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Two-Scale Convergence.
Some Remarks and Extensions.

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Abstract: We first study the fundamental ideas behind two-scale convergence to enhance an intuitive understanding of this notion. The classical definitions and ideas are motivated with geometrical arguments illustrated by illuminating figures. Then a version of this concept, very weak two-scale convergence, is discussed both independently and briefly in the context of homogenization. The main features of this variant are that it works also for certain sequences of functions which are not bounded in $L^2(\Omega)$ and at the same time is suited to detect rapid oscillations in some sequences which are strongly convergent in $L^2(\Omega)$. In particular, we show how very weak two-scale convergence explains in a more transparent way how the oscillations of the governing coefficient of the PDE to be homogenized causes the deviation of the G-limit from the weak $L^2(\Omega)^{N\times N}$-limit for the sequence of coefficients. Finally, we investigate very weak multiscale convergence and prove a compactness result for separated scales which extends a previous result which required well-separated scales.

Keywords: Two-scale convergence, multiscale convergence, very weak multiscale convergence, homogenization.

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1 Introduction

Let us consider the simplest example of a homogenization problem, the convergence of a sequence $\{u^\varepsilon\}$ of solutions to the elliptic problem

$$-\nabla \cdot \left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(x)\right) = f(x) \text{ in } \Omega, \tag{1}$$
$$u^\varepsilon(x) = 0 \text{ on } \partial\Omega$$

when ε passes to zero. Here a is periodic with respect to the unit cube Y in $\mathbb{R}^N$ and Ω is an open bounded set in $\mathbb{R}^N$ with a smooth boundary. Homogenization means that we look for a matrix b such that $\{u^\varepsilon\}$ converges to the solution u to

$$-\nabla \cdot (b\nabla u(x)) = f(x) \text{ in } \Omega,$$
$$u(x) = 0 \text{ on } \partial\Omega.$$

The origin and main source of inspiration for the development of homogenization theory is the study of heterogeneous materials, where b describes the effective properties of a homogeneous material with overall response similar to the composite in question, described by the oscillating coefficient in (1). In rare cases b coincides with the usual integral mean of a over Y, but usually b has to be found in more elaborative ways. Various techniques have been developed for this aim (see e.g. [3], [9] or [4]). Section 2 of this paper is devoted to a brief survey of the notion of so-called two-scale convergence of Nguetseng [11] and some of its essential features, see also [1] or [10]. A sequence $\{u^\varepsilon\}$ is said to two-scale converge to a limit u_0 if

$$\int_\Omega u^\varepsilon(x)\, v\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u_0(x,y)\, v(x,y)\, dydx$$

for all v in a quite large class of smooth test functions whose most important property is that they are Y-periodic in the second variable y.

Two-scale convergence appears, at least up to a subsequence (u.t.s.), if $\{u^\varepsilon\}$ is bounded in $L^2(\Omega)$. For $\{u^\varepsilon\}$ bounded in $H^1(\Omega)$ it is hence possible to find two-scale limits u.t.s. for the gradient, which is the most useful type of two-scale convergence for homogenization problems. We obtain

$$\int_\Omega \nabla u^\varepsilon(x) \cdot v\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y \left(\nabla u(x) + \nabla_y u_1(x,y)\right) \cdot v(x,y)\, dydx. \tag{2}$$

The corrector u_1 enables us to compute the deviation of b from $\int_Y a(y)\,dy$ and complete the homogenization procedure. The corrector u_1 can be found by means of the solutions to partial differential equations defined on Y which appear if we apply (2) to the weak form of (1) with a certain choice of oscillating test functions.

If $\{u^\varepsilon\}$ is strongly convergent in $L^2(\Omega)$, two-scale convergence fails to detect the micro-oscillations and no second scale appears in the two-scale limit. The concept of very weak two-scale convergence defined in Section 3 is designed to overcome this limitation in certain situations. See Remark 1. A more restrictive class of test functions allows a bounded sequence $\{u^\varepsilon\}$ in $H_0^1(\Omega)$ to be scaled to $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$ and the very weak two-scale limit turns out to coincide with the corrector u_1. It is noteworthy that very weak two-scale convergence provides us with a compactness result for sequences that do not have to be bounded in $L^2(\Omega)$. In this connection we discuss the distinction between the convergence properties of $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$ and $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ in the context of periodic homogenization in Section 3. Moreover, we show how very weak two-scale convergence explains in a quite transparent way how the oscillations of the governing coefficient of the PDE to be homogenized may cause a deviation between the G-limit and the weak $L^2(\Omega)^{N\times N}$-limit for $a\left(\frac{x}{\varepsilon}\right)$.

The natural extension of this notion to multiple scales, very weak multiscale convergence, is introduced in [5], where a compactness result is proven for well-separated scales. In Section 4 we prove a corresponding result for scales that are separated but not necessarily well-separated.

Remark 1 *Let us emphasize that this does not mean that a sequence has to be strongly convergent in $L^2(\Omega)$ to two-scale converge very weakly. In fact, as mentioned above, a sequence may two-scale converge very weakly without even being bounded in $L^2(\Omega)$.*

2 Two-scale convergence

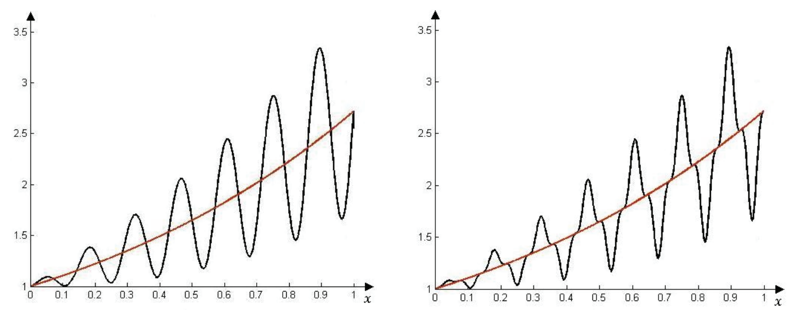
Let us consider a bounded sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$. A crude way to understand the limit behavior of such a sequence is weak convergence. It is well known that a bounded sequence in $L^2(\Omega)$ contains a weakly convergent subsequence, i.e., a subsequence such that

