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Crystals and D-Modules

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To the memory of A. Todorov.

Abstract: The goal of this paper is to develop the notion of crystal in the context of derived algebraic geometry, and to connect crystals to more classical objects such as D-modules.
Keywords: D-modules, crystals, derived algebraic geometry.

Contents

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Introduction

0.1. **Flat connections, D-modules and crystals.**

0.1.1. Let M be a smooth manifold with a vector bundle V. Recall that a flat connection on V is a map

$$\nabla : V \to V \otimes \Omega^1_M$$

satisfying the Leibniz rule, and such that the curvature $[\nabla, \nabla] = 0$. Dualizing the connection map, we obtain a map

$$T_M \otimes V \to V.$$

The flatness of the connection implies that this makes V into a module over the Lie algebra of vector fields. Equivalently, we obtain that V is module over the algebra Diff_M of differential operators on M.

This notion generalizes immediately to smooth algebraic varieties in characteristic zero. On such a variety a D-module is defined as a module over the sheaf of differential operators which is quasi-coherent as an $\mathcal{O}$-module. The notion of D-module on an algebraic variety thus generalizes the notion of vector bundle with a flat connection, and encodes the data of a system of linear differential equations with polynomial coefficients. The study of D-modules on smooth algebraic varieties is a very rich theory, with applications to numerous fields such as representation theory. Many of the ideas from the differential geometry of vector bundles with a flat connection carry over to this setting.

However, the above approach to D-modules presents a number of difficulties. For example, one needs to consider sheaves with a flat connection on singular schemes in addition to smooth ones. While the algebra of differential operators is well-defined on a singular variety, the category of modules over it is not *the* category that we are interested in (e.g., the algebra in question is not in general Noetherian). In another direction, even for a smooth algebraic variety, it is not clear how to define connections on objects that are not linear, e.g., sheaves of categories.

0.1.2. *Parallel transport.* The idea of a better definition comes from another interpretation of the notion of flat connection on a vector bundle in the context of differential geometry, namely, that of parallel transport:

Given a vector bundle with a flat connection V on a smooth manifold M, and a path $\gamma : [0, 1] \to M$, we obtain an isomorphism

$$\Pi_\gamma : V_{\gamma(0)} \simeq V_{\gamma(1)}$$

of the fibers of V at the endpoints, which only depends on the homotopy class of the path. We can rephrase this construction as follows. Let $B \subset M$ be a small ball inside M. Since the parallel transport isomorphism only depends on the homotopy class of the path, and B is contractible, we obtain a coherent identification of fibers of V

$$V_x \simeq V_y$$

for points $x, y \in B$ by considering paths inside B. So, roughly, the data of a connection gives an identification of fibers at "nearby" points of the manifold.

Building on this idea, Grothendieck [Gr] gave a purely algebraic analogue of the notion of parallel transport, using the theory of schemes (rather than just varieties) in an essential way: he introduced the relation of infinitesimal closeness for R-points of a scheme X. Namely, two R-points $x, y : \mathrm{Spec}(R) \to X$ are infinitesimally close if the restrictions to $\mathrm{Spec}({}^{red}R)$ agree, where ${}^{red}R$ is the quotient of R by its nilradical.

A *crystal* on X is by definition a quasi-coherent sheaf on X which is equivariant with respect to the relation of infinitesimal closeness. More preciesly, a crystal on X is a quasi-coherent sheaf $\mathcal{F}$ with the additional data of isomorphisms

$$x^*(\mathcal{F}) \simeq y^*(\mathcal{F})$$

for any two infinitesimally close points $x, y : \mathrm{Spec}(R) \to X$ satisfying a cocycle condition.

Grothendieck showed that on a smooth algebraic variety, the abelian category of crystals is equivalent to that of left modules over the ring of differential operators. In this way, crystals give a more fundamental definition of sheaves with a flat connection.

A salient feature of the category of crystals is that Kashiwara's lemma is built into its definition: for a closed embedding of schemes $i : Z \to X$, the category of crystals on Z is equivalent to the category of crystals on X, which are set-theoretically supported on Z. This observation allows us to reduce the study of crystals on schemes to the case of smooth schemes, by (locally) embedding a given scheme into a smooth one.

0.1.3. In this paper, we develop the theory of crystals in the context of derived algebraic geometry, where instead of ordinary rings one considers derived rings, i.e., E_∞ ring spectra. Since we work over a field k of characteristic zero, we shall use connective commutative DG k-algebras as our model of derived rings (accordingly, we shall use the term "DG scheme" rather than "derived scheme"). The key idea is that one should regard higher homotopy groups of a derived ring as a generalization of nilpotent elements.

Thus, following Simpson [Si], for a DG scheme X, we define its de Rham prestack X_{dR} to be the functor

$$X_{\mathrm{dR}} : R \mapsto X({}^{red,cl}R)$$

on the category of derived rings R, where

$${}^{red,cl}R := {}^{red}(\pi_0(R))$$

is the reduced ring corresponding to the underlying classical ring of R. I.e., X_{dR} is a *prestack* in the terminology of [GL:Stacks].

We define crystals on X as quasi-coherent sheaves on the prestack X_{dR}. See, [Lu1, Sect. 2] for the theory of quasi-coherent sheaves in prestacks, or [GL:QCoh, Sect. 1.1] for a brief review.

The above definition does not coincide with one of Grothendieck mentioned earlier: the latter specifies a map $\mathrm{Spec}(R) \to X$ up to an equivalence relation, and the former only a map $\mathrm{Spec}({}^{red,cl}R) \to X$. However, we will show that for X which is *eventually coconnective*, i.e., if its structure ring has only finitely many non-zero homotopy groups, the two definitions of a crystal are equivalent. [1]

0.1.4. Even though the category of crystals is equivalent to that of D-modules, it offers a more flexible framework in which to develop the theory. The definition immediately extends to non-smooth schemes, and the corresponding category is well-behaved (for instance, the category of crystals on any scheme is locally Noetherian).

Let $f : X \to Y$ be a map of DG schemes. We will construct the natural pullback functor

$$f^\dagger : \mathrm{Crys}(Y) \to \mathrm{Crys}(X).$$

In fact, we shall extend the assignment $X \mapsto \mathrm{Crys}(X)$ to a functor from the category $\mathrm{DGSch}^{\mathrm{op}}$ to that of stable ∞-categories. The latter will enable us to define crystals not just on DG schemes, but on arbitrary prestacks.

[1] When X is not eventually coconnective, the two notions are different, and the correct one is the one via X_{dR}.

Furthermore, the notion of crystal immediately extends to a non-linear and categorified setting. Namely, we can just as well define a crystal of schemes or a crystal of categories over X.

0.2. **Left crystals vs. right crystals.**

0.2.1. Recall that on a smooth algebraic variety X, in addition to usual (i.e., left) D-modules, one can also consider the category of right D-modules. The two categories are equivalent: the corresponding functor is given by tensoring with the dualizing line bundle ω_X over the ring of functions. However, this equivalence does not preserve the forgetful functor to quasi-coherent sheaves. For this reason, we can consider an abstract category of D-modules, with two different realization functors to quasi-coherent sheaves. In the left realization, the D-module pullback functor becomes the $*$-pullback functor on quasi-coherent sheaves, and in the right realization, it becomes the !-pullback functor.

It turns out that the "right" realization has several advantages over the "left" one. Perhaps the main advantage is that the "right" realization endows the category of D-modules with a t-structure with very favorable functorial properties. In particular, this t-structure becomes the perverse t-structure under the Riemann-Hilbert correspondence.

0.2.2. One can then ask whether there are also "left" and "right" crystals on arbitrary DG schemes. It turns out that indeed both categories are defined very generally.

Left crystals are what we defined in Sect. 0.1.3. However, in order to define right crystals, we need to replace the usual category of quasi-coherent sheaves by its renormalized version, the category of ind-coherent sheaves introduced in [IndCoh].

The category IndCoh(X) is well-behaved for (derived) schemes that are (almost) locally of finite type, so right crystals will only be defined on DG schemes, and subsequently, on prestacks with this property.

Let us recall from [IndCoh, Sect. 5] that for a map $f : X \to Y$ between DG schemes, we have the !-pullback functor

$$f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X).$$

The assignment $X \mapsto \mathrm{IndCoh}(X)$ is a functor from the category $\mathrm{DGSch}^{\mathrm{op}}$ to that of stable ∞-categories and thus can be extended to a functor out of the category of prestacks.

For a DG scheme X, we define the category of right crystals $\mathrm{Crys}^r(X)$ as $\mathrm{IndCoh}(X_{\mathrm{dR}})$. We can also reformulate this definition à la Grothendieck by saying that a right crystal on X is an object $\mathcal{F} \in \mathrm{IndCoh}(X)$, together with an identification

$$x^!(\mathcal{F}) \simeq y^!(\mathcal{F}) \tag{0.1}$$

for every pair of infinitesimally close points $x, y : \mathrm{Spec}(R) \to X$ satisfying (the ∞-category version of) the cocycle condition. It can be shown that, unlike in the case of left crystals, this does give an equivalent definition of right crystals without any coconnectivity assumptions.

0.2.3. Now that the category of right crystals is defined, we can ask whether it is equivalent to that of left crystals. The answer also turns out to be "yes." Namely, for any DG scheme X almost of finite type, tensoring by the dualizing complex ω_X defines a functor

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$$

that intertwines the $*$-pullback on quasi-coherent sheaves and the !-pullback on ind-coherent sheaves.

Although the functor Υ_S is not an equivalence for an individual S unless S is smooth, the totality of such maps for DG schemes mapping to the de Rham prestack of X define an equivalence between left and right crystals.

Thus, just as in the case of smooth varieties, to each DG scheme X we attach the category $\mathrm{Crys}(X)$ equipped with two "realization" functors

$$\begin{tikzcd} & \mathrm{Crys}(X). \arrow[dl, "\mathbf{oblv}^l_X"'] \arrow[dr, "\mathbf{oblv}^r_X"] & \\ \mathrm{QCoh}(X) \arrow[rr, "\Upsilon_X"] & & \mathrm{IndCoh}(X) \end{tikzcd}$$

However, in the case of non-smooth schemes, the advantages of the t-structure on $\mathrm{Crys}(X)$ that is associated with the "right" realization become even more pronounced.

0.2.4. *Historical remark.* To the best of our knowledge, the approach to D-modules via right crystals was first suggested by A. Beilinson in the early 90's, at the level of abelian categories.

For some time after that it was mistakenly believed that one cannot use left crystals to define D-modules, because of the incompatibility of the t-structures. However, it was explained by J. Lurie, that if one forgoes the t-structure and defines the corresponding stable ∞-category right away, left crystals work just as well.

0.3. **The theory of crystals/D-modules.** Let us explain the formal structure of the theory, as developed in this paper, and its sequel [GR2].

0.3.1. To each prestack (locally almost of finite type) $\mathcal{Y}$, we assign a stable ∞-category

$$\mathcal{Y} \rightsquigarrow \mathrm{Crys}(\mathcal{Y}).$$

This category has two realization functors: a left realization functor to $\mathrm{QCoh}(\mathcal{Y})$, and a right realization functor to $\mathrm{IndCoh}(\mathcal{Y})$ which are related via the following commutative diagram

$$
\begin{tikzcd}
& \mathrm{Crys}(\mathcal{Y}). \arrow[dl, "\mathbf{oblv}^l_{\mathcal{Y}}"'] \arrow[dr, "\mathbf{oblv}^r_{\mathcal{Y}}"] & \\
\mathrm{QCoh}(\mathcal{Y}) \arrow[rr, "\Upsilon_{\mathcal{Y}}"] & & \mathrm{IndCoh}(\mathcal{Y})
\end{tikzcd}
$$

where $\Upsilon_{\mathcal{Y}}$ is the functor $\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$ given by tensoring by the dualizing complex $\omega_{\mathcal{Y}}$.

0.3.2. The assignment of $\mathrm{Crys}(\mathcal{Y})$ to $\mathcal{Y}$ is functorial in a number of ways. For a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, there is a pullback functor

$$f^\dagger : \mathrm{Crys}(\mathcal{Y}_2) \to \mathrm{Crys}(\mathcal{Y}_1)$$

which is functorial in f; i.e., this assignment gives a functor

$$\mathrm{Crys}^\dagger_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

The pullback functor on D-modules is compatible with the realization functors. Namely, we have commutative diagrams

$$
\begin{tikzcd}
\mathrm{Crys}(\mathcal{Y}_1) \arrow[d, "\mathbf{oblv}^l_{\mathcal{Y}_1}"'] & \mathrm{Crys}(\mathcal{Y}_2) \arrow[l, "f^\dagger"'] \arrow[d, "\mathbf{oblv}^l_{\mathcal{Y}_2}"] \\
\mathrm{QCoh}(\mathcal{Y}_1) & \mathrm{QCoh}(\mathcal{Y}_2) \arrow[l, "f^*"']
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathrm{Crys}(\mathcal{Y}_1) \arrow[d, "\mathbf{oblv}^r_{\mathcal{Y}_1}"'] & \mathrm{Crys}(\mathcal{Y}_2) \arrow[l, "f^\dagger"'] \arrow[d, "\mathbf{oblv}^r_{\mathcal{Y}_2}"] \\
\mathrm{IndCoh}(\mathcal{Y}_1) & \mathrm{IndCoh}(\mathcal{Y}_2). \arrow[l, "f^!"']
\end{tikzcd}
$$

Furthermore, this compatibility is itself functorial in f; i.e. we have a naturally commutative diagram of functors

$$
\begin{tikzcd}
& \mathrm{Crys}^\dagger_{\mathrm{PreStk}} \arrow[dl, "\mathbf{oblv}^l"'] \arrow[dr, "\mathbf{oblv}^r"] & \\
\mathrm{QCoh}^*_{\mathrm{PreStk}} \arrow[rr, "\Upsilon"] & & \mathrm{IndCoh}^!_{\mathrm{PreStk}}
\end{tikzcd}
\quad .
$$

0.3.3. The above portion of the theory is constructed in the present paper. I.e., this paper is concerned with the assignment

$$\mathcal{Y} \rightsquigarrow \mathrm{Crys}(\mathcal{Y})$$

and the operation of pullback. Thus, in this paper, we develop the local theory of crystals/D-modules.

However, in addition to the functor $f^\dagger$, we expect to also have a pushforward functor $f_{\mathrm{dR},*}$, and the two must satisfy various compatibility relations. The latter will be carried out in [GR2]. However, let us indicate the main ingredients of the combined theory:

0.3.4. For a schematic quasi-compact map between prestacks $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, there is the de Rham pushforward functor

$$f_{\mathrm{dR},*} : \mathrm{Crys}(\mathcal{Y}_1) \to \mathrm{Crys}(\mathcal{Y}_2)$$

which is functorial in f. This assignment gives another functor

$$(\mathrm{Crys}_{\mathrm{dR},*})_{\mathrm{PreStk}_{\text{sch-qc}}} : \mathrm{PreStk}_{\text{sch-qc}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

where $\mathrm{PreStk}_{\text{sch-qc}}$ is the non-full subcategory of PreStk obtained by restricting 1-morpisms to schematic quasi-compact maps.

Let $\mathcal{Y} = X$ be a DG scheme[2]. In this case, the forgetful functor

$$\mathbf{oblv}^r_{\mathcal{Y}} : \mathrm{Crys}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$$

admits a left adjoint, denoted

$$\mathbf{ind}^r_{\mathcal{Y}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Crys}(\mathcal{Y}),$$

and called the induction functor.

The induction functor is compatible with de Rham pushforward. Namely, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathcal{Y}_1) \arrow[r, "f_*^{\mathrm{IndCoh}}"] \arrow[d, "\mathbf{ind}^r_{\mathcal{Y}_1}"'] & \mathrm{IndCoh}(\mathcal{Y}_2) \arrow[d, "\mathbf{ind}^r_{\mathcal{Y}_2}"] \\
\mathrm{Crys}(\mathcal{Y}_1) \arrow[r, "f_{\mathrm{dR},*}"] & \mathrm{Crys}(\mathcal{Y}_2).
\end{tikzcd}$$

[2]More generally, we can let $\mathcal{Y}$ be a prestack that *admits deformation theory.*

This compatibility is itself functorial, i.e. we have a natural transformation of functors

$$(\mathrm{IndCoh}_*)_{\mathrm{PreStk}_{\text{sch-qc}}} \xrightarrow{\mathbf{ind}^r} (\mathrm{Crys}_{\mathrm{dR},*})_{\mathrm{PreStk}_{\text{sch-qc}}}.$$

0.3.5. In the case when f is proper, the functors $(f_{\mathrm{dR},*}, f^\dagger)$ form an adjoint pair, and if f is smooth, the functors $(f^\dagger[-2n], f_{\mathrm{dR},*})$ form an adjoint pair for n the relative dimension of f.

In general, the two functors are not adjoint, but they satisfy a base change formula. As explained in [IndCoh, Sect. 5.1], a way to encode the functoriality of the base change formula is to consider a category of correspondences. Namely, let $(\mathrm{PreStk})_{\text{corr:all,sch-qc}}$ be the ∞-category whose objects are prestacks locally of finite type and morphisms from $\mathcal{Y}_1$ to $\mathcal{Y}_2$ are given by correspondences

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{Y}_1 \\ \mathcal{Y}_2 & \end{tikzcd}$$

such that f is schematic and quasi-compact, and g arbitrary. Composition in this category is given by taking Cartesian products of correspondences. A coherent base change formula for the functors $\mathrm{Crys}^\dagger$ and $\mathrm{Crys}_{\mathrm{dR},*}$ is then a functor

$$\mathrm{Crys}_{(\mathrm{PreStk})_{\text{corr:all,sch-qc}}} : (\mathrm{PreStk})_{\text{corr:all,sch-qc}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and an identification of the restriction to $(\mathrm{PreStk})^{\mathrm{op}}$ with $\mathrm{Crys}^\dagger_{\mathrm{PreStk}}$, and the restriction to $\mathrm{PreStk}_{\text{sch-qc}}$ with $(\mathrm{Crys}_{\mathrm{dR},*})_{\mathrm{PreStk}_{\text{sch-qc}}}$.

0.4. **Twistings.**

0.4.1. In addition to D-modules, it is often important to consider twisted D-modules. For instance, in representation theory, the localization theorem of Beilinson and Bernstein identifies the category of representations of a reductive Lie algebra $\mathfrak{g}$ with fixed central character χ with the category of twisted D-modules on the flag variety G/B, with the twisting determined by χ.

In the case of smooth varieties, the theory of twistings and twisted D-modules was introduced by Beilinson and Bernstein [BB]. Important examples of twistings are given by complex tensor powers of line bundles. For a smooth variety X, twistings form a Picard groupoid, which can be described as follows. Let $\mathcal{T}$ be the complex of sheaves, in degrees 1 and 2, given by

$$\mathcal{T} := \Omega^1 \to \Omega^{2,cl}$$

where Ω^1 is the sheaf of 1-forms on X, $\Omega^{2,cl}$ is the sheaf of closed 2-forms and the map is the de Rham differential. Then the space of objects of the Picard groupoid of twistings is given by $H^2(X, \mathcal{T})$ and, for a given object, the space of isomorphisms is $H^1(X, \mathcal{T})$.

0.4.2. The last two sections of this paper are concerned with developing the theory of twistings and twisted crystals in the derived (and, in particular, non-smooth) context. We give several equivalent reformulations of the notion of twisting and show that they are equivalent to that defined in [BB] in the case of smooth varieties.

For a prestack (almost locally of finite type) $\mathcal{Y}$, we define a twisting to be a $\mathbb{G}_m$-gerbe on the de Rham prestack $\mathcal{Y}_{\mathrm{dR}}$ with a trivialization of its pullback to $\mathcal{Y}$. A line bundle $\mathcal{L}$ on $\mathcal{Y}$ gives a twisting which is the trivial gerbe on $\mathcal{Y}_{\mathrm{dR}}$, but the trivialization on $\mathcal{Y}$ is given by $\mathcal{L}$.

Given a twisting T, the category of T-twisted crystals on $\mathcal{Y}$ is defined as the category of sheaves (ind-coherent or quasi-coherent) on $\mathcal{Y}_{\mathrm{dR}}$ twisted by the $\mathbb{G}_m$-gerbe given by T.

0.5. **Contents.** We now describe the contents of the paper, section-by-section.

0.5.1. In Section 1, for a prestack $\mathcal{Y}$, we define the de Rham prestack $\mathcal{Y}_{\mathrm{dR}}$ and establish some of its basic properties. Most importantly, we show that if $\mathcal{Y}$ is locally almost of finite type then so is $\mathcal{Y}_{\mathrm{dR}}$.

0.5.2. In Section 2, we define left crystals as quasi-coherent sheaves on the de Rham prestack and, in the locally almost of finite type case, right crystals as ind-coherent sheaves on the de Rham prestack. The latter is well-defined because, as established in Section 1, for a prestack locally almost of finite type its de Rham prestack is also locally almost of finite type. In this case, we show that the categories of left and right crystals are equivalent. Furthermore, we prove a version of Kashiwara's lemma in this setting.

0.5.3. In Section 3, we show that the category of crystals satisfies h-descent (and in particular, fppf descent). We also introduce the infinitesimal groupoid of a prestack $\mathcal{Y}$ as the Čech nerve of the natural map $\mathcal{Y} \to \mathcal{Y}_{\mathrm{dR}}$. Specifically, the infinitesimal groupoid of $\mathcal{Y}$ is given by

$$(\mathcal{Y} \times \mathcal{Y})^\wedge_{\mathcal{Y}} \rightrightarrows \mathcal{Y}$$

where $(\mathcal{Y} \times \mathcal{Y})^\wedge_{\mathcal{Y}}$ is the formal completion of $\mathcal{Y} \times \mathcal{Y}$ along the diagonal.

In much of Section 3, we specialize to the case that $\mathcal{Y}$ is an indscheme. Sheaves on the infinitesimal groupoid of $\mathcal{Y}$ are sheaves on $\mathcal{Y}$ which are equivariant with respect to the equivalence relation of infinitesimal closeness. In the case of ind-coherent sheaves, this category is equivalent to right crystals. However, quasi-coherent sheaves on the infinitesimal groupoid are, in general, not equivalent to left crystals. We show that quasi-coherent sheaves on the infinitesimal groupoid of $\mathcal{Y}$ are equivalent to left crystals if $\mathcal{Y}$ is an eventually coconnective DG scheme or a classically formally smooth prestack. Thus, in particular, this equivalence holds in the case of classical schemes.

We also define induction functors from $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(\mathcal{Y})$ to crystals on $\mathcal{Y}$. In the case of ind-coherent sheaves the induction functor is left adjoint to the forgetful functor, and we have that the category of right crystals is equivalent to the category of modules over the corresponding monad. The analogous result is true for QCoh and left crystals in the case that $\mathcal{Y}$ is an eventually coconnective DG scheme.

0.5.4. In Section 4, we show that the category of crystals has two natural t-structures: one compatible with the left realization to QCoh and another comaptible with the right realization to IndCoh. In the case of a quasi-compact DG scheme, the two t-structures differ by a bounded amplitude.

We also show that for an affine DG scheme, the category of crystals is equivalent to the derived category of its heart with respect to the right t-structure.

0.5.5. In Section 5 we relate the monad acting on IndCoh (resp., QCoh) on a DG scheme, responsible for the category of right (resp., left) crystals, to the sheaf of differential operators.

As a result, we relate the category of crystals to the derived category of D-modules.

0.5.6. In Section 6, we define the Picard groupoid of twistings on a prestack $\mathcal{Y}$ as that of $\mathbb{G}_m$-gerbes on the de Rham prestack $\mathcal{Y}_{\mathrm{dR}}$ which are trivialized on $\mathcal{Y}$. We then give several equivalent reformulations of this definition. For instance, using a version of the exponential map, we show that the Picard groupoid of twistings is equivalent to that of $\mathbb{G}_a$-gerbes on the de Rham prestack $\mathcal{Y}_{\mathrm{dR}}$ which are trivialized on $\mathcal{Y}$. In particular, this naturally makes twistings a k-linear Picard groupoid.

Furthermore, using the description of twistings in terms of $\mathbb{G}_a$-gerbes, we identify the ∞-groupoid of twistings as

$$\tau^{\leq 2}\left(H_{\mathrm{dR}}(\mathcal{Y}) \underset{H(\mathcal{Y})}{\times} \{*\}\right)[2]$$

where $H_{\mathrm{dR}}(Y)$ is the de Rham cohomology of $\mathcal{Y}$, and $H(\mathcal{Y})$ is the coherent cohomology of Y. In particular, for a smooth classical scheme, this shows that this notion of twisting agrees with that defined in [BB].

Finally, we show that the category of twistings on a DG (ind)scheme $\mathcal{X}$ locally of finite type can be identified with that of central extensions of its infinitesimal groupoid.

0.5.7. In Section 7, we define the category of twisted crystals and establish its basic properties. In particular, we show that most results about crystals carry over to the twisted setting.

0.6. **Conventions and notation.** Our conventions follow closely those of [GR1]. Let us recall the most essential ones.

0.6.1. *The ground field.* Throughout the paper we will work over a fixed ground field k of characteristic 0.

0.6.2. ∞-*categories.* By an ∞-category we shall always mean an $(\infty, 1)$-category. By a slight abuse of language, we will sometimes refer to "categories" when we actually mean ∞-categories. Our usage of ∞-categories is model independent, but we have in mind their realization as quasi-categories. The basic reference for ∞-categories as quasi-categories is [Lu0].

We denote by ∞-Grpd the ∞-category of ∞-groupoids, which is the same as the category $\mathcal{S}$ of spaces in the notation of [Lu0].

For an ∞-category $\mathbf{C}$, and $x, y \in \mathbf{C}$, we shall denote by $\mathrm{Maps}_{\mathbf{C}}(x, y) \in \infty\text{-Grpd}$ the corresponding mapping space. By $\mathrm{Hom}_{\mathbf{C}}(x, y)$ we denote the set $\pi_0(\mathrm{Maps}_{\mathbf{C}}(x, y))$, i.e., what is denoted $\mathrm{Hom}_{h\mathbf{C}}(x, y)$ in [Lu0].

A stable ∞-category $\mathbf{C}$ is naturally enriched in spectra. In this case, for $x, y \in \mathbf{C}$, we shall denote by $\mathcal{M}aps_{\mathbf{C}}(x, y)$ the spectrum of maps from x to y. In particular, we have that $\mathrm{Maps}_{\mathbf{C}}(x, y) = \Omega^\infty \mathcal{M}aps_{\mathbf{C}}(x, y)$.

When working in a fixed ∞-category $\mathbf{C}$, for two objects $x, y \in \mathbf{C}$, we shall call a point of $\mathrm{Maps}_{\mathbf{C}}(x, y)$ an *isomorphism* what is in [Lu0] is called an *equivalence.* I.e., an isomorphism is a map that admits a homotopy inverse. We reserve the word "equivalence" to mean a (homotopy) equivalence between ∞-categories.

