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Scalar Curvature Pinching for CMC Hypersurfaces in a Sphere

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Abstract: We consider an n-dimensional closed hypersurface M with constant mean curvature H in S^{n+1}, $3 \leq n \leq 8$. Denote by S and $\beta(n, H)$ the squared norm of the second fundamental form of M and $S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$ respectively, where $\mu = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2}$. We prove that there exist two positive constants $\gamma(n)$ and $\epsilon(n)$ such that if $|H| \leq \gamma(n)$ and $\beta(n, H) \leq S < \beta(n, H) + \epsilon(n)$, then $S \equiv \beta(n, H)$ and M is one of the following cases: (i) $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $k = 1, 2, \ldots, n-1$; (ii) $S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$. This result extends the scalar curvature pinching theorem for minimal hypersurfaces due to Peng-Terng, Wei-Xu and Zhang.
Keywords: hypersurfaces with constant mean curvature, rigidity, scalar curvature, Clifford torus.

1. Introduction

Since the appearance of the famous rigidity theorem for minimal submanifolds due to Simons [15], Lawson [6], and Chern-do Carmo-Kobayashi [4], various scalar curvature pinching problems for submanifolds have been studied by many geometers (see [3, 5, 7, 8, 10, 11, 12, 13, 18, 20, 21, 25, 29], etc.). Let M be an n-dimensional closed minimal hypersurface in an $(n + 1)$-dimensional unit sphere S^{n+1}. Denote by R and S the scalar curvature and the squared length of the second fundamental form of M, respectively. From Gauss equation, we have $R = n(n-1) - S$. The Simons-Lawson-Chern-do Carmo-Kobayashi rigidity

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theorem says that if $S \leq n$, then $S \equiv 0$ or $S \equiv n$, i.e., M must be the great sphere S^n or the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n-1$. Later, Peng and Terng [10, 11] found the second scalar curvature pinching phenomenon for closed minimal hypersurfaces in a unit sphere. In [10], they proved that if the scalar curvature of M is a constant and $n \leq S \leq n + \frac{1}{12n}$, then $S \equiv n$. When $n = 3$, Chang [1] gave the classification of all closed minimal hypersurfaces with constant scalar curvature in S^4. When $n \geq 4$, Peng-Terng's pinching constant $\frac{1}{12n}$ was improved to $\frac{n}{3}$ by Yang-Cheng [27, 28], and to $\frac{3n}{7}$ by Suh-Yang [16], respectively. The classification problem in dimension $n \geq 4$ is still open.

In [11], Peng and Terng obtained an important pinching theorem for $n(\leq 5)$-dimensional minimal hypersurfaces in a unit sphere without assumption that the scalar curvature is a constant.

Theorem 1.1. *Let M^n be an $n(\leq 5)$-dimensional closed minimal hypersurface in a unit sphere S^{n+1}. Then there exists a positive constant $\tau(n)$ depending only on n such that if $n \leq S \leq n + \tau(n)$, then $S \equiv n$, i.e., M is one of the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $k = 1, 2, \ldots, n-1$.*

In 1997, Ogiue and Sun [9] claimed that they had generalized Theorem 1.1 to the case for $n \geq 6$. In [17], Wei and Xu pointed out a fatal mistake in the paper by Ogiue and Sun [9]. Moreover, Wei and Xu [17] solved the second pinching problem for $n = 6, 7$. Later, Zhang [30] extended the second pinching theorem due to Peng-Terng [11] and Wei-Xu [17] to the case of $n = 8$. The second pinching theorem for minimal hypersurfaces due to Wei-Xu-Zhang [17, 30] is stated as follows.