$$\int_{\Omega} u^{\varepsilon}(x)\,\nu(x)\,dx \to \int_{\Omega} u(x)\,\nu(x)\,dx$$

for any $\nu \in L^2(\Omega)$. An alternative way to say that a bounded sequence in $L^2(\Omega)$ converges weakly to a certain limit u is to require that

$$\int_{I} u^{\varepsilon}(x)\,dx \to \int_{I} u(x)\,dx$$

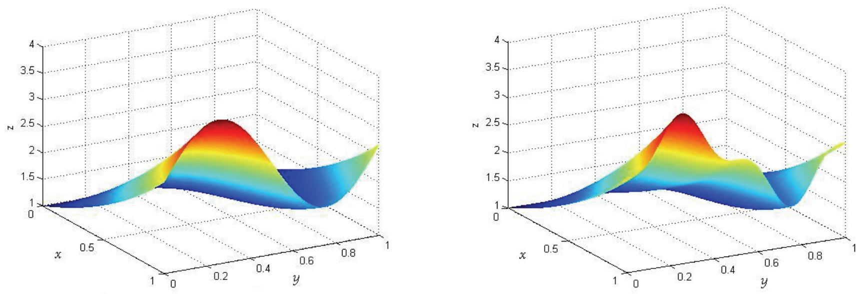
for any interval $I \subset \Omega$ which means that the weak limit adapts to the local mean of u^{ε} on any interval. This brings along that different sequences with different kinds of oscillations may attain the same weak limit as exhibited in Figure 1 below, where elements from two different sequences are shown together with the weak limits of the respective sequences.



(a) (b)

Figure 1

To capture the character of the high frequency oscillations we need a different concept of convergence. One way to tackle this is to introduce a limit with two scales; one representing the usual weak limit and the other taking care of the micro-oscillations. Figure 2a and 2b below display the two-scale limits of the sequences represented in Figure 1a and 1b respectively. If we compare the two-scale limits there is an obvious difference reflecting the different character of the micro-oscillations in the two sequences. This is how two-scale convergence is designed to work.



(a) (b)

Figure 2

Notation 2 *$F_\sharp(Y)$ means all functions in $F_{loc}(\mathbb{R}^N)$ that are periodical repetitions of some function in $F(Y)$, where F could mean L^p, H^1, C or C^∞.*

To avoid that the rapid oscillations are averaged away we need test functions which oscillate in time with $\{u^\varepsilon\}$. This means that we cannot choose one test function ν for all functions u^ε but need a sequence of oscillating test functions. In two-scale convergence this is arranged with quasi-periodic test functions $v\,(x, y)$, where the first variable x is defined over Ω and v is periodic with respect to the unit cube $Y \subset \mathbb{R}^N$ in the second variable y. If we substitute

$$y = \frac{x}{\varepsilon} \tag{3}$$

we obtain a sequence of functions

$$\nu^\varepsilon\,(x) = v\left(x, \frac{x}{\varepsilon}\right)$$

whose high frequency oscillations will have period ε and thus will be in resonance with the oscillations of u^ε of the same frequency and hence these oscillations are not hidden by averaging while passing to the limit. We say that $\{u^\varepsilon\}$ two-scale converges to u_0 if

$$\int\limits_\Omega u^\varepsilon\,(x)\, v\left(x, \frac{x}{\varepsilon}\right) dx \to \int\limits_\Omega \int\limits_Y u_0\,(x, y)\, v\,(x, y)\, dydx \tag{4}$$

for all v in a sufficiently large class of test functions of the type described above. The standard class of test functions is $L^2(\Omega; C_\sharp(Y))$, which contains all functions

$$v : \Omega \to C_\sharp(Y)$$

such that

$$\int_\Omega \left(\sup_{y \in Y} |v(x,y)| \right)^2 dx < \infty.$$

There are some important features of these functions. They are Y-periodic for any fixed $x \in \Omega$ and are hence suitable to detect oscillations in a sense similar to how we find Fourier coefficients. Moreover, they agree with the Carathéodory conditions and thus allow that y is replaced with a function of x and hence (3) makes it possible to choose which frequency of oscillation that should be recognized.

We have the definition and theorem below.

Definition 3 *We say that a sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ two-scale converges to $u_0 \in L^2(\Omega \times Y)$ if*

$$\int_\Omega u^\varepsilon(x)\, v\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u_0(x,y)\, v(x,y)\, dy dx \tag{5}$$

for any $v \in L^2(\Omega; C_\sharp(Y))$. We write

$$u^\varepsilon \overset{2}{\rightharpoonup} u_0.$$

The compactness result for two-scale convergence below is found in, e.g., [11], [1] and [10].

Theorem 4 *Any bounded sequence in $L^2(\Omega)$ two-scale converges u.t.s..*

To earn a deeper understanding of how a two-scale limit manages to capture the oscillations of $\{u^\varepsilon\}$ we introduce a special class of test functions. Let

$$v(x,y) = v_I(x)\, v_J(y),$$

where v_I and v_J are characteristic functions for small intervals $I \subset \Omega$ and $J \subset Y$ respectively and v_J is extended to its periodical repetition over $\mathbb{R}$. Consider a sequence $\{u^\varepsilon\}$ whose micro-oscillations appear with a period ε, as illustrated in Figure 3 for a fixed ε.

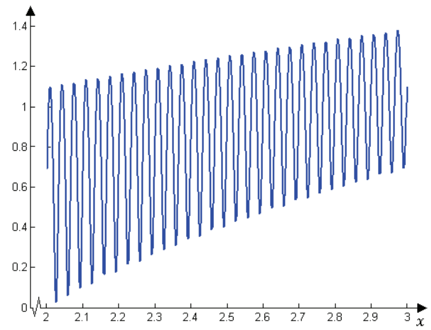


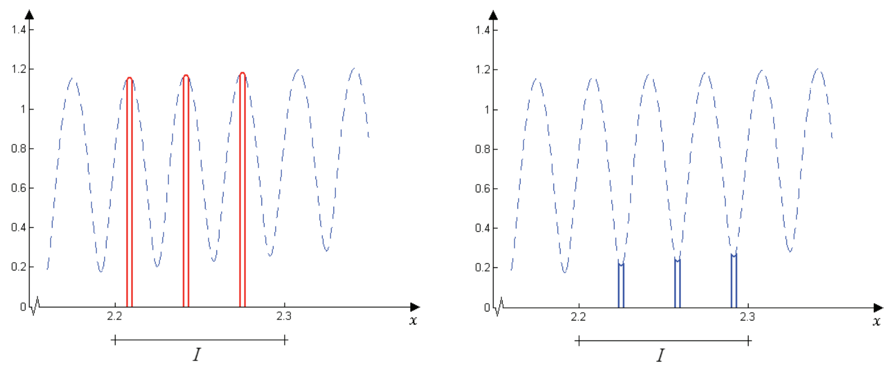
Figure 3

When ε goes to zero we have

$$\int_\Omega u^\varepsilon(x)\, v_I(x)\, v_J\left(\frac{x}{\varepsilon}\right) dx \tag{6}$$

