$ the set of all removable nodes of $\mathcal{N}(\lambda)$. Then $N = \#\mathcal{R}_\lambda = \sum_{i=1}^{r} \#\mathcal{R}_{\lambda^{(i)}}$.

A partial ordering " $\succ$ " on $\mathcal{R}_\lambda$ will be fixed from top to bottom and from left to right, that is, it satisfies that

$$(i, j, k) \succ (i', j', k') \text{ if } k < k', \text{ or if } k = k' \text{ and } i < i'.$$

Then, we have $\mathcal{R}_\lambda = \{\mathfrak{n}_1, \cdots, \mathfrak{n}_N\}$, with the property that $\mathfrak{n}_i \succ \mathfrak{n}_j$ for $i > j$. Let $j_{\mathfrak{n}}$, $\mathfrak{n} \in \mathcal{R}_\lambda$, be the number at the node $\mathfrak{n}$ in $\mathfrak{t}_\lambda$. For example, for $\lambda = \big((31), (22), (1)\big)$, $\mathcal{R}_\lambda = \{(1, 3, 1), (2, 1, 1), (2, 2, 2), (1, 1, 3)\}$.

Also, we define a partial order $\succeq$ on $\mathbb{Z}_{>0} \times \{1, \ldots, r\}$ by

$$(i, k) \succ (i', k') \text{ if } (i, 1, k) \succ (i', 1, k').$$

In the next proposition, we use the basis $\{\varphi^{1_A}_{\mu\lambda} z_\lambda\}_{\mu\in\Lambda^{\gamma}_{n+1,r}(\mathbf{m}), A\in\mathcal{T}^{ss}_{\mu}(\lambda)}$ in q-Schur modules instead of the cellular basis in Weyl modules. By using already existing formulae, we can easily reach the following consequence:

Proposition 4.2. *Let $\lambda \in \Lambda^{+}_{n+1,r}$, $\mu \in \Lambda^{\gamma}_{n+1,r}(\boldsymbol{m})$, $A \in \mathcal{T}^{ss}_{\mu}(\lambda)$. For $(i, k) \in \Gamma'(\boldsymbol{m}')$, we have the following*

$$(4.2)\quad E_{(i,k)} \cdot \varphi^{1_A}_{\mu\lambda} z_\lambda = \sum_{\substack{B\in\mathcal{T}^{ss}_{\mu+\alpha_{(i,k)}}(\lambda) \\ shape(B\backslash(m_r,r))\trianglerighteq shape(A\backslash(m_r,r))}} r_B \varphi^{1_B}_{\mu+\alpha_{(i,k)},\lambda} z_\lambda \ (r_B \in R);$$

$$(4.3)\quad F_{(i,k)} \cdot \varphi^{1_A}_{\mu\lambda} z_\lambda = \sum_{\substack{B\in\mathcal{T}^{ss}_{\mu-\alpha_{(i,k)}}(\lambda) \\ shape(B\backslash(m_r,r))\trianglerighteq shape(A\backslash(m_r,r))}} r_B \varphi^{1_B}_{\mu-\alpha_{(i,k)},\lambda} z_\lambda \ (r_B \in R).$$

Proof. Following from the notations of (5.8), (5.9) in [7], one shows that $\varphi^{1_A}_{\mu\lambda} = \Psi_{AT^\lambda}$. On the other hand, by a general theory of cellular algebras together with Wada's paper [13] (Proposition 3.3), it implies for $(i, k) \in \Gamma'(\mathbf{m}')$,

$$(4.4)\quad E_{(i,k)} \cdot \varphi^{1_A}_{\mu\lambda} \equiv \sum_{\substack{B\in\mathcal{T}^{ss}_{\mu+\alpha_{(i,k)}}(\lambda) \\ \text{shape}(B\backslash(m_r,r))\trianglerighteq\text{shape}(A\backslash(m_r,r))}} r_B \varphi^{1_B}_{\mu+\alpha_{(i,k)},\lambda} \mod \mathscr{S}^{\rhd\lambda}_{n+1,r},$$

where $r_B \in R$.

By definitions, $z_\lambda := \varphi^{1}_{\lambda\omega} T_w y_{\lambda'}$ and $\mathscr{S}^{\rhd\lambda}_{n+1,r}$ is linearly generated by Ψ_{ST} for $S, T \in \mathcal{T}_\Lambda(\nu)$ with $\nu \rhd \lambda$. It follows that $\mathscr{S}^{\rhd\lambda}_{n+1,r} \cdot z_\lambda = 0$. On the other hand, suppose that there exists some $S, T \in \mathcal{T}_\Lambda^{ss}(\nu)$ such that $\Psi_{ST} z_\lambda \neq 0$, which means $\lambda = \nu$ due to the proof of Theorem 3.1. This contradicts the fact $\nu \rhd \lambda$. Finally, we obtain the consequence of the first statement after multiplying the element z_λ on the two sides of (4.4).

