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On the Asymptotic Stability of Stationary Lines in the Curve Shortening Problem

Xiao-Liu Wang and Wei-Feng Wo*

Abstract: The asymptotic stability of the stationary lines in the curve shortening problem was studied in Nara-Taniguchi [NT2] when the initial curve is a graph over the stationary line. The result is extended to a class of non-graphic initial curves.

Keywords: curve shortening problem, curvature flow, asymptotic behavior, foliation, Sturm oscillation theorem.

1. Introduction

In the curve shortening problem (CSP) a family of plane curves γ_t evolves under the equation

$$\frac{\partial \gamma_t}{\partial t} = k\nu,$$

where the unit normals ν and the curvature k are defined by $\ddot{\gamma} = k\nu$ as usual in $\mathbb{R}^2$ and $\ddot{\gamma}$ denotes the second order derivative of γ with respect to arclength. As is well-known, aside from being the simplest case of curvature flows, the (CSP) also arises in applied areas such as phase transitions, Visintin [V] and image processing, Sapiro [S]. The (CSP) starting with a closed, embedded curve was studied by Gage-Hamilton [GH] who showed that it shrinks to a point in finite

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*Corresponding author

time, which, after blowing up, becomes a circle. To complete the study, Grayson [G] showed that any closed, embedded curve flows into a convex one in finite time under the (CSP). Thus the problem for closed, embedded curves is completely understood. For complete, non-compact embedded curves the (CSP) was studied subsequently in several works. In Ecker-Huisken [EH1],[EH2] it was shown that when the initial curve is an entire graph, this problem has a global solution which remains as graphs for all time. (In fact, in these works the higher dimensional version of the (CSP), the mean curvature flow, is studied.) They further established an asymptotic result asserting that the curve evolves to an expanding self-similar solution when the initial graph satisfies some growth conditions. See Stavrou [St] and Ishimura [I] for related results.

A straight line, usually taken to be the x-axis, is a stationary solution of the (CSP). The convergence to straight lines was studied by several authors. In Polden [P] (see also Huisken [H]) it is shown that any embedded curve whose ends are C^1-asymptotic to the x-axis converges to the x-axis smoothly as $t\to\infty$. In Nara-Taniguchi [NT1], assuming γ_0 is an entire graph $(x, u_0(x))$, it is established that $u(x,t)$, where $\gamma_t = (x, u(x,t))$, converges uniformly to zero provided $u_0 \in C^{2,\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$ and u_0 tends to zero uniformly at the ends. Under the additional assumption that $u_0 \in L^1(\mathbb{R})$, they also obtained the decay estimate

$$\sup_x |u(x,t)| \le C(1+t)^{-1/2}, \ t > 0.$$

In all of these works, the convergence to straight lines or self-similar solutions had been established under certain uniform growth or decay conditions of the initial curve at the ends. Thus it is surprising that Nara-Taniguchi [NT2] proved the following result: Assuming that the initial graph $u_0 \in C^{2,\alpha}(\mathbb{R})$ (for some $\alpha \in (0,1)$) and satisfies

$$\lim_{R\to\infty} \sup_{x_0} \frac{1}{2R} \left| \int_{x_0-R}^{x_0+R} u_0(x)\, dx \right| = 0, \tag{1}$$

they showed that $u(x,t)$ converges to zero uniformly. Moreover, the condition (1) is also necessary for an initial $C^{2,\alpha}$-function u_0 whose solution $u(t)$ of the (CSP) decays to 0 uniformly.

In this paper we extend this result to embrace certain classes of complete, non-compact embedded curves. To formulate our result we first express (1) in a form which is valid for non-graphic curves. Let $\gamma(p) = (x(p), y(p))$ be a smooth, properly embedded curve contained in $\mathbb{R} \times (-M, M)$ for some positive M. As it is properly embedded, we can choose a parametrization so that $\gamma(p) \to \pm\infty$ as $p \to \pm\infty$. Without loss of generality we may also assume $x(0) = 0$. For any given number $l > 0$, divide $\mathbb{R}$ into intervals, $I_n = [(2n-1)l, (2n+1)l],\ n \in \mathbb{Z}$. Starting from $\gamma(0)$, as p increases the curve $\gamma(p)$ will eventually leave I_0 by entering I_1 at p_1. Similarly, we define p_{-1} the last time the curve touches $x = -l$ and enters I_{-1}. We denote $\gamma^0 = \gamma|_{[p_{-1}, p_1]}$. Similarly, we can define $\gamma^n = \gamma|_{[p_{2n-1}, p_{2n+1}]}$. For each γ^n we extend it to be a closed curve c^n by adjoining to it the line segments in $x = (2n \pm 1)l$ and the $x-$axis (See Figure 1).