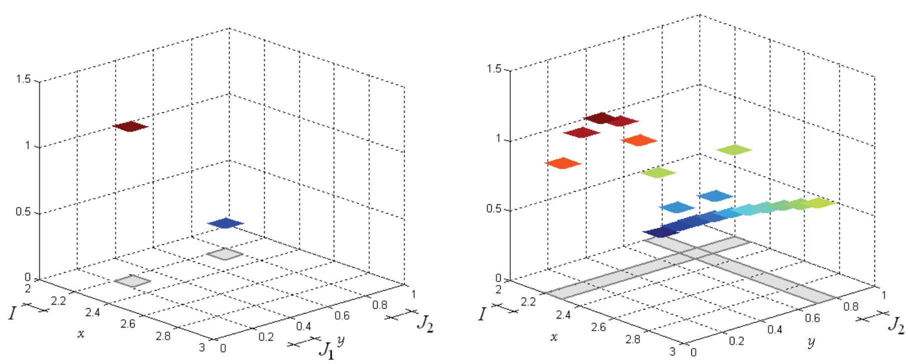
$$\to \int_\Omega \int_Y u_0(x,y)\, v_I(x)\, v_J(y)\, dydx = \int_I \int_J u_0(x,y)\, dydx, \tag{7}$$

where u_0 is the two-scale limit for $\{u^\varepsilon\}$. Figure 4 below displays the product $u^\varepsilon(x)\, v_I(x)\, v_J\left(\frac{x}{\varepsilon}\right)$ together with u^ε (dashed) for a fixed ε for the same interval I and two different choices of J, J_1 and J_2 respectively. In this example the test functions hit u^ε at the same phase of its oscillations all the time. Thus for these choices of J, we consequently collect large contributions to the integral in (6) in Figure 4a and small contributions in Figure 4b. This means that if we fix I and change J the limit for (6) that u_0 in (7) has to adapt to changes even though x is still contained in the same small interval I. This is illustrated in Figure 5a, where the obtained mean values of u_0 on $I \times J_1$ and $I \times J_2$ respectively are shown. Continuing with some more choices of J we get an idea about how the two-scale limit is shaped in the y direction on the interval I, see Figure 5b. As also illustrated in Figure 5b for the interval J_2, changing I, i.e. studying a different part of the domain Ω, may also affect the limit of (6), which in turn governs the two-scale limit. Thus, both the mean over Y of u_0 and the character of the oscillations captured by y may change with x.



(a) (b)

Figure 4



(a) (b)

Figure 5

When the micro-oscillations of u^ε are perfectly in time with those of the test functions, as in our example, they are fully exhibited in the two-scale limit as we can see in Figure 6.

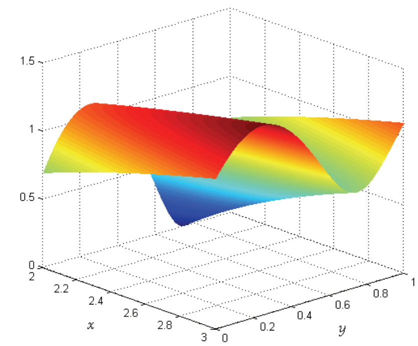


Figure 6

If this is not so the influence of the rapid changes of u^ε may be lost partially or completely. In Figure 7 below the test functions oscillate with a period 1.2ε while u^ε is still the same as above. Clearly, the test functions strike u^ε in completely different phases of its micro-oscillations for different repetitions and thus the y-dependence of u_0, if any, will not describe the character of the rapid oscillations of u^ε.

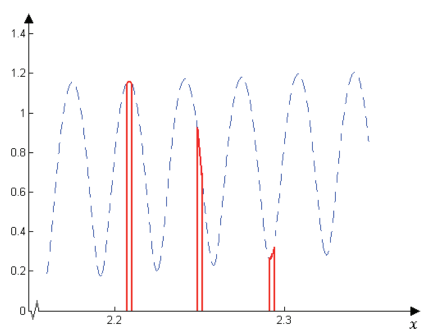


Figure 7

Remark 5 *In the discussion above we introduced a class of test functions which is not included in the standard space* $L^2\left(\Omega; C_\sharp(Y)\right)$ *of admissible test functions.*

However, in Theorem 16 in [10] it is proven that (4) holds with the same u_0 *for any*

$$v\left(x,y\right)=v_1\left(x\right)v_2\left(y\right),$$

where $v_1\in L^s\left(\Omega\right), v_2\in L^t_\sharp\left(Y\right), 2\le s,t\le\infty$ *and* $\frac{1}{s}+\frac{1}{t}=\frac{1}{2}$*, if* $\{u^\varepsilon\}$ *two-scale converges to* u_0*.*