Theorem 1.2. *Let M be an n-dimensional closed minimal hypersurface in a unit sphere S^{n+1}, $n = 6, 7, 8$. Then there exists a positive constant $\tau(n)$ depending only on n such that if $n \leq S \leq n + \tau(n)$, then $S \equiv n$, i.e., M is the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n-1$.*

The pinching phenomenon for constant mean curvature hypersurfaces, i.e., CMC hypersurfaces, is much more complicated than the case of minimal hypersurfaces (see [14, 18, 19, 22, 23, 29], etc.). Set

$$\alpha(n, H) = n + \frac{n^3 H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1) H^2},$$

$$\beta(n,H) = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2},$$

$$\lambda = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}, \quad \mu = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2}.$$

In [18], Xu proved the following pinching theorem for submanifolds with parallel mean curvature in a sphere.

Theorem 1.3. *Let M be an n-dimensional compact submanifold with parallel mean curvature vector ($H \neq 0$) in an $(n+p)$-dimensional unit sphere S^{n+p}. If $S \leq \alpha(n,H)$, then either M is pseudo-umbilical, or $S \equiv \alpha(n,H)$ and M is the isoparametric hypersurface $S^{n-1}\big(\frac{1}{\sqrt{1+\lambda^2}}\big) \times S^1\big(\frac{\lambda}{\sqrt{1+\lambda^2}}\big)$ in a great sphere S^{n+1}. In particular, if M is a compact hypersurface with constant mean curvature $H(\neq 0)$ in S^{n+1}, then M is either a totally umbilical sphere $S^n\big(\frac{1}{\sqrt{1+H^2}}\big)$, or a Clifford hypersurface $S^{n-1}\big(\frac{1}{\sqrt{1+\lambda^2}}\big) \times S^1\big(\frac{\lambda}{\sqrt{1+\lambda^2}}\big)$.*

In [23], Xu and Tian generalized Suh-Yang's pinching theorem [16] to the case where M is a compact hypersurface with constant scalar curvature and small constant mean curvature in S^{n+1}. The following second pinching theorem for hypersurfaces with small constant mean curvature was proved by Cheng-He-Li [2] and Xu-Zhao [24], respectively.

Theorem 1.4. *Let M be an n-dimensional closed hypersurface in a unit sphere S^{n+1} with constant mean curvature $H(\neq 0)$, $3 \leq n \leq 7$. There exist two positive constants $\gamma(n)$ and $\epsilon(n)$ such that if $|H| \leq \gamma(n)$, and $\beta(n,H) \leq S < \beta(n,H) + \epsilon(n)$, then $S \equiv \beta(n,H)$ and $M = S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$.*

In this paper, we obtain the following theorem which extends Theorem 1.2 and Theorem 1.4.

Theorem 1.5 (Main Theorem)**.** *Let M be an n-dimensional closed hypersurface in a unit sphere S^{n+1} with constant mean curvature H, $3 \leq n \leq 8$. There exist two positive constants $\gamma(n)$ and $\epsilon(n)$ depending only on n such that if $|H| \leq \gamma(n)$, and $\beta(n,H) \leq S < \beta(n,H) + \epsilon(n)$, then $S \equiv \beta(n,H)$ and M is one of the following cases: (i) $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n-1$; (ii) $S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$.*

2. Preliminaries

Let M^n be an n-dimensional closed hypersurface with constant mean curvature in a unit sphere S^{n+1}. We shall make use of the following convention on the range of indices.

$$1 \leq A, B, C, \ldots, \leq n+1,$$

$$1 \leq i, j, k, \ldots, \leq n.$$

Choose an orthonormal frame field $\{e_A\}$ in a neighborhood of $p \in M$ such that the e_i's are tangent to M at p. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ be the connection 1-forms of S^{n+1}. Restricting to M, we have

$$\omega_{n+1j} = \sum_i h_{ji}\omega_i, \quad h_{ji} = h_{ij}. \tag{2.1}$$

Let R and h be the scalar curvature and the second fundamental form of M, respectively. Denote by S the squared length of h and H the mean curvature of M, and assume $H \geq 0$. Then we have

