Bergman Kernel and Kähler Tensor Calculus

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Abstract: Fefferman [22] initiated a program of expressing the asymptotic expansion of the boundary singularity of the Bergman kernel for strictly pseudoconvex domains explicitly in terms of boundary invariants. Hirachi, Komatsu and Nakazawa [30] carried out computations of the first few terms of Fefferman's asymptotic expansion building partly on Graham's work on CR invariants and Nakazawa's work on the asymptotic expansion of the Bergman kernel for strictly pseudoconvex complete Reinhardt domains. In this paper, we prove a formula for coefficients in Nakazawa's asymptotic expansion as explicit summations over strongly connected graphs, and a formula expressing partial derivatives of Kähler metrics (resp. functions) as summations over rooted trees encoding covariant derivatives of curvature tensors (resp. functions). These formulae shall be useful in studying general patterns of Fefferman's asymptotic expansion.
Keywords: Bergman kernel, asymptotic expansion, Laplace integral.

1. INTRODUCTION

The heat kernel on Riemannian manifolds plays an important role in index theory, general relativity and cosmology. A central object of study is the short-time expansion of the heat kernel of the Laplacian operator,

$$K(t,x,x) \sim \sum_{j=0}^{\infty} a_j(x) t^{j-d/2}, \quad t \to 0+,$$

where d is the manifold's dimension and the coefficients $a_j(x)$ are invariant polynomials of jets of metrics. By H. Weyl's work on the invariants of the orthogonal

group, any invariant polynomial can be formed from Riemannian curvature tensor by covariant differentiations, multiplications and contractions. It is a very difficult problem to put a tensor expressions into a canonical form on a general Riemmanian manifold. While the asymptotic expansion of the heat kernel has important applications in spectral geometry (see [58] for a recent survey), it seems hopeless to detect any meaningful structure of its combinatorially explosive coefficients.

The ubiquitousness of heat kernels can be seen most clearly in Liu's work [39, 40, 41], where the heat kernel was used as a unifying tool to solve problems in algebra, geometry and topology.

Let Ω be a (bounded) domain in $\mathbb{C}^n$ and $A^2(\Omega)$ the Bergman space of holomorphic functions in $L^2(\Omega)$. The (unweighted) Bergman kernel of Ω is a real analytic function given by $K(z) = \sum_j |\phi_j(z)|^2$ for $z \in \Omega$, where $\{\phi_j\}$ is an arbitrary orthonormal basis of $A^2(\Omega)$. In [22], Fefferman initiated a program of studying the Bergman kernel of a strictly pseudoconvex domain as an analogy of the heat kernel, with the time variable t replaced by a defining function of the domain. Fefferman's program opened up the subject [3, 25, 27, 30] and has been extended to conformal geometry; a major recent breakthrough is Alexakis' proof [1] of the Deser-Schwimmer conjecture.

Yau [55, p.679] proposed to classify pseudoconvex domains whose Bergman metrics are Kähler-Einstein. A conjecture of S.-Y. Cheng says that if the Bergman metric of a strictly pseudoconvex domain is Kähler-Einstein, then the domain is biholomorphic to the ball (cf. [23]). The deep works of Cheng-Yau [11] and Mok-Yau [47] showed that on a bounded domain of holomorphy, there exists a unique biholomorphic invariant complete Kähler-Einstein metric with scalar curvature -1.

The complete asymptotic expansion of the weighted Bergman kernel has been established by Engliš [18] for bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundary. For compact Kähler manifolds, in the 1980's, Yau [56, p.139] initiated the program of approximating Kähler-Einstein metrics by embeddings into complex projective space by higher power sections of an ample holomorphic line bundle. The corresponding complete asymptotic expansion of Bergman kernels was established independently by Zelditch [57] and Catlin [9], which has important applications in the study of extremal metrics [15, 56]. It

can be viewed as the local version of the Hirzebruch-Riemann-Roch theorem. Thanks to the Kähler condition $d\omega = 0$, the space of Weyl invariant polynomials on Kähler manifolds has a canonical basis represented by multi-digraphs. A closed formula has been discovered [51] for coefficients in the asymptotic expansion of the weighted Bergman kernel. We expect that a similar formula should exist for the heat kernel of the Laplacian operator on Kähler manifolds.

Karabegov and Schlichenmaier [35] showed that the logarithm of the diagonal value of the weighted Bergman kernel is the Karabegov classifying form of the Berezin quantization. There has been much work on deformation quantization of Kähler manifolds and its applications (see the recent survey [50]).

We summarize the definitions and properties of curvature tensors on Kähler manifolds. Let (M, g) be a Kähler manifold of dimension n. Locally the Kähler form is given by $\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$. We will use the Einstein summation convention. The indices $i, j, k, \ldots$ run from 1 to n, while Greek indices α, β, γ may represent either i or $\bar{i}$. Let $\det g$ be the determinant of the Hermitian matrix $(g_{i\bar{j}})$ and $(g^{i\bar{j}})$ be the inverse of the matrix $(g_{i\bar{j}})$. We also use the notations

$$g_{j\bar{k}\alpha_1\alpha_2\ldots\alpha_m} := \partial_{\alpha_1\alpha_2\ldots\alpha_m} g_{j\bar{k}}, \qquad f_{\alpha_1\alpha_2\ldots\alpha_m} := \partial_{\alpha_1\alpha_2\ldots\alpha_m} f,$$

$$(R_{i\bar{j}k\bar{l}/\alpha_1\cdots\alpha_m})_{\beta_1\cdots\beta_r} := \partial_{\beta_1\cdots\beta_r} R_{i\bar{j}k\bar{l}/\alpha_1\cdots\alpha_m}.$$

The curvature tensor is defined as

$$(1) \qquad R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}k\bar{l}} + g^{m\bar{p}} g_{m\bar{j}\bar{l}} g_{i\bar{p}k}.$$

The Ricci tensor is given by $R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log(\det g)$ and the trace of the Ricci curvature is the scalar curvature $\rho = g^{i\bar{j}} R_{i\bar{j}}$.

The covariant derivative of a covariant tensor field $T_{\beta_1\ldots\beta_p}$ is defined by

$$(2) \qquad T_{\beta_1\ldots\beta_p/\gamma} = \partial_\gamma T_{\beta_1\ldots\beta_p} - \sum_{i=1}^{p} \Gamma^\delta_{\gamma\beta_i} T_{\beta_1\ldots\beta_{i-1}\delta\beta_{i+1}\ldots\beta_p},$$

where the Christoffel symbols $\Gamma^\alpha_{\beta\gamma} = 0$ except for $\Gamma^i_{jk} = g^{i\bar{l}} g_{j\bar{l}k}$, $\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{l\bar{i}} g_{l\bar{j}\bar{k}}$.

Lemma 1.1. *On Kähler manifolds, we have $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$, $\partial_{\bar{l}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{l}}$ and*

$$(3) \qquad \partial_\alpha g^{m\bar{l}} = -g^{p\bar{l}} g^{m\bar{q}} g_{p\bar{q}\alpha}, \quad \partial_\alpha \det g = \det g \cdot g^{m\bar{l}} g_{m\bar{l}\alpha},$$

$$(4) \qquad R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}} \text{ (first Bianchi)},$$

(5) $R_{i\bar{j}k\bar{l}/m} = R_{m\bar{j}k\bar{l}/i} = R_{i\bar{j}m\bar{l}/k}, \quad R_{i\bar{j}k\bar{l}/\bar{p}} = R_{i\bar{p}k\bar{l}/\bar{j}} = R_{i\bar{j}k\bar{p}/\bar{l}}$ (second Bianchi)

(6) $T_{\beta_1\ldots\beta_p/ij} - T_{\beta_1\ldots\beta_p/ji} = 0, \qquad T_{\beta_1\ldots\beta_p/\bar{i}\bar{j}} - T_{\beta_1\ldots\beta_p/\bar{j}\bar{i}} = 0,$

(7) $T_{\beta_1\ldots\beta_p/i\bar{j}} - T_{\beta_1\ldots\beta_p/\bar{j}i} = \sum_{k=1}^{p} R^{\gamma}_{\beta_k i\bar{j}} T_{\beta_1\ldots\beta_{k-1}\gamma\beta_{k+1}\ldots\beta_p}$ (Ricci formula),

where $R^{\bar{k}}_{\bar{l}i\bar{j}} = -g^{m\bar{k}}R_{m\bar{l}i\bar{j}}$, $R^{k}_{li\bar{j}} = g^{k\bar{m}}R_{l\bar{m}i\bar{j}}$ and $R^{k}_{\bar{l}i\bar{j}} = R^{\bar{k}}_{li\bar{j}} = 0.$

Recall that around each point x on a Kähler manifold M, there exists a normal coordinate system such that $g_{i\bar{j}}(x) = \delta_{ij}$, $g_{i\bar{j}k_1\ldots k_r}(x) = g_{i\bar{j}\bar{l}_1\ldots\bar{l}_r}(x) = 0$ for all $r \leq N \in \mathbb{N}$, where N can be chosen arbitrary large.

Fix a normal coordinate system around x on M. By (1) and (2), it is not difficult to see inductively that for any partial derivatives of metrics or curvature tensors, there exists a canonical tensor that coincides with it at x. We will use the operator D to denote this correspondence. For example, $D(g_{i\bar{j}k\bar{l}}) = -R_{i\bar{j}k\bar{l}}$, $D(g_{i\bar{j}k\bar{l}\alpha}) = -R_{i\bar{j}k\bar{l}/\alpha}$ and

(8) $D\Big((R_{i\bar{j}k\bar{l}/p_1\cdots p_m})_{\bar{q}_1\cdots\bar{q}_r} \Big) = R_{i\bar{j}k\bar{l}/p_1\cdots p_m\bar{q}_1\cdots\bar{q}_r}.$

As shown in [51, 52], tensor calculus on Kähler manifolds could be naturally formulated in terms of graphs. In particular, the Weyl invariants [22] can be represented by multi-digraphs. Each vertex in the graph is a partial derivatives of metrics in local coordinates. Note such graphs are not in one-to-one correspondence with tensorial vertices due to the Ricci formula.

In §2, we review Fefferman's program, especially the works of [25, 30]. More detailed and nice expositions can be found in [29]. In §3, building on work of Engliš [17], we prove a graph-theoretic formula for Nakazawa's asymptotic expansion. In §4-§6, we will prove closed formulae for the above operator D as a summation over rooted trees with external legs, by canonically associate a tensor expression to each rooted tree. As an illustration of the efficiency of these formulae, we rederive the weight 3 coefficient in the asymptotic expansion of the weighted Bergman kernel.

2. Fefferman's program and CR invariants: A review

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. If $r \in C^\infty(\overline{\Omega})$ is a defining function in the sense $\Omega = \{r > 0\}$ with $dr \neq 0$ on $\partial\Omega$, then by the pioneering work of Fefferman [20] (see also [8]), the boundary singularity of the Bergman kernel has the form

$$(9) \qquad K(z) = \frac{n!}{\pi^n}\left(\frac{\varphi(z)}{r(z)^{n+1}} + \psi(z)\log r(z)\right), \qquad \varphi, \psi \in C^\infty(\overline{\Omega}).$$

A weighted analogue of Fefferman's expansion was obtained by Engliš [18].

Hörmander [32] proved that $r(z)^{n+1}K(z) \to \frac{n!}{\pi^n}J[r](z_0)$ as $z \to z_0 \in \partial\Omega$, where $J[r]$ denotes the complex Monge-Ampère operator defined by

$$(10) \qquad J[r] = (-1)^n \det \begin{pmatrix} r & \partial r/\partial \bar{z}_j \\ \partial r/\partial z_i & \partial^2 r/\partial z_i \partial \bar{z}_j \end{pmatrix}_{1 \le i,\, j \le n}.$$

By (9), we have $\varphi = J[r]$ on $\partial\Omega$. Starting from an arbitrary smooth defining function of Ω, Fefferman [21] devised a recursive algorithm to explicitly construct another defining function $r^F \in C^\infty(\overline{\Omega})$ satisfying

$$(11) \qquad J[r^F] = 1 + O^{n+1}(r^F), \quad r^F > 0 \text{ in } \Omega, \quad r^F|_{\partial\Omega} = 0,$$

where $O^{n+1}(r^F)$ denotes a term of the form $(r^F)^{n+1}f$ with $f \in C^\infty(\overline{\Omega})$.