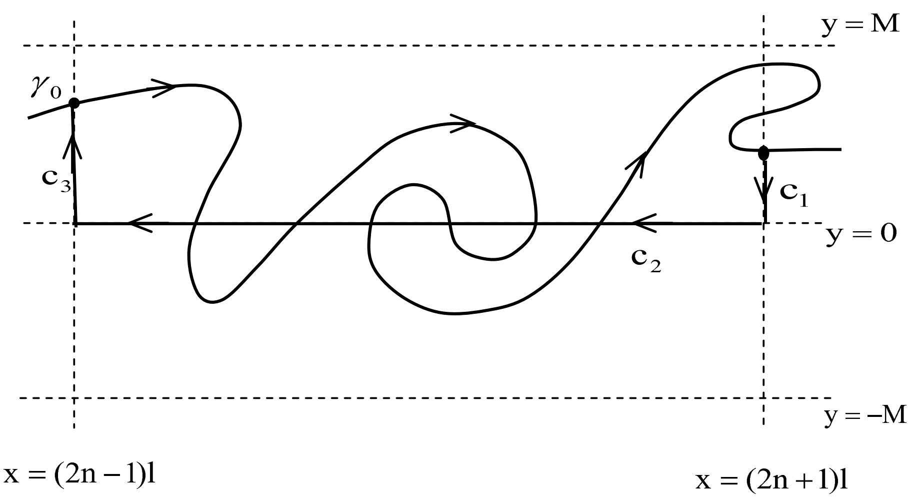

FIGURE 1. The closed curve c^n.

We define the "signed area" of γ over I_n, $A_n(\gamma)$, to be

$$A_n(\gamma) = \frac{1}{2} \int_{c^n} (y dx - x dy).$$

Note that when $\gamma^n = (x, u(x))$ is a graph over $[(2n-1)l, (2n+1)l]$,

$$A_n(\gamma) = \int_{(2n-1)l}^{(2n+1)l} u(x)\, dx.$$

For each $N \in \mathbb{Z}$, denote $A_{N,n}(\gamma) = \sum_{j=N-n}^{N+n} A_j(\gamma)$ the signed area of γ over $[(2N-2n-1)l, (2N+2n+1)l]$. We shall replace (1) by the following condition

$$\lim_{n\to\infty} \sup_N \frac{1}{(4n+2)l} |A_{N,n}(\gamma_0)| = 0. \tag{2}$$

When $\gamma_0 = (x, u_0(x))$ is confined to a strip and u_0 is uniformly continuous on the real line, it is clear that (1) holds if and only if (2) holds (for any fixed l). Now taking $M = 1$ for simplicity, we state

Theorem 1. *Suppose $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_l$ ($l \in \mathbb{N}^*$) are given embedded $C^{2,\alpha}$-curves, which are bounded and contained in $\mathbb{R} \times (-1,1)$ for some $\alpha \in (0,1)$. Let γ_0 be a properly embedded $C^{2,\alpha}$-curve contained in $\mathbb{R} \times (-1,1)$. Assume further that there exist a sequence of disjoint intervals $\{(a_n, b_n)\}_{n=1}^N (N < \infty$ or $= \infty)$, and a small number $\varepsilon_0 > 0$ such that*

γ_0 has exactly the same shape with one of $\mathcal{C}_i$ $(1 \le i \le l)$ over every interval (a_n, b_n), $\forall\ 1 \le n \le N$,

and

γ_0 is graphic over the set $\mathbb{R} \setminus (\cup_{n=1}^{\infty} [a_n + \varepsilon_0, b_n - \varepsilon_0])$.

Let γ_t be the global solution of the (CSP) starting at γ_0 .Then if (2) holds, there is a time $T > 0$ such that γ_t evolves into an entire graph $(x, u^*(x))$ at time T and (1) holds for $u_0 = u^*$. Consequently, γ_t converges to x-axis uniformly as $t \to \infty$.

The existence and uniqueness of a global solution to the (CSP) for smooth initial curves satisfying the assumptions in the theorem is established in Chou-Zhu [CZ1]. Our proof of this theorem consists of two parts. First, we show that the condition (2) is preserved under the (CSP). Second, we show that γ_t will evolve into an entire graph in finite time. After showing these, the convergence result follows from Nara-Taniguchi [NT2].