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j, \quad S = \sum_{i,j} h_{ij}^2, \tag{2.2}$$

$$H = \frac{1}{n}\sum_i h_{ii}, \tag{2.3}$$

$$R = n(n-1) + n^2H^2 - S. \tag{2.4}$$

Denote by h_{ijk}, h_{ijkl} and h_{ijklm} the first, second and third covariant derivatives of the second fundamental tensor h_{ij}. Then

$$\nabla h = \sum_{i,j,k} h_{ijk}\omega_i \otimes \omega_j \otimes \omega_k, h_{ijk} = h_{ikj}, \tag{2.5}$$

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{2.6}$$

$$h_{ijklm} = h_{ijkml} + \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}. \tag{2.7}$$

For an arbitrary fixed point $x \in M$, we take an orthonormal frame $\{e_i\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ for all i, j. Then $\sum\limits_i \lambda_i = nH$, and $\sum\limits_i \lambda_i^2 = S$. By a similar computation as in [11], we obtain

$$\frac{1}{2}\Delta S = S(n-S) - n^2H^2 + nHf_3 + |\nabla h|^2, \tag{2.8}$$

and

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla h|^2 &= (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 + |\nabla^2 h|^2 \\
&\quad + \sum_{i,j,k,l,m}(6h_{ijk}h_{ilm}h_{jl}h_{km} - 3h_{ijk}h_{ijl}h_{km}h_{ml}) \\
&\quad + 3nH\sum_{i,j,k,l} h_{ijk}h_{jlk}h_{li} \\
&= (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 + |\nabla^2 h|^2 \\
&\quad + 3(2B-A) + 3nHC,
\end{aligned} \tag{2.9}$$

where

$$f_k = \sum_i \lambda_i^k,\ A = \sum_{i,j,k} h_{ijk}^2\lambda_i^2,\ B = \sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_j,\ C = \sum_{i,j,k} h_{ijk}^2\lambda_i.$$

From [10], we have

$$h_{ijij} = h_{jiji} + t_{ij}, \tag{2.10}$$

where $t_{ij} = (\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j)$. This implies

$$|\nabla^2 h|^2 \geq \frac{3}{4}\sum_{i\neq j} t_{ij}^2 = \frac{3}{4}\sum_{i,j} t_{ij}^2. \tag{2.11}$$

Hence

$$|\nabla^2 h|^2 \geq \frac{3}{2}\Big[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3\Big]. \tag{2.12}$$

Now we give a similar computation as in [11].

$$
\begin{aligned}
\frac{1}{3}\sum_{i,j} h_{ij}(f_3)_{ij} &= \frac{1}{3}\sum_{k} \lambda_k (f_3)_{kk} \\
&= \sum_k \lambda_k \Big(\sum_i h_{iikk}\lambda_i^2 + 2\sum_{i,j} h_{ijk}^2 \lambda_i \Big) \\
&= \sum_{i,k} h_{iikk}\lambda_k\lambda_i^2 + 2\sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_k \\
&= \sum_{i,k} \Big[h_{kkii} + (\lambda_i - \lambda_k)(1+\lambda_i\lambda_k) \Big] \lambda_k\lambda_i^2 + 2B \\
&= \sum_i \Big(\frac{S_{ii}}{2} - \sum_{j,k} h_{ijk}^2 \Big)\lambda_i^2 + \sum_{i,k} \lambda_i^2\lambda_k(\lambda_i-\lambda_k)(1+\lambda_i\lambda_k) + 2B \\
&= \sum_{i,j,k} \frac{h_{ik}h_{kj}}{2} S_{ij} + nHf_3 - S^2 - f_3^2 + Sf_4 - (A-2B). \qquad (2.13)
\end{aligned}
$$

Since $\int_M \sum_{i,j} h_{ij}(f_3)_{ij} dM = 0$, we drive following integral equation of $A-2B$.