Let r^F be a Fefferman's defining function of Ω. Define a Lorentz-Kähler metric

$$g = \sum_{0 \le i,j \le n} \frac{\partial^2 r_\#}{\partial z_i \partial \bar{z}_j}\, dz_i\, d\bar{z}_j, \quad r_\#(z_0, z) = |z_0|^2 r^F(z) \text{ on } \mathbb{C}^* \times \overline{\Omega},$$

called Fefferman's ambient metric associated with $\partial\Omega$. From the curvature tensor R of g, Fefferman [22] constructed Weyl invariants by complete contractions of covariant derivatives $R^{(p,q)} := R_{a_1 \bar{b}_1 a_2 \bar{b}_2 / a_3 \cdots a_p \bar{b}_3 \cdots \bar{b}_q}$, e.g. the following Weyl invariant

$$(12) \qquad W_\# = \mathrm{contr}(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}),$$

is defined to be of weight $\sum_{j=1}^s (p_j + q_j)/2 - s$ and gives rise to a function $W = W_\#|_{z_0=1}$ on $\overline{\Omega}$. Fefferman proposed a program [22] to express φ, ψ in (9) as linear combinations of Weyl invariants W_k of weight k such that

$$(13) \qquad \varphi = \sum_{k=0}^n W_k\, r^k + O^{n+1}(r), \quad \psi = \sum_{k=0}^\infty W_{k+n+1}\, r^k + O^\infty(r),$$

where $O^\infty(r)$ means that $\psi = \sum_{k=0}^m W_{k+n+1}[r]\, r^k + O^{m+1}(r)$ for any $m \geq 0$. The expansion of φ was proved by Fefferman [22] and Bailey et al. [3] for any Fefferman's defining function $r = r^F$. The expansion of ψ was proved by Hirachi [27] for more refined defining functions than (11). Both are very deep works.

According to Fefferman [22], the restriction of W_k to $\partial\Omega$ gives a CR invariant of weight k. CR invariants for strictly pseudoconvex hypersurfaces can be more conveniently defined using Moser's normal form in analogy to the normal coordinate system in Riemannian geometry. Let $(z', z_n) = (z_1, \ldots, z_n) \in \mathbb{C}^n$. A hypersurface $0 \in \partial\Omega \subset \mathbb{C}^n$ with local equation

$$(14) \qquad 2u = |z'|^2 + \sum_{|\alpha|,|\beta|\geq 2,\, k\geq 0} A^k_{\alpha\bar\beta}(v) v^k z'_\alpha \bar z'_\beta, \qquad z_n = u + iv$$

is said to be in Moser's normal form if the coefficients $A^k_{\alpha\bar\beta}$ satisfy: (i) $A^k_{\alpha\bar\beta} = \overline{A^k_{\beta\bar\alpha}}$;

(ii) $\operatorname{tr}(A_{2\bar2}) = 0$, i.e. $\sum_{p=1}^{n-1} A^k_{pi\bar p\bar j} = 0$ for all k, i, j;

(iii) $\operatorname{tr}(A_{2\bar3}) = 0$, i.e. $\sum_{p,q=1}^{n-1} A^k_{pq\bar p\bar q\bar j} = 0$ for all k, j;

(iv) $\operatorname{tr}(A_{3\bar3}) = 0$, i.e. $\sum_{p,q,r=1}^{n-1} A^k_{pqr\bar p\bar q\bar r} = 0$ for all k.

A classical result of Chern and Moser [12] says that any real analytic hypersurface may be placed in Moser's normal form through a biholomorphic map.

Definition 2.1 ([22, 25, 30]). Denote by $N(A^k_{\alpha\bar\beta})$ a real hypersurface in normal form (14). A polynomial P in variables $A^k_{\alpha\bar\beta}$ is said to be a CR invariant of weight $w \in \mathbb{N}_{\geq 0}$ if it satisfies the transformation law $P(A^k_{\alpha\bar\beta}) = |\det \Phi'(0)|^{2w/(n+1)} P(B^k_{\alpha\bar\beta})$ for any biholomorphic mapping $\Phi : N(A^k_{\alpha\bar\beta}) \to N(B^k_{\alpha\bar\beta})$ preserving the origin.

Let I_w denote the set of CR invariants of weight w. Then every $P \in I_w$ is a homogeneous polynomial of weight w if we define the weight of $A^k_{\alpha\bar\beta}$ to be $(|\alpha| + |\beta|)/2 + k - 1$. Graham [25] proved the following:

Theorem 2.2 ([25]). *(i) Let $n = 2$. Then $I_1 = I_2 = \{0\}$ and $\dim I_3 = \dim I_4 = 1$. Moreover, I_3 and I_4 are respectively spanned by $A^0_{4\bar4}$ and $|A^0_{2\bar4}|^2$.*

(ii) Let $n \geq 3$. Then $I_1 = \{0\}$ and $\dim I_2 = 1$. Moreover, I_2 is spanned by $\|A^0_{2\bar2}\|^2 = \sum |A^0_{\alpha\bar\beta}|^2$, where the summation runs over $|\alpha| = |\beta| = 2$.

When $n = 2$, a basis of $\dim I_5 = 2$ has been determined in [25, 30] and a basis of $\dim I_6 = 3$ has been determined by Hirachi [28].

For the Dirichlet problem of the complex Monge-Ampère equation

$$(15) \qquad J[u] = 1, \quad u > 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

Cheng-Yau [11] proved that there exists a unique solution $u \in C^\infty(\Omega) \cap C^{n+3/2-\epsilon}(\overline{\Omega})$ for any $\epsilon > 0$. Lee-Melrose [37] proved that for any smooth defining function r, Cheng-Yau's solution has an asymptotic expansion

$$(16) \qquad u \sim r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k, \quad \eta_k \in C^\infty(\overline{\Omega}),$$

which implies that $u \in C^{n+2-\epsilon}(\overline{\Omega})$ for any $\epsilon > 0$ improving Cheng-Yau's estimate. However, the solution to (15) is not C^∞ smooth up to the boundary, so we have to use Fefferman's defining function r^F when studying the invariant expansions (13).

Let us fix $r = r^{\mathrm{F}}$ and $a \in C^\infty(\partial\Omega)$ locally near $0 \in \partial\Omega$. Then Graham [26] proved that there exists a unique formal series u of the form (16) satisfying

$$(17) \qquad J[u] = 1 + O(r^\infty), \qquad \eta_0 = 1 + ar^{n+1} + O(r^{n+2})$$

near $0 \in \partial\Omega$. For any $k \geq 1$, η_k modulo $O(r^{n+1})$ is independent of $r = r^{\mathrm{F}}$ and a. Each $\eta_k|_{\partial\Omega}$ modulo $O(r^{n+1})$ is a CR invariant of weight $k(n+1)$. From Theorem 2.2, Graham [25] proved that:

Theorem 2.3 ([25]). *(i) Let $n = 2$. Then $\eta_1 = 4A^0_{4\bar{4}}$ and the singularity of the Bergman kernel* (9) *has the expansions*

$$(18) \qquad \varphi = 1 + O(r^3), \qquad \psi = -3\eta_1 + c|A^0_{2\bar{4}}|^2 r + O(r^2),$$

where c is a constant independent on Ω.

(ii) Let $n \geq 3$. There is a constant c_n depending only on n such that

$$(19) \qquad \varphi = 1 + c_n \|A^0_{2\bar{2}}\|^2 r^2 + O(r^3).$$

Hirachi, Komatsu and Nakazawa [30] gave two different methods of identifying the above universal constants and proved similar expansions for the Szegö kernel. In [31], they extended the expansion of ψ in dimension 2 to weight 5. We outline their proof using Nakazawa's explicit asymptotic expansion for Reinhardt domains.

Theorem 2.4 ([30])**.** *The constants in* (18) *and* (19) *are given by* $c = 24/5$ *and* $n(n-1)c_n = 2/3$.

In the rest of the section, we assume $\Omega \subset \mathbb{C}^n$ is a bounded strictly pseudoconvex complete Reinhardt domain. Its logarithmic real representation domain is given by

$$-\log|\Omega| = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid (e^{-x_1}, \ldots, e^{-x_{n-1}}, e^{-y}) \in \Omega\}.$$

First we assume $n = 2$. Let $f(x) := \inf\{y \in \mathbb{R} \mid (x,y) \in -\log|\Omega|\}$. Then $\lambda = y - f(x)(> 0)$ is a defining function of $\partial\Omega \cap \{z_1 z_2 \neq 0\}$. We make change of variables $(x,y) \to (\lambda, v)$ with $v = f'(x)$ and set $p(v) = f''(x)$, the hodograph transformation. The following asymptotic expansion in dimension 2 was proved by Nakazawa [48] improving work of Boichu and Coeuré [7].

Theorem 2.5 ([48, 30])**.** *Let* $n = 2$. *Near* $\partial\Omega \cap \{z_1 z_2 \neq 0\}$, *we have*

(20) $$K(z) = \frac{2}{\pi^2} J[\lambda] \left(\frac{\widetilde{\varphi}(v,\lambda)}{\lambda^3} + \widetilde{\psi}(v,\lambda) \log \lambda \right),$$

where $J[\lambda] = \frac{p}{|4z_1 z_2|^2}$. *Let* $e_1 = p''$, $e_2 = (pp^{(3)})'$, $e_3 = (p^2 p^{(4)})''$, $e_{41} = e_1 e_3$, $e_{42} = (pe_3')'$ *and* $e_{43} = (pp^{(4)})^2$. *Then*

$$\widetilde{\varphi}(v,\lambda) = 1 + \frac{\lambda}{4}e_1 + \frac{\lambda^2}{12}e_2, \quad \widetilde{\psi}(v,\lambda) = \frac{-e_3}{48} + \frac{\lambda}{480}(2e_{42} + e_{43} - e_{41}) + O(\lambda^2).$$

Theorem 2.5 and the following lemma immediately implies $c = 24/5$ in (18).

Lemma 2.6 ([30])**.** *Under the notation of the above theorem, we have* $|A_{2\bar{4}}^0|^2 = J[\lambda]^{4/3} e_{43}/48^2$, $r^F = J[\lambda]^{-1/3}(\widetilde{r} + O(\lambda^4))$ *and* $\eta_1 = J[\lambda](\widetilde{\eta}_1 + O(\lambda^2))$, *where*

$$\widetilde{r} = \lambda - \frac{\lambda^2}{12}e_1 - \frac{\lambda^3}{36}\left(e_2 - \frac{e_1^2}{2}\right), \quad \widetilde{\eta}_1 = \frac{e_3}{144} - \frac{\lambda}{720}\left(e_{42} - \frac{e_{41}}{2}\right).$$

Next we assume $n \geq 3$. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex complete Reinhardt domain satisfying $-\log|\Omega| = \{\lambda := y - (f_1(x) + \cdots + f_{n-1}(x)) > 0\}$ with hodograph variables $v_j = f_j'(x_j)$ and $p_j(v_j) = f_j''(x_j)$. We introduce

$$e_1 = \sum_{j=1}^{n-1} p_j'', \quad e_{21} = \sum_{j=1}^{n-1}(p_j p_j''')', \quad e_{22} = \sum_{j=1}^{n-1}(p_j'')^2, \quad e_{23} = \sum_{j \neq k} p_j'' p_k''.$$

Theorem 2.7 ([30])**.** *Under the above notation, we have*

$$\|A_{2\bar{2}}^0\|^2 = \frac{J[\lambda]^{2/(n+1)}}{16n(n+1)}((n-2)(n-1)e_{22} + 2e_{23}),$$

$$r^F = J[\lambda]^{\frac{-1}{n+1}} \left(\lambda - \frac{e_1\lambda^2}{2n(n+1)} + \frac{-n(n+1)e_{21} + (n^2-1)e_{22} - e_{23}}{6(n-1)n^2(n+1)^2}\lambda^3 + O(\lambda^4) \right).$$

The Bergman kernel has the expansion

(21) $$K(z) = \frac{n!}{\pi^n} J[\lambda] \left(\frac{\widetilde{\varphi}(v,\lambda)}{\lambda^{n+1}} + \widetilde{\psi}(v,\lambda) \log \lambda \right),$$

where $J[\lambda] = \frac{p}{4^n |z_1\cdots z_n|^2}$ *and*

(22) $$\widetilde{\varphi}(v,\lambda) = 1 + \frac{\lambda}{2n}e_1 + \frac{\lambda^2}{n(n-1)} \left(\frac{1}{6}e_{21} + \frac{1}{8}e_{23} \right) + O(\lambda^3).$$

Theorem 2.7 immediately implies $c_n = \frac{2}{3n(n-1)}$ in (19). Explicit graph theoretic formulae for the coefficients of (20) and (21) will be proved in §3.

3. The asymptotic expansion of Bergman kernels

There has been much interest in explicit formulae of Fefferman's asymptotic expansions for (unweighted) Bergman kernels and Szegö kernels (see [4, 34, 36, 48]). In [17], Engliš studied the asymptotic expansion of a Laplace integral and proved a recursion relation for its coefficients. As an application, he derived a remarkable formula (Theorem 3.8) of Fefferman's invariants for the (unweighted) Bergman kernel of strictly pseudoconvex Hartogs domains using the Forelli-Rudin construction. We will show that some key quantities in Engliš' formula can be expressed as explicit summations over strongly connected graphs and using an observation of Nakazawa (Lemma 3.9), Engliš' formula can be drastically simplified when the domain is complete Reinhardt. The resulting graph-theoretic formula (Theorem 3.10) is also more computationally efficient. Similar formula also exists for Szegö kernels [19].