Remark 6 *Let us briefly comment on the relation between the concept of two-scale convergence and the classical div-curl lemma, which both deal with the pairing of weakly convergent sequences in* $L^2\left(\Omega\right)$*. The div-curl lemma says that for* $\left\{u^h\right\},\left\{\nu^h\right\}$ *bounded in* $L^2\left(\Omega\right)^N$*,* $\left\{\nabla\cdot u^h\right\}$ *bounded in* $L^2\left(\Omega\right)$ *and* $\left\{\nabla\times\nu^h\right\}$ *bounded in* $L^2\left(\Omega\right)^{N\times N}$ *we have, up to a subsequence,*

$$\int_\Omega u^h(x)\cdot\nu^h(x)\varphi(x)dx\to\int_\Omega u(x)\cdot\nu(x)\varphi(x)dx$$

for any $\varphi\in D\left(\Omega\right)$*. Clearly, the conditions for this kind of so-called compensated compactness are designed to neutralize the effect of the oscillations of the involved sequences and hence no second scale is needed. Two-scale convergence was established for the opposite aim. Here the mission of the sequence* $\left\{v(x,\frac{x}{\varepsilon})\right\}$ *is to detect oscillations of* $\{u^\varepsilon\}$ *with period* ε *and exhibit them in a separate scale. In [14] Visintin introduces a modified type of compensated compactness, where the two-scale limits of* $\left\{u^h\right\}$ *and* $\left\{\nu^h\right\}$ *replaces the weak limits* u *and* ν *and oscillating test functions* $\varphi(x,hx)$ *of a similar kind as those used in two-scale convergence replaces the test functions* $\varphi(x)$ *in order to detect micro oscillations.*

3 Very weak two-scale convergence

Let us again consider a sequence of functions with rapid oscillations, illustrated in Figure 8a for a fixed ε. For two-scale convergence we have a rich class of test functions and the limit reveals the global tendency as well as the micro-oscillations with period ε, see Figure 8b. For weak convergence in $L^2(\Omega)$ we have a smaller class of test functions and we capture only the global trend as seen in Figure 8c. In the kind of limit we see in the last picture, 8d, the class of test functions is chosen such that the micro-oscillations are visible while the global trend is not.

Compared to usual two-scale convergence we have made some restriction on the class of test functions, in particular that the integral mean value over the second variable of the test functions is zero. This third type of convergence, which could be regarded as two-scale convergence in a certain quite weak sense, we name by very weak two-scale convergence.

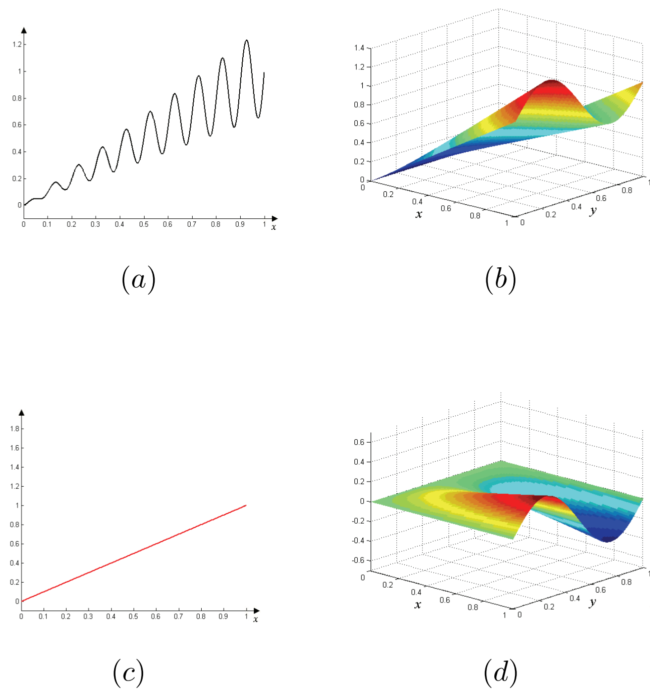


Figure 8

In the example above we obtain a conventional two-scale limit which requires that $\{u^\varepsilon\}$ is bounded in $L^2(\Omega)$ and hence the very weak limit is simply the difference between the two-scale limit and the weak $L^2(\Omega)$-limit. However, a significant property of very weak two-scale convergence, which provides us with more interesting examples, is that we get convergence up to a subsequence also for certain types of sequences that do not have to be bounded in $L^2(\Omega)$. Let us consider a sequence $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$, where ε passes to zero. Clearly, $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$ is not bounded in $L^2(\Omega)$ unless $\{u^\varepsilon\}$ passes to zero in a quite powerful way and hence such sequences normally neither two-scale converges nor converges weakly in $L^2(\Omega)$. However, for a more restrictive choice of test functions, in line with the discussion in connection with Figure 8d, it is possible to establish a limit of two-scale type also for certain sequences of this kind. Assume that $\{u^\varepsilon\}$ is bounded in $H_0^1(\Omega)$. Then, up to a subsequence,

$$\int_\Omega \frac{u^\varepsilon(x)}{\varepsilon} v_1(x)\, v_2\left(\frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u_1(x,y)\, v_1(x)\, v_2(y)\, dy dx \tag{8}$$

for all $v_1 \in D(\Omega)$ and $v_2 \in C_\sharp^\infty(Y)/\mathbb{R}$, where u_1 is the same as in (2), see [13]. A similar result has also been proven by Nguetseng and Woukeng [12] in the more general setting of Σ-convergence.

Originally, this type of convergence was established in [8] in a slightly different form. It was proven that

$$\int_\Omega \frac{u^\varepsilon(x) - u(x)}{\varepsilon} v_1(x)\, v_2\left(\frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u_1(x,y)\, v_1(x)\, v_2(y)\, dy dx \tag{9}$$

holds under the same assumptions on $\{u^\varepsilon\}$ and the test functions as above and with u the weak $H_0^1(\Omega)$-limit for $\{u^\varepsilon\}$.

Remark 7 *Results of the type in (9) and (8) were originally established in [8] and [12] respectively for the purpose of homogenizing parabolic equations with rapid oscillations in both space and time and were thus proven in a more general form adapted to this aim. Further results, where versions of the concept of very weak two-scale convergence has recently been applied in the context of parabolic homogenization, are found in for example [7], [15], [16] and [6].*

3.1 The concept of very weak two-scale convergence

Obviously, very weak two-scale convergence enables us to detect rapid oscillations of vanishing amplitude by means of a suitable scaling and making a choice of test functions such that the rapid oscillations are not drowned in the corresponding amplification of the macro-scale tendency while passing to the limit. Moreover, there is a clear connection between the very weak two-scale limit for $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$ and the two-scale limit for $\{\nabla u^\varepsilon\}$. These are essential features of very weak two-scale convergence.

We give the formal definition of very weak two-scale convergence.