$$
\begin{aligned}
\int_M (A-2B)dM &= \int_M \Big[nHf_3 - S^2 - f_3^2 + Sf_4 + \sum_{i,j,k} \frac{h_{ik}h_{kj}}{2} S_{ij} \Big] dM \\
&= \int_M \Big[nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} (h_{ik}h_{kj})_j \frac{S_i}{2} \Big] dM \\
&= \int_M \Big[nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} h_{ikj}h_{kj} \frac{S_i}{2} \\
&\quad - \sum_{i,j,k} h_{ik}h_{kjj} \frac{S_i}{2} \Big] dM \\
&= \int_M \Big[nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} h_{ikj}h_{kj} \frac{S_i}{2} \Big] dM \\
&= \int_M \Big[nHf_3 - S^2 - f_3^2 + Sf_4 - \frac{|\nabla S|^2}{4} \Big] dM. \qquad (2.14)
\end{aligned}
$$

From (2.8) (2.9) (2.12) (2.14), we have

$$\begin{aligned}\int_M |\nabla^2 h|^2 dM &= \int_M \Big[(S-2n-3)|\nabla h|^2 + \frac{3}{2}|\nabla S|^2 + 3(A-2B) - 3nHC\Big] dM\\ &\geq \int_M \frac{3}{2}\Big[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3\Big] dM\\ &= \int_M \Big[\frac{3}{2}(A-2B) - \frac{3}{2}|\nabla h|^2 + \frac{3}{8}|\nabla S|^2\Big] dM. \qquad (2.15)\end{aligned}$$

Hence

$$\int_M \Big[(S-2n-\frac{3}{2})|\nabla h|^2 + \frac{3}{2}(A-2B) + \frac{9}{8}|\nabla S|^2 - 3nHC\Big] dM \geq 0. \qquad (2.16)$$

3. Proof of main theorem

To simplify the computation, we use the tracefree second fundamental form $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$, where $\phi_{ij} = h_{ij} - H\delta_{ij}$. If $h_{ij} = \lambda_i \delta_{ij}$, then $\phi_{ij} = \mu_i \delta_{ij}$, where $\mu_i = \lambda_i - H$. Putting $\Phi = |\phi|^2$ and $\bar{f}_k = \sum_i \mu_i^k$, we get $\Phi = S - nH^2$ and $f_3 = \bar{f}_3 + 3H\Phi + nH^3$. Then, we rewrite (2.16) as follows.

$$\int_M \Big[(\Phi + nH^2 - 2n - \frac{3}{2})|\nabla \phi|^2 + \frac{3}{2}(A-2B) + \frac{9}{8}|\nabla \Phi|^2 - 3nHC\Big] dM \geq 0. \qquad (3.1)$$

On the other hand, since $\int_M \frac{1}{4}\Delta\Phi^2 dM = 0$, from (2.8), we have

$$-\frac{1}{2}\int_M |\nabla\Phi|^2 dM = \int_M \Big[\Phi|\nabla\phi|^2 - \Phi^3 + n\Phi^2 + nH^2\Phi^2 + nH\bar{f}_3\Phi\Big] dM. \qquad (3.2)$$

From (3.1) and (3.2), we have

$$\begin{aligned}0 \leq &\int_M \Big[(\Phi + nH^2 - 2n - \frac{3}{2})|\nabla\phi|^2 + \frac{3}{2}(A-2B)\\ &-\frac{9}{4}(\Phi|\nabla\phi|^2 - \Phi^3 + n\Phi^2 + nH^2\Phi^2 + nH\Phi\sum_i \mu_i^3) - 3nHC\Big] dM\\ = &\int_M \Big[(-\frac{5}{4}\Phi + nH^2 - 2n - \frac{3}{2})|\nabla\phi|^2 + \frac{3}{2}(A-2B)\\ &+\frac{9}{4}\Phi F(\Phi) - 3nHC\Big] dM, \qquad (3.3)\end{aligned}$$

where

$$F(\Phi) = \Phi^2 - n\Phi - nH^2\Phi - nH\sum_i \mu_i^3.$$

Lemma 3.1 ([18]). *Let $a_1,a_2,\ldots,a_n$ be real numbers satisfying $\sum_i a_i = 0$ and $\sum_i a_i^2 = a$. Then*

$$|\sum_i a_i^3| \leq \frac{n-2}{\sqrt{n(n-1)}} a^{\frac{3}{2}},$$

and the equality holds if and only if at least $n-1$ numbers of a_i's are same with each other.