0.6.3. *DG categories.* Our conventions regarding DG categories follow [IndCoh, Sect. 0.6.4]. By a DG category we shall understand a presentable DG category over k; in particular, all our DG categories will be assumed cocomplete. Unless specified otherwise, we will only consider continuous functors between DG categories (i.e., exact functors that commute with direct sums, or equivalently, with all colimits). In other words, we will be working in the category $\mathrm{DGCat}_{\mathrm{cont}}$ in the notation of [GL:DG]. [3]

We let Vect denote the DG category of complexes of k-vector spaces. The category $\mathrm{DGCat}_{\mathrm{cont}}$ has a natural symmetric monoidal structure, for which Vect is the unit.

[3] One can replace $\mathrm{DGCat}_{\mathrm{cont}}$ by (the equivalent) $(\infty, 1)$-category of stable presentable ∞-categories tensored over Vect, with colimit-preserving functors.

For a DG category $\mathbf{C}$ equipped with a t-structure, we denote by $\mathbf{C}^{\leq n}$ (resp., $\mathbf{C}^{\geq m}$, $\mathbf{C}^{\leq n, \geq m}$) the corresponding full subcategory of $\mathbf{C}$ spanned by objects x, such that $H^i(x) = 0$ for $i > n$ (resp., $i < m$, $(i > n) \wedge (i < m)$). The inclusion $\mathbf{C}^{\leq n} \hookrightarrow \mathbf{C}$ admits a right adjoint denoted by $\tau^{\leq n}$, and similarly, for the other categories.

There is a fully faithful functor from $\mathrm{DGCat}_{\mathrm{cont}}$ to that of stable ∞-categories and continuous exact functors. A stable ∞-category obtained in this way is enriched over the category Vect. Thus, we shall often think of the spectrum $\mathcal{M}aps_{\mathbf{C}}(x, y)$ as an object of Vect; the former is obtained from the latter by the Dold-Kan correspondence.

0.6.4. *(Pre)stacks and DG schemes.* Our conventions regarding (pre)stacks and DG schemes follow [GL:Stacks]:

Let $\mathrm{DGSch}^{\mathrm{aff}}$ denote the ∞-category opposite to that of *connective* commutative DG algebras over k.

The category PreStk of prestacks is by definition that of all functors

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}\,.$$

Let ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}$ be the full subcategory of $\mathrm{DGSch}^{\mathrm{aff}}$ given by eventually coconnective objects.

Recall that an eventually coconnective affine DG scheme $S = \mathrm{Spec}(A)$ is *almost of finite type* if

- $H^0(A)$ is finite type over k.
- Each $H^i(A)$ is finitely generated as a module over $H^0(A)$.

Let ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ denote the full subcategory of ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}$ consisting of schemes almost of finite type, and let $\mathrm{PreStk}_{\mathrm{laft}}$ be the category of all functors

$${}^{<\infty}(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \infty\text{-Grpd}\,.$$

As explained in [GL:Stacks, Sect. 1.3.11], $\mathrm{PreStk}_{\mathrm{laft}}$ is naturally a subcategory of PreStk via a suitable Kan extension.

In order to apply the formalism of ind-coherent sheaves developed in [IndCoh], we assume that the prestacks we consider are locally almost of finite type for most of this paper. We will explicitly indicate when this is not the case.

0.6.5. *Reduced rings.* Let $({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \subset (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}$ denote the category of reduced discrete rings. The inclusion functor has a natural left adjoint

$$^{cl,red}(-) : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to ({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$$

given by

$$S \mapsto H^0(S)/\operatorname{nilp}(H^0(S))$$

where $\operatorname{nilp}(H^0(S))$ is the ideal of nilpotent elements in $H^0(S)$.

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1. The de Rham prestack

For a prestack $\mathcal{Y}$, crystals are defined as sheaves (quasi-coherent or ind-coherent) on the de Rham prestack $\mathcal{Y}_{\mathrm{dR}}$ of $\mathcal{Y}$. In this section, we define the functor $\mathcal{Y} \mapsto \mathcal{Y}_{\mathrm{dR}}$ and establish a number of its basic properties.

Most importantly, we will show that if $\mathcal{Y}$ is locally almost of finite type, then so is $\mathcal{Y}_{\mathrm{dR}}$. In this case, we will also show that $\mathcal{Y}_{\mathrm{dR}}$ is classical, i.e., it can be studied entirely within the realm of "classical" algebraic geometry without reference to derived rings.

As the reader might find this section particularly abstract, it might be a good strategy to skip it on first pass, and return to it when necessary when assertions established here are applied to crystals.

1.1. **Definition and basic properties.**

1.1.1. Let $\mathcal{Y}$ be an object of PreStk. We define the de Rham prestack of $\mathcal{Y}$, $\mathcal{Y}_{\mathrm{dR}} \in \mathrm{PreStk}$ as

$$\mathcal{Y}_{\mathrm{dR}}(S) := \mathcal{Y}({}^{cl,red}S) \tag{1.1}$$

for $S \in \mathrm{DGSch}^{\mathrm{aff}}$.

1.1.2. More abstractly, we can rewrite

$$\mathcal{Y}_{\mathrm{dR}} := \mathrm{RKE}_{{}^{red}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}({}^{cl,red}\mathcal{Y}),$$

where ${}^{cl,red}\mathcal{Y} := \mathcal{Y}|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}}$ is the restriction of $\mathcal{Y}$ to reduced classical affine schemes, and

$$\mathrm{RKE}_{{}^{red}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}$$

is the right Kan extension of a functor along the inclusion ${}^{red}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$.

1.1.3. The following (obvious) observation will be useful in the sequel.

Lemma 1.1.4. *The functor* $\mathrm{dR} : \mathrm{PreStk} \to \mathrm{PreStk}$ *commutes with limits and colimits.*

Proof. Follows from the fact that limits and colimits in

$$\mathrm{PreStk} = \mathrm{Funct}((\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}, \infty\text{-Grpd})$$

are computed object-wise. □

As a consequence, we obtain:

Corollary 1.1.5. *The functor* $\mathrm{dR} : \mathrm{PreStk} \to \mathrm{PreStk}$ *is the left Kan extension of the functor*

$$\mathrm{dR}|_{\mathrm{DGSch}^{\mathrm{aff}}} : \mathrm{DGSch}^{\mathrm{aff}} \to \mathrm{PreStk}$$

along $\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$.

Proof. This is true for any colimit-preserving functor out of PreStk to an aribitrary ∞-category. □

1.1.6. Furthermore, we have:

Lemma 1.1.7. *The functor* $\mathrm{dR}|_{\mathrm{DGSch}^{\mathrm{aff}}} : \mathrm{DGSch}^{\mathrm{aff}} \to \mathrm{PreStk}$ *is isomorphic to the left Kan of the functor*

$$\mathrm{dR}|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}} : {}^{red}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{PreStk}$$

along ${}^{red}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$.

Proof. For any target category $\mathbf{D}$ and any functor $\Phi : \mathrm{DGSch}^{\mathrm{aff}} \to \mathbf{D}$, the map

$$\mathrm{LKE}_{{}^{red}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\Phi|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}}) \to \Phi$$

is an isomorphism if and only if the natural transformation

$$\Phi({}^{cl,red}S) \to \Phi(S), \quad S \in \mathrm{DGSch}^{\mathrm{aff}}$$

is an isomorphism. The latter is the case, by definition, for $\mathbf{D} = \mathrm{PreStk}$ and Φ the functor $S \mapsto S_{\mathrm{dR}}$. $\square$

1.1.8. Let $\mathbf{C}_1 \subset \mathbf{C}_2$ be a pair of categories from the following list of full subcategories of PreStk:

$${}^{red}\mathrm{Sch}^{\mathrm{aff}},\ \mathrm{Sch}^{\mathrm{aff}},\ \mathrm{DGSch}^{\mathrm{aff}},\ \mathrm{DGSch}_{\text{qs-qc}},\ \mathrm{DGSch},\ \mathrm{PreStk}$$

(here the subscript "qs-qc" means "quasi-separated and quasi-compact").

From Lemma 1.1.7 and Corollary 1.1.5 we obtain:

Corollary 1.1.9. *The functor* $\mathbf{C}_2 \to \mathrm{PreStk}$ *given by* $\mathrm{dR}|_{\mathbf{C}_2}$ *is isomorphic to the left Kan extension along* $\mathbf{C}_1 \hookrightarrow \mathbf{C}_2$ *of the functor* $\mathrm{dR}|_{\mathbf{C}_1} : \mathbf{C}_1 \to \mathrm{PreStk}$.

1.2. **Relation between $\mathcal{Y}$ and $\mathcal{Y}_{\mathrm{dR}}$.**

1.2.1. The functor $\mathrm{dR} : \mathrm{PreStk} \to \mathrm{PreStk}$ comes equipped with a natural transformation

$$p_{\mathrm{dR}} : \mathrm{Id} \to \mathrm{dR},$$

i.e., for every $\mathcal{Y} \in \mathrm{PreStk}$ we have a canonical map

$$p_{\mathrm{dR},\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_{\mathrm{dR}}.$$

1.2.2. Let $\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}$ be the Čech nerve of $p_{\mathrm{dR},\mathcal{Y}}$, regarded as a simplicial object of PreStk. It is augmented by $\mathcal{Y}_{\mathrm{dR}}$.

Note that each $\mathcal{Y}^i/\mathcal{Y}_{\mathrm{dR}}$ is the formal completion of $\mathcal{Y}^i$ along the main diagonal. (We refer the reader to [GR1, Sect. 6.1.1], for our conventions regarding formal completions).

We have a canonical map

$$|\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}| \to \mathcal{Y}_{\mathrm{dR}}. \tag{1.2}$$

1.2.3. *Classically formally smooth prestacks.* We shall say that a prestack $\mathcal{Y}$ is classically formally smooth, if for $S \in \mathrm{DGSch}^{\mathrm{aff}}$, the map

$$\mathrm{Maps}(S, \mathcal{Y}) \to \mathrm{Maps}({}^{cl,red}S, \mathcal{Y})$$

induces a surjection on π_0.

The following results from the definitions:

Lemma 1.2.4. *If $\mathcal{Y}$ is classically formally smooth, the map*

$$|\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}| \to \mathcal{Y}_{\mathrm{dR}}$$

is an isomorphism in PreStk.

1.3. **The locally almost of finite type case.**

1.3.1. Recall that PreStk contains a full subcategory $\mathrm{PreStk}_{\mathrm{laft}}$ of prestacks locally almost of finite type, see [GL:Stacks, Sect. 1.3.9]. By definition, an object $\mathcal{Y} \in \mathrm{PreStk}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$ if:

- $\mathcal{Y}$ is *convergent*; i.e., for $S \in \mathrm{DGSch}^{\mathrm{aff}}$, the natural map

  $$\mathrm{Maps}(S, \mathcal{Y}) \to \underset{n \geq 0}{lim}\, \mathrm{Maps}({}^{\leq n}S, \mathcal{Y})$$

  is an isomorphism, where ${}^{\leq n}S$ denotes th n-coconnective truncation of S.

- For every n, the restriction ${}^{\leq n}\mathcal{Y} := \mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$; i.e., the functor

  $$S \mapsto \mathrm{Maps}(S, \mathcal{Y}), \quad ({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}$$

commutes with filtered colimits (equivaently, is a left Kan extension form the full subcategory ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$).

1.3.2. The following observation will play an important role in this paper.

Proposition 1.3.3. *Assume that* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$. *Then:*

(a) $\mathcal{Y}_{\mathrm{dR}} \in \mathrm{PreStk}_{\mathrm{laft}}$.

(b) $\mathcal{Y}_{\mathrm{dR}}$ *is classical, i.e., belongs to the full subcategory* ${}^{cl}\mathrm{PreStk} \subset \mathrm{PreStk}$.

1.3.4. *Proof of point (a).*

We need to verify two properties:

(i) $\mathcal{Y}_{\mathrm{dR}}$ is convergent;

(ii) Each truncation ${}^{\leq n}(\mathcal{Y}_{\mathrm{dR}})$ is locally of finite type.

Property (i) follows tautologically; it is true for any $\mathcal{Y} \in \mathrm{PreStk}$. To establish property (ii), we need to show that the functor $\mathcal{Y}_{\mathrm{dR}}$ takes filtered limits in ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ to colimits in ∞-Grpd. Since $\mathcal{Y}$ itself has this property, it suffices to show that the functor

$$S \mapsto {}^{cl,red}S : \mathrm{DGSch}^{\mathrm{aff}} \to \mathrm{DGSch}^{\mathrm{aff}}$$

preserves filtered limits, which is evident.

$\square$

1.3.5. *Proof of point (b).*

By Corollary 1.1.9, we need to prove that the colimit

$$\underset{S \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}}{colim} S_{\mathrm{dR}} \in \mathrm{PreStk}$$

is classical. By part (a), the functor

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal; hence,

$$\underset{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}}}{colim} S_{\mathrm{dR}} \to \underset{S \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}}{colim} S_{\mathrm{dR}}$$

is an isomorphism.

Therefore, since the full subcategory ${}^{cl}\mathrm{PreStk} \subset \mathrm{PreStk}$ is closed under colimits, we can assume without loss of generality that $\mathcal{Y}$ is a classical affine scheme of finite type.

More generally, we will show that for $X \in \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, the prestack X_{dR} is classical. Let $i : X \hookrightarrow Z$ be a closed embedding, where Z is a *smooth* classical affine scheme of finite type. Let Y denote the formal completion $Z^{\wedge}_X$ of Z along X (see [GR1, Sects. 6.1.1 or 6.5]). The map $X \to Y$ induces an isomorphism $X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$. Hence, it suffices to show that Y_{dR} is classical.

Consider $Y^{\bullet}/Y_{\mathrm{dR}}$ (see Sect. 1.2.2 above). Note that Y is formally smooth, since Z is (see [GR1, Sect. 8.1]). In particular, Y is classically formally smooth. Since the subcategory ${}^{cl}\mathrm{PreStk} \subset \mathrm{PreStk}$ is closed under colimits and by Lemma 1.2.4, it suffices to show that each term Y^i/Y_{dR} is classical as a prestack.

Note that Y^i/Y_{dR} is isomorphic to the formal completion of Z^i along the diagonally embedded copy of X. Hence, Y^i/Y_{dR} is classical by [GR1, Proposition 6.8.2].

□

1.3.6. From Proposition 1.3.3 we obtain:

Corollary 1.3.7. *Let* $\mathbf{C}_1 \subset \mathbf{C}_2$ *be any of the following full subcategories of* $\mathrm{DGSch}^{\mathrm{aff}}$*:*

$$\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}},\ \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}},\ \mathrm{Sch}^{\mathrm{aff}},\ \mathrm{DGSch}^{\mathrm{aff}}.$$

Then for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$*, the functor*

$$(\mathbf{C}_1)_{/\mathcal{Y}_{\mathrm{dR}}} \to (\mathbf{C}_2)_{/\mathcal{Y}_{\mathrm{dR}}}$$

is cofinal.

Proof. It suffices to prove the assertion for the inclusions

$$\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}.$$

For right arrow, the assertion follows from point (b) of Proposition 1.3.3, and for the left arrow from point (a). □

1.3.8. Now, consider the following full subcategories

$$({1.3})\qquad {}^{red}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}},\ \mathrm{DGSch}_{\mathrm{aft}},\ \mathrm{DGSch}_{\mathrm{laft}},\ \mathrm{PreStk}_{\mathrm{laft}}\,.$$

of the categories appearing in Sect. 1.1.8.

Corollary 1.3.9. *The restriction of the functor* dR *to* $\mathrm{PreStk}_{\mathrm{laft}}$ *is isomorphic to the left Kan extension of this functor to* $\mathbf{C}$*, where* $\mathbf{C}$ *is one of the subcategories in* (1.3).

Proof. It suffices to prove the corollary for $\mathbf{C} = {}^{red}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. By Corollary 1.1.9, it is enough to show that for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, the functor

$$({}^{red}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal.

By Proposition 1.3.3(a), the functor

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal. Now, the assertion follows from the fact that the inclusion ${}^{red}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ admits a right adjoint. □

2. Definition of crystals

In this section we will define left crystals (for arbitrary objects of PreStk), and right crystals for objects of $\mathrm{PreStk}_{\mathrm{laft}}$. We will show that in the latter case, the two theories are equivalent.

2.1. **Left crystals.**

2.1.1. For $\mathcal{Y} \in \mathrm{PreStk}$ we define

$$\mathrm{Crys}^l(\mathcal{Y}) := \mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}).$$

I.e.,

$$\mathrm{Crys}^l(\mathcal{Y}) = \underset{S \in (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}}{lim} \mathrm{QCoh}(S).$$

Informally, an object $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})$ is an assignment of a quasi-coherent sheaf $\mathcal{F}_S \in \mathrm{QCoh}(S)$ for every affine DG scheme $S \in \mathrm{DGSch}^{\mathrm{aff}}$ with a map ${}^{red,cl}S \to \mathcal{Y}$, as well as an isomorphism

$$f^*(\mathcal{F}_S) \simeq \mathcal{F}_{S'} \in \mathrm{QCoh}(S')$$

for every morphism $f : S' \to S$ of affine DG schemes.

2.1.2. More functorially, let $\mathrm{Crys}^l_{\mathrm{PreStk}}$ denote the functor $(\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ defined as

$$\mathrm{Crys}^l_{\mathrm{PreStk}} := \mathrm{QCoh}^*_{\mathrm{PreStk}} \circ \mathrm{dR},$$

where

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is the functor which assigns to a prestack the corresponding category of quasi-coherent sheaves [GL:QCoh, Sect. 1.1.5].

For a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk, let $f^{\dagger,l}$ denote the corresponding pullback functor

$$\mathrm{Crys}^l(\mathcal{Y}_2) \to \mathrm{Crys}^l(\mathcal{Y}_1).$$

By construction, if f induces an isomorphism of the underlying reduced classical prestacks ${}^{cl,red}\mathcal{Y}_1 \to {}^{cl,red}\mathcal{Y}_2$, then it induces an isomorphism of de Rham prestacks $\mathcal{Y}_{1,\mathrm{dR}} \to \mathcal{Y}_{2,\mathrm{dR}}$ and in particular $f^{\dagger,l}$ is an equivalence.

2.1.3. Recall that the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ is by definition the right Kan extension of the functor

$$\mathrm{QCoh}^*_{\mathrm{DGSch}^{\mathrm{aff}}} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

along $(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}$.

In particular, it takes colimits in PreStk to limits in $\mathrm{DGCat}_{\mathrm{cont}}$. Therefore, by Corollary 1.1.9, for $\mathcal{Y} \in \mathrm{PreStk}$ we obtain:

Corollary 2.1.4. *Let* $\mathbf{C}$ *be any of the categories from the list of Sect. 1.1.8. Then for* $\mathcal{Y} \in \mathrm{PreStk}$*, the functor*

$$\mathrm{Crys}^l(\mathcal{Y}) \to \underset{X \in (\mathbf{C}_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{Crys}^l(X)$$

is an equivalence.

Informally, this corollary says that the data of an object $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})$ is equivalent to that of $\mathcal{M}_S \in \mathrm{Crys}^l(S)$ for every $S \in \mathbf{C}_{/\mathcal{Y}}$, and for every $f : S' \to S$, an isomorphism

$$f^{\dagger,l}(\mathcal{M}_S) \simeq \mathcal{M}_{S'} \in \mathrm{Crys}^l(S').$$

2.1.5. Recall the natural transformation $p_{\mathrm{dR}} : \mathrm{Id} \to \mathrm{dR}$. It induces a natural transformation

$$\mathbf{oblv}^l : \mathrm{Crys}^l_{\mathrm{PreStk}} \to \mathrm{QCoh}^*_{\mathrm{PreStk}}.$$

I.e., for every $\mathcal{Y} \in \mathrm{PreStk}$, we have a functor

$$\mathbf{oblv}^l_{\mathcal{Y}} : \mathrm{Crys}^l(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}), \tag{2.1}$$

and for every morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, a commutative diagram:

$$\begin{array}{ccc} \mathrm{Crys}^l(\mathcal{Y}_1) & \xrightarrow{\mathbf{oblv}^l_{\mathcal{Y}_1}} & \mathrm{QCoh}(\mathcal{Y}_1) \\ {\scriptstyle f^{\dagger,l}}\Big\uparrow & & \Big\uparrow{\scriptstyle f^*} \\ \mathrm{Crys}^l(\mathcal{Y}_2) & \xrightarrow{\mathbf{oblv}^l_{\mathcal{Y}_2}} & \mathrm{QCoh}(\mathcal{Y}_2). \end{array} \tag{2.2}$$

2.1.6. Recall the simplicial object $\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}$ of Sect. 1.2.2.

From Lemma 1.2.4 we obtain:

Lemma 2.1.7. *If $\mathcal{Y}$ is classically formally smooth, then the functor*

$$\mathrm{Crys}^l(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{QCoh}(\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}))$$

is an equivalence.

Remark 2.1.8. Our definition of left crystals on $\mathcal{Y}$ is what in Grothendieck's terminology is *quasi-coherent sheaves on the infinitesimal site of* $\mathcal{Y}$. The category $\mathrm{Tot}(\mathrm{QCoh}(\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}))$ is what in Grothendieck's terminology is *quasi-coherent sheaves on the stratifying site of* $\mathcal{Y}$. Thus, Lemma 2.1.7 says that the two are equivalent for classically formally smooth prestacks. We shall see in Sect. 3.4 that the same is also true when $\mathcal{Y}$ is an eventually coconnective DG scheme locally almost of finite type. However, the equivalence fails for DG schemes that are not eventually coconnective (even ones that are locally almost of finite type).

2.2. **Left crystals on prestacks locally almost of finite type.** For the rest of this section, unless specified otherwise, we will restrict ourselves to the subcategory $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$.

So, unless explicitly stated otherwise, by a prestack/DG scheme/affine DG scheme, we shall mean one which is locally almost of finite type.

Let $\mathrm{Crys}^l_{\mathrm{PreStk}_{\mathrm{laft}}}$ denote the restriction of $\mathrm{Crys}^l_{\mathrm{PreStk}}$ to $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$.

2.2.1. The next corollary says that we "do not need to know" about schemes of infinite type or derived algebraic geometry in order to define $\mathrm{Crys}^l(\mathcal{Y})$ for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$. In other words, to define crystals on a prestack locally almost of finite type, we can stay within the world of classical affine schemes of finite type.

Indeed, from Corollary 1.3.7 we obtain:

Corollary 2.2.2. *Let* $\mathbf{C}$ *be one of the full subcategories*

$$\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}},\ \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}},\ \mathrm{Sch}^{\mathrm{aff}}$$

of $\mathrm{DGSch}^{\mathrm{aff}}$*. Then for* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ *the natural functor*

$$\mathrm{Crys}^l(\mathcal{Y}) \to \underset{S \in (\mathbf{C}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}}{\mathit{lim}} \mathrm{QCoh}(S)$$

is an equivalence.

2.2.3. Recall that according to Corollary 2.1.4, the category $\mathrm{Crys}^l(\mathcal{Y})$ can be recovered from the functor

$$\mathrm{Crys}^l : \mathbf{C}_{/\mathcal{Y}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

where $\mathbf{C}$ is any one of the categories

$${}^{red}\mathrm{Sch}^{\mathrm{aff}},\ \mathrm{Sch}^{\mathrm{aff}},\ \mathrm{DGSch}^{\mathrm{aff}},\ \mathrm{DGSch}_{\mathrm{qs\text{-}qc}},\ \mathrm{DGSch} \subset \mathrm{PreStk}\,.$$

We now claim that the above categories can be also replaced by their full subcategories in the list (1.3):

$${}^{red}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}},\ \mathrm{DGSch}_{\mathrm{aft}},\ \mathrm{DGSch}_{\mathrm{laft}},\ \mathrm{PreStk}_{\mathrm{laft}}\,.$$

Corollary 2.2.4. *For* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ *and* $\mathbf{C}$ *being one of the categories in* (1.3), *the functor*

$$\mathrm{Crys}^l(\mathcal{Y}) \to \underset{X \in (\mathbf{C}_{/\mathcal{Y}})^{\mathrm{op}}}{\mathit{lim}} \mathrm{Crys}^l(X)$$

is an equivalence.

Proof. Follows from Corollary 1.3.9.

□

Informally, the above corollary says that an object $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})$ can be recovered from an assignment of $\mathcal{M}_S \in \mathrm{Crys}^l(S)$ for every $S \in \mathbf{C}_{/\mathcal{Y}}$, and for every $f : S' \to S$ of an isomorphism

$$f^{\dagger,l}(\mathcal{M}_S) \simeq \mathcal{M}_{S'} \in \mathrm{Crys}^l(S').$$

2.2.5. Consider again the functor

$$\mathbf{oblv}^l_{\mathcal{Y}} : \mathrm{Crys}^l(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})$$

of (2.1). We have:

Lemma 2.2.6. *For* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, *the functor* $\mathbf{oblv}^l_{\mathcal{Y}}$ *is conservative.*

The proof is deferred until Sect. 2.4.7.

2.3. **Right crystals.**

2.3.1. Recall that $\mathrm{PreStk}_{\mathrm{laft}}$ can be alternatively viewed as the category of all functors

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \infty\text{-Grpd},$$

see [GL:Stacks, Sect. 1.3.11].

Furthermore, we have the functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

of [IndCoh, Sect. 10.1.2], which is defined as the right Kan extension of the corresponding functor

$$\mathrm{IndCoh}^!_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}} : ({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

along

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

In particular, the functor $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ takes colimits in $\mathrm{PreStk}_{\mathrm{laft}}$ to limits in $\mathrm{DGCat}_{\mathrm{cont}}$.

2.3.2. We define the functor

$$\mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

as the composite

$$\mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}} := \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} \circ \mathrm{dR}.$$

In the above formula, Proposition 1.3.3(a) is used to make sure that dR is defined as a functor $\mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}$.

Remark 2.3.3. In defining $\mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}}$ we "do not need to know" about schemes of infinite type: we can define the endo-functor $\mathrm{dR} : \mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}$ directly by the formula

$$\mathrm{Maps}(S, \mathcal{Y}_{\mathrm{dR}}) = \mathrm{Maps}({}^{red,cl}S, \mathcal{Y})$$

for $S \in {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$.

2.3.4. For a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in $\mathrm{PreStk}_{\mathrm{laft}}$, we shall denote by $f^{\dagger,r}$ the corresponding functor $\mathrm{Crys}^r(\mathcal{Y}_2) \to \mathrm{Crys}^r(\mathcal{Y}_1)$.

If f induces an equivalence ${}^{cl,red}\mathcal{Y}_1 \to {}^{cl,red}\mathcal{Y}_2$, then the map $\mathcal{Y}_{1,\mathrm{dR}} \to \mathcal{Y}_{2,\mathrm{dR}}$ is an equivalence, and in particular, so is $f^{\dagger,r}$.