Definition 8 *Let $\{g^\varepsilon\}$ be a sequence of functions in $L^1(\Omega)$. We say that $\{g^\varepsilon\}$ two-scale converges very weakly to $g_0 \in L^1(\Omega \times Y)$ if*

$$\int_\Omega g^\varepsilon(x)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y g_0(x,y)\, v_1(x)\, v_2(y)\, dy dx$$

for all $v_1 \in D(\Omega)$ and all $v_2 \in C_\sharp^\infty(Y)/\mathbb{R}$. A unique limit is provided by requiring that

$$\int_Y g_0(x,y)\, dy = 0.$$

We write

$$g^\varepsilon \underset{vw}{\overset{2}{\rightharpoonup}} g_0.$$

The theorem below provides us with a compactness result for very weak two-scale convergence. At first sight it may be surprising that it is possible to establish a compactness result of two-scale type for a sequence like $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$ which can not be assumed to be bounded in any Lebesgue space. However, if we instead consider the expression

$$F^\varepsilon(u^\varepsilon) = \int_\Omega \frac{u^\varepsilon(x)}{\varepsilon} v\left(x, \frac{x}{\varepsilon}\right) dx$$

as a bounded sequence of functionals F^ε in $H^{-1}(\Omega)$ defined by $\left\{\frac{1}{\varepsilon} v(x, \frac{x}{\varepsilon})\right\}$ (see Theorem 3.2 in [2]) acting on a bounded sequence $\{u^\varepsilon\}$ of functions in $H_0^1(\Omega)$ it is easier to see that the concept makes sense. See also [7].

Theorem 9 *Let $\{u^\varepsilon\}$ be a bounded sequence in $H_0^1(\Omega)$. Then there exists a subsequence such that*

$$u^\varepsilon \rightharpoonup u \text{ in } H_0^1(\Omega),$$

$$u^\varepsilon \overset{2}{\rightharpoonup} u, \tag{10}$$

$$\nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla u + \nabla_y u_1, \tag{11}$$

$$\frac{u^{\varepsilon}-u}{\varepsilon} \underset{vw}{\overset{2}{\rightharpoonup}} u_1 \tag{12}$$

and

$$\frac{u^{\varepsilon}}{\varepsilon} \underset{vw}{\overset{2}{\rightharpoonup}} u_1, \tag{13}$$

where $u \in H_0^1(\Omega)$ *and* $u_1 \in L^2(\Omega; H_\sharp^1(Y)/\mathbb{R})$.

Proof. The first conclusion follows immediately from the reflexivity of $H_0^1(\Omega)$. The limits in (10) and (11) are some of the earliest results on two-scale convergence and were originally proven in [11]. The result (12) is a special case of Corollary 3.3 in [8]. A proof of (13) is found in Corollary 1.3 in [13] and can also be concluded with some effort from Lemma 3.4 in [12]. ■

3.2 Very weak two-scale convergence and homogenization

The essential difficulty in homogenization is to handle the fact that the G-limit b for $\left\{a\left(\frac{x}{\varepsilon}\right)\right\}$ usually does not coincide with any conventional limit such as, e.g., the weak $L^2(\Omega)^{N\times N}$-limit $\widetilde{a}$ of $\left\{a\left(\frac{x}{\varepsilon}\right)\right\}$. Below we demonstrate how very weak two-scale convergence can provide transparency in how this deviation appears as a result of the oscillations of the coefficient $a\left(\frac{x}{\varepsilon}\right)$. Applying usual two-scale convergence to the weak form

$$\sum_{i,j=1}^{N}\int_{\Omega} a_{ij}\left(\frac{x}{\varepsilon}\right)\partial_{x_j}u^{\varepsilon}(x)\,\partial_{x_i}v(x)\;dx=\int_{\Omega} f(x)v(x)\;dx, \tag{14}$$

$v \in H_0^1(\Omega)$, of (1) we arrive at

$$\sum_{i,j=1}^{N}\int_{\Omega}\int_{Y} a_{ij}(y)\left(\partial_{x_j}u(x)+\partial_{y_j}u_1(x,y)\right)\partial_{x_i}v(x)\;dydx=\int_{\Omega} f(x)v(x)\;dx.$$

The difference between b and $\widetilde{a}$ is contained in the term $\nabla_y u_1$which emerges through an abstract orthogonality argument in the characterization of two-scale limits for sequences of gradients but has no directly visible connection with the limit behavior of $a\left(\frac{x}{\varepsilon}\right)$. Let us instead study the limit of (14) by means of very weak two-scale convergence. Integrating by parts we arrive at the alternative weak formulation

$$-\sum_{i,j=1}^{N}\int_{\Omega}\frac{u^{\varepsilon}(x)}{\varepsilon}\partial_{x_i}v(x)\partial_{y_j}a_{ij}\left(\frac{x}{\varepsilon}\right)+a_{ij}\left(\frac{x}{\varepsilon}\right)u^{\varepsilon}(x)\,\partial^2_{x_ix_j}v(x)\;dx \tag{15}$$

$$= \int_{\Omega} f(x)v(x)dx,$$

$v \in H_0^1(\Omega)$. Letting ε go to zero in (15) with a completed passage to the limit in the second term in the left hand side and integrating this term back by parts we obtain

$$-\lim_{\varepsilon\to 0}\sum\nolimits_{i,j=1}^{N}\int_{\Omega}\frac{u^{\varepsilon}(x)}{\varepsilon}\partial_{x_i}v(x)\partial_{y_j}a_{ij}\left(\frac{x}{\varepsilon}\right)dx \tag{16}$$

$$+\sum\nolimits_{i,j=1}^{N}\int_{\Omega}\widetilde{a}_{ij}\partial_{x_j}u(x)\,\partial_{x_i}v(x)\;dx = \int_{\Omega} f(x)v(x)\;dx.$$

Clearly, the deviation of the G-limit b from the weak limit $\widetilde{a}$ is contained in

$$r = -\lim_{\varepsilon\to 0}\sum\nolimits_{i=1}^{N}\left(\int_{\Omega}\frac{u^{\varepsilon}(x)}{\varepsilon}\partial_{x_i}v(x)\sum\nolimits_{j=1}^{N}\partial_{y_j}a_{ij}\left(\frac{x}{\varepsilon}\right)dx\right) \tag{17}$$

and $\partial_{x_i}v(x)\sum_{j=1}^{N}\partial_{y_j}a_{ij}(y)$, $i = 1,2,...,N$, are admissible test function for very weak two-scale convergence. The concept of very weak two-scale convergence hence makes it possible to pass to the limit in this expression and we obtain a new viewpoint upon how the oscillations of the coefficient $a\left(\frac{x}{\varepsilon}\right)$ generates the difference between b and $\widetilde{a}$.