From Lemma 3.1, we get

$$\begin{aligned} F(\Phi) &\geq \Phi^2 - n\Phi - nH^2\Phi - nH\frac{(n-2)\Phi^{\frac{3}{2}}}{\sqrt{n(n-1)}} \\ &= \Phi\Big[\Phi - nH\frac{(n-2)\Phi^{\frac{1}{2}}}{\sqrt{n(n-1)}} - n(1+H^2)\Big] \\ &\geq 0, \end{aligned} \tag{3.4}$$

where

$$\Phi \geq \beta_0(n,H) := n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2} - nH^2.$$

Moreover, if $F(\Phi) = 0$, then $\Phi = \beta_0(n,H)$ and at least $n-1$ numbers of μ_i's are same with each other.

We consider the case for $\beta_0(n,H) \leq \Phi < \beta_0(n,H) + \epsilon(n)$, where $\epsilon(n)$ is to be determined. From (2.8), we have

$$\int_M F(\Phi)dM = \int_M |\nabla\phi|^2 dM. \tag{3.5}$$

Then

$$\int_M \Phi F(\Phi)dM \leq \int_M \Big(\beta_0(n,H) + \epsilon(n)\Big)|\nabla\phi|^2 dM. \tag{3.6}$$

Since $|C| \leq \sqrt{S}|\nabla h|^2 \leq S|\nabla h|^2$, from (3.3) and (3.6), we have

$$\begin{aligned} 0 \leq &\int_M \Big[(-\frac{5}{4}\Phi + nH^2 - 2n - \frac{3}{2} + 3nH\Phi + 3n^2H^3)|\nabla\phi|^2 + \frac{3}{2}(A-2B) \\ &+ \frac{9}{4}\Big(\beta_0(n,H) + \epsilon(n)\Big)|\nabla\phi|^2\Big]dM. \end{aligned} \tag{3.7}$$

Lemma 3.2. *Let M be an n-dimensional closed hypersurface in a unit sphere S^{n+1} with constant mean curvature, $n \geq 4$. Suppose that*

$$3(A-2B) \leq t(n)S|\nabla h|^2,$$

where $t(n)$ is a number depending only on n, satisfying $0 \le t(n) < 2 + \frac{3}{n}$. Then there exist positive constants $\gamma(n)$ and $\epsilon(n)$ such that if $|H| \le \gamma(n)$, and $\beta(n,H) \le S < \beta(n,H) + \epsilon(n)$, then $S \equiv \beta(n,H)$. Here

$$\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

Proof. Putting $D(n,H) = \beta_0(n,H) + nH^2 - n$, from (3.7) and the condition of the lemma, we have