2.3.5. By definition, for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, we have:

$$\mathrm{Crys}^r(\mathcal{Y}) = \underset{S \in (({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S).$$

Informally, an object $\mathcal{M} \in \mathrm{Crys}^r(\mathcal{Y})$ is an assignment for every $S \in {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ and a map ${}^{red,cl}S \to \mathcal{Y}$ of an object $\mathcal{F}_S \in \mathrm{IndCoh}(S)$, and for every $f : S' \to S$ of an isomorphism

$$f^!(\mathcal{F}_S) \simeq \mathcal{F}_{S'} \in \mathrm{IndCoh}(S').$$

2.3.6. As in Sect. 2.2.1, we "do not need to know" about DG schemes in order to recover $\mathrm{Crys}^r(\mathcal{Y})$:

Corollary 2.3.7. *For* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, *the functor*

$$\mathrm{Crys}^r(\mathcal{Y}) \to \underset{S \in ((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}}{\mathit{lim}} \mathrm{IndCoh}(S)$$

is an equivalence.

Proof. Follows readily from Corollary 1.3.7. □

Informally, the above corollary says that an $\mathcal{M} \in \mathrm{Crys}^r(\mathcal{Y})$ can be recovered from an assignment for every $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and a map ${}^{red,cl}S \to \mathcal{Y}$ of an object $\mathcal{F}_S \in \mathrm{IndCoh}(S)$, and for every $f : S' \to S$ of an isomorphism

$$f^!(\mathcal{F}_S) \simeq \mathcal{F}_{S'} \in \mathrm{IndCoh}(S').$$

2.3.8. Furthermore, the analogue of Corollary 2.2.4 holds for right crystals as well:

Corollary 2.3.9. *Let* $\mathbf{C}$ *be any of the categories from* (1.3). *Then the functor*

$$\mathrm{Crys}^r(\mathcal{Y}) \to \underset{X \in (\mathbf{C}_{/\mathcal{Y}})^{\mathrm{op}}}{\mathit{lim}} \mathrm{Crys}^r(X)$$

is an equivalence.

Proof. Follows from Corollary 1.3.9. □

Informally, this corollary says that we can recover an object $\mathcal{M} \in \mathrm{Crys}^r(\mathcal{Y})$ from an assignment of $\mathcal{M}_S \in \mathrm{Crys}^r(S)$ for every $S \in \mathbf{C}_{/\mathcal{Y}}$, and for every $f : S' \to S$ of an isomorphism

$$f^{\dagger,r}(\mathcal{M}_S) \simeq \mathcal{M}_{S'} \in \mathrm{Crys}^r(S').$$

2.3.10. The natural transformation $p_{\mathrm{dR}} : \mathrm{Id} \to \mathrm{dR}$ induces a natural transformation

$$\mathbf{oblv}^r : \mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}.$$

I.e., for every $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, we have a functor

$$\mathbf{oblv}^r_{\mathcal{Y}} : \mathrm{Crys}^r(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}),$$

and for every morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, a commutative diagram:

$$\begin{array}{ccc} \mathrm{Crys}^r(\mathcal{Y}_1) & \xrightarrow{\mathbf{oblv}^r_{\mathcal{Y}_1}} & \mathrm{IndCoh}(\mathcal{Y}_1) \\ {\scriptstyle f^{\dagger,r}}\uparrow & & \uparrow{\scriptstyle f^!} \\ \mathrm{Crys}^r(\mathcal{Y}_2) & \xrightarrow{\mathbf{oblv}^r_{\mathcal{Y}_2}} & \mathrm{IndCoh}(\mathcal{Y}_2). \end{array} \tag{2.3}$$

We have:

Lemma 2.3.11. *If $\mathcal{Y}$ is classically formally smooth, then the functor*

$$\mathrm{Crys}^r(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}))$$

is an equivalence.

Proof. Same as that of Lemma 2.1.7, i.e. follows from Lemma 1.2.4. □

Lemma 2.3.12. *For any $\mathcal{Y}$, the functor $\mathbf{oblv}^r_{\mathcal{Y}}$ is conservative.*

Proof. By Corollary 2.3.9 and the commutativity of (2.3), we can assume without loss of generality that $\mathcal{Y} = X$ is an affine DG scheme locally almost of finite type. Let $i : X \to Z$ be a closed embedding of X into a smooth classical finite type scheme Z, and let Y be the formal completion of Z along X. Let $'i$ denote the resulting map $X \to Y$.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Crys}^r(Y) & \xrightarrow{\mathbf{oblv}^r_{Y}} & \mathrm{IndCoh}(Y) \\ {\scriptstyle 'i^{\dagger,r}}\downarrow & & \downarrow{\scriptstyle 'i^!} \\ \mathrm{Crys}^r(X) & \xrightarrow{\mathbf{oblv}^r_{X}} & \mathrm{IndCoh}(X). \end{array}$$

In this diagram the left vertical arrow is an equivalence since $'i_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ is an isomorphism. The top horizontal arrow is conservative by Lemma 2.3.11, since Y is formally smooth (and, in particular, classically formally smooth).

Hence, it remains to show that the functor $'i^!$ is conservative. This follows, e.g., by combining [GR1, Proposition 7.4.5] and [IndCoh, Proposition 4.1.7(a)].

□

2.4. **Comparison of left and right crystals.** We remind the reader that we assume that all prestacks and DG schemes are locally almost of finite type.

2.4.1. Recall (see [IndCoh, Sect. 5.7.5]) that for any $S \in \mathrm{DGSch}_{\mathrm{aft}}$ there is a canonically defined functor

$$\Upsilon_S : \mathrm{QCoh}(S) \to \mathrm{IndCoh}(S),$$

given by tensoring with the duaizing sheaf $\omega_S \in \mathrm{IndCoh}(S)$, such that for $f : S_1 \to S_2$, the diagram

$$\begin{CD} \mathrm{QCoh}(S_1) @>{\Upsilon_{S_1}}>> \mathrm{IndCoh}(S_1) \\ @A{f^*}AA @AA{f^!}A \\ \mathrm{QCoh}(S_2) @>{\Upsilon_{S_2}}>> \mathrm{IndCoh}(S_2) \end{CD}$$

canonically commutes. In fact, the above data upgrades to a natural transformation of functors

$$\Upsilon_{\mathrm{DGSch}_{\mathrm{aft}}} : \mathrm{QCoh}^*_{\mathrm{DGSch}_{\mathrm{aft}}} \to \mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}},$$

and hence gives rise to a natural transformation

$$\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}} : \mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}},$$

[IndCoh, Sect. 10.3.3].

For an individual object $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, we obtain a functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}).$$

2.4.2. Applying Υ to $\mathcal{Y}_{\mathrm{dR}}$ for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, we obtain a canonically defined functor

$$\Upsilon_{\mathcal{Y}_{\mathrm{dR}}} : \mathrm{Crys}^l(\mathcal{Y}) \to \mathrm{Crys}^r(\mathcal{Y}), \tag{2.4}$$

making the diagram

$$\begin{CD} \mathrm{Crys}^l(\mathcal{Y}) @>{\Upsilon_{\mathcal{Y}_{\mathrm{dR}}}}>> \mathrm{Crys}^r(\mathcal{Y}) \\ @V{\mathbf{oblv}^l_{\mathcal{Y}}}VV @VV{\mathbf{oblv}^r_{\mathcal{Y}}}V \\ \mathrm{QCoh}(\mathcal{Y}) @>{\Upsilon_{\mathcal{Y}}}>> \mathrm{IndCoh}(\mathcal{Y}) \end{CD} \tag{2.5}$$

commute.

In fact we obtain a natural transformation

$$\Upsilon_{\mathrm{PreStk}_{\mathrm{laft}}} \circ \mathrm{dR} : \mathrm{Crys}^l_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}}.$$

In particular, for $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ the diagram

$$\begin{array}{ccc} \mathrm{Crys}^l(\mathcal{Y}_1) & \xrightarrow{\Upsilon_{\mathcal{Y}_{1\mathrm{dR}}}} & \mathrm{Crys}^r(\mathcal{Y}_1) \\ {\scriptstyle f^{\dagger,l}}\uparrow & & \uparrow{\scriptstyle f^{\dagger,r}} \\ \mathrm{Crys}^l(\mathcal{Y}_2) & \xrightarrow{\Upsilon_{\mathcal{Y}_{2\mathrm{dR}}}} & \mathrm{Crys}^r(\mathcal{Y}_2) \end{array}$$

commutes.

2.4.3. We claim:

Proposition 2.4.4. *For* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$*, the functor* (2.4) *is an equivalence.*

Proof. By Corollaries 2.2.4 and 2.3.9, the statement reduces to one saying that

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X)$$

is an equivalence for an affine DG scheme X almost of finite type.

Let $i : X \hookrightarrow Z$ be a closed embedding, where Z is a smooth classical scheme, and let Y be the formal completion of Z along X. Since $X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ is an isomorphism, the functors

$$f^{\dagger,l} : \mathrm{Crys}^l(Y) \to \mathrm{Crys}^l(X) \text{ and } f^{\dagger,r} : \mathrm{Crys}^r(Y) \to \mathrm{Crys}^r(X)$$

are both equivalences. Hence, it is enough to prove the assertion for Y.

Let $Y^\bullet/Y_{\mathrm{dR}}$ be the Čech nerve of $\mathrm{PreStk}_{\mathrm{laft}}$ corresponding to the map

$$p_{\mathrm{dR},Y} : Y \to Y_{\mathrm{dR}}.$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Crys}^l(Y) & \xrightarrow{\Upsilon_{Y_{\mathrm{dR}}}} & \mathrm{Crys}^r(Y) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(\mathrm{QCoh}(Y^\bullet/Y_{\mathrm{dR}})) & \xrightarrow{\mathrm{Tot}(\Upsilon_{Y^\bullet/Y_{\mathrm{dR}}})} & \mathrm{Tot}(\mathrm{IndCoh}(Y^\bullet/Y_{\mathrm{dR}})). \end{array}$$

By Lemmas 2.1.7 and 2.3.11, the vertical arrows in the diagram are equivalences. Therefore, it suffices to show that for every i,

$$\Upsilon_{Y^i/Y_{\mathrm{dR}}} : \mathrm{QCoh}(Y^i/Y_{\mathrm{dR}}) \to \mathrm{IndCoh}(Y^i/Y_{\mathrm{dR}})$$

is an equivalence.

Recall (also from the proof of Proposition 1.3.3) that Y^i/Y_{dR} is the completion of the smooth classical scheme Z^i along the diagonal copy of X. Let us denote by $U_i \subset Z^i$ the complementary open substack.

From [GR1, Propositions 7.1.3 and 7.4.5 and Diagram (7.16)], we obtain that we have a map of "short exact sequences" of DG categories

$$\begin{CD}
0 @>>> \mathrm{QCoh}(Y^i/Y_{\mathrm{dR}}) @>>> \mathrm{QCoh}(Z^i) @>>> \mathrm{QCoh}(U_i) @>>> 0 \\
@. @V{\Upsilon_{Y^i/Y_{\mathrm{dR}}}}VV @VV{\Upsilon_{Z^i}}V @VV{\Upsilon_{U_i}}V \\
0 @>>> \mathrm{IndCoh}(Y^i/Y_{\mathrm{dR}}) @>>> \mathrm{IndCoh}(Z^i) @>>> \mathrm{IndCoh}(U_i) @>>> 0.
\end{CD}$$

Now, the functors

$$\Upsilon_{Z^i} : \mathrm{QCoh}(Z^i) \to \mathrm{IndCoh}(Z^i) \text{ and } \Upsilon_{U_i} : \mathrm{QCoh}(U_i) \to \mathrm{IndCoh}(U_i)$$

are both equivalences, since Z^i and U_i are smooth:

Indeed, by [IndCoh, Proposition 9.3.3], for any $S \in \mathrm{DGSch}_{\mathrm{aft}}$, the functor Υ_S is the dual of $\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$, and the latter is an equivalence if S is smooth by [IndCoh, Lemma 1.1.6].

$\square$

2.4.5. Proposition 2.4.4 allows us to identify left and right crystals for objects $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$.

In other words, we can consider the category $\mathrm{Crys}(\mathcal{Y})$ equipped with two realizations: "left" and "right", which incarnate themselves as forgetful functors $\mathbf{oblv}^l_{\mathcal{Y}}$ and $\mathbf{oblv}^r_{\mathcal{Y}}$ from $\mathrm{Crys}(\mathcal{Y})$ to $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(\mathcal{Y})$, respectively.

The two forgetful functors are related by the commutative diagram

$$\begin{tikzcd} & \mathrm{Crys}(\mathcal{Y}). \arrow[dl, "\mathbf{oblv}^l_{\mathcal{Y}}"'] \arrow[dr, "\mathbf{oblv}^r_{\mathcal{Y}}"] & \\ \mathrm{QCoh}(\mathcal{Y}) \arrow[rr, "\Upsilon_{\mathcal{Y}}"'] & & \mathrm{IndCoh}(\mathcal{Y}) \end{tikzcd} \tag{2.6}$$

For a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ we have a naturally defined functor

$$f^\dagger : \mathrm{Crys}(\mathcal{Y}_2) \to \mathrm{Crys}(\mathcal{Y}_1),$$

which makes the following diagrams commute

$$\begin{tikzcd} \mathrm{Crys}(\mathcal{Y}_1) \arrow[d, "\mathbf{oblv}^l_{\mathcal{Y}_1}"'] & \mathrm{Crys}(\mathcal{Y}_2) \arrow[l, "f^\dagger"'] \arrow[d, "\mathbf{oblv}^l_{\mathcal{Y}_2}"] \\ \mathrm{QCoh}(\mathcal{Y}_1) & \mathrm{QCoh}(\mathcal{Y}_2) \arrow[l, "f^*"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Crys}(\mathcal{Y}_1) \arrow[d, "\mathbf{oblv}^r_{\mathcal{Y}_1}"'] & \mathrm{Crys}(\mathcal{Y}_2) \arrow[l, "f^\dagger"'] \arrow[d, "\mathbf{oblv}^r_{\mathcal{Y}_2}"] \\ \mathrm{IndCoh}(\mathcal{Y}_1) & \mathrm{IndCoh}(\mathcal{Y}_2). \arrow[l, "f^!"'] \end{tikzcd}$$

2.4.6. In the sequel, we shall use symbols $\mathrm{Crys}(\mathcal{Y})$, $\mathrm{Crys}^r(\mathcal{Y})$ and $\mathrm{Crys}^l(\mathcal{Y})$ interchangeably with the former emphasizing that the statement is independent of realization (left or right) we choose, and the latter two, when a choice of the realization is important.

2.4.7. *Proof of Lemma 2.2.6.* Follows by combining Lemma 2.3.12 and Proposition 2.4.4.

□

2.5. **Kashiwara's lemma.** A feature of the assignment $\mathcal{Y} \mapsto \mathrm{Crys}(\mathcal{Y})$ is that Kashiwara's lemma becomes nearly tautological.

We will formulate and prove it for the incarnation of crystals as right crystals. By Proposition 2.4.4, this implies the corresponding assertion for left crystals.

However, one could easily write the same proof in the language of left crystals as well.

2.5.1. Recall that a map $i : \mathcal{X} \to \mathcal{Z}$ in PreStk is called a closed embedding if it is such at the level of the underlying classical prestacks. I.e., if for every $S \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Z}}$ the base-changed map

$$^{cl}(S \underset{\mathcal{Z}}{\times} \mathcal{X}) \to S$$

is a closed embedding; in particular, $^{cl}(S \underset{\mathcal{Z}}{\times} \mathcal{X})$ is a classical affine scheme.

If $\mathcal{X}, \mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$, it suffices to check the above condition for $S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Z}}$.

2.5.2. For $i : \mathcal{X} \hookrightarrow \mathcal{Z}$ a closed embedding of objects of $\mathrm{PreStk}_{\mathrm{laft}}$, let $j : \overset{\circ}{\mathcal{Z}} \hookrightarrow \mathcal{Z}$ be the complementary open embedding. The induced map

$$j : \overset{\circ}{\mathcal{Z}}_{\mathrm{dR}} \to \mathcal{Z}_{\mathrm{dR}}$$

is also an open embedding of prestacks. Consider the restriction functor

$$j^{\dagger,r} : \mathrm{Crys}^r(\mathcal{Z}) \to \mathrm{Crys}^r(\overset{\circ}{\mathcal{Z}}).$$

It follows from [IndCoh, Lemma 4.1.1], that the above functor admits a *fully faithful* right adjoint, denoted $j_{\mathrm{dR},*}$, such that for every $S \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}}$ and

$$\overset{\circ}{S} := S \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \overset{\circ}{\mathcal{Z}}_{\mathrm{dR}} \overset{j_S}{\hookrightarrow} S,$$

the natural transformation in the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(S) & \mathrm{Crys}^r(\mathcal{Z}) \arrow[l, "\mathbf{oblv}^r_{\mathcal{Z}}"'] \\
\mathrm{IndCoh}(\overset{\circ}{S}) \arrow[u, "(j_S)^{\mathrm{IndCoh}}_*"] & \mathrm{Crys}^r(\overset{\circ}{\mathcal{Z}}) \arrow[l, "\mathbf{oblv}^r_{\overset{\circ}{S}}"'] \arrow[u, "j_{\mathrm{dR},*}"']
\end{tikzcd}$$

arising by adjunction from the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(S) \arrow[d, "(j_S)^!"'] & \mathrm{Crys}^r(\mathcal{Z}) \arrow[l, "\mathbf{oblv}^r_{\mathcal{Z}}"'] \arrow[d, "j^{\dagger,r}"] \\
\mathrm{IndCoh}(\overset{\circ}{S}) & \mathrm{Crys}^r(\overset{\circ}{\mathcal{Z}}), \arrow[l, "\mathbf{oblv}^r_{\overset{\circ}{S}}"']
\end{tikzcd}$$

is an isomorphism.

In particular, the natural transformation

$$\mathbf{oblv}^r_{\mathcal{Z}} \circ j_{\mathrm{dR},*} \to j^{\mathrm{IndCoh}}_* \circ \mathbf{oblv}^r_{\overset{\circ}{\mathcal{Z}}}$$

is an isomorphism.

2.5.3. Let $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$ denote the full subcategory of $\mathrm{Crys}^r(\mathcal{Z})$ equal to $\ker(j^{\dagger,r})$.

Clearly, an object $\mathcal{M} \in \mathrm{Crys}^r(\mathcal{Z})$ belongs to $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$ if and only if for every $S \in \mathrm{DGSch}_{\mathrm{aft}}$, equipped with a map ${}^{cl,red}S \to \mathcal{Z}$, the corresponding object $\mathcal{F}_S \in \mathrm{IndCoh}(S)$ lies in

$$\mathrm{IndCoh}(S)_{S-\overset{\circ}{S}} := \ker\left(j_S^! : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(\overset{\circ}{S})\right).$$

2.5.4. The functor $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}} \hookrightarrow \mathrm{Crys}^r(\mathcal{X})$ admits a right adjoint, given by

$$\mathcal{M} \mapsto \mathrm{Cone}(\mathcal{M} \to j_{\mathrm{dR},*} \circ j^{\dagger,r}(\mathcal{M}))[-1].$$

Hence, we can think of $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$ as a co-localization of $\mathrm{Crys}^r(\mathcal{Z})$.

2.5.5. Since the composite $i^{\dagger,r} \circ j_{\mathrm{dR},*}$ is zero, the functor $i^{\dagger,r} : \mathrm{Crys}^r(\mathcal{Z}) \to \mathrm{Crys}^r(\mathcal{X})$ factors through the above co-localization:

$$\mathrm{Crys}^r(\mathcal{Z}) \to \mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}} \xrightarrow{'i^{\dagger,r}} \mathrm{Crys}^r(\mathcal{X}).$$

Kashiwara's lemma says:

Proposition 2.5.6. *The above functor*

$$'i^{\dagger,r} : \mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}} \to \mathrm{Crys}^r(\mathcal{X})$$

is an equivalence.

Proof. Note that we have an isomorphism in $\mathrm{PreStk}_{\mathrm{laft}}$:

$$\underset{S \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}}}{colim} S \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{X}_{\mathrm{dR}} \simeq \mathcal{X}_{\mathrm{dR}}.$$

Furthermore, $S^\wedge := S \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{X}_{\mathrm{dR}}$ identifies with the formal completion of S along

$${}^{red,cl}S \underset{{}^{cl,red}\mathcal{Z}}{\times} {}^{cl,red}\mathcal{X}.$$

Hence, the category $\mathrm{Crys}^r(\mathcal{X})$ can be described as

$$\underset{S\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}})^{\mathrm{op}}}{lim} \mathrm{IndCoh}(S^\wedge).$$

By definition, the category $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$ is given by

$$\underset{S\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}})^{\mathrm{op}}}{lim} \ker\left(j_S^! : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(\overset{\circ}{S})\right).$$

Now, [GR1, Proposition 7.4.5] says that for any S as above, !-pullback gives an equivalence

$$\ker\left(j_S^! : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(\overset{\circ}{S})\right) \to \mathrm{IndCoh}(S^\wedge),$$

as desired.

□

Remark 2.5.7. If we phrased the above proof in terms of left crystals instead of right crystals, we would have used [GR1, Proposition 7.1.3] instead of [GR1, Proposition 7.4.5].

3. Descent properties of crystals

In this section all prestacks, including DG schemes and DG indschemes are assumed locally almost of finite type, unless explicitly stated otherwise.

The goal of this section is to establish a number of properties concerning the behavior of crystals on DG schemes and DG indschemes. These properties include: an interpretation of crystals (right and left) via the infinitesimal groupoid; h-descent; a monadic description of the category of crystals; induction functors for right and left crystals.

3.1. **The infinitesimal groupoid.** In this subsection, we let $\mathcal{X}$ be a DG indscheme locally almost of finite, see [GR1, Sect. 1.7.1].

3.1.1. Consider the simplicial prestack $\mathcal{X}^\bullet/\mathcal{X}_{\mathrm{dR}}$, i.e., the Čech nerve of the map $\mathcal{X} \to \mathcal{X}_{\mathrm{dR}}$. As was remarked already, each $\mathcal{X}^i/\mathcal{X}_{\mathrm{dR}}$ is the formal completion of $\mathcal{X}^i$ along the main diagonal. In particular, all $\mathcal{X}^i/\mathcal{X}_{\mathrm{dR}}$ also belong to $\mathrm{DGindSch}_{\mathrm{laft}}$.

We shall refer to

$$\mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X} \rightrightarrows \mathcal{X}$$

as the infinitesimal groupoid of $\mathcal{X}$.

3.1.2. Consider the cosimplicial category $\mathrm{IndCoh}(\mathcal{X}^\bullet/\mathcal{X}_{\mathrm{dR}})$.

Proposition 3.1.3. *The functor*

$$\mathrm{Crys}^r(\mathcal{X}) \to \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet/\mathcal{X}_{\mathrm{dR}})),$$

defined by the augmentation, is an equivalence.

Remark 3.1.4. Note that by Lemma 2.3.11, the assertion of the proposition holds also for $\mathcal{X}$ replaced any classically formally smooth object $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$.

Proof. It suffices to show that for any $S \in \mathrm{DGSch}_{\mathrm{aft}}$ and a map $S \to \mathcal{X}_{\mathrm{dR}}$, the functor

$$\mathrm{IndCoh}(S) \to \mathrm{Tot}\left(\mathrm{IndCoh}(S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}^\bullet/\mathcal{X}_{\mathrm{dR}})\right)$$

is an equivalence.

Note that the simplicial prestack $S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} (\mathcal{X}^\bullet/\mathcal{X}_{\mathrm{dR}})$ is the Čech nerve of the map

$$S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X} \to S. \tag{3.1}$$

Note that $S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}$ identifies with the formal completion of $S \times \mathcal{X}$ along the map ${}^{red,cl}S \to S \times \mathcal{X}$, where ${}^{red,cl}S \to \mathcal{X}$ is the map corresponding to $S \to \mathcal{X}_{\mathrm{dR}}$. In particular, we obtain that the map in (3.1) is *ind-proper* (see [GR1, Sect. 2.7.4], where the notion of ind-properness is introduced) and surjective.

Hence, our assertion follows from [GR1, Lemma 2.10.3].

□

3.2. **Fppf and h-descent for crystals.**

3.2.1. Recall the h-topology on the category $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, [IndCoh, Sect. 8.2]. It is generated by Zariski covers and proper-surjective covers.

Consider the functor

$$\mathrm{Crys}^r_{\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}} := \mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}}|_{\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}} : (\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}\,.$$

We will prove:

Proposition 3.2.2. *The functor* $\mathrm{Crys}^r_{\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}$ *satisfies h-descent.*

Proof. We will show that $\mathrm{Crys}^r_{\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}$ satisfies étale descent and proper-surjective descent.

The étale descent statement is clear: if $S' \to S$ is an étale cover in $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ then the corresponding map $S'_{\mathrm{dR}} \to S_{\mathrm{dR}}$ is a schematic, étale and surjective map in $\mathrm{PreStk}_{\mathrm{laft}}$. In particular, it is a cover for the fppf topology, and the statement follows from the fppf descent for IndCoh, see [IndCoh, Corollary 10.4.5].

Thus, let $S' \to S$ be a proper surjective map. Consider the bi-simplicial object of $\mathrm{PreStk}_{\mathrm{laft}}$ equal to

$$(S'^{\bullet}/S)^{\star}/({S'_{\mathrm{dR}}}^{\bullet}/S_{\mathrm{dR}}),$$

i.e., the term-wise infinitesimal groupoid of the Čech nerve of $S' \to S$. Namely, it is the bi-simplicial object whose (p,q) simplices are given by the q-simplices of Cech nerve of the map $S'^p/S \to {S'_{\mathrm{dR}}}^p/S_{\mathrm{dR}}$; so $\star$ stands for the index q, and $\bullet$ for the index p.

By Proposition 3.1.3, it is enough to show that the composite functor

$$\begin{aligned} \mathrm{Crys}^r(S) := \mathrm{IndCoh}(S_{\mathrm{dR}}) \to \mathrm{Tot}\left(\mathrm{IndCoh}((S'^{\bullet}/S)_{\mathrm{dR}})\right) \to \\ \to \mathrm{Tot}\left(\mathrm{IndCoh}((S'^{\bullet}/S)^{\star}/(S'^{\bullet}/S)_{\mathrm{dR}})\right). \end{aligned} \tag{3.2}$$

is an equivalence.