Theorem 3.1 (Engliš [17]). *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary. Then there is an asymptotic expansion for the Laplace integral as $\alpha \to \infty$,*

$$\int_\Omega f(y)e^{-\alpha(\Phi(x)+\Phi(y)-\Phi(x,y)-\Phi(y,x))} \frac{|\det g(x,y)|^2}{\det g(x)} dy \sim \pi^n \sum_{j\geq 0} \alpha^{-n-j} \mathcal{R}_j(f)(x),$$

where $\Phi(x,y)$ and $\det g(x,y)$ are the almost analytic extensions of the Kähler potential $\Phi(x)$ and $\det g(x)$ respectively, and $\mathcal{R}_j : C^\infty(\Omega) \to C^\infty(\Omega)$ are explicit

differential operators defined by

$$(23) \qquad \mathcal{R}_j f(x) = \frac{1}{(\det g)^2} \sum_{k=j}^{3j} \frac{1}{k!(k-j)!} L^k \big[f(y)|\det g(x,y)|^2 S(x,y)^{k-j} \big]\big|_{y=x},$$

where L is the (constant-coefficient) differential operator

$$Lf(y) = g^{i\bar{j}}(x)\partial_i \partial_{\bar{j}} f(y)$$

and the function S(x,y) satisfies

$$S = \partial_\alpha S = \partial_{\alpha\beta} S = \partial_{i_1 i_2 \ldots i_m} S = \partial_{\bar{i}_1 \bar{i}_2 \ldots \bar{i}_m} S = 0 \qquad at\ y = x,$$

$$\partial_{i\bar{j}\alpha_1\alpha_2\ldots\alpha_m} S|_{y=x} = -\partial_{\alpha_1\alpha_2\ldots\alpha_m} g_{i\bar{j}}(x), \qquad m \geq 1.$$

Denote by $\mathcal{K}_\alpha(x,y)$ the reproducing kernel of the weighted Bergman space of all holomorphic functions on Ω square-integrable with respect to the measure $e^{-\alpha\Phi}dx$. According to Engliš [17], $\mathcal{K}_\alpha(x,y)$ has an asymptotic expansion in a small neighborhood of the diagonal as $\alpha \to \infty$,

$$(24) \qquad \mathcal{K}_\alpha(x,y) = e^{\alpha\Phi(x,y)} \det g(x,y) \frac{1}{\pi^n} \sum_{k=0}^{\infty} \mathcal{B}_k(x,y)\alpha^{n-k}.$$

The corresponding Berezin transform is given by

$$(25) \qquad \mathcal{I}_\alpha f(x) = \int_\Omega f(y) \frac{|\mathcal{K}_\alpha(x,y)|^2}{\mathcal{K}_\alpha(x,x)} e^{-\alpha\Phi(y)} dy,$$

which has an asymptotic expansion as $\alpha \to \infty$,

$$(26) \qquad \mathcal{I}_\alpha f(x) = \sum_{k=0}^{\infty} \mathcal{Q}_k f(x)\alpha^{-k}.$$

The following lemma is the key result we will use, which slightly refines the formulae of Engliš [17].

Lemma 3.2. *We have $\mathcal{Q}_0 = id$ and $\mathcal{B}_0 = 1$. For $k \geq 1$,*

(27)
$$\mathcal{Q}_k f(x) = \sum_{j=0}^{k} \sum_{i=0}^{k-j} \mathcal{R}_j(\mathcal{B}_i(x,y)\mathcal{B}_{k-j-i}(y,x)f(y))|_{y=x} - \sum_{m=1}^{k} \mathcal{B}_m(x)\mathcal{Q}_{k-m}f(x),$$

$$(28) \qquad \mathcal{B}_k(x) = - \sum_{\substack{i+j=k \\ i,j\geq 1}} \mathcal{B}_i(x)\mathcal{B}_j(x) - \sum_{\substack{\ell+i+j=k \\ 1\leq\ell\leq k}} \mathcal{R}_\ell(\mathcal{B}_i(x,y)\mathcal{B}_j(y,x))\big|_{y=x}.$$

Proof. By multiplying $\mathcal{K}_\alpha(x, x)$ to both sides of (25) and using (24) and (26), we get

$$(29) \quad \sum_{m=0}^{\infty} \mathcal{B}_m(x, y) \sum_{i=0}^{\infty} \mathcal{Q}_i f(x) \alpha^{n-m-i} = \frac{1}{\pi^n} \int_\Omega f(y) e^{-\alpha(\Phi(x)+\Phi(y)-\Phi(x,y)-\Phi(y,x))}$$

$$\times \frac{|\det g(x, y)|^2}{\det g(x)} \sum_{i,m=0}^{\infty} \mathcal{B}_i(x, y) \mathcal{B}_m(y, x) \alpha^{2n-m-i} dy.$$

By applying Theorem 3.1 to the right-hand side of the above equation and equating the coefficients of α^{n-k}, we get (27).

Since $Q_0 = id$ and $Q_k(f) = 0$ when $k \geq 1$ and f is either holomorphic or antiholomorphic, by substituting $f = 1$ in (27), we get (28). $\qquad\square$

As noticed by Engliš [17], in a normal coordinate system around x, the operators $\mathcal{R}_j$ in (23) simplify to

$$(30) \qquad \mathcal{R}_j f(x) = \sum_{k=j}^{2j} \frac{1}{k!(k-j)!} L^k (f S^{k-j})\big|_{y=x}.$$

Before proceeding we need to introduce parallel notions for graphs and pointed graphs representing Weyl invariant polynomials in jets of metrics and functions.

A *digraph* or simply a graph $G = (V, E)$ is defined to be a finite directed multigraph which may have multi-edges and loops. A vertex v of a digraph G is called *stable* if $\deg^-(v) \geq 2$, $\deg^+(v) \geq 2$, i.e. both the inward and outward degrees of v are no less than 2. A vertex v is called *semistable* if we have

$$\deg^-(v) \geq 1, \ \deg^+(v) \geq 1, \ \deg^-(v) + \deg^+(v) \geq 3.$$

The weight of a digraph G is defined to be the integer $w(G) = |E| - |V|$. A digraph G is *stable* (*semistable*) if each vertex of G is stable (semistable). The set of semistable and stable graphs of weight k will be denoted by $\mathcal{G}^{ss}(k)$ and $\mathcal{G}(k)$ respectively. A directed edge uv of a semistable digraph is called *contractible* if $u \neq v$ and at least one of the following two conditions holds: (i) $\deg^+(u) = 1$; (ii) $\deg^-(v) = 1$. A semistable graph G is called *stabilizable* if after contractions of a finite number of contractible edges of G, the resulting graph becomes stable, which is called the *stabilization graph* of G and denoted by G^s.

A *(one-)pointed graph* $\Gamma = (V \cup \{\bullet\}, E)$ is defined to be a digraph with a distinguished vertex labeled by f. G or Γ is called *semistable* (*stable*) if each ordinary vertex $v \in V$ is semistable (stable). The weight of a pointed graph $\Gamma = (V \cup \{\bullet\}, E)$ is defined to be $w(\Gamma) = |E| - |V|$. By abuse of notation, we denote $V(\Gamma) = V \cup \{\bullet\}$. The set of semistable and stable pointed graphs of weight k will be denoted by $\mathcal{G}_1^{ss}(k)$ and $\mathcal{G}_1(k)$ respectively. Denote by $\mathrm{Aut}(\Gamma)$ the set of all automorphisms of the pointed graph Γ fixing the distinguished vertex. A directed edge uv of a semistable pointed graph is called *contractible* if $u \neq v$ and at least one of the following two conditions holds: (i) $u \in V$ and $\deg^+(u) = 1$; (ii) $v \in V$ and $\deg^-(v) = 1$. A semistable pointed graph Γ is called *stabilizable* if after contractions of a finite number of contractible edges of Γ, the resulting graph becomes stable, which is called the *stabilization graph* of Γ and denoted by Γ^s.

We can canonically associate a polynomial in the variables $\{g_{i\bar{j}\,\alpha}\}_{|\alpha| \geq 1}$ or $\{f_\alpha\}_{|\alpha| \geq 0}$ to a semistable graph or pointed graph, such that each ordinary vertex represents a partial derivative of $g_{i\bar{j}}$, the distinguished vertex represents a partial derivative of f and each edge represents the contraction of a pair of indices.

Remark 3.3. Note that the definition of stablizable semistable (pointed) graphs is independent of the order of edge-contractions (cf. [53, Lem. 3.1]). Engliš [17] showed that $\mathcal{B}_k, \mathcal{R}_k, \mathcal{Q}_k$ are Weyl invariant polynomials in the variables $\{g_{i\bar{j}\,\alpha}\}$ or $\{f_\alpha\}$, i.e. invariant under transformation of coordinates. Assume that the following two Weyl invariant polynomials are expressed, in a normal coordinate system, as summations over stable graphs and pointed graphs respectively,

$$(31) \qquad P_1 = \sum_{G \in \mathcal{G}(k)} c(G)\, G \quad \text{and} \quad P_2 = \sum_{\Gamma \in \mathcal{G}_1(k)} c(\Gamma)\, \Gamma.$$

The main result of [53] shows that they may be invariantly expressed as summations over stabilizable semistable (pointed) graphs,

$$(32) \qquad P_1 = \sum_{G \in \mathcal{G}^{ss}(k)}^{\text{stabilizable}} \frac{(-1)^{|V(G)| - |V(G^s)|} |\mathrm{Aut}(G^s)| c(G^s)}{|\mathrm{Aut}(G)|}\, G,$$

$$(33) \qquad P_2 = \sum_{\Gamma \in \mathcal{G}_1^{ss}(k)}^{\text{stabilizable}} \frac{(-1)^{|V(\Gamma)| - |V(\Gamma^s)|} |\mathrm{Aut}(\Gamma^s)| c(\Gamma^s)}{|\mathrm{Aut}(\Gamma)|}\, \Gamma,$$

which are valid in arbitrary holomorphic coordinates.

In the following, we use the notations

$$(34) \qquad \mathcal{B}_k = \sum_{G \in \mathcal{G}(k)} \mathcal{B}_G G, \qquad \mathcal{R}_k f = \sum_{\Gamma \in \mathcal{G}_1(k)} \mathcal{R}_\Gamma \Gamma, \qquad \mathcal{Q}_k f = \sum_{\Gamma \in \mathcal{G}_1(k)} \mathcal{Q}_\Gamma \Gamma.$$

Lemma 3.4. *Let* $\Gamma = (V \cup \{\bullet\}, E) \in \mathcal{G}_1$ *be a stable pointed graph. Then*

$$(35) \qquad \mathcal{R}_\Gamma = \frac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|}.$$

Proof. It follows from (30) if we regard L^k as k edges, S^{k-j} as $k-j$ vertices and $k!(k-j)!$ the symmetry factor. $\qquad\square$

Corollary 3.5. *In any holomorphic coordinates, we have*

$$(36) \qquad \mathcal{R}_k f = \sum_{\Gamma \in \mathcal{G}_1^{ss}(k)}^{stabilizable} \frac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} \cdot \Gamma.$$

Proof. It follows from (33). $\qquad\square$

In the following, we call a graph *strong* if it is strongly connected. We call a graph *quasi-strong* if all of its connected components are strong. A strongly connected component (SCC) of a digraph G is called a *source* (*sink*) if it has only outward (inward) edges in G. A connected graph is strong if and only it has no proper source or sink.

Theorem 3.6. *Let* $G \in \mathcal{G}$ *and* $\Gamma \in \mathcal{G}_1$. *Then*

$$(37) \qquad \mathcal{B}_G = \begin{cases} \dfrac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} & \text{if } G \text{ is quasi-strong with } n(G) \text{ components,} \\ 0 & \text{otherwise.} \end{cases}$$

$$(38) \qquad \mathcal{Q}_\Gamma = \begin{cases} \dfrac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} & \text{if } \Gamma \text{ is strong,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First assume that G is strongly connected. Let us look at the right-hand side of (28). The first term $\mathcal{B}_i(x)\mathcal{B}_j(x)$ only contributes disconnected graphs. In the second term $R_\ell(\mathcal{B}_i(x,y)\mathcal{B}_j(y,x))|_{y=x}$, the two factors $\mathcal{B}_i(x,y)$ and $\mathcal{B}_j(y,x)$ are sink and source respectively. Since G is strong, we must have $i = j = 0$. So

it is not difficult to see from (28) and (35) that

$$(39) \qquad \mathcal{B}_G = -\mathcal{R}_{G\coprod\{\bullet\}} = \frac{(-1)^{|V(G)|+1}}{|\mathrm{Aut}(G)|},$$

where $G\coprod\{\bullet\}$ is the disjoint union of G and the distinguished vertex $\bullet$.

Denote $\underline{n} = \{1,\cdots,n\}$. If $G = G_1\coprod\cdots\coprod G_n$ is disjoint union of strongly connected subgraphs, by inducting on $w(G)$ and using (28), we have

$$(40) \quad \mathcal{B}_G = -\sum_{\substack{I\coprod J=\underline{n}\\ I,J\neq\emptyset}} \frac{(-1)^{|V(G)|+|I|+|J|}}{|\mathrm{Aut}(G)|} - \sum_{\substack{K\coprod I\coprod J=\underline{n}\\ K\neq\emptyset}} \frac{(-1)^{|V(G)|+|I|+|J|}}{|\mathrm{Aut}(G)|}$$

$$= -\frac{(-2)^{|V(G)|+n}-2(-1)^{|V(G)|+n}}{|\mathrm{Aut}(G)|} - \frac{(1-1-1)^{|V(G)|+n}-(-2)^{|V(G)|+n}}{|\mathrm{Aut}(G)|}$$

$$= \frac{(-1)^{|V(G)|+n}}{|\mathrm{Aut}(G)|}.$$

If some G_i in $G = G_1\coprod\cdots\coprod G_n$ is not strongly connected, then G_i has a proper sink C. In order to prove $\mathcal{B}_G = 0$, we note that in $R_\ell(\mathcal{B}_i(x,y)\mathcal{B}_j(y,x))\big|_{y=x}$, the sink C may either belong to $B_i(x,y)$ or R_ℓ, actually the contributions of these two cases to G exactly cancel out. The argument is similar to the proof of [52, Prop. 3.3]. We omit the details. So we conclude the proof of the formula (37).