To conclude the introduction we point out some works on the asymptotic stability of other invariant solutions of the (CSP). One may consult Altschuler-Wu [AW] for traveling waves and Hungerbühler-Smoczyk [HS] for spiral waves.

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2. Area Evolution

Let γ_0 be a smooth, embedded curve confined to $\mathbb{R} \times (-1,1)$, which are graphs over some open intervals containing R_1 and R_2 respectively with $R_1 < R_2$. By the Sturm oscillation theorem (see, e.g. [A1], or chapter 1 in Chou-Zhu [CZ2]), γ_t remains as graphs over the same intervals. Moreover, the vertical lines $x = R_1, R_2$ are transversal to the graph of γ_t for all time.

Denote by the same notation γ_0 the portion of γ_0 inside $[R_1, R_2] \times (-1,1)$. Then γ_0 is regarded as a curve starting at a point on $x = R_1$ and ending up at a point on $x = R_2$. With this orientation, together with the x-axis and the two vertical lines $x = R_1, R_2$, we get a closed curve c. See Figure 1 for a similar situation. Here γ_0 is completely contained inside $[R_1, R_2] \times (-1,1)$. We denote the restriction of γ_t in this region by the same notation. Then the curve $\gamma_t(p) = (x(p,t), y(p,t))$ intersects $x = R_1, R_2$ respectively at $p = p_*(t)$ and $p = p^*(t)$. We have

$$x(p_*(t), t) = R_1,$$
$$x(p^*(t), t) = R_2.$$

By transversality, p_* and p^* are differentiable and

$$x_p \frac{dp_*}{dt} + x_t = 0, \quad x_p \frac{dp^*}{dt} + x_t = 0, \tag{2.1}$$

respectively at the intersection points. We use the parametrization $p \in [p_*(t), p^*(t)]$ to describe γ_t. For the other boundaries, we use the following parameterizations,

$$c_1(y) : y \text{ runs from } y(p^*(t), t) \text{ to } 0;$$
$$c_2(x) : x \text{ runs from } R_2 \text{ to } R_1;$$
$$c_3(y) : y \text{ runs from } 0 \text{ to } y(p_*(t), t).$$

It follows that

$$\begin{aligned} 2A(t) &= \int_c (ydx - xdy) \\ &= (\int_{\gamma_t} + \int_{c_1} + \int_{c_2} + \int_{c_3})(ydx - xdy) \\ &= \int_{p_*(t)}^{p^*(t)} (yx_p - xy_p)dp + R_2 y(p^*(t), t) - R_1 y(p_*(t), t), \end{aligned}$$

where $A(t)$ is the signed area of γ_t over $[R_1, R_2]$. We would like to compute dA/dt at any $t_0 \geq 0$. To simplify the computations we assume γ_{t_0} is parameterized by the arc length. We have

$$\begin{aligned}
&\frac{d}{dt}\int_{p_*(t)}^{p^*(t)} (yx_p - xy_p)dp\Big|_{t=t_0} \\
&= \int_{p_*(t_0)}^{p^*(t_0)} \langle (x_t, y_t), (-y_p, x_p)\rangle + \langle (x, y), (-y_{pt}, x_{pt}\rangle \, dp + \\
&\qquad (yx_p - xy_p)(p^*(t_0), t_0)p_t^*(t_0) - (yx_p - xy_p)(p_*(t_0), t_0)p_{*t}(t_0) \\
&= 2\int_{p_*(t_0)}^{p^*(t_0)} kdp - k(xx_p + yy_p)\Big|_{(p_*(t_0),t_0)}^{(p^*(t_0),t_0)} + \\
&\qquad (yx_p - xy_p)(p^*(t_0), t_0)p_t^*(t_0) - (yx_p - xy_p)(p_*(t_0), t_0)p_{*t}(t_0) \\
&= 2\int_{p_*(t_0)}^{p^*(t_0)} k\, dp - R_2\frac{k(p^*(t_0), t_0)}{x_p(p^*(t_0), t_0)} + R_1\frac{k(p_*(t_0), t_0)}{x_p(p_*(t_0), t_0)}
\end{aligned}$$

after using $x_t = -ky_p,\ y_t = kx_p$ and (2.1) at $t = t_0$. On the other hand,

$$\begin{aligned}
&\frac{d}{dt}(R_2 y(p^*(t), t) - R_1 y(p_*(t), t))\Big|_{t=t_0} \\
&= R_2(y_p p_t^* + y_t)\Big|_{(p^*(t_0),t_0)} - R_1(y_p p_{*t} + y_t)\Big|_{(p_*(t_0),t_0)} \\
&= \left(\frac{R_2 k(p^*(t_0), t_0)}{x_p(p^*(t_0), t_0)} - \frac{R_1 k(p_*(t_0), t_0)}{x_p(p_*(t_0), t_0)}\right)
\end{aligned}$$