Note, however, that we have a canonical isomorphism of bi-simplicial objects of $\mathrm{PreStk}_{\mathrm{laft}}$

$$(S'^{\bullet}/S)^{\star}/({S'_{\mathrm{dR}}}^{\bullet}/S_{\mathrm{dR}}) \simeq (S'^{\star}/S'_{\mathrm{dR}})^{\bullet}/(S^{\star}/S_{\mathrm{dR}}),$$

where the latter is the term-wise Čech nerve of the map of cosimplicial objects

$$(S'^{\star}/S'_{\mathrm{dR}}) \to (S^{\star}/S_{\mathrm{dR}}).$$

The map in (3.2) can be rewritten as

$$\mathrm{Crys}^r(S) := \mathrm{IndCoh}(S_{\mathrm{dR}}) \to \mathrm{Tot}\left(\mathrm{IndCoh}(S^{\bullet}/S_{\mathrm{dR}})\right) \to \\ \mathrm{Tot}\left(\mathrm{IndCoh}((S'^{\star}/S'_{\mathrm{dR}})^{\bullet}/(S^{\star}/S_{\mathrm{dR}}))\right).$$

Applying Proposition 3.1.3 again, we obtain that it suffices to show that for every i, the map

$$\mathrm{IndCoh}(S^i/S_{\mathrm{dR}}) \to \mathrm{Tot}\left(\mathrm{IndCoh}((S'^i/S'_{\mathrm{dR}})^{\bullet}/(S^i/S_{\mathrm{dR}}))\right)$$

is an equivalence.

However, we note that the map

$$S'^i/S'_{\mathrm{dR}} \to S^i/S_{\mathrm{dR}}$$

is ind-proper and surjective. Hence, the assertion follows from [GR1, Lemma 2.10.3].

□

3.2.3. Consider the fppf topology on the category $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, induced from the fppf topology on $\mathrm{DGSch}^{\mathrm{aff}}$ (see [GL:Stacks, Sect. 2.2]). Note that every fppf covering is in particular an h-covering. Therefore, we obtain,

Corollary 3.2.4. *The functor* $\mathrm{Crys}^r_{\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}$ *satisfies fppf descent.*

As in [IndCoh, Theorem 8.3.2], fppf descent is a combination of Nisnevich descent and finite-flat descent[4]. In particular, we established fppf descent in the proof of Proposition 3.2.2 without appealing to the fact that every fppf covering is also an h-covering.

[4]This observation was explained to us by J. Lurie.

3.2.5. Fppf (resp. h-) topology on $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ induces the fppf (resp. h-) topology on the full subcategory

$$^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}} \subset \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}.$$

Proposition 3.2.2 implies:

Corollary 3.2.6. *The functor*

$$\mathrm{Crys}^r_{^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}} := \mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}}|_{^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}} : (^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}$$

on $^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ *satisfies h-descent and, in particular, fppf descent.*

Thus by [Lu0, Corollary 6.2.3.5], we obtain:

Corollary 3.2.7. *Let* $\mathcal{Y}_1 \to \mathcal{Y}_2$ *be a map in* $\mathrm{PreStk}_{\mathrm{laft}}$ *which is a surjection in the h-topology. Then the natural map*

$$\mathrm{Crys}(\mathcal{Y}_2) \to \mathrm{Tot}(\mathrm{Crys}(\mathcal{Y}_1^\bullet/\mathcal{Y}_2))$$

is an equivalence.

3.3. **The induction functor for right crystals.**

3.3.1. Let p_s, p_t denote the two projections

$$\mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X} \rightrightarrows \mathcal{X}.$$

Note that the maps p_i, $i = s, t$ are ind-proper. Hence, the functors $p_i^!$ admit left adjoints, $(p_i)_*^{\mathrm{IndCoh}}$, see [GR1, Corollary 2.8.3].

Proposition 3.3.2.

(a) *The forgetful functor*

$$\mathbf{oblv}^r_{\mathcal{X}} : \mathrm{Crys}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})$$

admits a left adjoint, to be denoted $\mathbf{ind}^r_{\mathcal{X}}$.

(b) *We have a canonical isomorphism of functors*

$$\mathbf{oblv}^r_{\mathcal{X}} \circ \mathbf{ind}^r_{\mathcal{X}} \simeq (p_t)_*^{\mathrm{IndCoh}} \circ (p_s)^!.$$

(c) *The adjoint pair*

$$\mathbf{ind}^r_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X}) \rightleftarrows \mathrm{Crys}^r(\mathcal{X}) : \mathbf{oblv}^r_{\mathcal{X}}$$

is monadic, i.e., the natural functor from $\mathrm{Crys}^r(\mathcal{X})$ *to the category of modules in* $\mathrm{IndCoh}(\mathcal{X})$ *over the monad* $\mathbf{oblv}^r_{\mathcal{X}} \circ \mathbf{ind}^r_{\mathcal{X}}$ *is an equivalence.*

Proof. By Proposition 3.1.3 and [Lu2, Theorem 6.2.4.2], it suffices to show that the co-simplicial category

$$\mathrm{IndCoh}(\mathcal{X}^\bullet/\mathcal{X}_{\mathrm{dR}})$$

satisfies the Beck-Chevalley condition, i.e. for each n, the coface map

$$d^0 : \mathrm{IndCoh}(\mathcal{X}^n/\mathcal{X}_{\mathrm{dR}}) \to \mathrm{IndCoh}(\mathcal{X}^{n+1}/\mathcal{X}_{\mathrm{dR}})$$

admits a left adjoint, to be denoted by $\mathfrak{t}^0$, and for every map $[m] \to [n]$ in Δ, the diagram

$$\begin{tikzcd} \mathrm{IndCoh}(\mathcal{X}^m/\mathcal{X}_{\mathrm{dR}}) \arrow[d] & \mathrm{IndCoh}(\mathcal{X}^{m+1}/\mathcal{X}_{\mathrm{dR}}) \arrow[l, "\mathfrak{t}^0"'] \arrow[d] \\ \mathrm{IndCoh}(\mathcal{X}^n/\mathcal{X}_{\mathrm{dR}}) & \mathrm{IndCoh}(\mathcal{X}^{n+1}/\mathcal{X}_{\mathrm{dR}}) \arrow[l, "\mathfrak{t}^0"'] \end{tikzcd}$$

which, a priori, commutes up to a natural transformation, actually commutes.

In this case, the Beck-Chevalley condition amounts to the adjunction and base change between $*$-pushforwards and !-pullbacks for ind-proper morphisms between DG indschemes, and is given by [GR1, Proposition 2.9.2]. □

Corollary 3.3.3. *The category* $\mathrm{Crys}^r(\mathcal{X})$ *is compactly generated.*

Proof. The set of compact generators is obtained by applying $\mathbf{ind}^r_{\mathcal{X}}$ to the compact generators $\mathrm{Coh}(\mathcal{X}) \subset \mathrm{IndCoh}(\mathcal{X})$ (see [GR1, Corollary 2.4.4]). □

3.4. **The induction functor and infinitesimal groupoid for left crystals.**

3.4.1. It follows from Lemma 1.2.4 that for a smooth classical scheme X, the analogue of Proposition 3.1.3 holds for left crystals, i.e., the functor

$$\mathrm{Crys}^l(X) = \mathrm{QCoh}(X_{\mathrm{dR}}) \to \mathrm{Tot}\left(\mathrm{QCoh}(X^\bullet/X_{\mathrm{dR}})\right) \tag{3.3}$$

is an equivalence.

By Proposition 3.1.3, the analogous statement for right crystals is true for any DG scheme X (and even a DG indscheme). However, this is not the case for left crystals.

3.4.2. We claim:

Proposition 3.4.3. *If a DG scheme X is eventually coconnective, then the functor* (3.3) *is an equivalence.*

Remark 3.4.4. One can show that the statement of the proposition holds for any DG scheme X locally almost of finite type. But the proof is more involved. In addition, Lemma 2.1.7, the statement of the proposition holds for any prestack which is classically formally smooth.

Example 3.4.5. Consider the DG scheme $X = \mathrm{Spec}(k[\alpha])$, where α is in degree -2. This is a good case to have in mind to produce counterexamples for assertions involving $\mathrm{Crys}^l(X)$.

Proof of Proposition 3.4.3. We have a commutative diagram of functors

$$\begin{CD}
\mathrm{Crys}^l(X) @>>> \mathrm{Tot}\left(\mathrm{QCoh}(X^\bullet/X_{\mathrm{dR}})\right) \\
@V{\Upsilon_X}VV @VV{\mathrm{Tot}(\Upsilon_{X^\bullet/X_{\mathrm{dR}}})}V \\
\mathrm{Crys}^r(X) @>>> \mathrm{Tot}\left(\mathrm{IndCoh}(X^\bullet/X_{\mathrm{dR}})\right).
\end{CD}$$

with the left vertical map and the bottom horizontal map being equivalences. Hence, we obtain that $\mathrm{Crys}^l(X)$ is a retract of $\mathrm{Tot}\left(\mathrm{QCoh}(X^\bullet/X_{\mathrm{dR}})\right)$.

Recall that if Z is an eventually coconnective DG scheme, the functor

$$\Upsilon_Z : \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(Z)$$

is fully faithful (see [IndCoh, Corollary 9.6.3]. Hence, by [GR1, Propositions 7.1.3 and 7.4.5], the same is true for the completion of an eventually coconnective DG scheme along a Zariski-closed subset. Hence, the functors

$$\Upsilon_{X^i/X_{\mathrm{dR}}} : \mathrm{QCoh}(X^i/X_{\mathrm{dR}}) \to \mathrm{IndCoh}(X^i/X_{\mathrm{dR}})$$

are fully faithful. Thus, the functor $\mathrm{Tot}(\Upsilon_{X^\bullet/X_{\mathrm{dR}}})$ in the above commutative diagram is also fully faithful. But it is also essentially surjective since the identity functor is its retract.

$\square$

3.4.6. For a DG scheme X, define the functor

$$\mathbf{ind}^l_X : \mathrm{QCoh}(X) \to \mathrm{Crys}^l(X)$$

as

$$\mathbf{ind}^l_X := (\Upsilon_{X_{\mathrm{dR}}})^{-1} \circ \mathbf{ind}^r_X \circ \Upsilon_X.$$

We claim:

Lemma 3.4.7. *If X is an eventually coconnective DG scheme, the functors $(\mathbf{ind}^l_X, \mathbf{oblv}^l_X)$ are adjoint.*

Remark 3.4.8. The assertion of the lemma would be false if we dropped the assumption that X be eventually coconnective. Indeed, in this case the functor $\mathbf{ind}^l_X$ fails to preserve compactness.

Proof of Lemma 3.4.7. Recall (see [IndCoh, Sect. 9.6.6]) that for X eventually coconnective, the functor Υ_X admits a right adjoint, denoted $\Xi^\vee_X$; moreover, the functor Υ_X itself is fully faithful.

We obtain that the right adjoint to $\mathbf{ind}^l_X$ is given by

$$\Xi^\vee_X \circ \mathbf{oblv}^r_X \circ \Upsilon_{X_{\mathrm{dR}}} \simeq \Xi^\vee_X \circ \Upsilon_X \circ \mathbf{oblv}^l_X \simeq \mathbf{oblv}^l_X,$$

as required.

$\square$

In the course of the proof of Lemma 3.4.7 we have also seen:

Lemma 3.4.9. *The functor $\mathbf{oblv}^l_X$ is canonically isomorphic to*

$$\Xi^\vee_X \circ \mathbf{oblv}^r_X \circ \Upsilon_{X_{\mathrm{dR}}}.$$

3.4.10. We now claim:

Proposition 3.4.11. *Let X be an eventually coconnective DG scheme. Then the adjoint pair*

$$\mathbf{ind}^l_X : \mathrm{QCoh}(X) \rightleftarrows \mathrm{Crys}^l(X) : \mathbf{oblv}^l_X$$

is monadic, i.e., the natural functor from $\mathrm{Crys}^l(X)$ *to the category of modules in* $\mathrm{QCoh}(X)$ *over the monad* $\mathbf{oblv}^l_X \circ \mathbf{ind}^l_X$ *is an equivalence.*

Proof. We need to show that the conditions of the Barr-Beck-Lurie theorem hold. The functor $\mathbf{oblv}^l_X$ is continuous, and hence commutes with all colimits. The fact that $\mathbf{oblv}^l_X$ is conservative is given by Lemma 2.2.6. □

4. t-structures on crystals

The category of crystals has two natural t-structures, which are compatible with the left and right realizations respectively. One of the main advantages of the right realization is that the t-structure compatible with it is much better behaved.

In this section, we will define the two t-structures and prove some of their basic properties. These include: results on left/right t-exactness and boundedness of cohomological amplitude of the induction/forgetful functors; the left-completness property of Crys of a DG scheme; relation to the derived category of the heart of the t-structure.

4.1. **The left t-structure.** In this subsection, we do not make the assumption that prestacks be locally almost of finite type.

4.1.1. Recall [GL:QCoh, Sec. 1.2.3] that for any prestack $\mathcal{Z}$, the category $\mathrm{QCoh}(\mathcal{Z})$ has a canonical t-structure characterized by the following condition: an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$ belongs to $\mathrm{QCoh}(\mathcal{Z})^{\leq 0}$ if and only if for every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and a map $\phi : S \to \mathcal{Z}$, we have

$$\phi^*(\mathcal{F}) \in \mathrm{QCoh}(S)^{\leq 0}.$$

In particular, taking $\mathcal{Z} = \mathcal{Y}_{\mathrm{dR}}$ for some prestack $\mathcal{Y}$, we obtain a canonical t-structure on $\mathrm{Crys}^l(\mathcal{Y})$, which we shall call the "left t-structure."

By definition, the functor

$$\mathbf{oblv}^l : \mathrm{Crys}^l(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{X})$$

is right t-exact for the left t-structure.

4.1.2. In general, the left t-structure is quite poorly behaved. However, we have the following assertion:

Proposition 4.1.3. *Let $\mathcal{Y}$ be a classically formally smooth prestack. Then*

$$\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})^{\leq 0} \Leftrightarrow \mathbf{oblv}^l_{\mathcal{Y}}(\mathcal{M}) \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}.$$

Proof. We need to show that if $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})$ is such that $\mathbf{oblv}^l_X(\mathcal{M}) \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$ then $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})^{\leq 0}$. I.e., we need to show that for every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and $\phi : S \to \mathcal{Y}_{\mathrm{dR}}$, $\phi^*(\mathcal{M}) \in \mathrm{QCoh}(S)^{\leq 0}$.

Let $\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}$ be the Čech nerve of the map $p_{\mathrm{dR},\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_{\mathrm{dR}}$. By Lemma 1.2.4, there exists a map $\phi' : S \to \mathcal{Y}$ and an isomorphism $\phi \simeq p_{\mathrm{dR},\mathcal{Y}} \circ \phi'$. The assertion now follows from the fact that ϕ'^* is right t-exact. □

4.2. **The right t-structure.** From this point until the end of this section we reinstate the assumption that all prestacks are locally almost of finite type, unless explicitly stated otherwise.

In this subsection we shall specialize to the case of DG schemes.

4.2.1. Let X be a DG scheme. Recall that the category $\mathrm{IndCoh}(X)$ has a natural t-structure, compatible with filtered colimits, see [IndCoh, Sect. 1.2].

It is characterized by the property that an object of $\mathrm{IndCoh}(X)$ is connective (i.e., lies in $\mathrm{IndCoh}(X)^{\leq 0}$) if and only if its image under the functor $\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ is connective.

4.2.2. We define the right t-structure on $\mathrm{Crys}^r(X)$ by declaring that

$$\mathcal{M} \in \mathrm{Crys}^r(X)^{\geq 0} \ \Leftrightarrow\ \mathbf{oblv}^r_X(\mathcal{M}) \in \mathrm{IndCoh}(X)^{\geq 0}.$$

In what follows, we shall refer to the right t-structure on $\mathrm{Crys}^r(X)$ as "the" t-structure on crystals. In other words, by default the t-structure we shall consider will be the right one. By construction, this t-structure is also compatible with filtered colimits, since $\mathbf{oblv}^r_X$ is continuous.

4.2.3. We claim that the right t-structure on $\mathrm{Crys}^r(X)$ is Zariski-local, i.e., an object is connective/coconnective if and only if its restriction to a Zariski cover has this property. Indeed, this follows from the corresponding property of the t-structure on $\mathrm{IndCoh}(X)$, see [IndCoh, Corollaty 4.2.3].

4.2.4. *The right* t-*structure and Kashiwara's lemma.* Let $i : X \to Z$ be a closed embedding of DG schemes. Let $i_{\mathrm{dR},*}$ denote the functor $\mathrm{Crys}^r(X) \to \mathrm{Crys}^r(Z)$ equal to the composition

$$\mathrm{Crys}^r(X) \overset{('i^{\dagger,r})^{-1}}{\longrightarrow} \mathrm{Crys}^r(Z)_X \hookrightarrow \mathrm{Crys}^r(Z),$$

which, by construction, is the left adjoint of $i^{\dagger,r}$.

We have:

Proposition 4.2.5. *The functor* $i_{\mathrm{dR},*} : \mathrm{Crys}^r(X) \to \mathrm{Crys}^r(Z)$ *is t-exact.*

Proof. Note that the full subcategory

$$\mathrm{Crys}^r(Z)_X \subset \mathrm{Crys}^r(Z)$$

is compatible with the t-structure, since it is the kernel of the functor $j^{\dagger,r}$, which is t-exact (here j denotes the open embedding $Z - X \hookrightarrow Z$).

Hence, it remains to show that the functor

$$'i^{\dagger,r} : \mathrm{Crys}^r(Z)_X \to \mathrm{Crys}^r(X)$$

is t-exact.

Thus, we need to show that for $\mathcal{M} \in \mathrm{Crys}^r(Z)_X$ we have:

$$\mathbf{oblv}^r_Z(\mathcal{M}) \in \mathrm{IndCoh}(Z)^{>0} \Leftrightarrow \mathbf{oblv}^r_X(i^{\dagger,r}(\mathcal{M})) \in \mathrm{IndCoh}(X)^{>0}.$$

Recall the notation

$$\mathrm{IndCoh}(Z)_X := \ker\left(j^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z - X)\right).$$

It suffices to show the following:

Lemma 4.2.6. *Let $i : X \to Z$ be a closed embedding. Then for $\mathcal{F} \in \mathrm{IndCoh}(Z)_X$ we have:*

$$\mathcal{F} \in \mathrm{IndCoh}(Z)^{>0} \Leftrightarrow i^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{>0}.$$

$\square$

Proof of Lemma 4.2.6. The $\Rightarrow$ implication follows from the fact that $i^!$ is left t-exact (being the right adjoint of the t-exact functor, namely, i_*^{IndCoh}).

For the converse implication, note that the full subcategory

$$\mathrm{IndCoh}(Z)_X \subset \mathrm{IndCoh}(Z)$$

is also compatible with the t-structure, since it is the kernel of the t-exact functor $j^!$. Furthermore, it follows from [IndCoh, Proposition 4.1.7(b)] that the t-structure on $\mathrm{IndCoh}(Z)_X$ is generated by the t-structure on

$$\mathrm{Coh}(Z)_X := \ker\left(j^* : \mathrm{Coh}(Z) \to \mathrm{Coh}(Z - X)\right).$$

Let $\mathcal{F} \in \mathrm{IndCoh}(Z)_X$ be such that $i^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{>0}$. We need to show that $\mathcal{F}$ is right-orthogonal to $(\mathrm{Coh}(Z)_X)^{\leq 0}$. By assumption, $\mathcal{F}$ is right-orthogonal to the essential image of $\mathrm{Coh}(X)^{\leq 0}$ under

$$\mathrm{Coh}(X) \xrightarrow{i_*} \mathrm{Coh}(Z)_X \to \mathrm{IndCoh}(Z)_X.$$

However, it is easy to see that every object of $(\mathrm{Coh}(Z)_X)^{\leq 0}$ can be obtained as a finite successive extension of objects in the essential image of $\mathrm{Coh}(X)^{\leq 0}$, which implies the required assertion.

$\square$

Corollary 4.2.7. *If a map $X_1 \to X_2$ of DG schemes induces an isomorphism*

$$^{cl,red}X_1 \to {}^{cl,red}X_2,$$

then the corresponding t-structures on $\mathrm{Crys}^r(X_1) \simeq \mathrm{Crys}^r(X_2)$ coincide.

4.2.8. Let X be a DG scheme. By construction, the forgetful functor $\mathbf{oblv}^r_X$ is left t-exact. Hence, by adjunction, the functor $\mathbf{ind}^r_X$ is right t-exact.

We now claim:

Proposition 4.2.9. *The functor* $\mathbf{ind}^r_X$ *is t-exact.*

Proof. It suffices to show that the composition $\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X$ is left t-exact. We deduce this from Proposition 3.3.2(b):

The functor $p_s^!$ is left t-exact (e.g., by Lemma 4.2.6 applied to $\Delta_X : X \to X \times X$). The functor $(p_t)_*^{\mathrm{IndCoh}}$ is left t-exact (in fact, t-exact) by [GR1, Lemma 2.7.11].

□

4.2.10. We now claim:

Proposition 4.2.11.

(a) *If X is a smooth classical scheme, then* $\mathbf{oblv}^r_X$ *is t-exact.*

(b) *For a quasi-compact DG scheme X, the functor* $\mathbf{oblv}^r_X$ *is of bounded cohomological amplitude.*

Proof. Let X be a smooth classical scheme. By the definition of the t-structure on $\mathrm{Crys}^r(X)$, the essential image of $\mathrm{IndCoh}(X)^{\leq 0}$ under $\mathbf{ind}^r_X$ generates $\mathrm{Crys}^r(X)^{\leq 0}$ by taking colimits. Hence, in order to show that $\mathbf{oblv}^r_X$ is right t-exact, it suffices to show the same for the functor $\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X$. We will deduce this from Proposition 3.3.2(b):

We can write

$$(X \times X)^\wedge_X \simeq \underset{n}{colim}\, X_n,$$

where $X_n \overset{i_n}{\to} X \times X$ is the n-th infinitesimal neighborhood of the diagonal. Hence, by [GR1, Equation (2.2)],

$$(p_t)_*^{\mathrm{IndCoh}} \circ p_s^! \simeq \underset{n}{colim}\, (p_t \circ i_n)_*^{\mathrm{IndCoh}} \circ (p_s \circ i_n)^!.$$

Now, each of the functors $(p_t \circ i_n)_*^{\mathrm{IndCoh}}$ is t-exact by [GR1, Lemma 2.7.11], and each of the functors $(p_s \circ i_n)^!$ is t-exact because $p_s \circ i_n : X_n \to X$ is finite and flat.

Now, let X be a quasi-compact DG scheme, and let us show that $\mathbf{oblv}^r_X$ is of bounded cohomological amplitude. The question readily reduces to the case when X is affine, and let $i : X \hookrightarrow Z$ be a closed embedding, where Z is smooth. By Proposition 4.2.5 and point (a), it suffices to show that the functor $i^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(X)$ is of bounded cohomological amplitude, but the latter follows easily from the fact that Z is regular.

□

4.3. **Right t-structure on crystals on indschemes.**

4.3.1. Let $\mathcal{X}$ be a DG indscheme. Fix a presentation of $\mathcal{X}$

$$\mathcal{X} = \underset{\alpha}{colim}\ X_\alpha \tag{4.1}$$

as in [GR1, Prop. 1.7.6]. For each α, let i_α denote the corresponding closed embedding $X_\alpha \to \mathcal{X}$, and for each $\alpha_1 \to \alpha_2$ let i_{α_1,α_2} denote the closed embedding $X_{\alpha_1} \to X_{\alpha_2}$.

We have:

$$\mathrm{Crys}^r(\mathcal{X}) \simeq \underset{\alpha}{lim}\ \mathrm{Crys}^r(X_\alpha),$$

where for $\alpha_1 \to \alpha_2$, the functor $\mathrm{Crys}^r(X_{\alpha_2}) \to \mathrm{Crys}^r(X_{\alpha_1})$ is given by $i^{\dagger,r}_{\alpha_1,\alpha_2}$.

Hence, by [GL:DG, Sect. 1.3.3], we have that

$$\mathrm{Crys}^r(\mathcal{X}) \simeq \underset{\alpha}{colim}\ \mathrm{Crys}^r(X_\alpha),$$

where for $\alpha_1 \to \alpha_2$, the functor $\mathrm{Crys}^r(X_{\alpha_1}) \to \mathrm{Crys}^r(X_{\alpha_2})$ is given by $(i_{\alpha_1,\alpha_2})_{\mathrm{dR},*}$.

In particular, for each α, we obtain a pair of adjoint functors

$$(i_\alpha)_{\mathrm{dR},*} : \mathrm{Crys}^r(X) \rightleftarrows \mathrm{Crys}^r(\mathcal{X}) : i^{\dagger,r}_\alpha.$$

4.3.2. Recall from [GR1, Sect. 2.5] that $\mathrm{IndCoh}(\mathcal{X})$ has a natural t-structure compatible with filtered colimits.

Using this t-structure on $\mathrm{IndCoh}(\mathcal{X})$, we can define the right t-structure on $\mathrm{Crys}^r(\mathcal{X})$. Namely, we have

$$\mathcal{M} \in \mathrm{Crys}^r(\mathcal{X})^{\geq 0} \;\Leftrightarrow\; \mathbf{oblv}^r_{\mathcal{X}}(\mathcal{M}) \in \mathrm{IndCoh}(\mathcal{X})^{\geq 0}$$

Since $\mathbf{oblv}^r_{\mathcal{X}}$ preserves colimits, this t-structure is compatible with filtered colimits. We can describe this t-structure more explicitly using the presentation (4.1), in a way analogous to [GR1, Lemma 2.5.3] for the t-structure on $\mathrm{IndCoh}(\mathcal{X})$:

Lemma 4.3.3. *Under the above circumstances, we have:*

(a) *An object* $\mathcal{F} \in \mathrm{Crys}^r(\mathcal{X})$ *belongs to* $\mathrm{Crys}^r(\mathcal{X})^{\geq 0}$ *if and only if for every* α*, the object* $i_\alpha^{\dagger,r}(\mathcal{F}) \in \mathrm{Crys}^r(X_\alpha)$ *belongs to* $\mathrm{Crys}^r(X_\alpha)^{\geq 0}$*.*

(b) *The category* $\mathrm{Crys}^r(\mathcal{X})^{\leq 0}$ *is generated under colimits by the essential images of the functors* $(i_\alpha)_{\mathrm{dR},*}\big(\mathrm{Crys}^r(X_\alpha)^{\leq 0}\big)$*.*

Proof. Point (a) follows from the definition and [GR1, Lemma 2.5.3(a)]. Point (b) follows formally from point (a). ☐

4.3.4. Suppose that $i : X \to \mathcal{X}$ is a closed embedding of a DG scheme into a DG indscheme. By the exact same argument as in [GR1, Lemma 2.5.5], we have:

Lemma 4.3.5. *The functor* $i_{\mathrm{dR},*}$ *is t-exact.*

4.3.6. As an illustration of the behavior of the above t-structure on right crystals over a DG indscheme, let us consider the following situation. Let $i : X \to Z$ be a closed embedding of quasi-compact DG schemes. Let Y denote the formal completion of X in Z, considered as an object of $\mathrm{DGindSch}_{\mathrm{laft}}$; let $'i$ denote the resulting map $X \to Y$.