The formula for $\mathcal{Q}_\Gamma$ follows from (27), (35) and (37) by using the same argument as [52, Thm. 3.4]. $\hfill\square$

Corollary 3.7. *In any holomorphic coordinates, we have*

$$(41) \qquad \mathcal{B}_k = \sum_{G\in\mathcal{G}^{ss}(k)}^{quasi\text{-}strong} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} G,$$

$$(42) \qquad \mathcal{Q}_k f = \sum_{\Gamma\in\mathcal{G}_1^{ss}(k)}^{strong} \frac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} \Gamma.$$

Proof. In [53, Lem. 3.4 & Lem. 4.5], we proved that the stabilization graph of a semistable (pointed) graph G is strong if and only if G is strong. So the corollary follows from (32) and (33). $\hfill\square$

It is not difficult to see that (41) is equivalent to

$$\sum_{k=0}^{\infty} \mathcal{B}_k \nu^k = \exp\left(\sum_{k=1}^{\infty} \sum_{G \in \mathcal{G}^{ss}(k)}^{\text{strong}} \nu^k \frac{(-1)^{|V(G)|+1}}{|\text{Aut}(G)|} G\right),$$

where in the right-hand side, G runs over only strong graphs.

Theorem 3.8 (Engliš [17])**.** *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^N$ with real-analytic boundary, Φ a strictly plurisubharmonic real-analytic defining function for Ω, $g_{i\bar{j}}$ the Kähler metric defined by the potential Φ, d a positive integer, and $\widetilde{K}(z,w)$ the (ordinary unweighted) Bergman kernel of the Hartogs domain*

$$(43) \qquad \widetilde{\Omega} = \{z = (z_1, z_2) \in \Omega \times \mathbb{C}^d : \|z_2\|^2 < e^{-\Phi(z_1)}\}.$$

Then (i) as z approaches a point of $\partial\widetilde{\Omega} \backslash \{z_2 = 0\}$, the reproducing kernel $\widetilde{K}(z,z)$. admits an asymptotic expansion

$$(44) \qquad \widetilde{K}(z,z) = \sum_{l=0}^{\infty} c_l(z_1) \cdot u_{d+N-l}(\|z_2\|^2 e^{\Phi(z_1)}),$$

in the sense that the partial sum of the first l terms of the right-hand side differs from the left-hand side by a function which is $O(u_{d+N-l}(\|z_2\|^2 e^{\Phi(z_1)}))$ if $l \leqq d+N+1$, and is in $C^{l-(d+N+2)}(\widetilde{\Omega} \backslash \{z_2 = 0\})$ if $l \geqq d+N+2$. Here the function $u_l(w)$ is given by

$$(45) \qquad u_l(w) = \sum_{k=\max(0,-l)}^{\infty} \frac{(k+l)!}{k!} w^k$$

$$= \begin{cases} \dfrac{l!}{(1-w)^{l+1}}, & l \geq 0, \\[2mm] \dfrac{(-w)^{-l} + w(1-w)^{-l-1} - (w-1)^{-l-1}\log(1-w)}{(-l-1)!}, & l < 0. \end{cases}$$

(ii) the coefficients $c_l(z_1)$ in (44) are given by the formula

$$(46) \qquad c_l(z_1) = \pi^{-N-d} \det g(z_1) e^{d\Phi(z_1)} \sum_{j=0}^{l} a_{N-j,l-j} \mathcal{B}_j(z_1),$$

where $a_{m,n}(m \in \mathbb{Z}, n \geqq 0)$ are universal constants with $a_{m0} = 1$, and $\mathcal{B}_j$ are the scalar invariants of $g_{i\bar{j}}$ from (24).

Taking $N = d = 1$ and $\Phi(z_1)$ depending only on $|z_1|$ in Engliš' Theorem 3.8, then $\widetilde{\Omega}$ is a complete Reinhardt domain in $\mathbb{C}^2$, which implies Nakazawa's Theorem 2.5,

$$(47) \qquad \widetilde{K} = \frac{p}{8\pi^2|z_1 z_2|^2}\left(\frac{L_0}{\lambda^3} + \frac{L_1}{\lambda^2} + \frac{L_2}{\lambda} + \sum_{k=3}^{\infty} L_k \lambda^{k-3}\log\lambda\right).$$

We need the following lemma of Nakazawa.

Lemma 3.9 ([48, Prop. 0]). *Each coefficient L_k is a linear combination of*

$$p^{(\eta_1)}\cdots p^{(\eta_{2k})}/p^k \quad \text{with } \eta_1 + \cdots + \eta_{2k} = 2k.$$

Namely L_k is homogeneous of degree k and order $2k$.

We can now prove a closed formula of L_k by using (41).

Theorem 3.10. *Let $k \geq 0$. Define a function $W_k(p)$ by*

$$(48) \qquad W_k(p) = \frac{1}{p^k}\sum_{G\in\mathcal{G}^{ss}(k)}^{quasi\text{-}strong} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \prod_{v\in V(G)} h(\deg(v) - 2),$$

where G runs over all quasi-strong (i.e. all connected components are strongly connected) semistable graphs of weight k and $n(G)$ is the number of components of G; the function h is defined recursively by

$$h(1) = p', \quad h(k) = [p \cdot h(k-1)]', \; k \geq 2.$$

Then the coefficients of (47) are given by

$$(49) \qquad L_k = \begin{cases} \dfrac{(2-k)!}{2}W_k(p), & 0 \leq k \leq 2, \\ \dfrac{(-1)^k}{2(k-3)!}W_k(p), & k \geq 3. \end{cases}$$

Proof. In the notations of Theorem 2.5, for $(z_1, z_2) \in \mathbb{C}^2$, we have

$$x = -\log|z_1| = -\frac{1}{2}(\log z_1 + \log \bar{z}_1), \quad y = -\log|z_2|, \quad f(x) = \frac{1}{2}\Phi,$$

$$e^{-2\lambda} = |z_2|^2 e^{\Phi(z_1)}, \qquad \frac{\partial^2\Phi}{\partial z_1\bar\partial z_1} = \frac{1}{2|z_1|^2}\frac{\partial^2 f}{\partial x^2} = \frac{p}{2|z_1|^2}.$$

By using these equations, (44) becomes

$$(50) \qquad \widetilde{K} = \frac{1}{\pi^2}\frac{p}{2|z_1|^2}\frac{e^{-2\lambda}}{|z_2|^2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}a_{1-j,k-j}\mathcal{B}_j(z_1)u_{2-k}(e^{-2\lambda}).$$

By (45), the singular part of $u_{2-k}(e^{-2\lambda})$ is given by

$$
(51) \quad u_{2-k}(e^{-2\lambda}) = \begin{cases} \dfrac{(2-k)! + O(\lambda)}{2^{3-k}\lambda^{3-k}}, & 0 \le k \le 2, \\[3mm] \dfrac{[(-1)^k 2^{k-3}\lambda^{k-3} + O(\lambda^{k-2})]\log(\lambda)}{(k-3)!}, & k \ge 3. \end{cases}
$$

Note that the derivatives of p satisfy

$$
(52) \qquad \frac{\partial p}{\partial z_1} = -\frac{pp'}{2z_1}, \quad \frac{\partial p}{\partial \bar{z}_1} = -\frac{pp'}{2\bar{z}_1}.
$$

By (41), we express $\mathcal{B}_j(z_1)$ as a summation of rational differential functions of p,

$$
(53) \quad \mathcal{B}_j(z_1) = \sum_{G \in \mathcal{G}^{ss}(j)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|}
$$

$$
\times \prod_{v \in V(G)} \frac{\partial^{\deg(v)-2}}{\partial z_1^{\deg^+(v)-1} \partial \bar{z}_1^{\deg^-(v)-1}} \left[\frac{p}{2|z_1|^2} \right] \Bigg|_{|z_1|^2 = p/2}.
$$

Note that $\mathcal{B}_j$ is of degree no more than j. The top degree is achieved only when all derivatives are taken on the numerator p. It is not difficult to see from (53) that

$$
\mathcal{B}_j(z_1) = \sum_{G \in \mathcal{G}^{ss}(j)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \frac{1}{2^j p^j} \prod_{v \in V(G)} \frac{1}{p} \cdot \frac{\partial^{\deg(v)-2} p}{\partial z_1^{\deg^+(v)-1} \partial \bar{z}_1^{\deg^-(v)-1}} + Low
$$

$$
= \sum_{G \in \mathcal{G}^{ss}(j)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \frac{1}{2^j p^j} \prod_{v \in V(G)} h(\deg(v) - 2) + Low
$$

$$
= \frac{W_j(p)}{2^j} + Low,
$$

where *Low* denotes the terms of rational differential functions of p with degree strictly less than j, which may be discarded according to Lemma 3.9. It also implies that in the summation (50), we can discard all terms except when $j = k$, i.e. the term $a_{1-k,0}B_k(z_1) = B_k(z_1)$. In view of (51), Equation (49) follows immediately. $\square$

**Remark 3.11.** The formula (48) may be reformulated as

$$
(54) \qquad W_k(p) = \frac{1}{p^k} \sum_{H}^{\text{quasi-strong}} \frac{(-1)^{|V(H)|+n(H)}}{|V(H)|! \, (|V(H)|+k)!} \prod_{v \in V(H)} h(\deg(v) - 2),
$$

where H runs over isomorphism classes of *labeled* semistable quasi-strong graphs of weight k, i.e. the vertices and edges of H are labeled by $\{1, \ldots, |V(H)|\}$ and $\{1, \ldots, |E(H)|\}$ respectively. Denote by $N(d_1, \ldots, d_j)$ the number of labeled strong graphs with given degree sequence $(d_1, \ldots, d_j)$. The computation with (54) will be greatly eased if one can find a recursive formula for $N(d_1, \ldots, d_j)$, e.g. using the method of [38].

Example 3.12. Obviously $L_0 = W_0(p) = 1$. Note that

$$h(1) = p', \qquad h(2) = (p')^2 + pp'',$$
$$h(3) = (p')^3 + 4pp'p'' + p^2 p^{(3)},$$
$$h(4) = (p')^4 + 11p(p')^2 p'' + 7p^2 p' p^{(3)} + 4p^2 (p'')^2 + p^3 p^{(4)}.$$

We now compute L_1, L_2, L_3 by using Theorem 3.10. There are two quasi-strong graphs in $\mathcal{G}^{ss}(1)$,
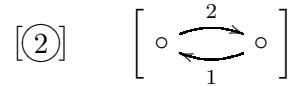
So $W_1(p) = \frac{1}{p}(\frac{h(2)}{2} - \frac{h(1)^2}{2}) = \frac{1}{2}p''$, which implies $L_1 = \frac{1}{4}p''$.

There are 19 quasi-strong graphs in $\mathcal{G}^{ss}(2)$ as depicted in Table 0, among which 4 are stable. They are grouped according to their stabilization graphs. It is a routine calculation that $W_2(p) = \frac{1}{6}(pp^{(3)})'$, which implies $L_2 = \frac{1}{12}(pp^{(3)})'$.