Let us allow ourselves one further observation. Watching closely we would see that the oscillations of u^{ε} are not uniformly distributed around u but contains a trend on a slower scale, see e.g. [7]. Neither of the compactness results in (12) and (13) are based on any assumption that $\left\{\frac{u^{\varepsilon}-u}{\varepsilon}\right\}$ or $\left\{\frac{u^{\varepsilon}}{\varepsilon}\right\}$ should be bounded in $L^2(\Omega)$. The presence of the term u does however make it possible to establish such bounds on $\left\{\frac{u^{\varepsilon}-u}{\varepsilon}\right\}$, which will help us to understand the slow tendency of the oscillations of u^{ε} discussed above.

Still assume that u^{ε} solves (1). Under modest assumptions of smoothness it is possible to prove that, for some $C > 0$,

$$\|u^{\varepsilon} - u\|_{L^2(\Omega)} < C\varepsilon,$$

see Section 1.4 in [9]. Hence

$$\left\|\frac{u^{\varepsilon} - u}{\varepsilon}\right\|_{L^2(\Omega)} < C$$

and the existence of a subsequence to $\left\{\frac{u^{\varepsilon}-u}{\varepsilon}\right\}$ that two-scale converges in the usual sense follows. Applying very weak two-scale convergence allows us to give a further characterization of the two-scale limit. The result below splits the limit into one term that describes the global tendency of $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ and one identical to the very weak two-scale limit. Note that a two-scale limit in the sense of Definition 3 can normally not be obtained for $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$ in this context because of the absence of boundedness in $L^2(\Omega)$. This means that we have found a situation, where there is a distinction between the limit processes of $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ and $\left\{\frac{u^\varepsilon}{\varepsilon}\right\}$. They have the same very weak two-scale limit, see (12) and (13), but under these circumstances there is more information to extract from $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ by using the richer class of test functions applied in conventional two-scale convergence. The result below follows easily from the assumptions on $\{u^\varepsilon\}$ and the definitions of the involved concepts of convergence.

Theorem 10 *Let $\{u^\varepsilon\}$ be a bounded sequence in $H_0^1(\Omega)$ which possesses a subsequence that converges weakly to a limit u. Further, assume that $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ is bounded in $L^2(\Omega)$. Then*

$$\frac{u^\varepsilon-u}{\varepsilon} \overset{2}{\rightharpoonup} \mu + u_1$$

u.t.s., where u_1 is the same as in the two-scale limit for $\{\nabla u^\varepsilon\}$ and μ is the weak $L^2(\Omega)$-limit to $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ for the chosen subsequence.

Proof. We have assumed that $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ is bounded in $L^2(\Omega)$ and thus there exists a subsequence such that

$$\frac{u^\varepsilon-u}{\varepsilon} \overset{2}{\rightharpoonup} \mu_0 \tag{18}$$

and

$$\frac{u^\varepsilon-u}{\varepsilon} \rightharpoonup \mu = \int_Y \mu_0(x,y)dy \text{ in } L^2(\Omega).$$

We apply (12) in Theorem 9 and obtain for any $\nu, v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y)/\mathbb{R}$

$$\int_\Omega \frac{u^\varepsilon(x)-u(x)}{\varepsilon}\left(\nu(x)+v_1(x)v_2\left(\frac{x}{\varepsilon}\right)\right)dx \to$$

$$\int_\Omega\int_Y \mu(x)\nu(x)+u_1(x,y)v_1(x)v_2(y)dydx.$$

Moreover, (18) means that

$$\int_\Omega \frac{u^\varepsilon(x)-u(x)}{\varepsilon}\left(\nu(x)+v_1(x)v_2\left(\frac{x}{\varepsilon}\right)\right)dx \to$$

$$\int_\Omega \int_Y \mu_0(x,y)(\nu(x) + v_1(x)v_2(y))dydx$$

and hence

$$\int_\Omega \int_Y \mu(x)\nu(x) + u_1(x,y)v_1(x)v_2(y)dydx = \tag{19}$$

$$\int_\Omega \int_Y \mu_0(x,y)(\nu(x) + v_1(x)v_2(y))dydx.$$

Any two-scale limit can be decomposed as (see Remark 5 in [11])

$$\mu_0(x,y) = \mu(x) + \widetilde{\mu}_0(x,y),$$

where μ is the weak $L^2(\Omega)$-limit and

$$\int_Y \widetilde{\mu}_0(x,y)dy = 0.$$

Choosing $\nu \equiv 0$ in (19) we obtain

$$\int_\Omega \int_Y \mu_0(x,y)v_1(x)v_2(y)dydx = \int_\Omega \int_Y \widetilde{\mu}_0(x,y)v_1(x)v_2(y)dydx =$$

$$\int_\Omega \int_Y u_1(x,y)v_1(x)v_2(y)dydx.$$

Applying the variational lemma (Proposition 18.36 and Corollary 18.37 in [17]) we obtain

$$\widetilde{\mu}_0(x,y) = u_1(x,y) \text{ a.e. in } \Omega \times Y.$$

The proof is complete. ■

Remark 11 *Already in the paper [11], where Nguetseng first introduced two-scale convergence, it was noted that a two-scale limit* u_0 *could be decomposed as*

$$u_0(x,y) = u(x) + \widetilde{u}_0(x,y),$$

where u *is the weak* $L^2(\Omega)$*-limit to the sequence* $\{u^\varepsilon\}$ *that generates the two-scale limit and* $\widetilde{u}_0$ *is some function in* $L^2(\Omega \times Y)$ *with average zero over* Y *in the second variable. This should not be confused with the result in Theorem 10, where*

u_1 is specifically the first corrector of the kind found in for example asymptotic expansions or the two-scale limit of gradients and μ is the weak $L^2(\Omega)$-limit to the sequence $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$ and not to $\{u^\varepsilon\}$. For the one-dimensional case it is possible to prove that μ is always a straight line. Note that the amplitude of the micro-oscillations of the sequence of solutions $\{u^\varepsilon\}$ to the homogenization problem (1) usually shrinks to zero when ε passes to zero and hence necessitates the upscaling of $\{u^\varepsilon - u\}$ to $\left\{\frac{u^\varepsilon-u}{\varepsilon}\right\}$.