$$\begin{aligned}
0 \le& \int_M \Big[(-\frac{5}{4}\Phi + nH^2 - 2n - \frac{3}{2} + 3nH\Phi + 3n^2H^3)|\nabla\phi|^2 + \frac{t(n)}{2}S|\nabla\phi|^2 \\
&+\frac{9}{4}\Big(\beta_0(n,H) + \epsilon(n)\Big)|\nabla\phi|^2\Big]dM \\
=& \int_M \Big[\Big(-\frac{5}{4} + 3nH + \frac{t(n)}{2}\Big)\Big(\Phi - \beta_0(n,H)\Big) \\
&+\Big(-\frac{5}{4} + 3nH + \frac{t(n)}{2} + \frac{9}{4}\Big)\beta_0(n,H) \\
&+nH^2 - 2n - \frac{3}{2} + 3n^2H^3 + \frac{t(n)}{2}nH^2 + \frac{9}{4}\epsilon(n)\Big]|\nabla\phi|^2 dM \\
=& \int_M \Big[\Big(-\frac{5}{4} + 3nH + \frac{t(n)}{2}\Big)\Big(\Phi - \beta_0(n,H)\Big) \\
&+\Big(1 + 3nH + \frac{t(n)}{2}\Big)\Big(n + D(n,H) - nH^2\Big) \\
&+nH^2 - 2n - \frac{3}{2} + 3n^2H^3 + \frac{t(n)}{2}nH^2 + \frac{9}{4}\epsilon(n)\Big]|\nabla\phi|^2 dM \\
=& \int_M \Big[\Big(-\frac{5}{4} + 3nH + \frac{t(n)}{2}\Big)\Big(\Phi - \beta_0(n,H)\Big) + \Big(1 + 3nH + \frac{t(n)}{2}\Big)D(n,H) \\
&+3n^2H + \Big(1 + \frac{t(n)}{2}\Big)n - 2n - \frac{3}{2} + \frac{9}{4}\epsilon(n)\Big]|\nabla\phi|^2 dM. \qquad (3.8)
\end{aligned}$$

Since $t(n) < 2 + \frac{3}{n}$, we have

$$\begin{aligned}
\Big(1 + \frac{t(n)}{2}\Big)n - 2n - \frac{3}{2} &< \Big(1 + \frac{2 + \frac{3}{n}}{2}\Big)n - 2n - \frac{3}{2} \\
&= 2n + \frac{3}{2} - 2n - \frac{3}{2} \\
&= 0. \qquad (3.9)
\end{aligned}$$

Letting $E(n,H) = (1 + 3nH + \frac{t(n)}{2})D(n,H) + 3n^2H$, we see that $E(n,t)$ is strictly increasing for $t \ge 0$ and $E(n,0) = 0$. Hence there exists a positive constant $\gamma'(n)$, such that if $H \le \gamma^{'}(n)$, then

$$\Big(1+3nH+\frac{t(n)}{2}\Big)D(n,H)+3n^2H \le \frac{1}{2}\Big[2n+\frac{3}{2}-\Big(1+\frac{t(n)}{2}\Big)n\Big].$$

Set $\gamma^{''}(n)=\frac{1-\frac{3}{n}}{12n}$ and

$$\epsilon(n,H)=-\frac{4}{9}\Big[\Big(1+3nH+\frac{t(n)}{2}\Big)D(n,H)+3n^2H+\Big(1+\frac{t(n)}{2}\Big)n-2n-\frac{3}{2}\Big].$$

When $n \ge 4$ and $H \le \gamma^{''}(n)$, we have

$$\begin{aligned} -\frac{5}{4}+3nH+\frac{t(n)}{2} &< -\frac{5}{4}+3nH+\frac{2+\frac{3}{n}}{2} \\ &= 3nH+\frac{\frac{3}{n}-1}{4} \\ &\le 0. \end{aligned} \tag{3.10}$$

Take $\gamma(n)=min\{\gamma^{'}(n),\gamma^{''}(n)\}$ and $\epsilon(n)=\epsilon(n,\gamma(n))$. When $H \le \gamma(n)$ and $\beta(n,H) \le S < \beta(n,H)+\epsilon(n)$, combining (3.8) and (3.10), we have

$$\int_M \Big(-\frac{5}{4}+3nH+\frac{t(n)}{2}\Big)\Big(\Phi-\beta_0(n,H)\Big)|\nabla\phi|^2 dM=0. \tag{3.11}$$