There are 300 quasi-strong graphs in $\mathcal{G}^{ss}(3)$, among which 14 are stable. With the help of Maple, we get $W_3(p) = \frac{1}{24}(p^2 p^{(4)})''$, which implies $L_3 = -\frac{1}{48}(p^2 p^{(4)})''$.
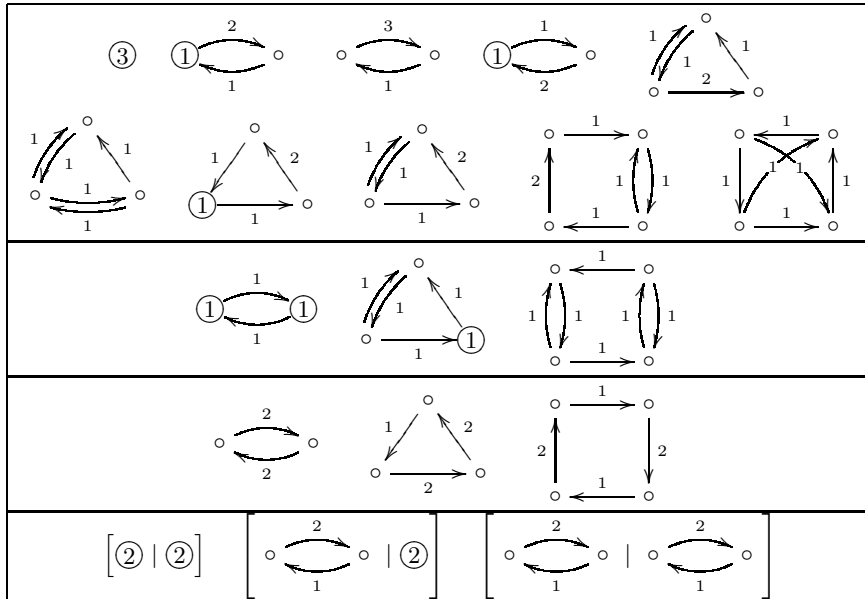
Now we consider the higher dimensional Reinhardt domains. Let $n \geq 2$. Under the notations of Theorem 2.7, denote

$$(55) \qquad \widetilde{K} = \frac{n! \, p_1 \cdots p_{n-1}}{4^n \pi^n |z_1 \cdots z_n|^2} \left(\sum_{k=0}^{n} \frac{L_k}{\lambda^{n+1-k}} + \sum_{k=n+1}^{\infty} L_k \lambda^{k-(n+1)} \log \lambda \right).$$

Theorem 3.13. *Let $k \geq 0$. Then the coefficients of* (55) *are given by*

$$(56) \qquad L_k = \begin{cases} \dfrac{(n-k)!}{n!} \displaystyle\sum_{\substack{k=m_1+\cdots+m_{n-1} \\ m_i \geq 0}} \prod_{i=1}^{n-1} W_{m_i}(p_i), & 0 \leq k \leq n, \\[3em] \dfrac{(-1)^{n-k}}{n!(k-1-n)!} \displaystyle\sum_{\substack{k=m_1+\cdots+m_{n-1} \\ m_i \geq 0}} \prod_{i=1}^{n-1} W_{m_i}(p_i), & k \geq n+1, \end{cases}$$

TABLE 0. The 19 quasi-strong semistable graphs of weight 2

where $W_{m_i}(p_i)$ is the function defined in (48) with p replaced by p_i.

Proof. Under the notations of Theorem 2.7, (44) becomes

$$(57) \qquad \widetilde{K} = \frac{1}{\pi^n} \frac{p_1 \cdot p_{n-1}}{2^{n-1} |z_1 \cdots z_{n-1}|^2} \frac{e^{-2\lambda}}{|z_n|^2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_{n-1-j,k-j} \mathcal{B}_j(z_1) u_{n-k}(e^{-2\lambda}).$$

Note that we have the analogue of Lemma 3.9 for any $n \geq 2$. So we can use the same argument as Theorem 3.10. The singular part of $u_{n-k}(e^{-2\lambda})$ is given by

$$(58) \quad u_{n-k}(e^{-2\lambda}) = \begin{cases} \dfrac{(n-k)! + O(\lambda)}{2^{n+1-k} \lambda^{n+1-k}}, & 0 \leq k \leq n, \\ \dfrac{[(-1)^{n-k} 2^{k-n-1} \lambda^{k-n-1} + O(\lambda^{k-n})] \log(\lambda)}{(k-n-1)!}, & k \geq n+1. \end{cases}$$

Finally, (56) follows from the multiplicativity of $\mathcal{B}_k$. $\qquad\qquad\square$

We can easily compute L_1, L_2 by using (56) and Example 3.12.

$$L_1 = \frac{1}{n} \sum_{i=1}^{n-1} W_1(p_i) = \frac{1}{2n} \sum_{i=1}^{n-1} p_i'',$$

$$L_2 = \frac{1}{n(n-1)} \left(\sum_{i=1}^{n-1} W_2(p_i) + \frac{1}{2} \sum_{i \neq j} W_1(p_i) W_1(p_j) \right)$$

$$= \frac{1}{n(n-1)} \left(\frac{1}{6} \sum_{i=1}^{n-1} (p_i p_i^{(3)})' + \frac{1}{8} \sum_{i \neq j} p_i'' p_j'' \right),$$

which agree with (22).

4. FROM PARTIAL TO COVARIANT DERIVATIVES

Definition 4.1. A *rooted tree* T is an oriented tree with a special vertex r, called the root, such that there is a unique directed path from r to any vertex v, i.e. all edges point away from r. We use $V(T)$ and $E(T)$ to denote the set of vertices and edges of T respectively. A *subtree* of T is a tree consisting of a vertex in T and all of its descendants. The subtree corresponding to r is T; a *proper subtree* is a subtree corresponding to any other vertex.

The number T_n of rooted trees on n nodes can be calculated from the following recursive equation

$$T_{n+1} = \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{d|i} d\, T_d \right) T_{n-i+1},$$

where d runs over the factors of i. In terms of generating functions

$$T(x) = \sum_{n=1}^{\infty} T_n x^n = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 48x^7 + 115x^8 + \cdots.$$

Asymptotically T_n satisfies

$$T_n \sim \frac{c}{n^{3/2}} \eta^{-n}, \quad n \to \infty,$$

where $\eta = 0.3383\ldots$ is the radius of convergence of $T(x)$ and $c = 0.4399\ldots$.

For trees in Figure 1, the root is either the leftmost or bottommost vertex.

Definition 4.2. A *decorated tree* T is a directed tree such that each vertex is decorated by a finite number of outward and inward external legs, corresponding to unbarred and barred indices respectively. T is called *stable* if each vertex is stable, i.e. has at least two outward half-edges and two inward half-edges. Note that a half-edge may refer to the head or tail of an edge of T or an external leg.
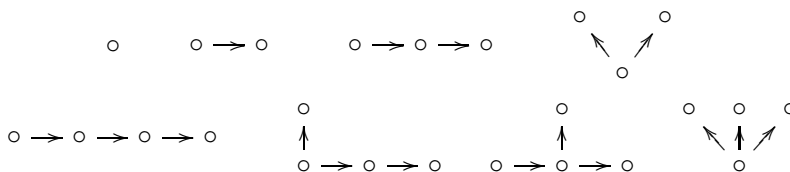
FIGURE 1. Rooted trees with no more than 4 vertices

Let (T, r) be a stable rooted tree decorated by $\{a_1 \cdots a_k, \bar{b}_1 \cdots \bar{b}_m\}$. For each vertex v of T, we define $i(v) = \min\{k \mid \bar{b}_k \in v\}$, which is well-defined since any vertex of T has at most one inward edge.

Definition 4.3. Denote by $\mathscr{T}_R(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$ the set of all decorated stable rooted trees T with root r such that $\bar{b}_1, \bar{b}_2 \in r$ and for each directed edge uv, we have $i(u) < i(v)$.

Define $\mathscr{T}_R(a_1 \cdots a_p | a_{p+1} \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$ to be the subset of $\mathscr{T}_R(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$ containing trees with the property that there does not exist a proper subtree which contains at least two indices from $\{a_1, \ldots, a_p\}$. Its complement is denoted by $\mathscr{T}_R^c(a_1 \cdots a_p | a_{p+1} \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$. The enumeration problem of these trees will be discussed in Appendix B.

Similar to the definition of Weyl invariants, for each $T \in \mathscr{T}_R(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$, we associate a unique monomial of curvature tensors R_T given by

$$(59) \qquad R_T = (-1)^{|V(T)|} \prod_{e \in E(T)} g^{a_e \bar{b}_e} \prod_{v \in V(T)} R_v,$$

where the indices $a_e, \bar{b}_e$ correspond to the head and tail of e respectively and R_v is the curvature tensor obtained by arranging the indices of half-edges of v as follows

$$(60) \qquad R_v = \begin{cases} R_{a_* \bar{b}_1 a_* \bar{b}_2 / a_* \cdots a_* \bar{b}_* \cdots \bar{b}_*}, & v = r, \\ R_{a_* \bar{b}_{i(v)} a_* \bar{b}_e / a_* \cdots a_* \bar{b}_* \cdots \bar{b}_*}, & v \neq r. \end{cases}$$

Note that when $v \neq r$, $\bar{b}_e$ corresponds to the tail of the unique inward edge $e \in E(T)$ of v. By the second Bianchi identity (5) and the Ricci formula (6), $R_{a_* \bar{b}_1 a_* \bar{b}_2 / a_* \cdots a_* \bar{b}_* \cdots \bar{b}_*}$ is separately symmetric in the indices a_* and $\bar{b}_*$, so their orders are irrelevant and thus R_T is well-defined.

The following theorem is one of the main results of this paper.

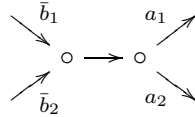**Theorem 4.4.** *Let $k \geq p \geq 2$ and $m \geq 2$. Then*

$$(61) \qquad D\Big((g^{r\bar{s}}g_{r\bar{b}_1\bar{b}_2}g_{a_1\bar{s}a_2})_{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m}\Big) = \sum_{T\in\mathscr{T}^c_R(a_1a_2|a_3\cdots a_k|\bar{b}_1\cdots\bar{b}_m)} R_T,$$

$$(62) \qquad -D\Big((R_{a_1\bar{b}_1a_2\bar{b}_2/a_3\cdots a_p})_{a_{p+1}\cdots a_k\bar{b}_3\cdots\bar{b}_m}\Big) = \sum_{T\in\mathscr{T}_R(a_1\cdots a_p|a_{p+1}\cdots a_k|\bar{b}_1\cdots\bar{b}_m)} R_T,$$

$$(63) \qquad D(g_{a_1\bar{b}_1a_2\bar{b}_2a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m}) = \sum_{T\in\mathscr{T}_R(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)} R_T.$$

Proof. First the theorem is obviously true when $k = m = 2$ or $p = k$. We will proceed by induction.

First we prove (61). Note that when applying $\partial_{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m}$ to $g^{r\bar{s}}g_{r\bar{b}_1\bar{b}_2}g_{a_1\bar{s}a_2}$, i.e. the decorated tree

we get a sum of paths decorated by $\{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m\}$. By induction, we may use (63) to convert the above partial derivatives to covariant derivatives and get a sum of decorated trees (T, r) with the property that a_1, a_2 are contained in a proper subtree T' of T. We need to compute the coefficient of T in the tensorial polynomial $D\Big((g^{r\bar{s}}g_{r\bar{b}_1\bar{b}_2}g_{a_1\bar{s}a_2})_{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m}\Big)$. It is without loss of generality to contract T' to a single vertex, so we may assume that a_1, a_2 are attached to the same vertex v. Let P_n, $n \geq 2$ be the unique path $(r = v_1, v_2, \ldots, v_{n-1}, v_n = v)$ from r to v. If $T \in \mathscr{T}^c_R(a_1a_2|a_3\cdots a_k|\bar{b}_1\cdots\bar{b}_m)$, i.e. $i(u) < i(v)$ whenever $uv \in E(T)$, the coefficient of T in $D\Big((g^{r\bar{s}}g_{r\bar{b}_1\bar{b}_2}g_{a_1\bar{s}a_2})_{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m}\Big)$ is equal to

$$\sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^{j+1} = 1.$$

On the other hand, if there are $k \geq 1$ edges uv in P_n that violate $i(u) < i(v)$, the coefficient of T in $D\Big((g^{r\bar{s}}g_{r\bar{b}_1\bar{b}_2}g_{a_1\bar{s}a_2})_{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m}\Big)$ is equal to

$$\sum_{j=0}^{n-1-k} \binom{n-1-k}{j}(-1)^{j+k+1} = 0,$$

which follows from $k < n - 1$. So we conclude the inductive proof of (61).

Next we prove (62). Note that (62) holds when $p = k$ by (8). From (2), we have

$$(64) \quad (R_{a_1\bar{b}_1 a_2\bar{b}_2/a_3\cdots a_p})_{a_{p+1}\cdots a_k\bar{b}_3\cdots\bar{b}_m} = (R_{a_1\bar{b}_1 a_2\bar{b}_2/a_3\cdots a_p a_{p+1}})_{a_{p+2}\cdots a_k\bar{b}_3\cdots\bar{b}_m}$$

$$+ (g^{r\bar{s}}g_{a_{p+1}\bar{s}a_1}R_{r\bar{b}_1 a_2\bar{b}_2/a_3\cdots a_p} + g^{r\bar{s}}g_{a_{p+1}\bar{s}a_2}R_{a_1\bar{b}_1 r\bar{b}_2/a_3\cdots a_p})_{a_{p+2}\cdots a_k\bar{b}_3\cdots\bar{b}_m}$$

$$+ \sum_{j=3}^{p}(g^{r\bar{s}}g_{a_{p+1}\bar{s}a_j}R_{a_1\bar{b}_1 a_2\bar{b}_2/a_3\cdots a_{j-1}ra_{j+1}\cdots a_p})_{a_{p+2}\cdots a_k\bar{b}_3\cdots\bar{b}_m}.$$

Then by induction, in the right-hand side of (64), the first term produces trees with the property that no two of $\{a_1, \ldots, a_p, a_{p+1}\}$ are contained in a proper subtree, the second term produces trees with the property that either a_1, a_{p+1} or a_2, a_{p+1} are contained in a proper subtree, the last term produces trees with the property that a_j, a_{p+1} $(3 \leq j \leq p)$ are contained in a proper subtree. The coefficient of each tree can be determined by exactly the same argument as in the above proof of (61). So we conclude the inductive proof of (62).