The case for $F_{(i,k)}$ with $(i,k) \in \Gamma'(\mathbf{m}')$ can be proved similarly with the above proof in the case for $E_{(i,k)}$. □

We denote the removable nodes set as $\mathcal{R}_\lambda = \{\mathfrak{n}_1, \cdots, \mathfrak{n}_N\}$, with the property that $\mathfrak{n}_i \succ \mathfrak{n}_j$ for $i > j$. By Theorem 3.5, let ${}_R M_i$ be an R-submodule of $\mathrm{Res}_n^{n+1}(\mathcal{A}^\lambda)$ spanned by

$$\{\varphi_{\mu\lambda}^{1_A} z_\lambda | A \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda) \text{ such that } A(\mathfrak{n}_j) = (m_r, r) \text{ for some } j \geq i\},$$

where we put $\mathcal{T}_\Lambda^\gamma(\lambda) := \bigcup_{\mu \in \Lambda_{n+1,r}^\gamma(\mathbf{m})} \mathcal{T}_\mu(\lambda)$. When there is no confusion about R, we also denote ${}_R M_i$ as M_i. Then we have a filtration of R-module

$$\mathrm{Res}_n^{n+1}(\mathcal{A}^\lambda) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0.$$

For $\lambda \in \Lambda_{n+1,r}^+$ and a removable node x of λ, define the semi-standard tableau $T_x^\lambda \in \mathcal{T}_\Lambda^{ss}(\lambda)$ by

$$T_x^\lambda(a,b,c) = \begin{cases} (a.c) & \text{if } (a,b,c) \neq x, \\ (m_r, r) & \text{if } (a,b,c) = x. \end{cases} \tag{4.5}$$

We see that $T_x^\lambda \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda)$, and $T_x^\lambda \setminus (m_r, r) = T^{\lambda \setminus x}$, where the tableau $T^{\lambda\setminus x}$ notes the unique element in the set $\mathcal{T}_{\lambda\setminus x}^{ss}(\lambda \setminus x)$.

From the definition, M_i/M_{i+1} has an free R-basis

$$\{\varphi_{\gamma(\mu)\lambda}^{1_A} z_\lambda + M_{i+1} | A \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda) \text{ such that } A(\mathfrak{n}_i) = (m_r, r) \text{ and } \mu \in \Lambda_{n,r}(\mathbf{m})\}.$$

For $A \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda)$ such that $A(\mathfrak{n}_i) = (m_r, r)$, we have $\mathrm{Shape}(A \setminus (m_r, r)) = \lambda \setminus \mathfrak{n}_i$ by the definition. Note that $\lambda \setminus \mathfrak{n}_j \rhd \lambda \setminus \mathfrak{n}_i$ if and only if $\mathfrak{n}_j \prec \mathfrak{n}_i$ (i.e., $j > i$). Then, by Proposition 4.2, we see that $\{M_i\}$ is a filtration of $\mathscr{S}_{n,r}$-module.

Now, we use the main result in Section 3 to give a new proof of the Branch rule of Weyl modules in [13].

Theorem 4.3. *[13] Assume that R is a field. For any $\lambda \in \Lambda_{n+1,r}^+(\boldsymbol{m})$, let $\mathfrak{n}_1, \cdots, \mathfrak{n}_k$ be the removable nodes of $\mathcal{N}(\lambda)$ counted from top to bottom,* *and define* M_t *as above for* $1 \le t \le k$*. Then, we have a filtration of* $\mathscr{S}_{n,1}$*-submodule for* $\mathcal{A}^\lambda$*:*

$$0 = M_{k+1} \subset M_k \subset \cdots \subset M_1 = \mathcal{A}^\lambda$$

with the sections of Weyl modules (or q-Schur modules): $M_t/M_{t-1} \cong W^{\lambda\setminus\mathfrak{n}_t}$*.*

Proof. First of all we set $\widehat{\mu} := \gamma(\mu)$, and consider the weight decomposition of $\mathscr{S}_{n,r}$-module $M_i/M_{i+1} = \bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} {}_{\mu}(M_i/M_{i+1}) = \bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} 1_\mu \cdot M_i/M_{i+1} = \bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} 1_{\widehat{\mu}}(M_i/M_{i+1})$, where $1_{\widehat{\mu}}(M_i/M_{i+1})$ is generated by

$$\{\varphi^{1_A}_{\widehat{\mu}\lambda} z_\lambda + M_{i+1} | A \in \mathcal{T}^\gamma_\Lambda(\lambda) \cap \mathcal{T}^{ss}_\Lambda(\lambda) \text{ such that } A(\mathfrak{n}_i) = (m_r, r)\}.$$

Since $A \setminus (m_r, r) \in \mathcal{T}^{ss}_\mu(\lambda \setminus \mathfrak{n}_i)$, we can find that ${}_\mu(M_i/M_{i+1}) \neq 0$ only if $\lambda \trianglerighteq \widehat{\mu}$, which implies that $\lambda \setminus \mathfrak{n}_i \trianglerighteq \mu$.