We claim:

Lemma 4.3.7. *The equivalence* $\mathrm{Crys}^r(X) \simeq \mathrm{Crys}^r(Y)$*, induced by the isomorphism* $'i_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$*, is compatible with the t-structures.*

Proof 1. Follows from Proposition 4.2.5 and the fact that the equivalence

$$\mathrm{IndCoh}(Y) \simeq \mathrm{IndCoh}(Z)_Y$$

of [GR1, Proposition 7.4.5] is compatible with the t-structures (see [GR1, Lemma 7.4.8]). □

Proof 2. From the commutative diagram

$$\begin{tikzcd}
\mathrm{Crys}^r(X) \arrow[d, "\mathbf{oblv}^r_X"'] & \mathrm{Crys}^r(Y) \arrow[l, "{}'i^{\dagger,r}"'] \arrow[d, "\mathbf{oblv}^r_Y"] \\
\mathrm{IndCoh}(X) & \mathrm{IndCoh}(Y). \arrow[l, "{}'i^!"']
\end{tikzcd}$$

we obtain that it suffices to show that for $\mathcal{F} \in \mathrm{IndCoh}(Y)$ we have

$$\mathcal{F} \in \mathrm{IndCoh}(Y)^{>0} \Leftrightarrow {}'i^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{>0},$$

which follows formally from [GR1, Lemma 7.4.8] and Lemma 4.2.6 (or can be easily proved directly). □

4.4. **Further properties of the left t-structure.**

4.4.1. First, let us describe the relation between the left and the right t-structures on crystals in the case of a smooth classical scheme.

Proposition 4.4.2. *Let X be a smooth classical scheme of dimension n. Then*

$$\mathcal{F} \in \mathrm{Crys}^l(X)^{\leq 0} \Leftrightarrow \mathcal{F} \in \mathrm{Crys}^r(X)^{\leq -n}.$$

I.e., the left t-structure agrees with the right t-structure up to a shift by the dimension of X.

Proof. Recall that the two forgetful functors are related by the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Crys}(X). \arrow[dl, "\mathbf{oblv}^l_X"'] \arrow[dr, "\mathbf{oblv}^r_X"] & \\
\mathrm{QCoh}(X) \arrow[rr, "\Upsilon_X"'] & & \mathrm{IndCoh}(X)
\end{tikzcd}$$

In the case that X is a smooth classical scheme of dimension n, the functior Υ_X is an equivalence and maps $\mathrm{QCoh}(X)^{\leq 0}$ isomorphically to $\mathrm{IndCoh}(X)^{\leq -n}$. The assertion now follows from Proposition 4.1.3 and Proposition 4.2.11(a), combined with the fact that $\mathbf{oblv}^r_X$ is conservative.

□

4.4.3. The next proposition compares the "left" and "right" t-structures on $\mathrm{Crys}(X)$ for an arbitrary DG scheme X.

Proposition 4.4.4. *Let X be quasi-compact. Then the identity functor*

$$\mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X)$$

has bounded amplitude, i.e. the difference between the left and right t-structures is bounded.

Proof. Without loss of generality, we can assume that X is affine. Let Z be a smooth classical scheme of dimension n; $i : X \hookrightarrow Z$ a closed embedding. We claim that for $\mathcal{M}^l \in \mathrm{Crys}^l(X)$ and the corresponding object $\mathcal{M}^r \in \mathrm{Crys}^r(X)$ we have

$$\begin{aligned} \mathcal{M}^l \in (\mathrm{Crys}^l(X))^{\leq 0} \;\Rightarrow\; \mathcal{M}^r \in (\mathrm{Crys}^r(X))^{\leq 0} \text{ and} \\ \mathcal{M}^r \in (\mathrm{Crys}^r(X))^{\leq 0} \;\Rightarrow\; \mathcal{M}^l \in (\mathrm{Crys}^l(X))^{\leq n}. \end{aligned} \tag{4.2}$$

Let $U \overset{j}{\hookrightarrow} Z$ denote the complementary open embedding. Let Y denote the formal completion of X in Z; let $\widehat{i}$ denote map $Y \to Z$.

The map $X \to Y$ defines an isomorphism $X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$, which allows to identify $\mathrm{Crys}^l(X) \simeq \mathrm{Crys}^l(Y)$. Applying Proposition 4.1.3, we have:

$$\mathcal{M}^l \in (\mathrm{Crys}^l(X))^{\leq 0} \;\Leftrightarrow\; \mathbf{oblv}^l_Y(\mathcal{M}^l) \in \mathrm{QCoh}(Y)^{\leq 0}, \tag{4.3}$$

where the t-structure on $\mathrm{QCoh}(Y)$ is that of Sect. 4.1.1.

Consider the subcategory $\mathrm{QCoh}(Z)_X \subset \mathrm{QCoh}(X)$ which is by definition equal to

$$\ker(j^* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(U)).$$

This subcategory is compatible with the t-structure on $\mathrm{QCoh}(Z)$, since the functor j^* is t-exact.

Recall (see [GR1, Proposition 7.1.3]) that the functor $\widehat{i}^*$ defines an equivalence

$$\mathrm{QCoh}(Z)_X \to \mathrm{QCoh}(Y).$$

Let $\mathcal{F}$ be the object of $\mathrm{QCoh}(Z)_X$ corresponding to $\mathbf{oblv}^l_Y(\mathcal{M}^l) \in \mathrm{QCoh}(Y)$. We have:

$$\Upsilon_Z(\mathcal{F}) \simeq \mathbf{oblv}^r_Z(i_{\mathrm{dR},*}(\mathcal{M}^r)).$$

Since the functor $i_{\mathrm{dR},*}$ is t-exact (Proposition 4.2.5), and since Υ_Z shifts cohomological degrees by $[-n]$, we have:

$$\mathcal{M}^r \in (\mathrm{Crys}^r(X))^{\leq 0} \Leftrightarrow \mathcal{F} \in (\mathrm{QCoh}(Z)_X)^{\leq n}. \tag{4.4}$$

Combining (4.3) and (4.4), the implications in (4.2) follow from the next assertion:

Lemma 4.4.5. *The equivalence* $\widehat{i}^* : \mathrm{QCoh}(Z)_X \simeq \mathrm{QCoh}(Y)$ *has the following properties with respect to the t-structure on* $\mathrm{QCoh}(Z)_X$ *inherited from* $\mathrm{QCoh}(Z)$ *and the t-structure on* $\mathrm{QCoh}(Y)$ *of Sect. 4.1.1:*

(a) *If* $\mathcal{F} \in (\mathrm{QCoh}(Z)_X)^{\leq 0}$ *then* $\widehat{i}^*(\mathcal{F}) \in \mathrm{QCoh}(Y)^{\leq 0}$.

(b) *If* $\widehat{i}^*(\mathcal{F}) \in \mathrm{QCoh}(Y)^{\leq 0}$, *then* $\mathcal{F} \in (\mathrm{QCoh}(Z)_X)^{\leq n}$.

□

Proof of Lemma 4.4.5. Point (a) follows from the fact that the functor $\widehat{i}^*$ is right t-exact.

To prove point (b), we note that the category $\mathrm{QCoh}(Y)^{\leq 0}$ is generated under taking colimits by the essential image of $\mathrm{QCoh}(Z)^{\leq 0}$ under the functor $\widehat{i}^*$, see [GR1, Proposition 7.3.5]. Hence, it is sufficient to show the the functor

$$\mathrm{QCoh}(Z) \xrightarrow{\widehat{i}^*} \mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(Z)_X$$

has cohomological amplitude bounded by n. However, the above functor is the right adjoint to the embedding

$$\mathrm{QCoh}(Z)_X \hookrightarrow \mathrm{QCoh}(Z),$$

and is given by

$$\mathcal{F}' \mapsto \mathrm{Cone}(\mathcal{F}' \to j_* \circ j^*(\mathcal{F}'))[-1].$$

Now, j^* is t-exact, and j_* is of cohomological amplitude bounded by $n-1$. This implies the required assertion.

□

4.4.6. Let X be an arbitrary quasi-compact DG scheme. We have:

Proposition 4.4.7.

(a) *The functor* $\mathbf{oblv}^l_X : \mathrm{Crys}(X) \to \mathrm{QCoh}(X)$ *has bounded cohomological amplitude.*

(b) *If X is eventually coconnective, the functor* $\mathbf{ind}^l_X : \mathrm{QCoh}(X) \to \mathrm{Crys}(X)$ *has cohomological amplitude bounded from above.*

Proof. For point (a) we can assume that X is affine and find a closed embedding $i : X \hookrightarrow Z$, where Z is a smooth classical scheme. In this case, the assertion follows from Proposition 4.2.5 and the fact that the functor

$$i^* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$$

has a bounded cohomological amplitude.

Point (b) follows from point (a) by the $(\mathbf{ind}^l_X, \mathbf{oblv}^l_X)$-adjunction.

□

Remark 4.4.8. The assumption that X be eventually coconnective in point (b) is essential; otherwise a counterexample can be provided by the DG scheme from Example 3.4.5. In addition, is it easy to show that $\mathbf{ind}^l_X$ has a cohomological amplitude bounded from below if and only if X is Gorenstein (see Lemma 4.6.12).

4.5. **Left completeness.**

4.5.1. Let X be an affine smooth classical scheme. We observe that in this case the category $\mathrm{Crys}^r(X)$ contains a canonical object

$$\mathbf{ind}^r_X(\mathcal{O}_X),$$

which lies in the heart of the t-structure (see Proposition 4.2.9), and is *projective*, i.e.,

$$H^0(\mathcal{N}) = 0 \Rightarrow \mathrm{Hom}_{\mathrm{Crys}^r(X)}(\mathbf{ind}^r_X(\mathcal{O}_X), \mathcal{N}) = 0.$$

Moreover, $\mathbf{ind}^r_X(\mathcal{O}_X)$ is a compact generator of $\mathrm{Crys}^r(X)$. This implies:

Corollary 4.5.2. *Let X be an affine smooth classical scheme. Then the category* $\mathrm{Crys}^r(X)$ *is left-complete in its t-structure.*

4.5.3. The above corollary implies left-completeness for any DG scheme X:

Corollary 4.5.4. *For any DG scheme X, the category* $\mathrm{Crys}^r(X)$ *is left-complete in the "right" t-structure.*

Proof. First, we note that the property of being left-complete is Zariski-local (proved by the same argument as [GL:QCoh, Proposition 5.2.4]). Hence, we can assume without loss of generality that X is affine. Choose a closed embedding $i : X \hookrightarrow Z$, where Z is a smooth classical scheme. Now the assertion follows formally from the fact that the functor $i_{\mathrm{dR},*}$ is continuous, fully faithful (by Proposition 2.5.6), t-exact (by Proposition 4.2.5), and the fact that $\mathrm{Crys}^r(Z)$ is left-complete (by the previous corollary).

Here is an alternative argument:

By Corollary 4.2.7, we can assume that X is eventually coconnective. In this case, the functor $\mathbf{oblv}^l_X$ commutes with limits, as it admits a left adjoint. Moreover, by Lemma 2.2.6, $\mathbf{oblv}^l_X$ is conservative, and by Proposition 4.4.7 it has bounded cohomological amplitude. Therefore, the fact that $\mathrm{QCoh}(X)$ is left-complete in its t-structure implies the corresponding fact for $\mathrm{Crys}^r(X)$.

□

Remark 4.5.5. The question of right completeness is not an issue: since our t-structures are compatible with filtered colimits, right completeness is equivalent to the t-structure being separated on the coconnective subcategory, which is evident since $\mathbf{oblv}^r_X$ is left t-exact and conservative, and the t-structure on $\mathrm{IndCoh}(X)^+ \overset{\Psi_X}{\simeq} \mathrm{QCoh}(X)^+$ has this property.

4.5.6. Combining Corollary 4.5.4 with Proposition 4.4.4, we obtain:

Corollary 4.5.7. *For a quasi-compact DG scheme X, the category* $\mathrm{Crys}(X)$ *is also left-complete in the "left" t-structure.*

4.6. **The "coarse" induction and forgetful functors.**

4.6.1. Let X be a DG scheme. Recall that the functor Ψ_X identifies the category $\mathrm{QCoh}(X)$ with the left-completion of $\mathrm{IndCoh}(X)$ (see [IndCoh, Proposition 1.3.4]).

Since the category $\mathrm{Crys}^r(X)$ is left-complete in its t-structure, and the functor $\mathbf{ind}^r_X$ is t-exact, by the universal property of left completions, we obtain:

Corollary 4.6.2. *The functor* $\mathbf{ind}^r_X$ *canonically factors as*

$$\mathrm{IndCoh}(X) \overset{\Psi_X}{\longrightarrow} \mathrm{QCoh}(X) \overset{'\mathbf{ind}^r_X}{\longrightarrow} \mathrm{Crys}^r(X).$$

4.6.3. We can also consider the functor

$$'\mathbf{oblv}^r_X : \mathrm{Crys}^r(X) \to \mathrm{QCoh}(X),$$

given by $\Psi_X \circ \mathbf{oblv}^r_X$, where $\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ is the functor of [IndCoh, Sect. 1.1.5].

It is clear that the functor $'\mathbf{oblv}^r_X$ has a finite cohomological amplitude. Indeed, the follows from the corresponding fact for $\mathbf{oblv}^r_X$ and the fact that Ψ_X is t-exact (see [IndCoh, Lemma 1.2.2]).

Proposition 4.6.4. *The functor* $'\mathbf{oblv}^r_X$ *is conservative.*

Proof. The assertion is Zariski-local, so we can assume that X is affine. Choose a closed embedding $i : X \to Z$, where Z is a smooth classical affine scheme.

Let $i^{\mathrm{QCoh},!} : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$ denote the right adjoint of $i_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Z)$.[5] It is easy to see that we have a canonical isomorphism of functors

$$\Psi_X \circ i^! \simeq i^{\mathrm{QCoh},!} \circ \Psi_Z.$$

Hence, for $\mathcal{M} \in \mathrm{Crys}^r(Z)$, we have

$$'\mathbf{oblv}^r_X(i^{\dagger,r}(\mathcal{M})) \simeq i^{\mathrm{QCoh},!}('\mathbf{oblv}^r_Z(\mathcal{M})).$$

Applying Kashiwara's lemma, the assertion of the proposition follows from the next lemma:

[5] Although this is irrelevant for us, we note that the $i^{\mathrm{QCoh},!}$ is continuous. This is because the functor $i_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Z)$ sends compact objects to compacts (since Z is regular, any coherent sheaf on it is perfect).

Lemma 4.6.5. *The functor* $i^{\mathrm{QCoh},!}:\mathrm{QCoh}(Z)\to\mathrm{QCoh}(X)$ *is conservative when restricted to* $\mathrm{QCoh}(Z)_X$*.*

□

Proof of Lemma 4.6.5. We need to show that the essential image of the functor $i_*:\mathrm{QCoh}(X)\to\mathrm{QCoh}(Z)$ generates $\mathrm{QCoh}(Z)_X$.

First, we claim that $\mathrm{QCoh}(Z)_X$ is generated by the subcategory of bounded objects, denoted $(\mathrm{QCoh}(Z)_X)^b$. This follows from the corresponding fact for $\mathrm{QCoh}(Z)$ and the fact that the inclusion $\mathrm{QCoh}(Z)_X\hookrightarrow\mathrm{QCoh}(Z)$ has a right adjoint of bounded cohomological amplitude. By devissage, we obtain that $\mathrm{QCoh}(Z)_X$ is generated by $(\mathrm{QCoh}(Z)_X)^\heartsuit$, and further by $(\mathrm{QCoh}(Z)_X)^\heartsuit\cap\mathrm{Coh}(Z)$.

However, it is clear that every object of $(\mathrm{QCoh}(Z)_X)^\heartsuit\cap\mathrm{Coh}(Z)$ is a finite extension of objects lying in the essential image of $\mathrm{Coh}(X)^\heartsuit$.

□

Remark 4.6.6. In the case when X is eventually coconnective we will give a cleaner proof of Proposition 4.6.4, below.

4.6.7. Assume now that X is eventually coconnective. Recall that in this case the functor Ψ_X admits a fully faithful left adjoint Ξ_X (see [IndCoh, Proposition 1.5.3]).

We observe:

Lemma 4.6.8. *There exists a canonical isomorphism* $'\mathbf{ind}^r_X\simeq\mathbf{ind}^r_X\circ\Xi_X$*.*

Proof. Follows from the isomorphisms $\mathbf{ind}^r_X\simeq{}'\mathbf{ind}^r_X\circ\Psi_X$ and $\Psi_X\circ\Xi_X\simeq\mathrm{Id}_{\mathrm{QCoh}(X)}$. □

Corollary 4.6.9. *The functors* $('\mathbf{ind}^r_X,{}'\mathbf{oblv}^r_X)$ *form an adjoint pair.*

Proof. Follows formally from Lemma 4.6.8 by adjunction. □

Remark 4.6.10. The functors $('\mathbf{ind}^r_X,{}'\mathbf{oblv}^r_X)$ are *not* adjoint unless X is eventually coconnective. Indeed, if X is not eventually coconnective, the functor $'\mathbf{ind}^r_X$

does not preserve compact objects: it sends $\mathcal{O}_X \in \mathrm{QCoh}(X)$ to a non-compact object of $\mathrm{Crys}^r(X)$.

Alternate proof of Proposition 4.6.4. By Corollary 4.6.9, the assertion of Proposition 4.6.4 (in the eventually coconnective case) is equivalent to the fact that the essential image of the functor $'\mathbf{ind}^r_X$ generates $\mathrm{Crys}^r(X)$. However, the latter is tautological from the corresponding fact for $\mathbf{ind}^r_X$.

□

4.6.11. Let X be an eventually coconnective DG scheme, and consider the pair of adjoint functors

$$\Xi_X : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) : \Psi_X$$

with Ξ_X being fully faithful (see [IndCoh, Sect. 1.4]).

We have seen that the functor $\mathbf{ind}^r_X$ factors through the colocalization functor Ψ_X. However, it is *not* true in general that the functor $\mathbf{oblv}^r_X$ factors through Ξ_X, i.e., that it takes values in $\mathrm{QCoh}(X)$, considered as a full subcategory of $\mathrm{IndCoh}(X)$ via Ξ_X.

In fact, the following holds:

Lemma 4.6.12 (Drinfeld). *The functor* $\mathbf{oblv}^r_X$ *factors through the essential image of* $\mathrm{QCoh}(X)$ *under* Ξ_X *if and only if* X *is Gorenstein.*

Recall that a DG scheme X is said to be Gorenstein if:

(a) $\omega_X \in \mathrm{Coh}(X)$ (which is equivalent to X being eventually coconnective, see [IndCoh, Proposition 9.6.11]);

(b) When considered as a coherent sheaf, ω_X is a graded line bundle (which is equivalent to $\omega_X \in \mathrm{QCoh}(X)^{\mathrm{perf}}$, see [IndCoh, Corollary 7.4.3]).

Proof. Suppose that $\mathbf{oblv}^r_X$ factors through $\mathrm{QCoh}(X)$. In particular, we obtain that $\omega_X \in \mathrm{Coh}(X)$ lies in the essential image of Ξ_X. Now the assertion follows from [IndCoh, Lemma 1.5.8].

For the opposite implication, we write $\mathbf{oblv}^r_X(\mathcal{M})$ as

$$\Upsilon_X(\mathbf{oblv}^l_X(\mathcal{M})) = \mathbf{oblv}^l_X(\mathcal{M}) \otimes \omega_X,$$

where the tensor product is understood in the sense of the action of $\mathrm{QCoh}(X)$ on $\mathrm{IndCoh}(X)$, see [IndCoh, Sect. 1.4]. Recall also that the functor Ξ_X is tautologically compatible with the above action of $\mathrm{QCoh}(X)$. Hence, if ω_X, being perfect, lies in the essential image of Ξ_X, then so does $\mathbf{oblv}^l_X(\mathcal{M}) \otimes \omega_X$

□

4.7. **Relation to the abelian category.** In this subsection we let X be an affine DG scheme. We will relate the category $\mathrm{Crys}^r(X)$ to a more familiar object.

4.7.1. Since the t-structure on $\mathrm{Crys}^r(X)$ is compatible with filtered colimits, we obtain that $\mathrm{Crys}^r(X)^\heartsuit$ is a Grothendieck abelian category.

Using the fact that $\mathrm{Crys}^r(X)$ is *right-complete* in its structure, by reversal of arrows in [Lu2, Theorem 1.3.2.2], we obtain a canonically defined t-exact functor

$$D\left(\mathrm{Crys}^r(X)^\heartsuit\right)^+ \to \mathrm{Crys}^r(X)^+, \tag{4.5}$$

where $D(-)^+$ denotes the eventually coconnective part of the derived category of the abelian category.

4.7.2. We are going to prove:

Proposition 4.7.3. *The functor* (4.5) *uniquely extends to an equivalence of categories*

$$D\left(\mathrm{Crys}^r(X)^\heartsuit\right) \to \mathrm{Crys}^r(X).$$

The rest of this subsection is devoted to the proof of Proposition 4.7.3. Without loss of generality, we can assume that X is classical.

4.7.4. *Step 1.* Assume first that X is a smooth classical scheme. In this case the assertion is obvious from the fact that

$$\mathbf{ind}^r_X(\mathcal{O}_X)$$

is a compact projective generator for both categories.

4.7.5. *Step 2.* Let us show that the functor

$$D\left(\mathrm{Crys}^r(X)^{\heartsuit}\right)^+ \to \mathrm{Crys}^r(X)^+$$

is an equivalence. For this, it suffices to show that every object $\mathcal{M} \in \mathrm{Crys}^r(X)^{\heartsuit}$ can be embedded in an *injective* object, i.e., an object $\mathcal{I} \in \mathrm{Crys}^r(X)^{\heartsuit}$ such that

$$H^0(\mathcal{N}) = 0 \Rightarrow \mathrm{Hom}_{\mathrm{Crys}^r(X)}(\mathcal{N}, \mathcal{I}) = 0.$$

Let $i : X \hookrightarrow Z$ be a closed embedding, where Z is a smooth classical scheme. Choose an embedding $i_{\mathrm{dR},*}(\mathcal{M}) \hookrightarrow \mathcal{J}$, where $\mathcal{J}$ is an injective object (in the same sense) in $\mathrm{Crys}^r(Z)$; it exists by Step 1.

Since the functor $\mathbf{ind}^r_Z$ is t-exact, we obtain that $\mathbf{oblv}^r_Z(\mathcal{J})$ is an injective object of $\mathrm{QCoh}(Z)^{\heartsuit}$. This implies that $\mathcal{I} := i^{\dagger,r}(\mathcal{J})$ belongs to $\mathrm{Crys}^r(X)^{\heartsuit}$ and has the required property.

4.7.6. *Step 3.* We note that by Corollary 4.5.4, the category $\mathrm{Crys}^r(X)$ identifies with the left completion of $\mathrm{Crys}^r(X)^+$. Hence, it is enough to show that the canonical embedding

$$D\left(\mathrm{Crys}^r(X)^{\heartsuit}\right)^+ \hookrightarrow D\left(\mathrm{Crys}^r(X)^{\heartsuit}\right)$$

identifies $D\left(\mathrm{Crys}^r(X)^{\heartsuit}\right)$ with the left completion of $D\left(\mathrm{Crys}^r(X)^{\heartsuit}\right)^+$.

For that it suffices to exhibit a generator $\mathcal{P}$ of $\mathrm{Crys}^r(X)^{\heartsuit}$ of *bounded Ext dimension.*

Consider the object

$$\mathcal{P} := \mathbf{ind}^r_X(\mathcal{O}_X).$$

It has the required property by Proposition 4.2.11(b).

□

Remark 4.7.7. A standard argument allows us to extend the statement of Proposition 4.7.3 to the case when X is a quasi-compact DG scheme with an affine diagonal.

Remark 4.7.8. Once we identify crystals with D-modules on smooth affine classical schemes, we will obtain many other properties of $\mathrm{Crys}^r(X)$ on quasi-compact

DG schemes: e.g., the fact that the abelian category $\mathrm{Crys}^r(X)^\heartsuit$ is locally Noetherian[6] and that $\mathrm{Crys}^r(X)$ has finite cohomological dimension with respect to its t-structure. [7] Note that by Proposition 4.2.5, in order to establish both these properties, it suffices to show them for smooth affine classical schemes.

5. Relation to D-modules

In this section we will relate the monads $\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X$ and $\mathbf{oblv}^l_X \circ \mathbf{ind}^l_X$ to the sheaf of differential operators. As a result we relate the category Crys^r over a DG scheme to the (derived) category of D-modules.

5.1. **Crystals via an integral transform.**

In this subsection we let $\mathcal{X}$ be a DG indscheme locally almost of finite type.

5.1.1. Recall that for $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$, the category $\mathrm{IndCoh}(\mathcal{X})$ is dualizable and canonically self-dual, see [GR1, Sect. 2.6].

Hence, for $\mathcal{X}, \mathcal{Y} \in \mathrm{DGindSch}$, the category $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{Y}))$ identifies with

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{Y}) \simeq \mathrm{IndCoh}(\mathcal{X} \times \mathcal{Y}).$$

Expilcitly, an object $\mathcal{Q} \in \mathrm{IndCoh}(\mathcal{X} \times \mathcal{Y})$ defines a functor $\mathsf{F}_{\mathcal{Q}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$ by

$$\mathcal{F} \mapsto (p_2)^{\mathrm{IndCoh}}_* \circ (\Delta_{\mathcal{X}} \times \mathrm{id}_{\mathcal{Y}})^!(\mathcal{F} \boxtimes \mathcal{Q}), \tag{5.1}$$

where $p_2 : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ is the projection map and $\Delta_{\mathcal{X}}$ is the diagonal map $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$,

In particular, the endo-functor $\mathbf{oblv}^r_{\mathcal{X}} \circ \mathbf{ind}^r_{\mathcal{X}}$ defines an object, denoted

$$\mathcal{D}^r_{\mathcal{X}} \in \mathrm{IndCoh}(\mathcal{X} \times \mathcal{X}).$$

We will identify this object.

---

[6]By this we mean that $\mathrm{Crys}^r(X)^\heartsuit$ is generated by its compact objects, and a subobject of a compact one is compact.

[7]By this we mean that there exists $N \in \mathbb{N}$ such that for $n > N$, $\mathrm{Hom}_{\mathrm{Crys}^r(X)}(\mathcal{M}_1, \mathcal{M}_2[n]) = 0$ for $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Crys}^r(X)^\heartsuit$.

5.1.2. Let $\widehat{\Delta}_{\mathcal{X}}$ denote the map

$$\mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X} \simeq (\mathcal{X} \times \mathcal{X})^{\wedge}_{\mathcal{X}} \to \mathcal{X} \times \mathcal{X}.$$

Proposition 5.1.3. *There is a canonical isomorphism in* $\mathrm{IndCoh}(\mathcal{X} \times \mathcal{X})$

$$\mathcal{D}^r_{\mathcal{X}} \simeq (\widehat{\Delta}_{\mathcal{X}})^{\mathrm{IndCoh}}_*(\omega_{\mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}}).$$

Proof. We begin with the following general observation.