4 Very weak multiscale convergence for separated scales

Two-scale convergence and the related concepts discussed this far can be extended to multiple scales of rapid oscillation. So-called multiscale convergence was first defined by Allaire and Briane in [2]. Let $Y^n = Y_1 \times Y_2 \times ... \times Y_n$, where $Y_k, k = 1, 2, ..., n$, are N-dimensional unit cubes associated to the local variables $y_1, y_2, ..., y_n$ respectively. In a similar way we define the corresponding variable $y^n = (y_1, y_2, ..., y_n)$ and denote $dy^n = dy_1 dy_2 ... dy_n$. Moreover, let $\varepsilon_k(\varepsilon)$, $k = 1, 2, ..., n$, be functions such that $\varepsilon_k(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Definition 12 *A sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ is said to $(n+1)$-scale converges to $u_0 \in L^2(\Omega \times Y^n)$ if*

$$\int_\Omega u^\varepsilon(x)\, v\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_n}\right) dx \to \int_\Omega \int_{Y^n} u_0(x, y^n)\, v(x, y^n)\, dy^n dx$$

for any $v \in L^2(\Omega; C_\sharp(Y^n))$. We write

$$u^\varepsilon \overset{n+1}{\rightharpoonup} u_0.$$

For this type of convergence we tell apart two different kinds of sets of scales. We call the scales separated if

$$\lim_{\varepsilon\to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$$

and we say that they are well-separated if

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon_k}\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^m = 0$$

for some positive integer m. If $\{u^\varepsilon\}$ is bounded in $H^1(\Omega)$ it is possible to describe multiscale limits for gradients in a similar way as for two-scale convergence and hence to obtain homogenization procedures for problems with more than two scales. The result in Theorem 13 below does not differ between separated and well-separated scales but the proofs of multiscale convergence compactness have to be performed in different ways.

Theorem 13 *Let $\{u^\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$ and assume that the scales are separated. Then there exists a subsequence such that*

$$u^\varepsilon \rightharpoonup u \text{ in } H^1(\Omega), \tag{20}$$

$$u^\varepsilon \overset{n+1}{\rightharpoonup} u \tag{21}$$

and

$$\nabla u^\varepsilon \overset{n+1}{\rightharpoonup} \nabla u + \sum_{k=1}^{n} \nabla_{y_k} u_k, \tag{22}$$

where $u \in H^1(\Omega)$, $u_1 \in L^2(\Omega; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_k \in L^2(\Omega \times Y^{k-1}; H^1_\sharp(Y_k)/\mathbb{R})$ for $k = 2, 3, ..., n$.

Proof. The conclusion in (20) is an obvious consequence of the reflexivity of $H^1(\Omega)$ and (21) and (22) are found in Theorem 2.6 in [2]. ■

For well-separated scales a multiscale correspondence of very weak two-scale convergence was introduced in [5]. It was proven that for $\{u^\varepsilon\}$ bounded in $H^1_0(\Omega)$

$$\int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_n} v_1\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_{n-1}}\right) v_2\left(\frac{x}{\varepsilon_n}\right) dx$$

$$\to \int_\Omega \int_{Y^n} u_n(x, y^n)\, v_1\left(x, y^{n-1}\right) v_2(y_n)\, dy^n dx$$

u.t.s. for all $v_1 \in D(\Omega; C^\infty_\sharp(Y^{n-1}))$ and $v_2 \in C^\infty_\sharp(Y_n)/\mathbb{R}$, where u_n is the same as in (22).

To obtain a similar result for separated scales we need to restrict ourselves to a smaller class of test functions. Let $\nabla^p_y v$ mean all partial derivatives of v with respect to $y_1, y_2, ..., y_{n-1}$ up to order p and $\nabla_y^p \nabla_x v$ all such partial derivatives of the components of $\nabla_x v$ and introduce the norm

$$\|\phi\| = \sup_{\Omega\times Y^n} |\phi| .$$

To obtain a space of admissible test functions we make the definition below.

Definition 14 *We say that $v \in F_n$ if $v \in D(\Omega; C_\sharp^\infty(Y^n))$, $\int_{Y_n} v \, dy_n = 0$ and there exists $\delta > 0$ such that for any $p \in \mathbb{Z}_+$*

$$\left\|\nabla_y^p v\right\| + \left\|\nabla_y^p \nabla_x v\right\| \leq \delta^p. \tag{23}$$

The result below is found in Theorem 3.9 in [2].

Theorem 15 *If $v \in F_n$ and the scales $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ are separated, then*

$$\left\| \frac{1}{\varepsilon_n} v\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_n}\right)\right\|_{H^{-1}(\Omega)} < C$$

for some C which depends only on v.

We are ready to define the appropriate admissible space of test functions to obtain a compactness result for very weak multiscale convergence for separated scales which do not have to be well-separated.

Definition 16 *We say that $v \in G_n$ if $v = v_1 v_2 \in F_n$, where $v_1 \in D(\Omega; C_\sharp^\infty(Y^{n-1}))$ and $v_2 \in C_\sharp^\infty(Y_n)/\mathbb{R}$.*

The observation in the remark below will be used in the proof of Theorem 18.

Remark 17 *Note that if $v = v_1 v_2 \in G_n$ for some $v_2 \in C_\sharp^\infty(Y_n)/\mathbb{R}$ that is not identically zero, then for any $r \in C_\sharp^\infty(Y_n)/\mathbb{R}$ we can find a constant $\alpha \in \mathbb{R}$ such that $\widetilde{v} = \alpha v_1 r \in G_n$. For r not identically zero we can choose e.g.*

$$\alpha = \sup_{y_n \in Y_n} |v_2| / \sup_{y_n \in Y_n} |r| .$$

For $r \equiv 0$ the conclusion is trivial for any $\alpha \in \mathbb{R}$.