Therefore,

$$\frac{d}{dt}A(t) = \int_{p_*(t)}^{p^*(t)} k\, dp.$$

The integral on the right of this formula is simply the difference in the tangent angles of γ_t at the intersection points at $x = R_2$ and R_1. It follows that

$$|\frac{d}{dt}A(t_0)| \leq \pi.$$

Since t_0 is arbitrary, for any fixed $t_1 \geq 0$, it holds that

$$|A(t_1)| \leq |A(0)| + \pi t_1. \tag{2.2}$$

We now show that (2) is preserved by the evolution. Consider a division of $\mathbb{R}$ into intervals of the form, $[(2n-1)R,(2n+1)R)],\ n \in \mathbb{Z}$. For $\varepsilon > 0$, there exists a large n_0, such that for all N,

$$\frac{1}{2n}|A_{N,n}(\gamma_0)| < \frac{\varepsilon}{2},\ \forall\ n \geq n_0.$$

Taking $R_1 = (2N-2n-1)R$ and $R_2 = (2N+2n+1)R$, we apply (2.2) to $A(t_1) = A_{N,n}(\gamma_{t_1})$ to obtain

$$\frac{1}{2n}|A_{N,n}(\gamma_{t_1})| < \frac{1}{2n}(|A_{N,n}(\gamma_0)| + \pi t_1) < \varepsilon,$$

if we further take $n_0 \geq \pi t_1/\varepsilon$. We conclude that (2) continues to hold at γ_{t_1} for any $t_1 > 0$.

Note that (2.2) still holds without the transversality condition at R_1 and R_2. Indeed, in the proof, if the vertical lines are not transversal at R_1 and R_2, we may find $\{\varepsilon_j\}_{j=1}^{\infty}$ with $\varepsilon_j \to 0$ such that $x = R_1 + \varepsilon_j, R_2 + \varepsilon_j$ are transversal to the initial curve by Sard's theorem. As above we have

$$|\tilde{A}(t_1)| \leq |\tilde{A}(0)| + \pi t_1,$$

where $\tilde{A}(t_1)$ is the area enclosed by the x-axis, the two vertical lines $x = R_1 + \varepsilon_j, R_2 + \varepsilon_j$ and γ_{t_1}. Then it's easy to observe that

$$|A(t_1)| \leq |A(0)| + \pi t_1 + 8\varepsilon_j,$$

and hence (2.2) continues to hold after letting $\varepsilon_j \to 0$.

3. Evolving into an Entire Graph

We shall use the method of foliations to show that the (CSP) for any initial curve satisfying the assumption of the theorem evolves into an entire graph in finite time.

For convenience, we may assume that $(a_1, b_1) = (-R, R)$ with $R > 2$ and $\varepsilon_0 = 1$. Then γ_t is a graph on the intervals $(-R-1, -R+1)$ and $(R-1, R+1)$. We focus on the behaviour of the solution on a typical cell $(-R, R) \times (-1, 1)$. Our aim is to construct a foliation $\mathcal{F}$ of $[-R-\frac{1}{2}, R+\frac{1}{2}] \times [-2, 2]$ consisting of "vertical leaves" $\Gamma_\mu,\ \mu \in [\mu_1, \mu_2]$, satisfying

(1) For each μ, the curve $\Gamma_\mu(y) : y \in [-1,1]$ is smooth, and $\mu \mapsto \Gamma_\mu$ is continuous from $[\mu_1, \mu_2]$ to $C^{2,\alpha}([-1,1])$,
(2) All Γ_μ are mutually disjoint and cover the set $[-R-\frac{1}{2}, R+\frac{1}{2}] \times [-2,2]$,
(3) $\Gamma^1_\mu(1) = \Gamma^1_\mu(-1)$ where $\Gamma_\mu = (\Gamma^1_\mu, \Gamma^2_\mu)$ for all μ, and
(4) each Γ_μ intersects γ_0 transversally.