Suppose that we have a functor $\mathsf{F} \in \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{Y}))$ given by a correspondence, i.e. we have a diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \mathcal{X} & & \mathcal{Y} \end{tikzcd}$$

of DG indschemes, and $\mathsf{F} := (q_2)^{\mathrm{IndCoh}}_* \circ q_1^!$. Let

$$i : \mathcal{Z} \to \mathcal{X} \times \mathcal{Y}$$

be the induced product map.

Lemma 5.1.4. *In the above situation, the functor*

$$(q_2)^{\mathrm{IndCoh}}_* \circ q_1^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$$

is given by the kernel $\mathcal{Q} = i^{\mathrm{IndCoh}}_*(\omega_{\mathcal{Z}})$.

Proof. We have a diagram, whose inner square is Cartesian

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "q_1 \times \mathrm{id}_{\mathcal{Z}}"] \arrow[d, "i"'] & \mathcal{X} \times \mathcal{Z} \arrow[r] \arrow[d, "\mathrm{id}_X \times i"] & \mathcal{X} \\ \mathcal{X} \times \mathcal{Y} \arrow[r, "\Delta_X \times \mathrm{id}_{\mathcal{Y}}"] \arrow[d, "p_2"'] & \mathcal{X} \times \mathcal{X} \times \mathcal{Y} & \\ \mathcal{Y}. & & \end{tikzcd}$$

For $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, we have

$$(q_2)^{\mathrm{IndCoh}}_* \circ q_1^!(\mathcal{F}) \simeq (p_2)^{\mathrm{IndCoh}}_* \circ i^{\mathrm{IndCoh}}_* \circ (q_1 \times \mathrm{id}_{\mathcal{Z}})^!(\mathcal{F} \boxtimes \omega_Z)$$

By [GR1, Proposition 2.9.2],[8]

$$i_*^{\mathrm{IndCoh}} \circ (q_1 \times \mathrm{id}_{\mathcal{Z}})^!(\mathcal{F} \boxtimes \omega_Z) \simeq (\Delta_{\mathcal{X}} \times \mathrm{id}_{\mathcal{Y}})^!(\mathcal{F} \boxtimes i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Z}})).$$

□

We apply this lemma to prove Proposition 5.1.3 as follows:

By Proposition 3.3.2(b), we have that the functor $\mathbf{oblv}^r_{\mathcal{X}} \circ \mathbf{ind}^r_{\mathcal{X}}$ is given by the correspondence

$$\begin{tikzcd} & (\mathcal{X} \times \mathcal{X})^\wedge_{\mathcal{X}} \arrow[dl, "p_s"'] \arrow[dr, "p_t"] & \\ \mathcal{X} & & \mathcal{X}. \end{tikzcd}$$

The assertion now follows from Lemma 5.1.4.

□

5.1.5. As a corollary of Proposition 5.1.3 we obtain:

Corollary 5.1.6. *There exists a canonical isomorphism* $\sigma(\mathcal{D}^r_{\mathcal{X}}) \simeq \mathcal{D}^r_{\mathcal{X}}$, *where* σ *is the transposition of factors acting on* $\mathcal{X} \times \mathcal{X}$.

5.2. **Explicit formulas for other functors.** In this subsection we let X be an eventually coconnective quasi-compact DG scheme almost of finite type.

5.2.1. Recall that the category $\mathrm{QCoh}(X)$ is also compactly generated and self-dual. Under the identifications

$$\mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X),$$

the dual of the functor Υ_X is the functor Ψ_X of [IndCoh, Sect. 1.1.5] (see [IndCoh, Proposition 9.3.3] for the duality statement).

In particular, for $\mathbf{C}' \in \mathrm{DGCat}_{\mathrm{cont}}$, we have

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathbf{C}') \simeq \mathrm{QCoh}(X) \otimes \mathbf{C}',$$

by a formula similar to (5.1).

[8]Strictly speaking, the base change isomorphism was stated in [GR1, Proposition 2.9.2] only in the case when the vertical arrow is ind-proper, which translates into i being proper. For the proof of Proposition 5.1.3 we will apply it in such a situation.

5.2.2. Let $\mathbf{C}$ be any of the categories

$$\mathrm{QCoh}(X\times X)\simeq \mathrm{QCoh}(X)\otimes\mathrm{QCoh}(X),\ \mathrm{IndCoh}(X\times X)\simeq \mathrm{IndCoh}(X)\otimes\mathrm{IndCoh}(X),$$

$$\mathrm{QCoh}(X)\otimes \mathrm{IndCoh}(X) \text{ or } \mathrm{IndCoh}(X)\otimes \mathrm{QCoh}(X).$$

Then $\mathbf{C}$ is a module over $\mathrm{QCoh}(X\times X)$, and we define an endo-functor of $\mathbf{C}$, denoted

$$\mathcal{F}\mapsto \mathcal{F}_{\{X\}}$$

given by tensor product with the object

$$\mathrm{Cone}(\mathcal{O}_{X\times X}\to j_*\circ j^*(\mathcal{O}_{X\times X}))[-1],$$

where j is the open embedding $X\times X-X\hookrightarrow X\times X$.

Note that by [IndCoh, Proposition 4.1.7 and Corollary 4.4.3], for $\mathbf{C} = \mathrm{IndCoh}(X\times X)$ this functor identifies with

$$(\widehat{\Delta}_X)_*^{\mathrm{IndCoh}}\circ(\widehat{\Delta}_X)^!,$$

where we recall that $\widehat{\Delta}_X$ denotes the map

$$X\underset{X_{\mathrm{dR}}}{\times} X\simeq (X\times X)^\wedge_X\to X\times X.$$

5.2.3. We claim:

Proposition 5.2.4.

(a) *The object of*

$$\mathrm{QCoh}(X)\otimes\mathrm{QCoh}(X)\simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X),\mathrm{QCoh}(X)),$$

corresponding to $\mathbf{oblv}^l_X\circ\mathbf{ind}^l_X$, *is canonically identified with*

$$(\Psi_X(\omega_X)\boxtimes\mathcal{O}_X)_{\{X\}}.$$

(b) *The object of*

$$\mathrm{QCoh}(X)\otimes\mathrm{IndCoh}(X)\simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X),\mathrm{IndCoh}(X)),$$

corresponding to $\mathbf{oblv}^r_X\circ\Upsilon_{X_{\mathrm{dR}}}\circ\mathbf{ind}^l_X$, *is canonically identified with*

$$(\Psi_X(\omega_X)\boxtimes\omega_X)_{\{X\}}.$$

(c) *The object of*

$$\mathrm{IndCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{QCoh}(X)),$$

corresponding to $\mathbf{oblv}^l_X \circ (\Upsilon_{X_{\mathrm{dR}}})^{-1} \circ \mathbf{ind}^r_X$, *is canonically identified with*

$$(\omega_X \boxtimes \mathcal{O}_X)_{\{X\}}.$$

(d) *The object of*

$$\mathrm{QCoh}(X) \boxtimes \mathrm{IndCoh}(X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{IndCoh}(X)),$$

corresponding to $\mathbf{oblv}^r_X \circ {}'\mathbf{ind}^r_X$, *is canonically identified with*

$$(\mathcal{O}_X \boxtimes \omega_X)_{\{X\}}.$$

(e) *The object of*

$$\mathrm{IndCoh}(X) \boxtimes \mathrm{QCoh}(X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{QCoh}(X)),$$

corresponding to ${}'\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X$, *is canonically identified with*

$$(\omega_X \boxtimes \Psi_X(\omega_X))_{\{X\}}.$$

(f) *The object of*

$$\mathrm{QCoh}(X \times X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

corresponding to ${}'\mathbf{oblv}^r_X \circ {}'\mathbf{ind}^r_X$, *is canonically identified with*

$$(\mathcal{O}_X \boxtimes \Psi_X(\omega_X))_{\{X\}}.$$

(g) *The object of*

$$\mathrm{QCoh}(X \times X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

corresponding to ${}'\mathbf{oblv}^r_X \circ \Upsilon_{X_{\mathrm{dR}}} \circ \mathbf{ind}^l_X$, *is canonically identified with*

$$(\Psi_X(\omega_X) \boxtimes \Psi_X(\omega_X))_{\{X\}}.$$

(h) *The object of*

$$\mathrm{QCoh}(X \times X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

corresponding to $\mathbf{oblv}^l_X \circ (\Upsilon_{X_{\mathrm{dR}}})^{-1} \circ {}'\mathbf{ind}^r_X$, *is canonically identified with*

$$(\mathcal{O}_X \boxtimes \mathcal{O}_X)_{\{X\}}.$$

Proof. Let $\mathbf{C}$ and $\mathbf{D}$ be objects of $\mathrm{DGCat}_{\mathrm{cont}}$ with $\mathbf{C}$ dualizable, so that

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C},\mathbf{D}) \simeq \mathbf{C}^\vee \otimes \mathbf{D}.$$

Let $\mathsf{F}:\mathbf{C}_1 \to \mathbf{C}$ and $\mathsf{G}:\mathbf{D}\to\mathbf{D}_1$ be continuous functors. Then the resulting functor

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C},\mathbf{D}) \to \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1,\mathbf{D}_1)$$

is given by

$$(\mathsf{F}^\vee \otimes \mathsf{G}) : \mathbf{C}^\vee \otimes \mathbf{D} \to \mathbf{C}_1^\vee \otimes \mathbf{D}_1.$$

With this in mind, we have:

Points (a) and (c) follow by combining Proposition 5.1.3, Lemma 3.4.9, and the following assertion:

Lemma 5.2.5. *The unit of the adjunction* $\mathrm{Id}_{\mathrm{QCoh}(X)} \to \Xi_X^\vee \circ \Upsilon_X$ *defines an isomorphism*

$$\mathcal{O}_X \to \Xi^\vee(\omega_X).$$

Point (b) follows from Proposition 5.1.3 using $\mathbf{ind}_X^l \simeq (\Upsilon_{X_{\mathrm{dR}}})^{-1} \circ \mathbf{ind}_X^r \circ \Upsilon_X$.

Point (d) follows from Proposition 5.1.3 using the isomorphism $'\mathbf{ind}_X^r \simeq \mathbf{ind}_X^r \circ \Xi_X^\vee$ and Lemma 5.2.5.

Point (e) follows from Proposition 5.1.3. Point (f) follows from point (d). Point (g) follows from point (b).

Point (h) follows from point (d) using Lemma 3.4.9 and Lemma 5.2.5.

□

5.2.6. Let $\mathcal{D}_X^l$, $\mathcal{D}_X^{l\to r'}$, $\mathcal{D}_X^{r'\to l}$ and $\mathcal{D}_X^{r'}$ denote the objects of

$$\mathrm{QCoh}(X)\otimes\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X\times X)$$

corresponding to the functors

$$\mathbf{oblv}_X^l \circ \mathbf{ind}_X^l,\ '\mathbf{oblv}_X^r \circ \Upsilon_{X_{\mathrm{dR}}} \circ \mathbf{ind}_X^l,\ \mathbf{oblv}_X^l \circ (\Upsilon_{X_{\mathrm{dR}}})^{-1} \circ {'\mathbf{ind}_X^r} \text{ and } '\mathbf{oblv}_X^r \circ {'\mathbf{ind}_X^r},$$

respectively.

We have:

Proposition 5.2.7. *The objects* $\mathcal{D}^r_X$, $\mathcal{D}^l_X$, $\mathcal{D}^{l\to r'}_X$, $\mathcal{D}^{r'\to l}_X$ *and* $\mathcal{D}^{r'}_X$ *are related by*

(i) $\mathcal{D}^l_X \simeq (\Psi_X \boxtimes \Xi^\vee_X)(\mathcal{D}^r_X) \in \mathrm{QCoh}(X \times X)$*;*

(ii) $(\Psi_X \boxtimes \mathrm{Id}_{\mathrm{IndCoh}(X)})(\mathcal{D}^r_X) \simeq (\mathrm{Id}_{\mathrm{QCoh}(X)} \boxtimes \Upsilon_X)(\mathcal{D}^l_X) \in \mathrm{QCoh}(X) \otimes \mathrm{IndCoh}(X)$*;*

(iii) $\mathcal{D}^{l\to r'}_X \simeq (\Psi_X \boxtimes \Psi_X)(\mathcal{D}^r_X) \in \mathrm{QCoh}(X \times X)$*;*

(iii') $\mathcal{D}^{l\to r'}_X \simeq (\mathrm{Id}_{\mathrm{QCoh}(X)} \boxtimes \Psi_X \circ \Upsilon_X)(\mathcal{D}^l_X) \simeq (\mathcal{O}_X \boxtimes \Psi_X(\omega_X)) \underset{\mathcal{O}_{X\times X}}{\otimes} \mathcal{D}^l_X \in \mathrm{QCoh}(X \times X)$*;*

(iv) $\mathcal{D}^{r'\to l}_X \simeq (\Xi^\vee_X \boxtimes \Xi^\vee_X)(\mathcal{D}^r_X)$*.*

(v) $\mathcal{D}^{r'} \simeq (\Xi^\vee_X \boxtimes \Psi_X)(\mathcal{D}^r_X) \in \mathrm{QCoh}(X \times X)$*.*

Proof. Point (i) follows from Lemma 3.4.9.

Point (ii) follows from the (tautological) isomorphism of functors

$$\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X \circ \Upsilon_X \simeq \mathbf{oblv}^r_X \circ \Upsilon_{X_{\mathrm{dR}}} \circ \mathbf{ind}^l_X \simeq \Upsilon_X \circ \mathbf{oblv}^l_X \circ \mathbf{ind}^l_X.$$

Point (iii) is tautological.

Point (iii') follows by combining points (ii) and (iii).

Point (iv) follows from Lemma 3.4.9.

Point (v) is tautological.

□

5.3. **Behavior with respect to the t-structure.** We continue to assume that X is a quasi-compact DG scheme almost of finite type.

5.3.1. We note:

Lemma 5.3.2. *The object* $\mathcal{D}^r_X$ *is bounded below, i.e., belongs to* $\mathrm{IndCoh}(X\times X)^+$*.*

Proof. Follows from Proposition 5.1.3, using the fact that $\omega_X \in \mathrm{IndCoh}(X)^+$, and the fact that the functor

$$\mathcal{F} \mapsto \mathcal{F}_{\{X\}}, \quad \mathrm{IndCoh}(X \times X) \to \mathrm{IndCoh}(X \times X)$$

is right t-exact. □

5.3.3. Assume now that X is eventually coconnective. We claim:

Proposition 5.3.4. *The objects $\mathcal{D}^l_X$, $\mathcal{D}^{l \to r'}_X$, $\mathcal{D}^{r' \to l}_X$ and $\mathcal{D}^{r'}_X$ of* $\mathrm{QCoh}(X \times X)$ *are all eventually coconnective, i.e., belong to* $\mathrm{QCoh}(X \times X)^+$.

Proof. Follows from Proposition 5.2.4, using the fact that $\Psi_X(\omega_X), \mathcal{O}_X \in \mathrm{QCoh}(X)^+$ and the fact that the functor

$$\mathcal{F} \mapsto \mathcal{F}_{\{X\}}, \quad \mathrm{QCoh}(X \times X) \to \mathrm{QCoh}(X \times X)$$

is right t-exact. □

5.3.5. Finally, let us assume that X is a smooth classical scheme. We claim:

Proposition 5.3.6. *The object $\mathcal{D}^l_X \in$* $\mathrm{QCoh}(X \times X)$ *lies in the heart of the t-structure.*

Proof. The assertion is Zariski-local, hence, we can assume that X is affine. It is sufficient to show that

$$(p_2)_*(\mathcal{D}^l_X) \in \mathrm{QCoh}(X)$$

lies in the heart of the t-structure. We have,

$$(p_2)_*(\mathcal{D}^l_X) \simeq \mathbf{oblv}^l_X \circ \mathbf{ind}^l_X(\mathcal{O}_X) \simeq \Xi^\vee_X \circ (\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X) \circ \Upsilon_X(\mathcal{O}_X).$$

Now, the functor $\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X$ is t-exact (see Proposition 4.2.11), the functor Υ_X is an equivalence that shifts degrees by $[n]$, and $\Xi^\vee_X$ is the inverse of Υ_X. □

5.4. **Relation to the sheaf of differential operators.** In this subsection we shall take X to be a smooth classical scheme. We are going to identify $\mathcal{D}^l_X$ with the object of $\mathrm{QCoh}(X \times X)$ underlying the classical sheaf of differential operators Diff_X.

5.4.1. For any $\mathcal{Q} \in \mathrm{QCoh}(X \times X)^{\heartsuit}$, which is set-theoretically supported on the diagonal, and $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(X)^{\heartsuit}$, a datum of a map

$$(p_2)_*(p_1^*(\mathcal{F}_1) \otimes \mathcal{Q}) \to \mathcal{F}_2$$

is equivalent to that of a map

$$\mathcal{Q} \to \mathrm{Diff}_X(\mathcal{F}_1, \mathcal{F}_2).$$

Furthermore, this assignment is compatible with the monoidal structure on $\mathrm{QCoh}(X \times X)^{\heartsuit}$, given by convolution and composition of differential operators.

5.4.2. Taking $\mathcal{Q} = \mathcal{D}^l_X$ and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{O}_X$, from the action of the monad $\mathbf{oblv}^l_X \circ \mathbf{ind}^l_X$ on $\mathcal{O}_X$, we obtain the desired map

$$\mathcal{D}^l_X \to \mathrm{Diff}_X, \tag{5.2}$$

compatible with the algebra structure.

We claim:

Lemma 5.4.3. *The map* (5.2) *is an isomorphism of algebras.*

Proof. It suffices to show that (5.2) is an isomorphism at the level of the underlying objects of $\mathrm{QCoh}(X \times X)$. The latter follows, e.g., from the description of $\mathcal{D}^l_X$ as a quasi-coherent sheaf given by Proposition 5.2.4. □

5.5. **Relation between crystals and D-modules.** Let X be a classical scheme of finite type. We will show that the category $\mathrm{Crys}^r(X)$ can be canonically identified with the (derived) category $\mathrm{D\text{-}mod}^r(X)$ of right D-modules on X.

Remark 5.5.1. The category $\mathrm{D\text{-}mod}^r(X)$ satisfies Zariski descent. Therefore, in what follows, by Proposition 3.2.2, it will suffice to establish a canonical equivalence for affine schemes.

5.5.2. Let Z be a smooth classical affine scheme, and let $i : X \hookrightarrow Z$ be a closed embedding. By the classical Kashiwara's lemma and Proposition 2.5.6, in order to construct an equivalence

$$\mathrm{Crys}^r(X) \simeq \mathrm{D\text{-}mod}^r(X),$$

it suffices to do so for Z.

Hence, we can assume that X itself is a smooth classical affine scheme. We shall construct the equivalence in question together with the commutative diagram of functors

$$\begin{tikzcd} \mathrm{Crys}^r(X) \arrow[r] \arrow[d, "\mathbf{oblv}^r_X"'] & \text{D-mod}^r(X) \arrow[d] \\ \mathrm{IndCoh}(X) \arrow[r, "\Psi_X"] & \mathrm{QCoh}(X), \end{tikzcd}$$

where the right vertical arrow is the natural forgetful functor, and the functor Ψ_X is the equivalence of [IndCoh, Lemma 1.1.6].

By Proposition 2.4.4, constructing an equivalence $\mathrm{Crys}^r(X) \simeq \text{D-mod}^r(X)$ as above is the same as constructing an equivalence between *left* crystals and *left* D-modules together with the commutative diagram of functors

$$\begin{tikzcd} \mathrm{Crys}^l(X) \arrow[rr] \arrow[dr, "\mathbf{oblv}^l_X"'] & & \text{D-mod}^l(X) \arrow[dl] \\ & \mathrm{QCoh}(X) & \end{tikzcd} \tag{5.3}$$

5.5.3. By Propositions 4.7.3, 2.4.4 and 4.4.2, the category $\mathrm{Crys}^l(X)$ identifies with the derived category of the heart of its t-structure. The category $\text{D-mod}^l(X)$ is by definition the derived category of $\text{D-mod}^l(X)^\heartsuit$. Moreover, the vertical arrows in diagram (5.3) are t-exact.

Hence, it suffices to construct the desired equivalence at the level of the corresponding abelian categories

$$\begin{tikzcd} \mathrm{Crys}^l(X)^\heartsuit \arrow[r] \arrow[d, "\mathbf{oblv}^l_X"'] & \text{D-mod}^l(X)^\heartsuit \arrow[d] \\ \mathrm{QCoh}(X)^\heartsuit \arrow[r, "\mathrm{Id}"] & \mathrm{QCoh}(X)^\heartsuit \end{tikzcd} \tag{5.4}$$

5.5.4. The latter is a classical calculation, due to Grothendieck:

Namely, one interprets $\mathrm{Crys}^l(X)^\heartsuit$ as the heart of the category of quasi-coherent sheaves on the truncated simplicial object

$$(X \times X \times X)^\wedge_X \overset{p_{12}}{\underset{p_{23}}{\overset{\xrightarrow{p_{13}}}{\rightrightarrows}}} (X \times X)^\wedge_X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \ .$$

I.e., explicitly, an object of $\mathrm{Crys}^l(X)^\heartsuit$ is a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)^\heartsuit$ together with an isomorphism

$$\phi : p_2^*(\mathcal{F}) \tilde{\to} p_1^*(\mathcal{F})$$

which restricts to the identity on the diagonal and satisfies the cocycle condition

$$p_{13}^*(\phi) = p_{12}^*(\phi) \circ p_{23}^*(\phi).$$

Below we give an alternative approach to establishing the equivalence in (5.4).

5.5.5. The abelian categories $\mathrm{Crys}^l(X)^\heartsuit$ and $\text{D-mod}^l(X)^\heartsuit$ are given as modules over the monads $\mathsf{M}_{\mathrm{Crys}^l(X)}$ and $\mathsf{M}_{\text{D-mod}(X)}$, respectively, acting on the category $\mathrm{QCoh}(X)^\heartsuit$.

By definition, $\mathsf{M}_{\text{D-mod}(X)}$ is given by the algebra of differential operators Diff_X. The monad $\mathsf{M}_{\mathrm{Crys}^l(X)}$ is given by $\mathbf{oblv}^l_X \circ \mathbf{ind}^l_X$. Now, the desired equivalence follows from Lemma 5.4.3.

Remark 5.5.6. It follows from the construction that the equivalence

$$\mathrm{Crys}^l(X) \to \text{D-mod}^l(X)$$

is compatible with pull-back for maps $f : Y \to X$ between smooth classical schemes.

6. Twistings

In this section, we do *not* assume that the prestacks and DG schemes that we consider are locally almost of finite type. We will reinstate this assumption in Sect. 6.7.

6.1. **Gerbes.**

6.1.1. Let $\mathrm{pt}/\mathbb{G}_m$ be the classifying stack of the group $\mathbb{G}_m$. In other words, $\mathrm{pt}/\mathbb{G}_m$ is the algebraic stack that represents the functor which assigns to an affine DG scheme S, the ∞-groupoid of line bundles on S.

In fact, since $\mathbb{G}_m$ is an abelian group, the stack $\mathrm{pt}/\mathbb{G}_m$ has a natural abelian group structure. The multiplication map on $\mathrm{pt}/\mathbb{G}_m$ represents tensor product of line bundles. This structure upgrades $\mathrm{pt}/\mathbb{G}_m$ to a functor from affine DG schemes to ∞-Picard groupoids, i.e. connective spectra.

For our purposes, a $\mathbb{G}_m$-gerbe will be a presheaf $\mathcal{G}$ of $\mathrm{pt}/\mathbb{G}_m$-torsors, which satisfies any of the following three (non-equivalent) conditions:

(i) $\mathcal{G}$ is locally non-empty in the étale topology [9].

(ii) $\mathcal{G}$ is locally non-empty in the Zariski topology.

(iii) $\mathcal{G}$ is globally non-empty.

Specifically, let $B^{\mathrm{naive}}(\mathrm{pt}/\mathbb{G}_m)$ be the classifying prestack of $\mathrm{pt}/\mathbb{G}_m$. It is given by the geometric realization of the simplicial prestack

$$B^{\mathrm{naive}}(\mathrm{pt}/\mathbb{G}_m) := \left| \cdots\ \mathrm{pt}/\mathbb{G}_m \times \mathrm{pt}/\mathbb{G}_m \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathrm{pt}/\mathbb{G}_m \rightrightarrows \mathrm{pt} \right|.$$

Let $B^{\mathrm{Zar}}(\mathrm{pt}/\mathbb{G}_m)$ (resp. $B^{\mathrm{et}}(\mathrm{pt}/\mathbb{G}_m)$) be the Zariski (resp. étale) sheafification of the prestack $B^{\mathrm{naive}}(\mathrm{pt}/\mathbb{G}_m)$.

The prestacks $B^{\mathrm{et}}(\mathrm{pt}/\mathbb{G}_m)$, $B^{\mathrm{Zar}}(\mathrm{pt}/\mathbb{G}_m)$ and $B^{\mathrm{naive}}(\mathrm{pt}/\mathbb{G}_m)$ represent $\mathbb{G}_m$-gerbes satisfying the above conditions (i), (ii), and (iii) respectively.

Let $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}^{\mathrm{aff}}}$ be the functor

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd}$$

that associates to an affine DG scheme S, the groupoid of $\mathbb{G}_m$-gerbes, where we consider any of the three notions of gerbe defined above.

Remark 6.1.2. While these three versions do not give equivalent notions of $\mathbb{G}_m$-gerbe, we will see shortly that they do lead to the same definition of twisting, since the relevant gerbes will be those whose restrictions to ${}^{cl,red}S$ are trivialized.

[9] By Toën's theorem, this is equivalent to local non-emptyness in the fppf topology.

6.1.3. We define the functor

$$(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \infty\text{-PicGrpd}$$

as the right Kan extension of $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}^{\mathrm{aff}}}$ along

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}.$$

I.e., for $\mathcal{Y} \in \mathrm{PreStk}$,

$$\mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y}) := \underset{S \in (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{Ge}_{\mathbb{G}_m}(S).$$

Equivalently,

$$\mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y}) = \mathrm{Maps}_{\mathrm{PreStk}}(\mathcal{Y}, B^?(\mathrm{pt}/\mathbb{G}_m))$$

for $? = \text{naive},\ \text{Zar}$ or et.

Thus, informally, a $\mathbb{G}_m$-gerbe on $\mathcal{Y}$ is an assignment of a $\mathbb{G}_m$-gerbe on every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ mapping to $\mathcal{Y}$, functorial in S.

For a subcategory $\mathbf{C} \subset \mathrm{PreStk}$, let $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathbf{C}}$ denote the restriction $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{PreStk}}|_{\mathbf{C}}$.

6.2. **The notion of twisting.**

6.2.1. Let $\mathcal{Y}$ be a prestack. The Picard groupoid of twistings on $\mathcal{Y}$ defined as

$$\mathrm{Tw}(\mathcal{Y}) := \ker\left(p^*_{\mathrm{dR},\mathcal{Y}} : \mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y}_{\mathrm{dR}}) \to \mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y})\right),$$

where $\mathrm{Ge}_{\mathbb{G}_m}$ is understood in any of the three versions: naive, Zar or et. As we shall see shortly (see Sect. 6.4), all three versions are equivalent.