Finally we prove (63). By (1), we have

$$g_{a_1\bar{b}_1 a_2\bar{b}_2 a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m} = -(R_{a_1\bar{b}_1 a_2\bar{b}_2})_{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m} + (g^{r\bar{s}}g_{r\bar{b}_1\bar{b}_2}g_{a_1\bar{s}a_2})_{a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m}.$$

So the expression of $D(g_{a_1\bar{b}_1 a_2\bar{b}_2 a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m})$ in (63) follows directly from (61) and the $p = 2$ case of (62). Therefore we conclude the inductive proof of the theorem. $\square$

Remark 4.5. See [54] for related results more general than Theorem 4.4.

We sort indices in alphabetical order, $i, j, k, l, p, q, \ldots$. Note that only the order of barred indices is essential.

Example 4.6. Let us compute $D(g_{i\bar{j}k\bar{l}p\bar{q}})$. Besides the one-vertex tree, there are three two-vertex trees in $\mathscr{T}_R(ikp \mid \bar{j}\bar{l}\bar{q})$,
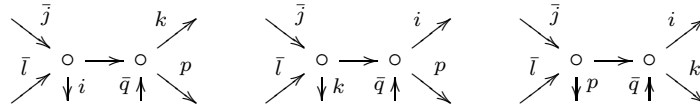
By (63), we get

$$(65) \qquad D(g_{i\bar{j}k\bar{l}p\bar{q}}) = -R_{i\bar{j}k\bar{l}/p\bar{q}} + g^{s\bar{t}}(R_{i\bar{j}s\bar{l}}R_{k\bar{q}p\bar{t}} + R_{k\bar{j}s\bar{l}}R_{i\bar{q}p\bar{t}} + R_{p\bar{j}s\bar{l}}R_{i\bar{q}k\bar{t}}).$$

Example 4.7. Since both $\mathscr{T}_R(ikp_1\ldots p_r|\bar{j}\bar{l})$ and $\mathscr{T}_R(ik|\bar{j}\bar{l}\bar{p}_1\ldots\bar{p}_r)$ contain only one tree, we get $D(g_{i\bar{j}k\bar{l}p_1\ldots p_r}) = -R_{i\bar{j}k\bar{l}/p_1\ldots p_r}$ and $D(g_{i\bar{j}k\bar{l}\bar{p}_1\ldots\bar{p}_r}) = -R_{i\bar{j}k\bar{l}/\bar{p}_1\ldots\bar{p}_r}$.

The next theorem shows the difference when we interchange two non-adjacent barred indices in a curvature tensor.

Theorem 4.8. *Let $m \geq 3$. Then*

$$(66) \qquad R_{a_1\bar{b}_1a_2\bar{b}_2/a_3\cdots a_k\bar{b}_3\cdots\bar{b}_m} - R_{a_1\bar{b}_1a_2\bar{b}_3/a_3\cdots a_k\bar{b}_2\bar{b}_4\cdots\bar{b}_m} = \sum_T \eta(T)R_T,$$

where $T = \{u \xrightarrow{e} v\}$ runs over stable two-vertex paths decorated by indices $\{a_1\cdots a_k, \bar{b}_1\cdots\bar{b}_m\}$ and

$$(67) \qquad \eta(T) = \begin{cases} 1, & \text{if } \bar{b}_1, \bar{b}_2 \in u \text{ and } \bar{b}_3 \in v, \\ -1, & \text{if } \bar{b}_1, \bar{b}_3 \in u \text{ and } \bar{b}_2 \in v, \\ 0, & \text{otherwise.} \end{cases}$$

The curvature tensor R_T corresponding to T is given by (cf. (80))

$$R_T = \begin{cases} R_{a_*\bar{b}_1a_*\bar{b}_2/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}R_{a_*\bar{b}_3a_*\bar{b}_e/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}, & \text{if } \bar{b}_1, \bar{b}_2 \in u \text{ and } \bar{b}_3 \in v, \\ R_{a_*\bar{b}_1a_*\bar{b}_3/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}R_{a_*\bar{b}_2a_*\bar{b}_e/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}, & \text{if } \bar{b}_1, \bar{b}_3 \in u \text{ and } \bar{b}_2 \in v. \end{cases}$$

Proof. The left-hand side of (66) is equal to

$$(68) \quad \sum_{j=3}^{k} \left(R_{a_1\bar{b}_1a_2\bar{b}_2/a_3\cdots a_j\bar{b}_3a_{j+1}\cdots a_k\bar{b}_4\cdots\bar{b}_m} - R_{a_1\bar{b}_1a_2\bar{b}_2/a_3\cdots a_{j-1}\bar{b}_3a_j\cdots a_k\bar{b}_4\cdots\bar{b}_m} \right)$$

$$- \sum_{j=3}^{k} \left(R_{a_1\bar{b}_1a_2\bar{b}_3/a_3\cdots a_j\bar{b}_2a_{j+1}\cdots a_k\bar{b}_4\cdots\bar{b}_m} - R_{a_1\bar{b}_1a_2\bar{b}_3/a_3\cdots a_{j-1}\bar{b}_2a_j\cdots a_k\bar{b}_4\cdots\bar{b}_m} \right)$$

By the Ricci formula (7), it is not difficult to see that any two-vertex path $T = \{u \xrightarrow{e} v\}$ appearing in (68) must satisfy one of the following conditions:

(i) $\bar{b}_1, \bar{b}_2 \in u$, $\bar{b}_3 \in v$; (ii) $\bar{b}_1, \bar{b}_3 \in u$, $\bar{b}_2 \in v$; (iii) $\bar{b}_2, \bar{b}_3 \in u$, $\bar{b}_1 \in v$.

Again by the Ricci formula (7), the contributions of the two summations in (68) to (iii) cancel out.

A two-vertex path T in (i) is contained in the first (resp. second) summation in (68) if and only if at least one of a_1, a_2 belongs to u (resp. both a_1, a_2 belong to v).

A two-vertex path T in (ii) is contained in the first (resp. second) summation in (68) if and only if both a_1, a_2 belong to v (resp. at least one of a_1, a_2 belongs to u).

Finally, (68) follows by noting that T in (i) and (ii) has multiplicity 1 and -1 respectively. $\qquad\square$

Example 4.9. By (66), we have

$$(69) \quad R_{i\bar{j}k\bar{l}/p\bar{q}} = R_{i\bar{j}k\bar{q}/p\bar{l}}$$

$$= R_{i\bar{j}k\bar{q}/p\bar{l}} + R_{i\bar{j}s\bar{l}}R_{k\bar{q}p\bar{s}} + R_{k\bar{j}s\bar{l}}R_{i\bar{q}p\bar{s}} + R_{p\bar{j}s\bar{l}}R_{i\bar{q}k\bar{s}}$$
$$- R_{i\bar{j}s\bar{q}}R_{k\bar{l}p\bar{s}} - R_{k\bar{j}s\bar{q}}R_{i\bar{l}p\bar{s}} - R_{p\bar{j}s\bar{q}}R_{i\bar{l}k\bar{s}}.$$

5. Local and global Bergman kernels: an explicit computation

First we briefly review the definitions of Bergman kernel in both local and global settings. Let Ω be an open subset of $\mathbb{C}^n$ with a Kähler potential $\Phi(x)$. Let $\Phi(x, y)$ be an almost analytic extension of $\Phi(x)$ to a neighborhood of the diagonal, i.e. $\bar{\partial}_x\Phi$ and $\partial_y\Phi$ vanish to infinite order for $x = y$. We can assume $\overline{\Phi(x, y)} = \Phi(y, x)$. For $m > 0$, consider the weighted Bergman space of all holomorphic functions on Ω square-integrable with respect to the measure $e^{-m\Phi}\frac{\omega_g^n}{n!}$. We denote by $K_m(x, y)$ the reproducing kernel. When Ω is a strongly pseudoconvex domain with real analytic boundary, Engliš [18] proved the existence of the asymptotic expansion

$$(70) \qquad K_m(x, y) \sim e^{m\Phi(x,y)}\sum_{k=0}^{\infty} B_k(x, y)m^{n-k}, \qquad m \to \infty,$$

uniformly on compact subsets. The asymptotic expansion (70) plays a crucial role in Berezin quantization [5]. The coefficients B_k along the diagonal were computed by Engliš [17] for $k \leq 3$ by a recursive formula of B_k derived from the asymptotics of Laplace integrals. Loi [43, 44] refined Engliš' recursive formula and gave a new proof of Engliš' asymptotic expansion. See also [10] for related

works. As shown in [43] (also cf. [51, §3]), B_k is equal to a_k of (71) in the global case.

For a compact Kähler manifold M of dimension n, instead of holomorphic functions, we consider holomorphic sections of holomorphic line bundles. Let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle and g be the polarized Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m \in \mathbb{N}$, h induces a Hermitian metric h_m on L^m. Let $\{S_1, \cdots, S_d\}$ be an orthonormal basis of $H^0(M, L^m)$ with respect to the inner product

$$(S_i, S_j)_{h_m} = \int_M h_m(S_i(x), S_j(x)) \frac{1}{n!} \omega_g^n.$$

Zelditch [57] and Catlin [9] independently proved that there is a complete asymptotic expansion:

$$(71) \qquad \sum_{i=0}^{d} \|S_i(x)\|_{h_m}^2 = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots$$

Various extensions to off-diagonal asymptotic expansion and generalizations to orbifolds and symplectic manifolds can be found in e.g. [6, 13, 14, 35, 46, 49]. See also recent works [2, 16, 24, 33, 42].

The a_k for $k \leq 3$ were computed by Lu [45] using peak section method. In particular,

$$a_0 = 1, \quad a_1 = \frac{\rho}{2}, \quad a_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}|R|^2 - \frac{1}{6}|Ric|^2 + \frac{1}{8}\rho^2.$$

The computation of a_3, independently done by Lu [45] and Engliš [17], is a marvelous feat; this requires technical and hard calculations occupying more than ten pages in both papers.

In the rest of the section, we use results proved in §4 to give a relatively easy derivation of the tensor expression for a_3. The following explicit closed formula of a_k was proved in [51, 52],

$$(72) \qquad B_k(x) = a_k(x) = \sum_G z(G) \cdot G = \sum_G \frac{(-1)^n \det(A - I)}{|\mathrm{Aut}(G)|} G,$$

where $G = G_1 \cup \cdots \cup G_n$ runs over stable (i.e. both the indegree and outdegree of each vertex are no less than 2) multi-digraphs of weight k (i.e. $|E(G)| - |V(G)| = k$) such that each component G_i is strongly connected and A is the adjacency matrix of G. Although (72) gives a closed formula of a_k as a summation of local

partial derivatives, it is still a quite hard task to convert it into tensor expressions when $k \geq 3$.

We will express a_3 in terms of the following basis as used by Engliš [17].

$$\sigma_1 = \rho^3, \qquad \sigma_2 = \rho R_{i\bar{j}}R_{j\bar{i}}, \qquad \sigma_3 = \rho R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k}},$$

$$\sigma_4 = R_{i\bar{j}}R_{k\bar{l}}R_{j\bar{i}l\bar{k}}, \qquad \sigma_5 = R_{i\bar{j}}R_{k\bar{i}l\bar{m}}R_{j\bar{k}m\bar{l}}, \qquad \sigma_6 = R_{i\bar{j}}R_{j\bar{k}}R_{k\bar{i}},$$

$$\sigma_7 = R_{i\bar{j}k\bar{l}}R_{j\bar{i}m\bar{n}}R_{l\bar{k}n\bar{m}}, \qquad \sigma_8 = \rho\Delta\rho, \qquad \sigma_9 = R_{i\bar{j}}R_{j\bar{i}/k\bar{k}},$$

$$\sigma_{10} = R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k}/m\bar{m}}, \qquad \sigma_{11} = \rho_{/i}\rho_{/\bar{i}}, \qquad \sigma_{12} = R_{i\bar{j}/k}R_{j\bar{i}/\bar{k}},$$

$$\sigma_{13} = R_{i\bar{j}k\bar{l}/m}R_{j\bar{i}l\bar{k}/\bar{m}}, \qquad \sigma_{14} = \Delta^2\rho, \qquad \sigma_{15} = R_{i\bar{j}k\bar{l}}R_{j\bar{m}l\bar{n}}R_{m\bar{i}n\bar{k}}.$$