It is evident that such foliation exists. However, in the following we sketch a construction in which the dependence of regularity of the foliation on the initial curve is carefully displayed. And the constructed foliations guarantee that γ_t evolves into a graph over $[-R, R]$ at a time T depending (but not only) on the $C^{2,\alpha}$-regularity of γ_0 in the cell $(-R,R) \times (-1,1)$.

Let $D = (-R-1, R+1) \times (-\frac{3}{2}, 2)$ and denote the restriction of γ_0 on D still by γ_0. The curve γ_0 divides D into two regions D_+ and D_- with D_+ lying on the left of γ_0 and D_- on its right. Denote the four corners of D_+ by $v_1 = \gamma_0 \cap \{x = -R-1\}, v_2 = \gamma_0 \cap \{x = R+1\}, v_3 = (R+1, 2)$ and $v_4 = (-R-1, 2)$. The following lemma displays the dependence of regularity of the foliation on the initial curve. It guarantees that a $C^{2,\alpha}$ curve will yield a $C^{2,\alpha}$ foliation.

Lemma 3.1. *There is a conformal diffeomorphism Φ_+ mapping D_+ onto the cube $(-1,1) \times (0,q)$ where $q > 0$ so that the corners v_1, v_2, v_3 and v_4 are mapped to $(-1,0), (1,0), (1,q)$ and $(-1,q)$ respectively. Φ_+ extends as a conformal diffeomorphism on $\overline{D_+}$ away from the corners. Moreover, the $C^{2,\alpha}$-norms of Φ_+(resp. Φ_+^{-1}) on any subdomain of $\overline{D_+}$(resp. of $[-1,1] \times [0,q]$) away from the corners are bounded by the $C^{2,\alpha}$-norm of γ_0 and the distance of the subdomain to the corners.*

Proof. By the generalized Schwarz-Christoffel formula (Pommmerenke [Po]), there is a conformal diffeomorphism f which maps the unit disk $\mathbb{D}$ to D_+. Let z_i be such that $f(z_i) = v_i, i = 1,2,3,4$. It is well-known that f extends as a conformal diffeomorphism on $\overline{\mathbb{D}} \setminus \{z_1, \ldots, z_4\}$. Next, by the Schwarz-Christoffel formula, there exists a conformal diffeomorphism g from $\mathbb{D}$ to the rectangle $C_+ = (-1,1) \times (0,q)$ where q is a fixed positive number determined by z_1, z_2, z_3 and z_4. Again, g is a conformal diffeomorphism on $\overline{\mathbb{D}} \setminus \{z_1, \ldots, z_4\}$. It follows that $\Phi_+ = g \circ f^{-1}$ is a conformal diffeomorphism satisfying the requirement of the lemma.

Let $\Phi_+ = u + iv$. The $v \in C(\overline{D_+})$ and $\triangle v = 0$ in D_+. As $v|_{\gamma_0} = 0$ and $v|_{(-R-1,R+1)\times\{2\}} = q$. By elliptic regularity, v has a $C^{2,\alpha}$-bound depending on the $C^{2,\alpha}$-norm of γ_0 away from the corners. By the Cauchy-Riemann equations $u_x = -v_y$ and $u_y = v_x$, we conclude that Φ_+ has the same regularity as v. Similar estimates hold for Φ_+^{-1} as the Jacobian of Φ_+ never vanishes on $\overline{D_+} \setminus \{v_1, v_2, v_3, v_4\}$. □

Consider the vertical lines $\{l_\mu(y) : y \in [0, q]\}, \mu \in (-1, 1)$, which foliate C_+. The pull-back $\Gamma_\mu^+ = (\Phi_+^{-1})(l_\mu)$ foliates $\overline{D_+} \setminus \{x = \pm(R+1)\}$. Each leaf Γ_μ^+ is a smooth curve starting at $(x, 2) = \Phi_+^{-1}(\mu, q)$ vertically and ending at $\Phi_+^{-1}(\mu, 0)$ vertically. The $C^{2,\alpha}$-norms of Γ_μ^+ are uniformly bounded for $\mu \in (-1+\varepsilon_1, 1-\varepsilon_1)$ for any fixed $\varepsilon_1 > 0$.