Informally, a twisting T on $\mathcal{Y}$ is the following data: for every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ equipped with a map ${}^{cl,red}S \to \mathcal{Y}$ we specify an object $\mathcal{G}_S \in \mathrm{Ge}_{\mathbb{G}_m}(S)$, which behaves compatibly under the maps $S_1 \to S_2$. Additionally, for every extension of the above map to a map $S \to \mathcal{Y}$ we specify a trivialization of $\mathcal{G}_S$, which also behaves functorially with respect to maps $S_1 \to S_2$.

Remark 6.2.2. When we write $\ker(\mathcal{A}_1 \to \mathcal{A}_2)$, where $\mathcal{A}_1 \to \mathcal{A}_2$ is a map in ∞-PicGrpd, we mean

$$\mathcal{A}_1 \underset{\mathcal{A}_2}{\times} \{*\},$$

where the fiber product is taken in ∞-PicGrpd. I.e., this the same as the connective truncation of the fiber product taken in the category of all (i.e., not necessarily connective) spectra.

6.2.3. *Example.* Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$. We define a twisting $T(\mathcal{L})$ on $\mathcal{Y}$ as follows: it assigns to every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ with a map ${}^{cl,red}S \to \mathcal{Y}$ the trivial $\mathbb{G}_m$-gerbe. For a map $S \to \mathcal{Y}$, we trivialize the above gerbe by multiplying the tautological trivialization by $\mathcal{L}$.

6.2.4. It is clear that twistings form a functor

$$\mathrm{Tw}_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \infty\text{-PicGrpd}\,.$$

For a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ we let f^* denote the corresponding functor

$$\mathrm{Tw}(\mathcal{Y}_2) \to \mathrm{Tw}(\mathcal{Y}_1).$$

If $\mathbf{C}$ is a subcategory of PreStk (e.g., $\mathbf{C} = \mathrm{DGSch}^{\mathrm{aff}}$ or DGSch), we let $\mathrm{Tw}_{\mathbf{C}}$ denote the restriction of $\mathrm{Tw}_{\mathrm{PreStk}}$ to $\mathbf{C}^{\mathrm{op}}$.

6.2.5. By construction, the functor $\mathrm{Tw}_{\mathrm{PreStk}}$ takes colimits in PreStk to limits in ∞-PicGrpd. Hence, from Corollary 1.1.5, we obtain:

Lemma 6.2.6. *The functor* $\mathrm{Tw}_{\mathrm{PreStk}}$ *maps isomorphically to the right Kan extension of* $\mathrm{Tw}_{\mathbf{C}}$ *along*

$$\mathbf{C}^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}$$

for $\mathbf{C}$ *being one of the categories*

$$\mathrm{DGSch}^{\mathrm{aff}},\ \mathrm{DGSch}_{\text{qs-qc}},\ \mathrm{DGSch}\,.$$

Concretely, this lemma says that the map

$$\mathrm{Tw}(\mathcal{Y}) \to \underset{S \in (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{Tw}(S)$$

is an isomorphism (and that $\mathrm{DGSch}^{\mathrm{aff}}$ can be replaced by $\mathrm{DGSch}_{\text{qs-qc}}$ or DGSch.)

Informally, this means that to specify a twisting on a prestack $\mathcal{Y}$ is equivalent to specifying a compatible family of twistings on affine DG schemes S mapping to $\mathcal{Y}$.

6.3. **Variant: other structure groups.**

6.3.1. Let S be an affine DG scheme. Consider the Picard groupoid

$$\mathrm{Ge}^{/red}_{\mathbb{G}_m}(S) := \ker\left(\mathrm{Ge}_{\mathbb{G}_m}(S) \to \mathrm{Ge}_{\mathbb{G}_m}({}^{cl,red}S)\right).$$

Let $(\mathrm{Ge}^{/red}_{\mathbb{G}_m})_{\mathrm{DGSch}^{\mathrm{aff}}}$ denote the resulting functor

$$(\mathrm{DGSch})^{\mathrm{op}} \to \infty\text{-PicGrpd}\,.$$

6.3.2. By definition, we can think of $\mathrm{Ge}^{/red}_{\mathbb{G}_m}(S)$ as gerbes (in any of the three versions of Sect. 6.1.1) with respect to the presheaf of abelian groups

$$(\mathcal{O}^\times)^{/red}_S := \ker(\mathcal{O}^\times_S \to \mathcal{O}^\times_{{}^{cl,red}S}).$$

6.3.3. In addition to $\mathbb{G}_m$-gerbes, we can also consider $\mathbb{G}_a$-gerbes. We have the functor

$$(\mathrm{Ge}_{\mathbb{G}_a})_{\mathrm{DGSch}^{\mathrm{aff}}} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd}$$

which assigns to an affine DG scheme S the groupoid of $\mathbb{G}_a$-gerbes on S.

Note that unlike the case of $\mathbb{G}_m$-gerbes, the three notions of gerbes discussed in Sect. 6.1.1 are equivalent for $\mathbb{G}_a$-gerbes. This is due to the fact that for an affine DG scheme S,

$$H^2_{\mathrm{Zar}}(S, \mathbb{G}_a) = H^2_{\mathrm{et}}(S, \mathbb{G}_a) = 0.$$

Thus, we have that $(\mathrm{Ge}_{\mathbb{G}_a})_{\mathrm{DGSch}^{\mathrm{aff}}}$ is represented by $B^2(\mathbb{G}_a)$, which is the geometric realization of the corresponding simplicial prestack.

6.3.4. By definition, for an affine DG scheme S, we have

$$\mathrm{Ge}_{\mathbb{G}_a}(S) = B^2(\mathrm{Maps}(S,\mathbb{G}_a)) \simeq B^2(\Gamma(S,\mathcal{O}_S)).$$

In particular, viewed as a connective spectrum, $\mathrm{Ge}_{\mathbb{G}_a}(S)$ has a natural structure of a module over the ground field k. This upgrades $(\mathrm{Ge}_{\mathbb{G}_a})_{\mathrm{DGSch}^{\mathrm{aff}}}$ to a functor

$$(\mathrm{DGSch})^{\mathrm{op}} \to \infty\text{-}\mathrm{PicGrpd}_k,$$

where $\infty\text{-}\mathrm{PicGrpd}_k$ denotes the category of k-modules in connective spectra. Note that by the Dold-Kan correspondence, we have

$$\infty\text{-}\mathrm{PicGrpd}_k \simeq \mathrm{Vect}^{\leq 0}.$$

We define the functor

$$(\mathrm{Ge}_{\mathbb{G}_a})_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \infty\text{-}\mathrm{PicGrpd}_k$$

as the right Kan extension of the functor $(\mathrm{Ge}_{\mathbb{G}_a})_{\mathrm{DGSch}^{\mathrm{aff}}}$ along

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}.$$

6.3.5. As with $\mathbb{G}_m$-gerbes, we can consider the Picard groupoid

$$\mathrm{Ge}^{/red}_{\mathbb{G}_a}(S) := \ker\left(\mathrm{Ge}_{\mathbb{G}_a}(S) \to \mathrm{Ge}_{\mathbb{G}_a}({}^{cl,red}S)\right),$$

and let $(\mathrm{Ge}^{/red}_{\mathbb{G}_a})_{\mathrm{DGSch}^{\mathrm{aff}}}$ denote the resulting functor

$$(\mathrm{DGSch})^{\mathrm{op}} \to \infty\text{-}\mathrm{PicGrpd}_k.$$

By definition, for an affine DG scheme S, $\mathrm{Ge}^{/red}_{\mathbb{G}_a}(S)$ is given by gerbes for the presheaf of connective spectra

$$\mathcal{O}^{/red}_S := \ker(\mathcal{O}_S \to \mathcal{O}_{{}^{cl,red}S}).$$

Explicitly,

$$\mathrm{Ge}^{/red}_{\mathbb{G}_a}(S) \simeq B^2(\Gamma(S,\mathcal{O}^{/red}_S)).$$

6.3.6. Recall from [GR1, Sect. 6.8.8] that the exponential map defines an isomorphism

$$\exp : \mathcal{O}_S^{/red} \to (\mathcal{O}^\times)_S^{/red}.$$

Hence, we obtain:

Corollary 6.3.7. *The exponential map defines an isomorphism of functors*

$$\exp : (\mathrm{Ge}_{\mathbb{G}_a}^{/red})_{\mathrm{DGSch}^{\mathrm{aff}}} \to (\mathrm{Ge}_{\mathbb{G}_m}^{/red})_{\mathrm{DGSch}^{\mathrm{aff}}} \tag{6.1}$$

for any of the three versions (naive, Zar *or* et) *of* $(\mathrm{Ge}_{\mathbb{G}_m}^{/red})_{\mathrm{DGSch}^{\mathrm{aff}}}$.

Thus, if we realize $\mathrm{Ge}_{\mathbb{G}_m}^{/red}(S)$ as gerbes in the étale or Zariski topology, this category has trivial π_0 and π_1. In other words, any such gerbe on an affine DG scheme is globally non-empty, and any automorphism is non-canonically isomorphic to identity.

6.3.8. The isomorphism (6.1) endows $\mathrm{Ge}_{\mathbb{G}_m}^{/red}(S)$, viewed as a connective spectrum, with a structure of module over the ground field k. This upgrades $(\mathrm{Ge}_{\mathbb{G}_m}^{/red})_{\mathrm{DGSch}^{\mathrm{aff}}}$ to a functor

$$(\mathrm{DGSch})^{\mathrm{op}} \to \infty\text{-}\mathrm{PicGrpd}_k\,.$$

We define the functor

$$(\mathrm{Ge}_{\mathbb{G}_m}^{/red})_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \infty\text{-}\mathrm{PicGrpd}_k$$

as the right Kan extension of the functor $(\mathrm{Ge}_{\mathbb{G}_m}^{/red})_{\mathrm{DGSch}^{\mathrm{aff}}}$ along

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}\,.$$

6.3.9. By definition, for $\mathcal{Y} \in \mathrm{PreStk}$

$$\mathrm{Ge}_{\mathbb{G}_m}^{/red}(\mathcal{Y}) := \underset{S \in (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{Ge}_{\mathbb{G}_m}^{/red}(S).$$

Informally, for $\mathcal{Y} \in \mathrm{PreStk}$, an object $\mathcal{G} \in \mathrm{Ge}_{\mathbb{G}_m}^{/red}(\mathcal{Y})$ is an assignment for every $S \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$ of an object $\mathcal{G}_S \in \mathrm{Ge}_{\mathbb{G}_m}^{/red}(S)$, and for every $S' \to S$ of an isomorphism

$$f^*(\mathcal{G}_S) \simeq \mathcal{G}_{S'}.$$

The following results from the definitions:

Lemma 6.3.10. *For $\mathcal{Y} \in \mathrm{PreStk}$, the natural map*

$$\mathrm{Ge}^{/red}_{\mathbb{G}_m}(\mathcal{Y}) \to \ker\left(\mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y}) \to \mathrm{Ge}_{\mathbb{G}_m}({}^{cl,red}\mathcal{Y})\right)$$

is an isomorphism, where

$${}^{cl,red}\mathcal{Y} := \mathrm{LKE}_{({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}}(\mathcal{Y}|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}}).$$

6.4. **Twistings: reformulations.** We are going to show that the notion of twisting can be formulated in terms of

$$(\mathrm{Ge}^{/red}_{\mathbb{G}_m})_{\mathrm{PreStk}}, (\mathrm{Ge}_{\mathbb{G}_a})_{\mathrm{PreStk}} \text{ or } (\mathrm{Ge}^{/red}_{\mathbb{G}_a})_{\mathrm{PreStk}},$$

instead of $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{PreStk}}$.

6.4.1. Consider the functors

$$\mathrm{Tw}^{/red}, \mathrm{Tw}_a, \mathrm{Tw}^{/red}_a : \mathrm{PreStk}^{\mathrm{op}} \to \infty\text{-PicGrpd}$$

given by

$$\mathrm{Tw}^{/red}(\mathcal{Y}) := \ker\left(p^*_{\mathrm{dR},\mathcal{Y}} : \mathrm{Ge}^{/red}_{\mathbb{G}_m}(\mathcal{Y}_{\mathrm{dR}}) \to \mathrm{Ge}^{/red}_{\mathbb{G}_m}(\mathcal{Y})\right),$$

$$\mathrm{Tw}_a(\mathcal{Y}) := \ker\left(p^*_{\mathrm{dR},\mathcal{Y}} : \mathrm{Ge}_{\mathbb{G}_a}(\mathcal{Y}_{\mathrm{dR}}) \to \mathrm{Ge}_{\mathbb{G}_a}(\mathcal{Y})\right)$$

and

$$\mathrm{Tw}^{/red}_a(\mathcal{Y}) := \ker\left(p^*_{\mathrm{dR},\mathcal{Y}} : \mathrm{Ge}^{/red}_{\mathbb{G}_a}(\mathcal{Y}_{\mathrm{dR}}) \to \mathrm{Ge}^{/red}_{\mathbb{G}_a}(\mathcal{Y})\right).$$

We have the following diagram of functors given by the exponential map and the evident forgetful functors.

$$\begin{tikzcd} \mathrm{Tw}^{/red}_a \arrow[r, "exp"] \arrow[d] & \mathrm{Tw}^{/red} \arrow[d] \\ \mathrm{Tw}_a & \mathrm{Tw} \end{tikzcd} \tag{6.2}$$

Proposition 6.4.2. *The functors in* (6.2) *are equivalences.*

Proof. The functor given by the exponential map is an equivalence by Sect. 6.3.6.

Let us show that the right vertical map in (6.2) is an equivalence. This is in fact tautological:

Both functors are right Kan extensions under

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to (\mathrm{PreStk})^{\mathrm{op}},$$

so it is enough to show that the map in question is an isomorphism when evaluated on objects $S \in \mathrm{DGSch}^{\mathrm{aff}}$.

We have:

$$\mathrm{Tw}^{/red}(S) = \mathrm{Ge}^{/red}_{\mathbb{G}_m}(S_{\mathrm{dR}}) \underset{\mathrm{Ge}^{/red}_{\mathbb{G}_m}(S)}{\times} \{*\} \overset{\text{Lemma 6.3.10}}{\simeq}$$

$$\simeq \ker\Big(\mathrm{Ge}_{\mathbb{G}_m}(S_{\mathrm{dR}}) \to \mathrm{Ge}_{\mathbb{G}_m}({}^{cl,red}(S_{\mathrm{dR}}))\Big) \underset{\ker\big(\mathrm{Ge}_{\mathbb{G}_m}(S) \to \mathrm{Ge}_{\mathbb{G}_m}({}^{cl,red}S)\big)}{\times} \{*\} =$$

$$= \ker\Big(\mathrm{Ge}_{\mathbb{G}_m}(S_{\mathrm{dR}}) \to \mathrm{Ge}_{\mathbb{G}_m}({}^{cl,red}S)\Big) \underset{\ker\big(\mathrm{Ge}_{\mathbb{G}_m}(S) \to \mathrm{Ge}_{\mathbb{G}_m}({}^{cl,red}S)\big)}{\times} \{*\} \simeq$$

$$\simeq \mathrm{Ge}_{\mathbb{G}_m}(S_{\mathrm{dR}}) \underset{\mathrm{Ge}_{\mathbb{G}_m}(S)}{\times} \{*\} = \mathrm{Tw}(S).$$

The fact that the left vertical arrow in (6.2) is an equivalence is proved similarly.

□

6.4.3. As a consequence of Proposition 6.4.2, we obtain:

Corollary 6.4.4. *The notions of twisting in all three versions:* naive*,* Zar *and* et *are equivalent.*

In addition:

Corollary 6.4.5. *The functor* $\mathrm{Tw} : (\mathrm{PreStk})^{\mathrm{op}} \to \infty\text{-}\mathrm{Grpd}$ *canonically upgrades to a functor*

$$(\mathrm{PreStk})^{\mathrm{op}} \to \infty\text{-}\mathrm{PicGrpd}_k\,.$$

6.4.6. *Example.* We can use the natural k-module structure on Tw to produce additional examples of twistings. Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$, and let $T(\mathcal{L})$ be the twisting of Sect. 6.2.3. Now, for $a \in k$, the k-module structure on $\mathrm{Tw}(\mathcal{Y})$ gives us a new twisting $T(\mathcal{L}^{\otimes a})$.

Remark 6.4.7. Note that it is *not* true that any twisting T on an affine DG scheme X is trivial, even locally in the Zariski or étale topology. It is true that for any $S \in \mathrm{DGSch}^{\mathrm{aff}}$ with a map $S \to X_{\mathrm{dR}}$, the corresponding $\mathbb{G}_m$-gerbe on S can be non-canonically trivialized; but such a trivialization can not necessarily be made compatible for the different choices of S.

An example of such a gerbe for a smooth classical X can be given by a choice of a closed 2-form (see Sect. 6.5.4) which is not étale-locally exact.

Note, however, that the gerbes described in Example 6.4.6 *are* Zariski-locally trivial, because of the corresponding property of line bundles.

6.4.8. *Convergence.* We now claim:

Proposition 6.4.9. *The functor* $\mathrm{Tw} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-PicGrpd}$ *is convergent.*[10]

Proof. We will show that the functor $\mathrm{Tw}_a^{/red}$ is convergent. For this, it is enough to show that the functors

$$S \mapsto \mathrm{Ge}_{\mathbb{G}_a}^{/red}(S_{\mathrm{dR}}) \text{ and } \mathrm{Ge}_{\mathbb{G}_a}^{/red}(S)$$

are convergent.

The convergence of $\mathrm{Ge}_{\mathbb{G}_a}^{/red}((-)_{\mathrm{dR}})$ is obvious, as this functor only depends on the underlying reduced classical scheme. Thus, it remains to prove the convergence of $\mathrm{Ge}_{\mathbb{G}_a}^{/red}(-)$.

We have:

$$\mathrm{Ge}_{\mathbb{G}_a}^{/red}(S) = \mathrm{Ge}_{\mathbb{G}_a}(S) \underset{\mathrm{Ge}_{\mathbb{G}_a}({}^{cl,red}S)}{\times} \{*\}.$$

Hence, it is sufficient to show that the functor $\mathrm{Ge}_{\mathbb{G}_a}(-)$ is convergent. The latter follows from the fact that

$$\mathrm{Ge}_{\mathbb{G}_a}(-) = B^2(\mathrm{Maps}(-, \mathbb{G}_a)),$$

while $\mathbb{G}_a$ is convergent, being a DG scheme.

□

We can reformulate Proposition 6.4.9 tautologically as follows:

Corollary 6.4.10. *The functor* $\mathrm{Tw}_{\mathrm{PreStk}}$ *maps isomorphically to the right Kan extension of*

$$\mathrm{Tw}_{<\infty\mathrm{DGSch}^{\mathrm{aff}}} := \mathrm{Tw}_{\mathrm{DGSch}^{\mathrm{aff}}} |_{<\infty\mathrm{DGSch}^{\mathrm{aff}}}$$

---

[10] See Sect. 1.3.1, where the notion of convergence is recalled.

along

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}.$$

Remark 6.4.11. We can use Proposition 6.4.9 to show that the functor $\mathrm{Ge}_{\mathbb{G}_m}$ is also convergent (in any of the three versions).

6.4.12. *Twistings in the locally almost of finite type case.* Corollary 6.4.10 implies that we "do not need to know" about DG schemes that are not locally almost of finite type in order to know what twistings on $\mathcal{Y} \in \mathrm{PreStk}$ if $\mathcal{Y}$ is locally almost of finite type.

Corollary 6.4.13.

(a) *For* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$*, the naturally defined map*

$$\mathrm{Tw}(\mathcal{Y}) \to \underset{S \in (({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{lim} \mathrm{Tw}(S)$$

is an equivalence.

(b) *The functor* $\mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{laft}}}$ *maps isomorphically to the right Kan extension of* $\mathrm{Tw}_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}}$ *along the inlcusions*

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

Proof. This is true for Tw replaced by any convergent prestack $(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}$. □

Remark 6.4.14. It follows from Remark 6.4.11 that the functor $\mathrm{Ge}_{\mathbb{G}_m}$ (in any of the three versions), viewed as a presheaf, belongs to $\mathrm{PreStk}_{\mathrm{laft}}$. Indeed, this is evident in the naive version, since $\mathrm{pt}/\mathbb{G}_m$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$. For the Zar and et versions, this follows from [GL:Stacks, Corollary 2.5.7] that says that the condition of being locally of finite type in the context of n-connective prestacks survives sheafification, once we restrict ourselves to *truncated* prestacks.

6.5. **Identification of the Picard groupoid of twistings.** We can use the description of twistings in terms of $\mathbb{G}_a$-gerbes to give a cohomological description of the groupoid of twistings.

6.5.1. *De Rham cohomology.* Let $\mathcal{Y}$ be a prestack. Recall that the coherent cohomology of $\mathcal{Y}$ is defined as

$$H(\mathcal{Y}) := \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}}).$$

We define the de Rham cohomology of $\mathcal{Y}$ to be the coherent cohomology of $\mathcal{Y}_{\mathrm{dR}}$; i.e.,

$$H_{\mathrm{dR}}(\mathcal{Y}) := H(\mathcal{Y}_{\mathrm{dR}}) = \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}})}(\mathcal{O}_{\mathcal{Y}_{\mathrm{dR}}}, \mathcal{O}_{\mathcal{Y}_{\mathrm{dR}}}).$$

Note that since $\mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}})$ is a stable ∞-category, the $\mathcal{M}aps$ above gives a (not necessarily connective) spectrum.

Let X be a smooth classical scheme. In this case, by Sect. 5.5, we have

$$H_{\mathrm{dR}}(X) = \mathcal{M}aps_{\mathrm{D\text{-}mod}^l(X)}(\mathcal{O}_X, \mathcal{O}_X).$$

In particular, our definition of de Rham cohomology agrees with the usual one for smooth classical schemes.

6.5.2. Consider the functor $B^2(\mathbb{G}_a)$, which represents $\mathbb{G}_a$-gerbes. By definition, for a prestack $\mathcal{Y}$, we have an isomorphism of connective spectra:

$$\mathrm{Maps}(\mathcal{Y}, B^2(\mathbb{G}_a)) \simeq \tau^{\leq 0}\left(\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}})[2]\right) \simeq \tau^{\leq 0}(H(\mathcal{Y})[2]).$$

Thus by Proposition 6.4.2, we obtain:

Corollary 6.5.3. *For a prestack $\mathcal{Y}$, groupoid of twistings is given by*

$$\mathrm{Tw}(\mathcal{Y}) \simeq \tau^{\leq 2}\left(H_{\mathrm{dR}}(\mathcal{Y}) \underset{H(\mathcal{Y})}{\times} \{*\}\right)[2].$$

6.5.4. Now, suppose that X is a smooth classical scheme. In this case, we have

$$H_{\mathrm{dR}}(X) \simeq \Gamma(X, \Omega^\bullet)$$

where $\Omega^\bullet$ is the complex of de Rham differentials on X. The natural map

$$H_{\mathrm{dR}}(X) \to H(X)$$

is given by global sections of the projection map $\Omega^\bullet \to \mathcal{O}_X$. Therefore, we have

$$\mathrm{Tw}(X) \simeq \tau^{\leq 2}\left(H_{\mathrm{dR}}(X) \underset{H(X)}{\times} \{*\}\right)[2] \simeq \tau^{\leq 2}\left(\Gamma(X, \Omega^\bullet \underset{\mathcal{O}_X}{\times} 0)\right)[2] \simeq$$
$$\simeq \tau^{\leq 2}\left(\Gamma(X, \tau^{\leq 2}(\Omega^\bullet \underset{\mathcal{O}_X}{\times} 0))\right)[2].$$

The complex $\tau^{\leq 2}(\Omega^\bullet \underset{\mathcal{O}_X}{\times} 0)$ identifies with the complex

$$\Omega^1 \to \Omega^{2,cl}$$

where $\Omega^{2,cl}$ is the sheaf of closed 2-forms (placed in cohomological degree 2) and the map is the de Rham differential.

Thus, we have that the Picard groupoid of twistings on X is given by

$$\mathrm{Tw}(X) \simeq \tau^{\leq 2}\left(\Gamma(X, \Omega^1 \to \Omega^{2,cl})\right)[2].$$

In particular, our definition of twistings agrees with the notion of TDO of [BB] for smooth classical schemes.

6.6. **Twisting and the infinitesimal groupoid.**

6.6.1. Let

$$\mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0$$

be a groupoid object in PreStk, and let $\mathcal{Y}^\bullet$ be the corresponding simplicial object. Let us recall the notion of central extension of this groupoid object by $\mathbb{G}_m$. (Here $\mathbb{G}_m$ can be replaced by any commutative group-object $H \in \mathrm{PreStk}$).

By definition, a central extension of $\mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0$ by $\mathbb{G}_m$ is an object of $\mathrm{Ge}_{\mathbb{G}_m}(|\mathcal{Y}^\bullet|)$, equipped with a trivialization of its restriction under

$$\mathcal{Y}^0 \to |\mathcal{Y}^\bullet|.$$

6.6.2. Informally, the data of such a central extension is a line bundle $\mathcal{L}$ on $\mathcal{Y}^1$, whose pullback under the degeneracy map $\mathcal{Y}^0 \to \mathcal{Y}^1$ is trivialized, and such that for the three maps

$$p_{1,2}, p_{2,3}, p_{1,2} : \mathcal{Y}^2 \to \mathcal{Y}^1,$$

we are given an isomorphism

$$p_{1,2}^*(\mathcal{L}) \otimes p_{2,3}^*(\mathcal{L}) \simeq p_{1,3}^*(\mathcal{L}),$$

and such that the further coherence conditions are satisfied.

6.6.3. For $\mathcal{Y} \in \mathrm{PreStk}$ consider its infinitesimal groupoid

$$\mathcal{Y} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y} \rightrightarrows \mathcal{Y}.$$

By definition, a twisting on $\mathcal{Y}$ gives rise to a central extension of its infinitesimal groupoid by $\mathbb{G}_m$.

Conversely, from Lemma 1.2.4, we obtain:

Corollary 6.6.4. *Assume that $\mathcal{Y}$ is classically formally smooth. Then the above functor*

$$\mathrm{Tw}(\mathcal{Y}) \to \{\text{Central extensions of the infinitesimal groupoid of } \mathcal{Y} \text{ by } \mathbb{G}_m\}$$

is an equivalence.

6.7. **Twistings on indschemes.**

6.7.1. Let $\mathcal{X}$ be an object of $\mathrm{DGindSch}_{\mathrm{laft}}$. We will show that the assertion of Corollary 6.6.4 holds for $\mathcal{X}$:

Proposition 6.7.2. *The functor*

$$\mathrm{Tw}(\mathcal{X}) \to \{\text{Central extensions of the infinitesimal groupoid of } \mathcal{X} \text{ by } \mathbb{G}_m\}$$

is an equivalence.