Let $\mathfrak{n}_i = (a, b, c)$. Note that $E_{(j,l)} \cdot \varphi^{1_A}_{\widehat{\mu}\lambda} z_\lambda$ is a linear combination of $\{\varphi^{1_B}_{\widehat{\mu}+\alpha_{(j,l)},\lambda} z_\lambda | B \in \mathcal{T}^{ss}_{\widehat{\mu}+\alpha_{(j,l)}}(\lambda)\}$ and that $\mathcal{T}^{ss}_{\widehat{\mu}+\alpha_{(j,l)}}(\lambda) = \emptyset$ unless $\lambda \trianglerighteq \widehat{\mu} + \alpha_{(j,l)}$.

We have $T^\lambda_{\mathfrak{n}_i} \in \mathcal{T}^{ss}_\tau(\lambda)$ in the case of $\tau := \widehat{\lambda \setminus \mathfrak{n}_i}$, i.e., $\tau = \lambda - (\alpha_{(a,c)} + \alpha_{(a+1,c)} + \cdots + \alpha_{(m_r-1,r)})$.

If $(j,l) \succ (a,c)$, we have $E_{(j,l)} \cdot \varphi^{1_A}_{\tau\lambda} z_\lambda = 0$ since $\lambda \not\trianglerighteq \tau + \alpha_{(j,l)}$ for any $A \in \mathcal{T}^{ss}_\tau(\lambda)$.

If $(j,l) \preceq (a,c)$, for any $S \in \mathcal{T}^{ss}_{\tau+\alpha_{(j,l)}}(\lambda)$ together with the definition of semi-standard tableaux, we can easily check that $S\big((a',b',c')\big) \succeq (j,l)$ for any $(a',b',c') \in \lambda$ satisfying $(a',c') \succeq (j,l)$. This implies that

$$|S \setminus (m_r, r)| \neq |\lambda \setminus \mathfrak{n}_i| \text{ for any } S \in \mathcal{T}^{ss}_{\tau+\alpha_{(j,l)}}(\lambda), \tag{4.6}$$

since $(a,c) \succeq (j,l)$ and $T^\lambda_{\mathfrak{n}_i}\big((a,b,c)\big) = (m_r, r) \preceq (j,l)$. From now on, we note the tableau $T^\lambda_{\mathfrak{n}_i}$ as X.

Thus, Proposition 4.2 together with (4.6) implies that

$$E_{(j,l)} \cdot \varphi^{1_X}_{\tau\lambda} \cdot z_\lambda = 0 \in M_{i+1} \text{ for any } (j,l) \in \Gamma'(\mathbf{m}').$$

Thus, $\varphi^{1_X}_{\tau\lambda} \cdot z_\lambda + M_{i+1}$ is a highest weight vector of weight $\lambda \setminus \mathfrak{n}_i$ of $\mathscr{S}_{n,r}$-module in sense of [14]. Moreover, since the Weyl modules are simple modules in category of ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-modules, due to the universality of Weyl modules in [14], we have an ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-isomorphism:

$$\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i}: \quad {}_{\mathcal{K}}\mathcal{A}^{\lambda\backslash\mathfrak{n}_i} \to {}_{\mathcal{K}}\mathscr{S}_{n,r} \cdot (\varphi_{\tau\lambda}^{1_X} \cdot z_\lambda) + {}_{\mathcal{K}}M_{i+1}. \tag{4.7}$$

Note that $\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i}$ is determined by $\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i}(\varphi^1_{\lambda\backslash\mathfrak{n}_i\lambda\backslash\mathfrak{n}_i} \cdot z_{\lambda\backslash\mathfrak{n}_i}) = \varphi_{\tau\lambda}^{1_X} \cdot z_\lambda + {}_{\mathcal{K}}M_{i+1}$. We see that $\theta_{\mathcal{A}}^{\lambda\backslash\mathfrak{n}_i}$ is a restriction of $\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i}$ which assigns the submodule ${}_{\mathcal{A}}\mathcal{A}^{\lambda\backslash\mathfrak{n}_i}$ onto the submodule ${}_{\mathcal{A}}\mathscr{S}_{n,r} \cdot (\varphi_{\tau\lambda}^{1_X} \cdot z_\lambda) + {}_{\mathcal{A}}M_{i+1}$. Then, we find that $\theta_{\mathcal{A}}^{\lambda\backslash\mathfrak{n}_i}$ is an isomorphism of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$-modules. Furthermore, by the argument of specialization to any arbitrary commutative ring, it follows that $\theta_R^{\lambda\backslash\mathfrak{n}_i} := \theta_{\mathcal{A}}^{\lambda\backslash\mathfrak{n}_i} \otimes_{\mathcal{A}} R$ is an isomorphism for the algebra ${}_R\mathscr{S}_{n,r}$.

R is assumed to be a field. Since $W^{\lambda\backslash\mathfrak{n}_i} \cong \mathcal{A}^{\lambda\backslash\mathfrak{n}_i} \cong {}_R\mathscr{S}_{n,r} \cdot (\varphi_{\tau\lambda}^{1_X} \cdot z_\lambda) + {}_RM_{i+1}$, which is a ${}_R\mathscr{S}_{n,r}$-submodule of M_i/M_{i+1}, we finally reach the consequence by comparing the dimensions of $\mathcal{A}^{\lambda\backslash\mathfrak{n}_i}$ and M_i/M_{i+1}. □

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