At each point on γ_0, there passes a unique leaf Γ_μ^+. So $\xi_0 = \frac{d}{dy}|_{y=0}\Gamma_\mu^+$ defines a $C^{1,\alpha}$-vector field along γ_0. Now, similar to the situation of D_+, we have a conformal diffeomorphism Φ_- which maps D_- onto $C_- = (-1, 1) \times (-r, 0)$ (for some $r > 0$) so that $\Phi_-(\gamma_0) = (-1, 1) \times \{0\}$. Φ_- enjoys the same regularity as Φ_+. Under Φ_-, its differential evaluating at ξ_0, $(d\Phi_-)(\xi_0)$, is a $C^{1,\alpha}$-vector field on $(-1, 1) \times \{0\}$. In fact, $d\Phi_-(\cdot)$ gives a map from the vector field along γ_0 to the vector field on $(-1, 1) \times \{0\}$. We could extend $(d\Phi_-)(\xi_0)$ to become the vertical vector field V on C_- by letting $V = (1 + \frac{y}{r})[(d\Phi_-)(\xi_0)(x, 0)] - \frac{y}{r}(0, 1)$ where $(x, y) \in C_-$. Then $\xi = (d\Phi_-^{-1})(V)$ is a $C^{1,\alpha}$-vector field on $\overline{D}_- \setminus \{x = \pm(R+1)\}$. At each point on γ_0 there passes a unique, vertical integral curve Γ_μ^- of ξ which stays inside D_- and leaves it by hitting a point on $(-R-1, R+1) \times \{-\frac{3}{2}\}$ vertically. Together with Γ_μ^+ we obtain a family of foliations of $\overline{D} \setminus \{x = \pm(R+1)\}$ where each leaf $\Gamma_\mu^+ \cup \Gamma_\mu^-$ is a $C^{1,1}$-curve which is smooth away from γ_0 (because the derivatives of the fields ξ and $\frac{d\Gamma_\mu^+}{dy}$ may have jump discontinuity along γ_0, $\Gamma_\mu^+ \cup \Gamma_\mu^-$ may not be $C^{2,\alpha}$.)

If we smooth out the discontinuities of $\Gamma_\mu^+ \cup \Gamma_\mu^-$ near γ_0 directly, the obtained smooth curves need not intersect γ_0 transversally. To obtain a smooth foliation satisfying requirement, we fix a tubular neighborhood $\mathcal{N}$ of γ_0, which guarantees that there exists a smooth diffeomorphism Φ_0 which maps $\mathcal{N}$ onto the cube $C_0 = (-1, 1) \times (-1, 1)$ so that $\Phi_0(\gamma_0) = (-1, 1) \times \{0\}$. (As the tubular neighborhood is constructed by using the map $(p, t) \mapsto \gamma_0(p) + t\nu(p)$, in order the diffeomorphism to be smooth we need the smoothness of γ_0. In general, the diffeomorphism is of class $C^{1,\alpha}$ if γ_0 is $C^{2,\alpha}$, and this would be sufficient for our purpose.) Note that the vertical lines $\{\tilde{l}_\mu(y) : y \in [-1,1]\}$ with $\mu \in (-1,1)$ foliates C_0. Then the pull-back $\Gamma^0_\mu = (\Phi_0^{-1})(\tilde{l}_\mu)$ foliates $\mathcal{N}$. Each leaf Γ^0_μ is a smooth curve starting at $(\Phi_0^{-1})(\mu,1)$, ending at $(\Phi_0^{-1})(\mu,-1)$ and intersecting γ_0 vertically. Moreover, for each $\mu \in (-1,1)$, there exists a unique μ_+ such that $(\Phi_0^{-1})(\mu,1)$ is one point of $\Gamma^+_{\mu_+}$ and a unique μ_- such that $(\Phi_0^{-1})(\mu,-1)$ is one point of $\Gamma^-_{\mu_-}$. Thus, for each $\mu \in (-1,1)$, we obtain a curve Γ_μ by linking $\Gamma^+_{\mu_+}$ and Γ^0_μ at the point $(\Phi_0^{-1})(\mu,1)$, and linking $\Gamma^-_{\mu_-}$ and Γ^0_μ at the point $(\Phi_0^{-1})(\mu,-1)$. After smoothing out the corners at $(\Phi_0^{-1})(\mu,1)$ and $(\Phi_0^{-1})(\mu,-1)$, we obtain a smooth foliation of $\overline{D} \cap \{-R-1+\delta \le x \le R+1-\delta\}$ (for any fixed small $\delta > 0$) so that each leaf $\Gamma_\mu(y), y \in [-r,q]$ (using the same notation) with μ in some subinterval of $(-1,1)$ intersects γ_0 transversally. The $C^{2,\alpha}$-norms of the leaves are uniformly bounded and the transversal angles are uniformly bounded away from zero on $\overline{D} \cap \{-R-1+\delta \le x \le R+1-\delta\}$.