The rest of this subsection is devoted to the proof of Proposition 6.7.2.

6.7.3. *Step 1.* By Corollary 6.4.13(b), we have to show the following:

For every $S \in (\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}_{\mathrm{dR}}}$ a datum of $\mathbb{G}_m$-gerbe on S, equipped with a trivialization of its pullback to $S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}$, is equivalent to that of a $\mathbb{G}_m$-gerbe on the simplicial prestack

$$S^\bullet := S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} (\mathcal{X}^\bullet/\mathcal{X}_{\mathrm{dR}}),$$

equipped with a trivialization over 0-simplices, i.e., $S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}$.

6.7.4. *Step 2.* Note that the simplicial prestack ${}^{cl,red}(S^\bullet)$ is constant with value ${}^{red,cl}S$. Hence, by Lemma 6.3.7, we can consider $\mathbb{G}_a$-gerbes instead of $\mathbb{G}_m$-gerbes.

Hence, it is enough to show that the map

$$\mathcal{M}aps_{\mathrm{QCoh}(S)}(\mathcal{O}_S, \mathcal{O}_S) \to \mathrm{Tot}\left(\mathcal{M}aps_{\mathrm{QCoh}(S^\bullet)}(\mathcal{O}_{S^\bullet}, \mathcal{O}_{S^\bullet})\right)$$

is an isomorphism in Vect.

6.7.5. *Step 3.* Note that for any $\mathcal{X}' \in \mathrm{DGindSch}_{\mathrm{laft}}$, the canonical map

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{X}')}(\mathcal{O}_{\mathcal{X}'}, \mathcal{O}_{\mathcal{X}'}) \to \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{X}')}(\omega_{\mathcal{X}'}, \omega_{\mathcal{X}'})$$

is an isomorphism. This follows, e.g., from the corresponding assertion for DG schemes, i.e., Lemma 5.2.5.

Hence, it is enough to show that the map

$$\mathcal{M}aps_{\mathrm{IndCoh}(S)}(\omega_S, \omega_S) \to \mathrm{Tot}\left(\mathcal{M}aps_{\mathrm{IndCoh}(S^\bullet)}(\omega_{S^\bullet}, \omega_{S^\bullet})\right)$$

is an isomorphism.

6.7.6. *Step 4.* Note that $S^\bullet$ identifies with the Čech nerve of the map

$$S \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X} \to S. \tag{6.3}$$

As in the proof of Proposition 3.1.3, all S^i belong to DGindSch, and the morphism (6.3) is ind-proper and surjective.

Now, the desired assertion follows from the descent for IndCoh under ind-proper and surjective maps of DG indschemes, see [GR1, Lemma 2.10.3].

7. Twisted crystals

In this section we will show how the data of a twisting gives a modification of the categories of left and right crystals. The main results say that "not much really changes."

7.1. **Twisted left crystals.** In this subsection we do not assume that our DG schemes and prestacks are locally almost of finite type.

7.1.1. Let $\mathcal{Y}$ be a prestack. Consider the category $\mathrm{PreStk}_{/\mathcal{Y}}$, and the functor

$$\mathrm{QCoh}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}} : (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

The group-stack $\mathrm{pt}\,/\mathbb{G}_m$ acts on QCoh via tensoring by line bundles.

Let $\mathcal{G}$ be a $\mathbb{G}_m$-gerbe on $\mathcal{Y}$. Then $\mathcal{G}$ gives a twist of the functor $\mathrm{QCoh}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}}$ via the action of $\mathrm{pt}\,/\mathbb{G}_m$ on QCoh. This defines a functor

$$\mathrm{QCoh}^{\mathcal{G}}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}} : (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

7.1.2. In particular, if T is a twisting on $\mathcal{Y}$, we obtain a functor

$$\mathrm{QCoh}^{T}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_{\mathrm{dR}}}} : (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Let $\mathrm{QCoh}^{T}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}}$ be its restriction along the map

$$(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}} \to (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}.$$

By construction, $\mathrm{QCoh}^{T}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}}$ is canonically isomorphic to $\mathrm{QCoh}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}}$.

7.1.3. More generally, we can consider the functor

$$\mathrm{QCoh}^{T}_{\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}} : (\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

which is the right Kan extension of $\mathrm{QCoh}^{T}_{\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_{\mathrm{dR}}}}$ along

$$(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}.$$

The restriction $\mathrm{QCoh}^{T}_{\mathrm{PreStk}_{/\mathcal{Y}}}$ of $\mathrm{QCoh}^{T}_{\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}}$ along

$$\mathrm{PreStk}_{/\mathcal{Y}} \to \mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}$$

is canonically isomorphic to $\mathrm{QCoh}_{\mathrm{PreStk}_{/\mathcal{Y}}}$.

7.1.4. For a twisting T on a prestack $\mathcal{Y}$, the category of T-twisted left crystals on $\mathcal{Y}$ is defined as

$$\mathrm{Crys}^{T,l}(\mathcal{Y}) := \mathrm{QCoh}^{T}(\mathcal{Y}_{\mathrm{dR}}).$$

Explicitly, we have

$$\mathrm{Crys}^{T,l}(\mathcal{Y}) = \underset{S\in(\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}}{\mathit{lim}} \mathrm{QCoh}^{T}(S).$$

7.1.5. More generally, we define the functor

$$\mathrm{Crys}^{T,l}_{\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}} : (\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

as the composite $\mathrm{QCoh}^{T}_{\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}} \circ \mathrm{dR}$. The analogue of Corollary 2.1.4 holds for this functor.

7.1.6. We have a canonical natural transformation

$$\mathbf{oblv}^{T,l} : \mathrm{Crys}^{T,l}_{\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}} \to \mathrm{QCoh}^{T}_{\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}}.$$

For an individual $\mathcal{Y}' \in \mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}$, we denote the resulting functor

$$\mathrm{Crys}^{T,l}(\mathcal{Y}') \to \mathrm{QCoh}^{T}(\mathcal{Y}')$$

by $\mathbf{oblv}^{T,l}(\mathcal{Y}')$.

7.1.7. Let $\mathrm{Crys}^{T,l}_{\mathrm{PreStk}_{/\mathcal{Y}}}$ denote the restriction of $\mathrm{Crys}^{T,l}_{\mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}}$ along $\mathrm{PreStk}_{/\mathcal{Y}} \to \mathrm{PreStk}_{/\mathcal{Y}_{\mathrm{dR}}}$.

By a slight abuse of notation we shall use the same symbol $\mathbf{oblv}^{T,l}$ to denote the resulting natural transformation

$$\mathrm{Crys}^{T,l}_{\mathrm{PreStk}_{/\mathcal{Y}}} \to \mathrm{QCoh}_{\mathrm{PreStk}_{/\mathcal{Y}}}.$$

7.2. **Twisted right crystals.** At this point, we reinstate the assumption that all DG schemes and prestacks are locally almost of finite type for the rest of the paper.

7.2.1. Let $\mathcal{Y}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$, and let $\mathcal{G}$ be a $\mathbb{G}_m$-gerbe on $\mathcal{Y}$.

The action of $\mathrm{QCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}$ on $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}$ (see [IndCoh, Sect. 10.3]) allows to define the functor

$$\mathrm{IndCoh}^{\mathcal{G}}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

with properties analogous to those of

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}} := \mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}|_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}}.$$

7.2.2. In particular, for $T \in \mathrm{Tw}(\mathcal{Y})$, we have the functor

$$\mathrm{Crys}^{T,r}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and the natural transformations $\mathbf{oblv}^{T,l}$

$$\mathrm{Crys}^{T,r}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}} \to \mathrm{IndCoh}^{T}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}}$$
$$\text{and } \mathrm{Crys}^{T,r}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}} \to \mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}}.$$

The analogues of Corollaries 2.3.7 and 2.3.9 and Lemmas 2.3.11 and 2.3.12 hold for $\mathcal{Y}' \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}$, with the same proofs.

7.3. **Properties of twisted crystals.** As was mentioned above, all DG schemes and prestacks are assumed locally almost of finite type.

Let $\mathcal{Y}$ be a fixed object of $\mathrm{PreStk}_{\mathrm{laft}}$, and $T \in \mathrm{Tw}(\mathcal{Y})$.

Remark 7.3.1. In general, results about crystals do not automatically hold for twisted crystals. In some of our proofs, we needed to embed a given affine DG scheme X into a smooth classical scheme Z. In the case of twisted crystals, the problem is that we might not be able to find such a Z which also maps to $\mathcal{Y}$ (or even $\mathcal{Y}_{\mathrm{dR}}$).

However, there is a large family of examples (which covers all the cases that have appeared in applications so far), where the extension of the results is automatic: namely, when T is such that its restriction to any $S \in (\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}}$ is locally trivial in the Zariski or étale topology (see also Remark 6.4.7). This is the case for twistings of the form $\mathcal{L}^{\otimes a}$ for $\mathcal{L} \in \mathrm{Pic}(\mathcal{Y})$ and $a \in k$, and tensor products thereof.

7.3.2. The analogues of Corollaries 2.2.2 and 2.2.4 and Lemma 2.1.7 hold for twisted left crystals, with the same proofs.

Furthermore, Kashiwara's lemma holds for both left and right twisted crystals, also with the same proof.

Finally, note that there exists a canonical natural transformation

$$\Upsilon : \mathrm{Crys}^{T,l}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}} \to \mathrm{Crys}^{T,r}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}}. \tag{7.1}$$

Proposition 7.3.3. *The natural transformation* (7.1) *is an equivalence.*

Proof. The argument is the same as that of Proposition 2.4.4:

We do not need the smooth classical scheme Z to map to $\mathcal{Y}$. Rather, we use the fact that if Y is the completion of a smooth classical scheme Z along a Zariski-closed subset, and $\mathcal{G}$ is a $\mathbb{G}_m$-gerbe on Y, which is trivial over ${}^{cl,red}Y$, then the functor

$$\Upsilon_Y : \mathrm{QCoh}^{\mathcal{G}}(Y) \to \mathrm{IndCoh}^{\mathcal{G}}(Y)$$

is an equivalence. The latter follows from the corresponding fact in the non-twisted situation (proved in the course of the proof of Proposition 2.4.4), since $\mathcal{G}$ is (non-canonically) trivial.

□

As a corollary, we obtain that the analog of Lemma 2.2.6 holds in the twisted case as well.

7.3.4. Hence, for $\mathcal{Y}' \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}$ we can regard crystals on $\mathcal{Y}'$ as a single category, $\mathrm{Crys}^T(\mathcal{Y}')$, endowed with two forgetful functors

(7.2)
$$\begin{tikzcd}
& \mathrm{Crys}^T(\mathcal{Y}'). \arrow[dl, "\mathbf{oblv}^{T,l}_{\mathcal{Y}'}"'] \arrow[dr, "\mathbf{oblv}^{T,r}_{\mathcal{Y}'}"] & \\
\mathrm{QCoh}^T(\mathcal{Y}') \arrow[rr, "\Upsilon_{\mathcal{Y}'}"'] & & \mathrm{IndCoh}^T(\mathcal{Y}')
\end{tikzcd}$$

For $\mathcal{Y}' \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$, the above forgetful functors map to non-twisted sheaves:

(7.3)
$$\begin{tikzcd}
& \mathrm{Crys}^T(\mathcal{Y}'). \arrow[dl, "\mathbf{oblv}^{T,l}_{\mathcal{Y}'}"'] \arrow[dr, "\mathbf{oblv}^{T,r}_{\mathcal{Y}'}"] & \\
\mathrm{QCoh}(\mathcal{Y}') \arrow[rr, "\Upsilon_{\mathcal{Y}'}"'] & & \mathrm{IndCoh}(\mathcal{Y}')
\end{tikzcd}$$

7.3.5. Let $\mathcal{X} \in (\text{DGindSch}_{\text{laft}})_{/\mathcal{Y}_{\text{dR}}}$. The analogue of Proposition 3.1.3 holds with no change. In particular, we obtain a functor

$$\mathbf{ind}^{T,r}_{\mathcal{X}} : \text{IndCoh}^T(\mathcal{X}) \to \text{Crys}^{T,r}(\mathcal{X})$$

left adjoint to $\mathbf{oblv}^{T,r}_{\mathcal{X}}$.

Similarly, the analogue of Proposition 3.4.3 holds in the present context as well.

7.3.6. The following observation will be useful in the sequel:

Let X be an affine DG scheme (or an ind-affine DG indscheme) over $\mathcal{Y}_{\text{dR}}$. Choose a trivialization of the resulting $\mathbb{G}_m$-gerbe on X. This choice defines an identification

$$\text{IndCoh}^T(X) \overset{\alpha}{\simeq} \text{IndCoh}(X).$$

Lemma 7.3.7. *The monad* $\mathbf{oblv}^{T,r}_X \circ \mathbf{ind}^{T,r}_X$*, regarded as a functor (without the monad structure)*

$$\text{IndCoh}(X) \overset{\alpha^{-1}}{\simeq} \text{IndCoh}^T(X) \to \text{IndCoh}^T(X) \overset{\alpha}{\simeq} \text{IndCoh}(X),$$

is non-canonically isomorphic to $\mathbf{oblv}^r_X \circ \mathbf{ind}^r_X$.

Proof. First, we observe that the analogue of Proposition 5.1.3 holds; namely, the object of $\text{IndCoh}(X \times X)$ that defines the functor $\mathbf{oblv}^{T,r}_X \circ \mathbf{ind}^{T,r}_X$ is given by

$$(\widehat{\Delta}_X)^{\text{IndCoh}}_* \left(\mathcal{L} \otimes (\omega_{X \underset{X_{\text{dR}}}{\times} X}) \right),$$

where $\mathcal{L}$ is the line bundle on $X \underset{X_{\text{dR}}}{\times} X$ corresponding to T and α as in Sect. 6.6.1.

By construction, $\mathcal{L}$ is trivial when restricted to $X \hookrightarrow X \underset{X_{\text{dR}}}{\times} X$. Now, since X is affine, this implies that $\mathcal{L}$ can be trivialized on all of $X \underset{X_{\text{dR}}}{\times} X$.

$\square$

7.4. **t-structures on twisted crystals.** As in the previous subsection, let $\mathcal{Y}$ be a fixed object of $\text{PreStk}_{\text{laft}}$, and $T \in \text{Tw}(\mathcal{Y})$.

7.4.1. If X is a DG scheme and $\mathcal{G}$ is a $\mathbb{G}_m$-gerbe on it, the twisted categories $\mathrm{QCoh}^{\mathcal{G}}(X)$ and $\mathrm{IndCoh}^{\mathcal{G}}(X)$ have natural t-structures with properties analogous to those of their usual counterparts $\mathrm{QCoh}(X)$ and $\mathrm{IndCoh}(X)$.

In particular, we have the "left" t-structure on $\mathrm{Crys}^{T,l}(\mathcal{Y}')$ for any $\mathcal{Y}' \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}$. (This t-structure can be defined without the locally almost of finite type assumption on either $\mathcal{Y}$ or $\mathcal{Y}'$.)

The t-structure on twisted IndCoh on DG schemes allows us to define a t-structure on $\mathrm{IndCoh}^{\mathcal{G}}(\mathcal{X})$, where $\mathcal{X}$ is a DG indscheme. We can then define the "right" t-structure on the category $\mathrm{Crys}^{T,r}(\mathcal{X})$.

7.4.2. We observe that Proposition 4.2.5 renders to the twisted context with no change. We now claim:

Proposition 7.4.3. *Let X be a quasi-compact DG scheme mapping to $\mathcal{Y}_{\mathrm{dR}}$.*

(a) *The functor $\mathbf{ind}_X^{T,r}$ is t-exact.*

(b) *For a quasi-compact scheme X, the functor $\mathbf{oblv}_X^{T,r}$ is of bounded cohomological amplitude.*

Proof. The functor $\mathbf{ind}_X^{T,r}$ is right t-exact, since its right adjoint $\mathbf{oblv}_X^{T,r}$ is left t-exact. By the definition of the "right" t-structure, the left t-exactness of $\mathbf{ind}_X^{T,r}$ is equivalent to the same property of the composition $\mathbf{oblv}_X^{T,r} \circ \mathbf{ind}_X^{T,r}$.

The assertion is Zariski-local, so we can assume that X is affine. Now, the fact that the functor $\mathbf{oblv}_X^{T,r} \circ \mathbf{ind}_X^{T,r}$ is left t-exact follows from Lemma 7.3.7 and the fact that the analogous assertion holds in the non-twisted case.

Since $\mathrm{IndCoh}^T(X)^{\leq 0}$ generates $\mathrm{Crys}^{T,l}(X)^{\leq 0}$ via the functor $\mathbf{ind}_X^{T,r}$, in order to show that the cohomological amplitude of $\mathbf{oblv}_X^{T,r}$ is bounded from above, it suffices to show the same for $\mathbf{oblv}_X^{T,r} \circ \mathbf{ind}_X^{T,r}$. Again, the assertion follows from Lemma 7.3.7.

□

7.4.4. We now claim:

Proposition 7.4.5. *Let X be a quasi-compact DG scheme mapping to $\mathcal{Y}_{\mathrm{dR}}$.*

(a) *The "left" and "right" t-structures on* $\mathrm{Crys}^T(X)$ *differ by finite cohomological amplitude.*

(b) *The functor* $\mathbf{oblv}_X^{T,l} : \mathrm{Crys}^T(X) \to \mathrm{QCoh}(X)$ *is of bounded cohomological amplitude.*[11]

7.4.6. We shall first prove the following:

Let $i : X \to Z$ be a closed embedding, where Z is a smooth classical scheme. Let Y be the formal completion of Z along X.

Lemma 7.4.7. *The functor*

$$\mathbf{oblv}_Y^{T,r} : \mathrm{Crys}^{T,r}(Y) \to \mathrm{IndCoh}^T(Y)$$

is t-exact.

Proof. As in the proof of Proposition 7.4.3, it suffices to show that the functor

$$\mathbf{oblv}_Y^{T,r} \circ \mathbf{ind}_Y^{T,r} : \mathrm{IndCoh}^T(Y) \to \mathrm{IndCoh}^T(Y)$$

is t-exact. The assertion is Zariski-local, so we can assume that X is affine. Now, as in the proof of Proposition 7.4.3, the functor in question is non-canonically isomorphic to the non-twisted version: $\mathbf{oblv}_Y^r \circ \mathbf{ind}_Y^r$, and the latter is known to be t-exact by Proposition 4.2.11(a).

□

7.4.8. *Proof of Proposition 7.4.5.* The assertion is Zariski-local, so we can assume that X is affine and embed it into a smooth classical scheme Z. Let Y denote the formal completion of X in Z. By definition, T defines a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on Y. Let $'i$ denote the corresponding map $X \to Y$.

To prove point (a), by Lemma 4.3.7 (whose 2nd proof is applicable in the twisted case), we can replace X by Y, and it suffices to show that the discrepancy

[11] By point (a) this statement does not depend on which of the two t-structures we consider on $\mathrm{Crys}^T(X)$.

between the two t-structures on $\mathrm{Crys}^T(Y)$ is finite. By Proposition 4.1.3 (applied in the twisted case) and Lemma 7.4.7, it suffices to show that the functor

$$\Psi_Y : \mathrm{QCoh}^T(Y) \to \mathrm{IndCoh}^T(Y)$$

is of bounded cohomological amplidude. This is equivalent to the corresponding fact for

$$\Psi_Y : \mathrm{QCoh}(Y) \to \mathrm{IndCoh}(Y),$$

which in turn follows from the corresponding fact for Z.

Point (b) follows from the fact that the functor

$$'i^* : \mathrm{QCoh}^T(Y) \to \mathrm{QCoh}^T(X)$$

is of bounded amplitude, which is again equivalent to the corresponding fact for

$$'i^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X),$$

and the latter follows from the corresponding fact for Z.

□

7.4.9. The results concerning the "coarse" forgetful and induction functors, established in Sect. 4.6 for untwisted crystals, render automatically to the twisted situation.

7.4.10. Our current goal is to show:

Proposition 7.4.11. *Let X be a quasi-compact DG scheme mapping to $\mathcal{Y}_{\mathrm{dR}}$.*

(a) *The "right" t-structure on $\mathrm{Crys}^{T,r}(X)$ is left-complete.*

(b) *For X affine, the natural functor $D(\mathrm{Crys}^{T,r}(X)^{\heartsuit})^+ \to \mathrm{Crys}^{T,r}(X)^+$, where the heart is taken with respect to the "right" t-structure, uniquely extends to an an equivalence*

$$D(\mathrm{Crys}^{T,r}(X)^{\heartsuit}) \to \mathrm{Crys}^{T,r}(X).$$

7.4.12. *Proof of Proposition 7.4.11(a).* Again, the assertion is Zariski-local, and we retain the setting of the proof of Proposition 7.4.5.

It suffices to exhibit a collection of objects

$$\mathcal{P}_\alpha \in \mathrm{Crys}^{T,r}(X)$$

that generate $\mathrm{Crys}^{T,r}(X)$ and that are of bounded Ext dimension, i.e., if for each α there exists an integer k_α such that

$$\mathrm{Hom}_{\mathrm{Crys}^T(X)}(\mathcal{P}_\alpha, \mathcal{M}) = 0 \text{ if } \mathcal{M} \in \mathrm{Crys}^T(X)^{<-k_\alpha}.$$

We realize $\mathrm{Crys}^{T,r}(X)$ as $\mathrm{Crys}^{T,r}(Y)$. By Lemma 4.3.7 and Lemma 7.4.7, the t-structure on $\mathrm{Crys}^{T,r}(X) \simeq \mathrm{Crys}^{T,r}(Y)$ is characterized by the property that

$$\mathcal{M} \in \mathrm{Crys}^{T,r}(Y)^{\geq 0} \Leftrightarrow \mathbf{oblv}_Y^{T,r}(\mathcal{M}) \in \mathrm{IndCoh}^T(\mathcal{Y})^{\geq 0}.$$

We take $\mathcal{P}_\alpha$ to be of the form $\mathbf{ind}^{T,r}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{Coh}^T(Y)^\heartsuit$. To prove the required vanishing of Exts, we need to show that for $\mathcal{M} \in \mathrm{Crys}^{T,r}(Y)^{\ll 0}$,

$$\mathrm{Hom}_{\mathrm{IndCoh}^T(Y)}(\mathcal{F}, \mathbf{oblv}_Y^{T,r}(\mathcal{M})) = 0.$$

However, this follows from the fact that the category $\mathrm{IndCoh}^T(Y)$ has finite cohomological dimension with respect to its t-structure:[12] indeed, the category in question in non-canonically equivalent to $\mathrm{IndCoh}(Y)$, and the cohomological dimension of the latter is bounded by that of $\mathrm{IndCoh}(Z)$.

7.4.13. *Proof of Proposition 7.4.11(b).* We keep the notations from the proof of point (a).

As in the proof of Proposition 4.7.3, given what we have shown in point (a), we only have to verify that for $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Crys}^T(X)^\heartsuit$ and any $k \geq 0$, the map

$$\mathrm{Ext}^k_{\mathrm{Crys}^T(X)^\heartsuit}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Hom}_{\mathrm{Crys}^T(X)}(\mathcal{M}_1, \mathcal{M}_2[k])$$

is an isomorphism.

[12]We refer the reader to the footnone in Remark 4.7.8 where we explain what we mean by this.

For that it suffices to show that the category $\mathrm{Crys}^{T,r}(X)^{\heartsuit}$ contains a pro-projective generator of $\mathrm{Crys}^{T,r}(X)$, i.e., that there exists a filtered inverse family with surjective maps $\mathcal{P}_\alpha \in \mathrm{Crys}^{T,r}(X)^{\heartsuit}$, such that the functor

$$\underset{\alpha}{colim}\, \mathcal{M}aps_{\mathrm{Crys}^{T,r}(X)}(\mathcal{P}_\alpha, -)$$

is t-exact and conservative on $\mathrm{Crys}^{T,r}(X)$.

We take $\mathcal{P}_\alpha$ to be

$$\mathbf{ind}_Y^{T,r}(\mathcal{O}_{X_n}) \in \mathrm{Crys}^{T,r}(Y)^{\heartsuit} \simeq \mathrm{Crys}^{T,r}(X)^{\heartsuit},$$

where X_n is the n-th infinitesimal neighborhood of ${}^{cl,red}X$ in Z. □

7.5. **Other results.**

7.5.1. *Twisted crystals and twisted D-modules.* Let X be a smooth classical scheme. We have seen in Sect. 6.5.4 that the Picard category of twistings on X is equivalent to that of TDO's on X.

Given a twisting T, and the corresponding TDO, denoted Diff_X^T, there exists a canonical equivalence

$$\mathrm{Crys}^{T,l}(X) \simeq \text{D-mod}^{T,l}(X),$$

which commutes with the forgetful functors to $\mathrm{QCoh}(X)$, and similarly for twisted right crystals. The proof is either an elaboration of the strategy indicated in Sect. 5.5.4, or one using Sect. 5.4.

7.5.2. The relation between twisted D-modules and modules over a TDO can be extended to the case when instead of a smooth classical scheme X, we are dealing with a formal completion Y of a DG scheme X inside a smooth classical scheme Z.

This allows to prove:

Proposition 7.5.3. *Let X be a quasi-compact DG scheme over $\mathcal{Y}_{\mathrm{dR}}$.*

(a) *The abelian category* $\mathrm{Crys}^{T,r}(X)^{\heartsuit}$ *is locally Noetherian.*

(b) $\mathrm{Crys}^{T,r}(X)$ *has finite cohomological dimension with respect to its t-structure.*

(We refer the reader to the footnotes in Remark 4.7.8 where we explain what we mean by the properties asserted in points (a) and (b) of the proposition.)

References

[BB] A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gelfand Seminar, 1–50, Adv. Sov. Math. 16, Part 1, AMS, 1993.

[GL:DG] Notes on Geometric Langlands, *Generalities on DG categories*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:Stacks] Notes on Geometric Langlands, *Stacks*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:QCoh] Notes on Geometric Langlands, *Quasi-coherent sheaves on stacks*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[IndCoh] D. Gaitsgory, *Ind-coherent sheaves*, arXiv:1105.4857.

[GR1] D. Gaitsgory and N. Rozenblyum, *DG indschemes*, arXiv:1108.1738.

[GR2] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry*, in preparation, preliminary version will gradually become available at http://www.math.harvard.edu/ gaitsgde/GL/.

[Gr] A. Grothendieck, *Crystals and the de Rham cohomology of schemes*, Dix exposés sur la cohomologie des schémas, 306–358.

[Lu0] J. Lurie, *Higher Topos Theory*, Princeton Univ. Press (2009).

[Lu1] J. Lurie, *DAG-VIII*, available at http://www.math.harvard.edu/ lurie.

[Lu2] J. Lurie, *Higher algebra*, available at http://www.math.harvard.edu/ lurie.

[Si] C. Simpson, *Homotopy over the complex numbers and generalized cohomology theory*, Moduli of vector bundles (Taniguchi Symposium, December 1994), M. Maruyama ed., Dekker Publ, 229–263, 1996.

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