Putting $X=\{x\in M|\Phi(x)>\beta_0(n,H)\}$ and $Y=\{x\in M|\Phi(x)=\beta_0(n,H)\}$, we have $M=X\cup Y$. We assert that $X=\emptyset$. Otherwise, from (3.11), we have $\nabla\phi(x)=\nabla h(x)=0$ for $x\in X$. Since Φ is a continuous function on M, $X=\{x\in M|\Phi(x)>\beta_0(n,H)\}$ is an open subset. Hence $\Phi(x)$ identically equals to a constant τ_0 on X. This shows that X is a closed subset. So, $X=M$. This together with (3.11) implies that $\nabla\phi(x)=\nabla h(x)=0$ for all $x\in M$. It follows from (3.4) and (3.5) that $\Phi(x)\equiv\beta_0(n,H)$. This is a contradiction. Therefore, $M=Y$, i.e., $\Phi\equiv\beta_0(n,H)$. This implies $S\equiv\beta(n,H)$. □

By using a similar method as in the proof of Lemma 3.2 in [30], we get the following lemma.

Lemma 3.3. *Let M be an n-dimensional closed hypersurface with constant mean curvature in a unit sphere S^{n+1}. If $\lambda_1^2-4\lambda_1\lambda_2 \ge tS$ for some $t\ge 2$, then $(\lambda_1^2-4\lambda_1\lambda_2)-(\lambda_1^2-4\lambda_1\lambda_i)\ge rS$, where $r=\frac{16t-8-12\sqrt{-2t^2+2t+8}}{17}$ and $i\ne 1,2$.*

The following lemma will be used in proof of Lemma 3.5.

Lemma 3.4 ([30]). *Let* $f_n(t) = 17[t-2-\delta(n)][3(n-2)t+(n+2)\delta(n)+10-4n]$ *and* $g_n(t) = [8+16\delta(n)](4t-2-3\sqrt{-2t^2+2t+8})$. *Then*

$$h_n(t) = f_n(t) - g_n(t) \le 0,$$

where $t \ge 2, n = 8, \delta(8) = 0.34$.

Putting $b_i = h_{ii1}$, $b = \sum\limits_{i\ne 1} b_i^2 + \frac{1}{3}b_1^2$, $f = \sum\limits_{i\ne 1}(\lambda_1^2 - 4\lambda_1\lambda_i)b_i^2 - \lambda_1^2 b_1^2$, we have

Lemma 3.5. *Let* M *be an* 8*-dimensional closed hypersurface with constant mean curvature in a unit sphere* S^9. *Then*

$$f \le \Big(2+\delta(n)\Big)Sb,$$

where $n = 8$, $\delta(8) = 0.34$.

Proof. If $\lambda_1^2 - 4\lambda_1\lambda_i \le (2+\delta(n))S$ for $2 \le i \le n$, then we have proven Lemma 3.5. Otherwise, without loss of generality, we assume that $\lambda_1^2 - 4\lambda_1\lambda_2 = tS$ for some $t \ge 2$, and $b_1 = xb_2$. Since H is a constant, we have $(b_1+b_2)^2 = (\sum\limits_{i\ne 1,2} b_i)^2$. Hence

$$\sum_{i\ne 1,2} b_i^2 \ge \frac{(1+x)^2}{n-2}b_2^2, \tag{3.12}$$

$$\lambda_1^2 \ge (t-2)S. \tag{3.13}$$

From (3.12), (3.13) and Lemma 3.3, we have

$$\begin{aligned} f - \Big(2+\delta(n)\Big)Sb \le & \Big(t-2-\delta(n)\Big)Sb_2^2 + \Big(t-r-2-\delta(n)\Big)S\sum_{i\ne 1,2} b_i^2 \\ & -\Big(t-2+\frac{2+\delta(n)}{3}\Big)Sb_1^2 \\ \le & \Big(t-2-\delta(n)\Big)Sb_2^2 + \frac{t-r-2-\delta(n)}{n-2}(1+x)^2Sb_2^2 \\ & -\Big(t-2+\frac{2+\delta(n)}{3}\Big)x^2Sb_2^2, \end{aligned} \tag{3.14}$$