We need two more tensors

$$\widetilde{\sigma}_9 = R_{i\bar{j}}\rho_{j\bar{i}}, \qquad \widetilde{\sigma}_{10} = R_{i\bar{j}k\bar{l}}R_{j\bar{i}/l\bar{k}}.$$
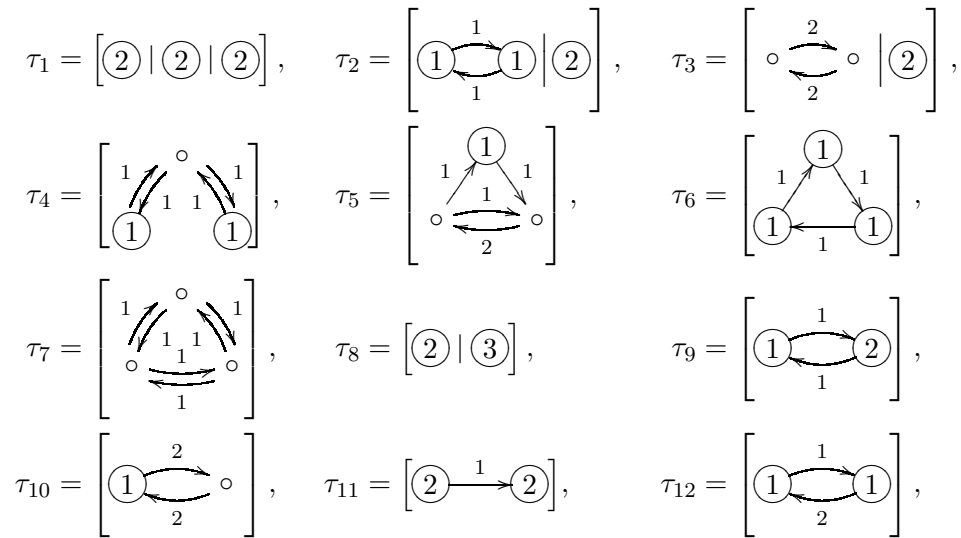
By (69), it is not difficult to get

(73) $$\widetilde{\sigma}_9 = \sigma_9 + \sigma_4 - \sigma_6, \qquad \widetilde{\sigma}_{10} = \sigma_{10} + 2\sigma_7 - \sigma_5 - \sigma_{15}.$$

We will compute the coefficients c_i, $1 \leq i \leq 15$, such that

(74) $$a_3 = c_1\sigma_1 + c_2\sigma_2 + \cdots + c_{15}\sigma_{15}.$$

There are 15 stable graphs of weight 3 in $\mathcal{G}(3)$.

$$\tau_{13} = \left[\; \circ \overset{3}{\underset{2}{\rightleftarrows}} \circ \;\right], \qquad \tau_{14} = \left[\;\boxed{4}\;\right], \qquad \tau_{15} = \left[\;\begin{array}{c} \circ \\ {}^2\nearrow \quad \searrow {}^2 \\ \circ \xleftarrow{\;2\;} \circ \end{array}\;\right].$$

The graphs τ_i are almost in one-to-one correspondence with σ_i. The only exceptions are that τ_9 may correspond to either σ_9 or $\widetilde{\sigma}_9$, and τ_{10} may correspond to either σ_{10} or $\widetilde{\sigma}_{10}$.

By (72), we have

$$(75) \qquad a_3 = \sum_{i=1}^{15} z(\tau_i)\tau_i = z_1\tau_1 + z_2\tau_2 + \cdots + z_{15}\tau_{15},$$

where the coefficients are given by

$$z_1 = -1/48, \quad z_2 = -1/4, \quad z_3 = -3/16, \quad z_4 = 0, \quad z_5 = -1,$$
$$(76) \quad z_6 = -1/3, \quad z_7 = -2/3, \quad z_8 = 1/6, \quad z_9 = 1/2, \quad z_{10} = 1,$$
$$z_{11} = 0, \quad z_{12} = 1, \quad z_{13} = 5/12, \quad z_{14} = -1/8, \quad z_{15} = -7/24.$$

We need to express each τ_i representing partial derivatives as a linear combination of σ_i, $1 \le i \le 15$. As stated in [51] without detailed computations,

$$\tau_i = -\sigma_i, \quad 1 \le i \le 7,$$
$$\tau_8 = -2\sigma_2 - \sigma_3 + \sigma_8, \quad \tau_9 = -\sigma_4 - \sigma_5 - \sigma_6 + \sigma_9, \quad \tau_{10} = -2\sigma_5 + \sigma_{10} - \sigma_{15},$$
$$(77) \qquad \tau_{11} = \sigma_{11}, \quad \tau_{12} = \sigma_{12}, \quad \tau_{13} = \sigma_{13},$$
$$\tau_{14} = -3\sigma_4 - 12\sigma_5 - 3\sigma_6 + 6\sigma_7 + 7\sigma_9 + 8\sigma_{10} + 10\sigma_{12} + 3\sigma_{13} - \sigma_{14} - 6\sigma_{15},$$
$$\tau_{15} = -\sigma_{15}.$$

While the terms $\tau_8, \tau_9, \tau_{10}$ can be computed easily using (65), the computation for τ_{14} is much more difficult. Here we present a graph-theoretic computation of τ_{14} by using Theorem 4.4. By (63), the unique one-vertex tree in $D(g_{i\bar{i}j\bar{j}k\bar{k}l\bar{l}})$ is equal to

$$(78) \qquad -\rho_{/kl\bar{k}\bar{l}} = -(\rho_{/k\bar{k}l\bar{l}} + R_{k\bar{s}l\bar{k}/l}\rho_{/s} + R_{k\bar{s}l\bar{k}}\rho_{/s\bar{l}})$$
$$= -(\sigma_{14} + \sigma_{11} + \widetilde{\sigma}_9)$$
$$= -(\sigma_{14} + \sigma_{11} + \sigma_9 + \sigma_4 - \sigma_6),$$

where we used the Ricci formula (7) in the first equation.

The 48 trees in $D(g_{i\bar{i}j\bar{j}kl\bar{k}\bar{l}})$ with at least two vertices are enumerated in Tables 1-6 in Appendix A. Note that each tree T is weighted by $(-1)^{|V(T)|}$. A tree is labeled by the dagger symbol † if and only if i, j are contained in a proper subtree.

From (1), we have

$$(79) \qquad \tau_{14} = D(g_{i\bar{i}j\bar{j}kl\bar{k}\bar{l}}) = D(-\partial_{kl\bar{k}\bar{l}}R_{i\bar{i}j\bar{j}}) + D\big(\partial_{kl\bar{k}\bar{l}}(g^{m\bar{n}}g_{m\bar{i}j}g_{i\bar{n}j})\big).$$

Denote by $I, II, \ldots, IV$ the sum of trees in the same numbered table such that i, j are not both contained in a proper subtree. By (62) and (78),

$$D(-\partial_{kl\bar{k}\bar{l}}R_{i\bar{i}j\bar{j}}) = -(\sigma_{14} + \sigma_{11} + \sigma_9 + \sigma_4 - \sigma_6) + I + II + III + IV + V + VI$$

$$= -(\sigma_{14} + \sigma_{11} + \sigma_9 + \sigma_4 - \sigma_6) + 2\sigma_9 + (\sigma_{11} + 4\sigma_{12}) + 4\sigma_{12}$$

$$+ (4\sigma_9 + 2\widetilde{\sigma}_9 + 4\widetilde{\sigma}_{10}) - (2\sigma_4 + 2\sigma_5 + 2\sigma_6) - (2\sigma_4 + 2\sigma_7)$$

$$= -3\sigma_4 - 6\sigma_5 - 3\sigma_6 + 6\sigma_7 + 7\sigma_9 + 4\sigma_{10} + 8\sigma_{12} - \sigma_{14} - 4\sigma_{15}.$$

Denote by $I', II', \ldots, IV'$ the sum of trees in the same numbered table such that i, j are contained in a proper subtree. By (61),

$$D\big(\partial_{kl\bar{k}\bar{l}}(g^{m\bar{n}}g_{m\bar{i}j}g_{i\bar{n}j})\big) = I' + II' + III' + IV' + V' + VI'$$

$$= (\sigma_{10} + \widetilde{\sigma}_{10}) + \sigma_{13} + (2\sigma_{12} + 2\sigma_{13}) + 2\sigma_{10} - (3\sigma_5 + 2\sigma_7 + \sigma_{15}) - 2\sigma_5$$

$$= -6\sigma_5 + 4\sigma_{10} + 2\sigma_{12} + 3\sigma_{13} - 2\sigma_{15}.$$

Adding up the above two identities, we get the desired tensor expression for τ_{14}. Substituting (77) into (75), we can get the coefficients in (74).

$$c_1 = 1/48, \qquad c_2 = -1/12, \qquad c_3 = 1/48, \qquad c_4 = -1/8, \qquad c_5 = 0,$$
$$c_6 = 5/24, \qquad c_7 = -1/12, \qquad c_8 = 1/6, \qquad c_9 = -3/8, \qquad c_{10} = 0,$$
$$c_{11} = 0, \qquad c_{12} = -1/4, \qquad c_{13} = 1/24, \qquad c_{14} = 1/8, \qquad c_{15} = 1/24.$$

6. Partial and covariant derivatives of functions

For any smooth function f defined on a Kähler manifold, we may derive from (7) the following well-known identities:

$$f_{/\alpha} = f_\alpha, \qquad f_{/i\bar{j}} = f_{/\bar{j}i} = f_{i\bar{j}},$$

$$f_{/ij} = f_{ij} - \sum_k \Gamma_{ij}^k f_k, \qquad f_{/\bar{i}\bar{j}} = f_{\bar{i}\bar{j}} - \sum_k \Gamma_{\bar{i}\bar{j}}^{\bar{k}} f_{\bar{k}},$$

$$f_{/i\bar{j}k} = f_{/k\bar{j}i} = f_{i\bar{j}k} - \sum_l \Gamma_{ik}^l f_{l\bar{j}}, \qquad f_{/ij\bar{k}} = (f_{/ij})_{\bar{k}}.$$

On the other hand, for any partial derivatives $f_{\alpha_1\alpha_2\ldots\alpha_m}$, there exists a canonical tensor, denoted by $D(f_{\alpha_1\alpha_2\ldots\alpha_m})$, that coincides with $f_{\alpha_1\alpha_2\ldots\alpha_m}$ at the center of any normal coordinate system. Below we will extend Theorem 4.4 to functions. First we slightly modify the definitions in §4.

Definition 6.1. Denote by $\mathscr{T}_f(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)$ the set of all decorated rooted trees T with root r such that all non-root vertices of T are stable and for each directed edge uv with $u,v \neq r$, we have $i(u) < i(v)$.

Define $\mathscr{T}_f(a_1\cdots a_p|a_{p+1}\cdots a_k|\bar{b}_1\cdots\bar{b}_m)$ to be the subset of $\mathscr{T}_f(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)$ containing trees with the property that there does not exist a proper subtree which contains at least two indices from $\{a_1,\ldots,a_p\}$.

For each $T \in \mathscr{T}_f(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)$, we associate a unique monomial of curvature tensors R_T given by

$$(80) \qquad R_T = (-1)^{|V(T)|-1} \prod_{e\in E(T)} g^{a_e\bar{b}_e} \prod_{v\in V(T)} R_v,$$

where the indices $a_e, \bar{b}_e$ correspond to the head and tail of e respectively and R_v is the curvature tensor obtained by arranging the indices of half-edges of v as follows

$$(81) \qquad R_v = \begin{cases} f_{/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}, & v = r, \\ R_{a_*\bar{b}_{i(v)}a_*\bar{b}_e/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}, & v \neq r. \end{cases}$$

Note that when $v \neq r$, $\bar{b}_e$ corresponds to the tail of the unique inward edge $e \in E(T)$ of v. By the second Bianchi identity (5) and the Ricci formula (6), $f_{/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}$ and $R_{a_*\bar{b}_1a_*\bar{b}_2/a_*\cdots a_*\bar{b}_*\cdots\bar{b}_*}$ are separately symmetric in the indices a_* and $\bar{b}_*$, so their orders are irrelevant and thus R_T is well-defined.