It remains to fulfill (3). We observe that for any point x on the upper boundary there passes a unique Γ_μ which ends up at a point in $[-R-1+\delta, R+1-\delta] \times \{-\frac{3}{2}\}$ whose x-coordinate equals to, say, $\phi(x)$. The map ϕ is a homeomorphism on $[-R-1+\delta, R+1-\delta]$. We can extend our foliation on $\overline{D} \cap \{-R-1+\delta \le x \le R+1-\delta\}$ to $[-R-1+\delta, R+1-\delta] \times [-2,2]$ by prolonging each Γ_μ by the line segment connecting $(\phi(x), -\frac{3}{2})$ and $(x,-2)$. After smoothing out the corners at $(\phi(x), -\frac{3}{2})$ we obtain, finally, a foliation on $[-R-1+\delta, R+1-\delta] \times [-2,2]$ whose leaves are still denoted by $\Gamma_\mu(y), y \in [-r-1,q]$. We may scale $[-r-1,q]$ to $[-1,1]$ and fix a small number $\varepsilon_2 > 0$ such that the foliation $\mathcal{F}$ consisting of all $\Gamma_\mu, \mu \in [\mu_1,\mu_2]$, where $\mu_1 = -1+\varepsilon_2, \mu_2 = 1-\varepsilon_2$, covers $[-R-\frac{1}{2}, R+\frac{1}{2}] \times [-2,2]$. All Γ_μ are uniformly bounded in $C^{2,\alpha}$-norm. This foliation satisfies (1)-(4) of our requirement.

According to Huisken [H], the (CSP) for fixed end points has global solutions which evolves to the line segment connecting the endpoints. So for each Γ_μ has a global solution $\Gamma_\mu(t)$ which evolves to the vertical line segment $\{(x,y) : y \in [-2,2]\}$, where x satisfies $\Phi_+(x,2) = (\mu,q)$. Given any large $K > 0$, for each Γ_μ there exists some time T_μ such that the slope of $\Gamma_\mu(t)$ has absolute value greater than K. As $\mu \mapsto \Gamma_\mu$ is continuous from $[\mu_1,\mu_2]$ to $C^{2,\alpha}([-1,1])$, a compactness argument shows that there is a uniform time T such that all slopes of $\Gamma_\mu(t)$ have absolute value greater than K, for all $\mu \in [\mu_1, \mu_2]$ and $t \geq T$.

To show that γ_t evolves into a graph over $(-R, R)$ at $t = T$ we use an argument from Angenent [A2]. We rotate the foliation $\mathcal{F}$ counterclockwise by a small angle so that the new leaves are still transversal to γ_0 in $(-R - \frac{1}{2}, R + \frac{1}{2}) \times [-2, 2]$. Denote this foliation by $\mathcal{F}^-$. Each leaf of $\mathcal{F}^-$ will evolve to a line segment with negative slope m_-. Similarly, we obtain a smooth foliation $\mathcal{F}^+$ by rotating $\mathcal{F}$ clockwise. The slopes of the leaves of $\mathcal{F}^+$ converge to a positive m_+. Then the initial γ_0 is pinched between Γ_μ^+ and Γ_μ^- at each point (see Figure 2).

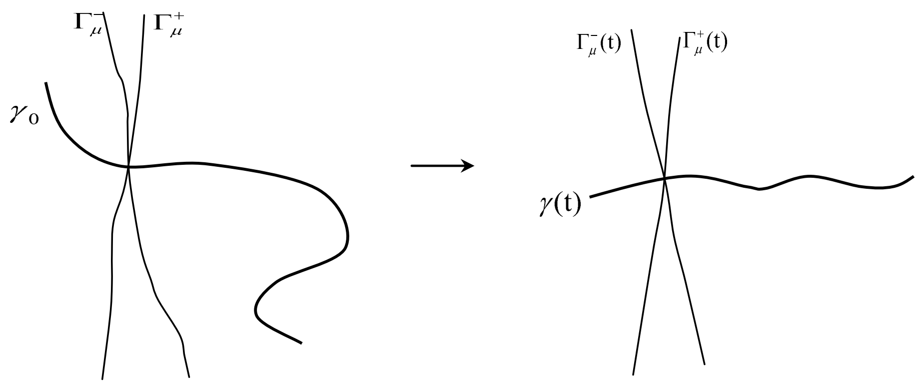

FIGURE 2. foliations.