where $r = \frac{16t-8-12\sqrt{-2t^2+2t+8}}{17}$. Let $F(n,t,x) = t-2-\delta(n) + \frac{t-r-2-\delta(n)}{n-2}(1+x)^2 - (t-2+\frac{2+\delta(n)}{3})x^2$, then (3.14) becomes

$$f - \Big(2+\delta(n)\Big)Sb \le F(n,t,x)Sb_2^2. \tag{3.15}$$

For fixed t, let x_0 be the maximum point of $F(n,t,x)$. Noting that $\frac{\partial F(n,t,x)}{\partial x}|_{x=x_0} = 0$, we have

$$F(n,t,x) \le F(n,t,x_0) = \frac{h_n(t)}{51G(n,t)},$$

where $G(n,t) = r + 2 + \delta(n) - t + (n-2)(t - 2 + \frac{2+\delta(n)}{3})$, and $h_n(t)$ is defined in Lemma 3.4. Therefore,

$$f \le \Big(2+\delta(n)\Big)Sb.$$

□

Lemma 3.6. *Let M be an 8-dimensional closed hypersurface with constant mean curvature in a unit sphere S^9. Then*

$$3(A-2B) \le 2.34S|\nabla h|^2,$$

where A, B are defined in (2.9).

Proof. Using Lemma 3.5, for any j, we have

$$\begin{aligned} f_j &= \sum_{i\neq j}(\lambda_j^2 - 4\lambda_j\lambda_i)h_{iij}^2 - \lambda_j^2 h_{jjj}^2 \\ &\le \Big(2+\delta(n)\Big)S\Big(\sum_{i\neq j} h_{iij}^2 + \frac{1}{3}h_{jjj}^2\Big). \end{aligned} \tag{3.16}$$

Therefore

$$\begin{aligned} 3(A-2B) = & \sum_{i,j,k \text{ distinct}} \Big[2(\lambda_i^2+\lambda_j^2+\lambda_k^2) - (\lambda_i+\lambda_j+\lambda_k)^2\Big]h_{ijk}^2 \\ & -3\sum_i \lambda_i^2 h_{iii}^2 + 3\sum_j\sum_{i\neq j}(\lambda_j^2 - 4\lambda_i\lambda_j)h_{iij}^2 \\ \le & \, 2S\sum_{i,j,k \text{ distinct}} h_{ijk}^2 + 3\sum_j\sum_{i\neq j}\Big[(\lambda_j^2-4\lambda_i\lambda_j)h_{iij}^2 - \lambda_j^2 h_{jjj}^2\Big] \\ \le & \Big(2+\delta(n)\Big)S\Big(\sum_{i,j,k \text{ distinct}} h_{ijk}^2 + 3\sum_j\sum_{i\neq j} h_{iij}^2 + \sum_j h_{jjj}^2\Big) \\ = & \Big(2+\delta(n)\Big)S\sum_{i,j,k} h_{ijk}^2. \end{aligned} \tag{3.17}$$

Since $\delta(8) = 0.34$, we have

$$3(A-2B) \le 2.34S|\nabla h|^2.$$

□

Now we are in a position to give the proof of Main Theorem.

Proof. i) When $3 \leq n \leq 7$, Main Theorem follows from Theorem 1.1, Theorem 1.2 and Theorem 1.4.
ii) When $n = 8$, from Lemma 3.2, Lemma 3.6 and the condition of Main Theorem, we have

$$S \equiv \beta(n, H).$$

Hence

$$F(\Phi) = 0.$$

From Lemma 3.1, we know that at least $n-1$ numbers of the λ_i's are same with each other. From the condition of Main Theorem, we have

$$\lambda_1 = \cdots = \lambda_{n-1} = H - \sqrt{\frac{\beta(n,H) - nH^2}{n(n-1)}},$$

$$\lambda_n = H + \sqrt{\frac{(n-1)(\beta(n,H) - nH^2)}{n}}.$$

Therefore M is the Clifford hypersurface

$$S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$$

in S^{n+1}, where $\mu = \frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$. This completes the proof of Main Theorem. □

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