Lemma 6.2. *A tree in $\mathscr{T}_f(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)$ has at most $\min(k,m)+1$ vertices.*

Proof. It is obvious, since all non-root vertices of trees in $\mathscr{T}_f(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)$ need to be stable. $\qquad\square$

Theorem 6.3. *Let $k \geq p \geq 0$ and $m \geq 0$. Then*

$$(82) \qquad D\Big((f_{/a_1\cdots a_p})_{a_{p+1}\cdots a_k \bar{b}_3 \cdots \bar{b}_m}\Big) = \sum_{T \in \mathscr{T}_f(a_1 \cdots a_p | a_{p+1} \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)} R_T.$$

Proof. Obviously (82) holds when $p = k$. From (2), we have

$$(83) \quad (f_{/a_1\cdots a_p})_{a_{p+1}\cdots a_k \bar{b}_1 \cdots \bar{b}_m} = (f_{/a_1 \cdots a_p a_{p+1}})_{a_{p+2}\cdots a_k \bar{b}_1 \cdots \bar{b}_m}$$

$$+ \sum_{j=1}^{p} (g^{r\bar{s}} g_{a_{p+1}\bar{s} a_j} f_{/a_1 \cdots a_{j-1} r a_{j+1} \cdots a_p})_{a_{p+2}\cdots a_k \bar{b}_1 \cdots \bar{b}_m}.$$

Then by induction, in the right-hand side of (83), the first term produces trees with the property that no two of $\{a_1, \ldots, a_p, a_{p+1}\}$ are contained in a proper subtree, the last term produces trees with the property that a_j, a_{p+1} ($1 \leq j \leq p$) are contained in a proper subtree. The coefficient of each tree can be determined by exactly the same argument as in the above proof of (61). So we conclude the inductive proof of (82). $\qquad \square$

Corollary 6.4. *Let $k, m \geq 0$. Then*

$$(84) \qquad D(f_{a_1 \cdots a_k \bar{b}_1 \cdots \bar{b}_m}) = \sum_{T \in \mathscr{T}_f(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)} R_T.$$

Proof. We get the equation by taking $p = 0$ in (82). $\qquad \square$

Example 6.5. Let us compute $D(f_{ij\bar{k}})$. Since there is a one-vertex tree and a two-vertex tree in $\mathscr{T}_f(ij \mid \bar{k})$, by (84), we get

$$(85) \qquad D(f_{ij\bar{k}}) = f_{/ij\bar{k}} - f_{/l}R_{i\bar{k}j\bar{l}}.$$

Example 6.6. Let us compute $D(f_{ij\bar{k}\bar{l}})$. Besides the one-vertex tree, there are three two-vertex trees in $\mathscr{T}_f(ij \mid \bar{k}\bar{l})$,
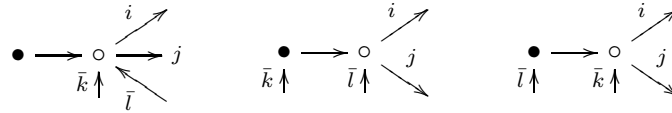
By (84), we get

$$(86) \qquad D(f_{ij\bar{k}\bar{l}}) = f_{/ij\bar{k}\bar{l}} - f_{/p}R_{i\bar{k}j\bar{p}/\bar{l}} - f_{/p\bar{k}}R_{i\bar{l}j\bar{p}} - f_{/p\bar{l}}R_{i\bar{k}j\bar{p}}.$$

Example 6.7. Since there is only one tree in $\mathscr{T}_f(p_1 \ldots p_r)$ and $\mathscr{T}_f(\bar{p}_1 \ldots \bar{p}_r)$, we get $D(f_{p_1 \ldots p_r}) = f_{/p_1 \ldots p_r}$ and $D(f_{\bar{p}_1 \ldots \bar{p}_r}) = f_{/\bar{p}_1 \ldots \bar{p}_r}$.

Appendix A. Tables for the computation of $D(g_{i\bar{i}j\bar{j}kl\bar{k}\bar{l}})$

Tables 1-6 contains 48 trees with at least two vertices in $D(g_{i\bar{i}j\bar{j}kl\bar{k}\bar{l}})$, as demonstrated by Theorem 4.4. The trees are grouped according to the distribution of the external legs. Tables 1-6 are in one-to-one correspondence with the following six types of trees, where the numbers denote the multiplicities of external legs.
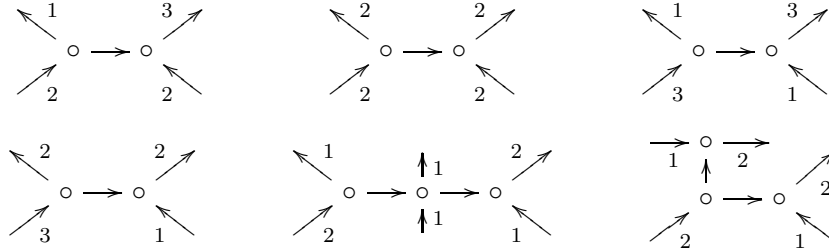
Note that each tree T is weighted by $(-1)^{|V(T)|}$ in the tensor expression. We label a tree by the dagger symbol † if and only if i,j are contained in a proper subtree. These trees add up to give $D\big(\partial_{kl\bar{k}\bar{l}}(g^{m\bar{n}}g_{m\bar{i}j}g_{i\bar{n}j})\big)$.

TABLE 1

| σ_9 | σ_9 | σ_{10} † | $\widetilde{\sigma}_{10}$ † |

TABLE 2

| σ_{11} | σ_{12} | σ_{12} |
| σ_{12} | σ_{12} | σ_{13} † |

TABLE 3

i jkl $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	j ikl $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	k ijl $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	l ijk $\overline{i}\overline{j}\overline{k}$ $\overline{l}$
σ_{12}	σ_{12}	σ_{12} †	σ_{13} †
i jkl $\overline{i}\overline{j}\overline{l}$ $\overline{k}$	j ikl $\overline{i}\overline{j}\overline{l}$ $\overline{k}$	k ijl $\overline{i}\overline{j}\overline{l}$ $\overline{k}$	l ijk $\overline{i}\overline{j}\overline{l}$ $\overline{k}$
σ_{12}	σ_{12}	σ_{13} †	σ_{12} †

TABLE 4

ij kl $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	ik jl $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	il jk $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	jk il $\overline{i}\overline{j}\overline{k}$ $\overline{l}$
$\widetilde{\sigma}_9$	σ_9	$\widetilde{\sigma}_{10}$	σ_9
jl ik $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	kl ij $\overline{i}\overline{j}\overline{k}$ $\overline{l}$	ij kl $\overline{i}\overline{j}\overline{l}$ $\overline{k}$	ik jl $\overline{i}\overline{j}\overline{l}$ $\overline{k}$
$\widetilde{\sigma}_{10}$	σ_{10} †	$\widetilde{\sigma}_9$	$\widetilde{\sigma}_{10}$
il jk $\overline{i}\overline{j}\overline{l}$ $\overline{k}$	jk il $\overline{i}\overline{j}\overline{l}$ $\overline{k}$	jl ik $\overline{i}\overline{j}\overline{l}$ $\overline{k}$	kl ij $\overline{i}\overline{j}\overline{l}$ $\overline{k}$
σ_9	$\widetilde{\sigma}_{10}$	σ_9	σ_{10} †

APPENDIX B. ENUMERATIONS OF DECORATED ROOTED TREES

Denote by $t_{k,m}(n)$ the number of n-vertex trees in $\mathscr{T}_R(a_1 \cdots a_k | \overline{b}_1 \cdots \overline{b}_m)$.

Let $s_{p,k,m}(n)$ and $s^c_{p,k,m}(n)$ denote respectively the numbers of n-vertex trees in $\mathscr{T}_R(a_1 \cdots a_p | a_{p+1} \cdots a_k | \overline{b}_1 \cdots \overline{b}_m)$ and $\mathscr{T}^c_R(a_1 \cdots a_p | a_{p+1} \cdots a_k | \overline{b}_1 \cdots \overline{b}_m)$. Namely $s_{p,k,m}(n)$ counts trees with the property that there does not exist a proper subtree which contains at least two indices from $\{a_1, \ldots, a_p\}$.

In order to enumerate the number of n-vertex decorated trees, we need to count all admissible decorations of inward external legs and outward external legs separately for each n-vertex rooted tree, i.e. each vertex should be stable and $i(u) < i(v)$ for any directed edge uv (see Definition 4.3).

TABLE 5

σ_4	σ_6	σ_5
σ_4	σ_6	σ_5
σ_5 †	σ_5 †	σ_{15} †
σ_7 †	σ_7 †	σ_5 †

TABLE 6

σ_4	σ_7	σ_4
σ_7	σ_5 †	σ_5 †

Lemma B.1. *Let* $k \geq p \geq 2$, $m \geq 2$.

(i) $t_{k,m}(n) = 0$ *when* $n \geq \min(k, m)$, *i.e. a tree in* $\mathscr{T}_R(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$ *has at most* $\min(k, m) - 1$ *vertices.*

(ii) For 2-vertex trees, we have $t_{k,m}(2) = (2^k - 2 - k)(2^{m-2} - 1)$ *and*

$$s_{p,k,m}(2) = ((p+1)2^{k-p} - k - 1)(2^{m-2} - 1),$$
$$s_{p,k,m}^c(2) = (2^k - (p+1)2^{k-p} - 1)(2^{m-2} - 1).$$

(ii) For 3-vertex trees, we have

$$t_{k,m}(3) = \left(2 \cdot 3^k - \frac{3k}{2} \cdot 2^k - 5 \cdot 2^k + k^2 + 3k + 4\right)\left(\frac{1}{18}3^m - \frac{1}{4}2^m + \frac{1}{2}\right),$$

$$s_{p,k,m}(3) = \left(3^{k-p}(p^2 + 3p + 2) - 2^{k-p-1}(3kp + 3k + p^2 + 7p + 8) + k^2 + 2k + 2\right)$$
$$\times \left(\frac{1}{18}3^m - \frac{1}{4}2^m + \frac{1}{2}\right).$$

(iii) For 4-vertex trees, we have

$$t_{k,m}(4) = \left(6 \cdot 4^k - 3^k(4k + 19) + 2^k\left(\frac{7}{4}k^2 + \frac{43}{4}k + 21\right) - (k^3 + 4k^2 + 9k + 9)\right)$$
$$\times \left(\frac{1}{96}4^m - \frac{1}{18}3^m + \frac{1}{8}2^m - \frac{1}{6}\right).$$

Proof. (i) is obvious, since each tree in $\mathscr{T}_R(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$ is stable.

Now we prove (ii). From the unique one 2-vertex tree, we have

$$t_{k,m}(2) = \sum_{i=2}^{k-1}\binom{k}{i}\sum_{j=1}^{m-2}\binom{m-2}{j} = (2^k - 2 - k)(2^{m-2} - 1),$$

$$s_{p,k,m}(2) = \left(\sum_{i=2}^{k-p}\binom{k-p}{i} + p\sum_{i=1}^{k-p}\binom{k-p}{i}\right)\sum_{j=1}^{m-2}\binom{m-2}{j}$$
$$= ((p+1)2^{k-p} - k - 1)(2^{m-2} - 1),$$

$$s_{p,k,m}^c(2) = \left(\sum_{i=2}^{p}\binom{p}{i}\sum_{j=0}^{k-p}\binom{k-p}{j} - 1\right)\sum_{j=1}^{m-2}\binom{m-2}{j}$$
$$= (2^k - (p+1)2^{k-p} - 1)(2^{m-2} - 1).$$

Next we prove (iii). There are two 3-vertex rooted trees

(87)
$$\underset{x}{\circ} \longrightarrow \underset{y}{\circ} \longrightarrow \underset{z}{\circ} \qquad \underset{u}{\circ} \longleftarrow \underset{v}{\circ} \longrightarrow \underset{w}{\circ}$$

First we count the number of admissible decorations of inward external legs for the two trees, which respectively equal to

$$\sum_{i=0}^{m-4} \binom{m-2}{i} \sum_{j=0}^{m-4-i} \binom{m-3-i}{j} = \frac{1}{18} 3^m - \frac{1}{4} 2^m + \frac{1}{2},$$

$$\frac{1}{2} \sum_{i=0}^{m-4} \binom{m-2}{i} \sum_{j=1}^{m-3-i} \binom{m-2-i}{j} = \frac{1}{18} 3^m - \frac{1}{4} 2^m + \frac{1}{2}.$$

We count the number of all admissible decorations of outward external legs $(\deg^+(v) \geq 2$ for all vertices) to the two 3-vertex trees of (87).

$$\sum_{i=0}^{k-2} \binom{k}{i} \sum_{j=1}^{k-i-1} \binom{k-i}{j} + \sum_{i=0}^{k-4} \binom{k}{i} \sum_{j=2}^{k-i-2} \binom{k-i}{j}$$

$$= 2 \cdot 3^k - \frac{3k}{2} \cdot 2^k - 5 \cdot 2^k + k^2 + 3k + 4,$$

which gives the formula of $t_{k,m}(3)$.

We count the number of all admissible decorations of outward external legs to the two 3-vertex trees of (87), such that there does not exist a proper subtree which contains at least two indices from $\{a_1, \ldots, a_p\}$. We call $\{a_1, \ldots, a_p\}$ the set of special indices. The following summation may be simplified to get the first factor in the formula of $s_{p,k,m}(3)$.

$$\sum_{i=0}^{k-p-3} \binom{k-p}{i} \sum_{j=1}^{k-p-i-2} \binom{k-p-i}{j} + p \sum_{i=0}^{k-p-2} \binom{k-p}{i} \sum_{j=0}^{k-p-i-2} \binom{k-p-i}{j}$$

$$+ p \sum_{i=0}^{k-p-2} \binom{k-p}{i} \sum_{j=1}^{k-p-i-1} \binom{k-p-i}{j} + \sum_{i=0}^{k-p-4} \binom{k-p}{i} \sum_{j=2}^{k-p-i-2} \binom{k-p-i}{j}$$

$$+ 2p \sum_{i=0}^{k-p-3} \binom{k-p}{i} \sum_{j=1}^{k-p-i-2} \binom{k-p-i}{j} + 2\binom{p}{2} \sum_{i=0}^{k-p-2} \binom{k-p}{i} \sum_{j=1}^{k-p-i-1} \binom{k-p-i}{j}$$

The first three terms come from the first tree of (87) corresponding to the three cases: (i) all special indices belong to x; (ii) all but one special indices belong to x and one special index belongs to y; (iii) all but one special indices belong to x and one special index belongs to z.

The last three terms come from the second tree of (87) corresponding to the three cases: (i) all special indices belong to v; (ii) all but one special indices belong to v and one special index belongs to u or w; (iii) all but two special indices belong to v, one special index belongs to u and one special index belongs to w .

The proof of the formula for $t_{k,m}(4)$ is similar to the above. We need to deal with the four 4-vertex rooted trees in Figure 1. $\qquad\square$

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