Now, let γ_0 and $\mathcal{F}^\pm$ evolve with time. As over $(-R - \frac{1}{2}, -R)$ and $(R, R + \frac{1}{2})$, γ_0 is always a graph and its intersection points with Γ_μ^+ (Γ_μ^- resp.) are exactly once and transversal, by the Sturm oscillation theorem γ_t intersects Γ_μ^+ (Γ_μ^- resp.) exactly once transversally at any later instance. At $t = T$ the slope of Γ_μ^+ is uniformly close to m_+ and the slope of Γ_μ^- is uniformly close to m_-. As γ_t is pinched between these two curves, its slope cannot become vertical. So it is a graph over $[-R, R]$, see Figure 2.

Proof of Theorem 1. Applying the above arguments to our solution of (CSP) on each cell $(a_n, b_n) \times (-2, 2), n \in \mathbb{N}^*$, we find a uniform time T such that γ_T becomes a graph $(x, u^*(x))$. By the conclusion in Section 1, (1) holds for

$u_0 = u^*$, that is,

$$\lim_{R\to\infty} \sup_{x_0} \frac{1}{2R} \left| \int_{x_0-R}^{x_0+R} u^*(x)\, dx \right| = 0.$$

Then Theorem 1 follows immediately from Nara-Taniguchi's result in [NT2]. □

4. Discussion

We note that some conditions in our main theorem look very artificial. A natural question is, how about the long time behavior of γ_t if $l = \infty$? Since there is no explicit estimate for the time a general bounded curve takes to evolve into a graph, we cannot obtain a uniform lower bound of the time for the curves $\mathcal{C}_i$ $(1 \le i \le \infty)$ to flow into a graph and hence cannot show that γ_t evolves into an entire graph at finite time. As a partial answer to the above question, we give an example showing that the regularity of the initial curve alone is not enough to ensure the curve becomes an entire graph under the (CSP). Additional information such as the geometry of the regions $D_\pm$ also plays a role.

Consider the following curve in Figure 3, which is formed by connecting the negative $x-$axis and a sequence of length-varying cells of the form (see Figure 4) where the rectangles

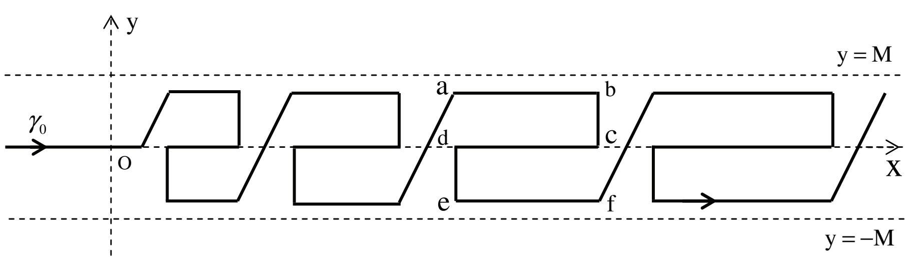

Figure 3. An example.

abcd and *cdef* are congruent. As p increases, the length $L = \overline{ab}$ of a cell increases by 1 unit each time as the cell prolongs along the x-axis. Clearly, (2) is satisfied, and by suitably smoothing out the corners, all C^k-norms of γ_0 can be made to be uniformly bounded. The (CSP) starting at the γ_0 will not evolve into an entire graph in finite time. To see this, consider an ellipse inside the rectangle *abcd* with area πL. By the curve shortening flow, the area decreases at rate 2π, [CZ2]. So

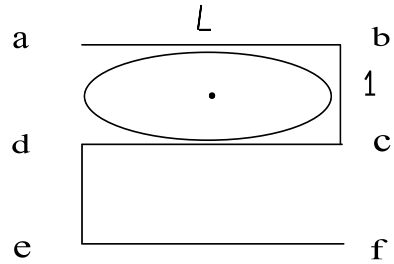

FIGURE 4. A cell.

the ellipse would shrink to a point at $T = \pi L/(2\pi) = L/2$. This means that the L-th cell of γ_0 cannot become a graph in $[0, L/2]$.

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Xiao-Liu Wang
Department of Mathematics
Southeast University
Nanjing, P.R.C., 210096.
E-mail: xlwang.seu@gmail.com

Wei-Feng Wo
Department of Mathematics
Ningbo University
Ningbo, P.R.C., 315211.
E-mail: woweifeng@nbu.edu.cn