We are now prepared to prove the following theorem.

Theorem 18 *Let $\{u^\varepsilon\}$ be a bounded sequence in $H_0^1(\Omega)$ and assume that the scales are separated. Then there exists a subsequence such that*

$$\int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_n} v\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_n}\right) dx \to \int_\Omega \int_{Y^n} u_n(x, y^n)\, v(x, y^n)\, dy^n dx$$

for any $v \in G_n$, where u_n is the same as in Theorem 13.

Proof. We want to prove that for any $v = v_1 v_2 \in G_n$

$$\int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_n} v_1\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_{n-1}}\right) v_2\left(\frac{x}{\varepsilon_n}\right) dx \tag{24}$$

$$\to \int_\Omega \int_{Y^n} u_n(x, y^n)\, v_1\left(x, y^{n-1}\right) v_2(y_n)\, dy^n dx$$

u.t.s.. The result is trivial for $v_2 \equiv 0$ and hence we only consider v_2 that is not identically zero in the sequel. First we note that any $v_2 \in C_\sharp^\infty(Y_n)/\mathbb{R}$ can be expressed as

$$v_2(y_n) = \Delta_{y_n} w(y_n) = \nabla_{y_n} \cdot (\nabla_{y_n} w(y_n))$$

for some $w \in C_\sharp^\infty(Y_n)/\mathbb{R}$ (see e.g. Remark 3.2 in [12]). Furthermore, let

$$r(y_n) = \nabla_{y_n} w(y_n)$$

and observe that

$$\int_{Y_n} r(y_n)\, dy_n = \int_{Y_n} \nabla_{y_n} w(y_n)\, dy_n = 0 \tag{25}$$

because of the Y_n-periodicity of w. Hence the left-hand side of (24) can be expressed as

$$\int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_n} v_1\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_{n-1}}\right) (\nabla_{y_n} \cdot r)\left(\frac{x}{\varepsilon_n}\right) dx$$

$$= \int_\Omega u^\varepsilon(x)\, v_1\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_{n-1}}\right) \nabla \cdot r\left(\frac{x}{\varepsilon_n}\right) dx.$$

Integrating by parts with respect to x the above equals

$$-\int_{\Omega}\nabla u^{\varepsilon}(x)\cdot v_{1}\left(x,\frac{x}{\varepsilon_{1}},\frac{x}{\varepsilon_{2}},...,\frac{x}{\varepsilon_{n-1}}\right)r\left(\frac{x}{\varepsilon_{n}}\right)$$

$$+u^{\varepsilon}(x)\nabla_{x}v_{1}\left(x,\frac{x}{\varepsilon_{1}},\frac{x}{\varepsilon_{2}},...,\frac{x}{\varepsilon_{n-1}}\right)\cdot r\left(\frac{x}{\varepsilon_{n}}\right) \tag{26}$$

$$+\sum_{k=1}^{n-1}u^{\varepsilon}(x)\varepsilon_{k}^{-1}\nabla_{y_{k}}v_{1}\left(x,\frac{x}{\varepsilon_{1}},\frac{x}{\varepsilon_{2}},...,\frac{x}{\varepsilon_{n-1}}\right)\cdot r\left(\frac{x}{\varepsilon_{n}}\right)dx.$$

Passing to the multiscale limit in the first term of (26) using (22) we arrive u.t.s. at

$$-\int_{\Omega}\int_{Y^{n}}\left(\nabla u(x)+\nabla_{y_{1}}u_{1}(x,y_{1})+...+\nabla_{y_{n}}u_{n}(x,y^{n})\right)\cdot v_{1}\left(x,y^{n-1}\right)r(y_{n})\,dy^{n}dx$$

$$=-\int_{\Omega}\int_{Y^{n}}\nabla_{y_{n}}u_{n}(x,y^{n})\cdot v_{1}\left(x,y^{n-1}\right)r(y_{n})\,dy^{n}dx, \tag{27}$$

where all but the last term have vanished due to (25).

Moreover, (21) means that the second term of (26) u.t.s. approaches

$$-\int_{\Omega}\int_{Y^{n}}u(x)\nabla_{x}v_{1}\left(x,y^{n-1}\right)\cdot r(y_{n})\,dy^{n}dx$$

$$=-\int_{\Omega}\int_{Y^{n-1}}u(x)\nabla_{x}v_{1}\left(x,y^{n-1}\right)dy^{n-1}\cdot\left(\int_{Y_{n}}r(y_{n})\,dy_{n}\right)dx=0$$

as a result of (25).

It remains to investigate the last term of (26). We write

$$\sum_{k=1}^{n-1}\int_{\Omega}u^{\varepsilon}(x)\varepsilon_{k}^{-1}\nabla_{y_{k}}v_{1}\left(x,\frac{x}{\varepsilon_{1}},\frac{x}{\varepsilon_{2}},...,\frac{x}{\varepsilon_{n-1}}\right)\cdot r\left(\frac{x}{\varepsilon_{n}}\right)dx$$

$$=\sum_{k=1}^{n-1}\frac{\varepsilon_{n}}{\varepsilon_{k}}\int_{\Omega}u^{\varepsilon}(x)\varepsilon_{n}^{-1}\nabla_{y_{k}}v_{1}\left(x,\frac{x}{\varepsilon_{1}},\frac{x}{\varepsilon_{2}},...,\frac{x}{\varepsilon_{n-1}}\right)\cdot r\left(\frac{x}{\varepsilon_{n}}\right)dx. \tag{28}$$

If $v = v_1 v_2 \in G_n$ for some $v_2 \in C^\infty_\sharp (Y_n) /\mathbb{R}$ that is not identically zero, then $\alpha_i v_1 r_i \in G_n$ for

$$\alpha_i = \sup_{y_n \in Y_n} |v_2| / \sup_{y_n \in Y_n} |r_i|$$

and $i = 1, 2, ..., N$, see Remark 16. Moreover, (23) holds with p replaced with $p + 1$ if $v \in F_n$ is substituted with $\nabla_{y_{k,i}} v$, $k = 1, 2, ..., n-1$, $i = 1, 2, ..., N$. From the definition of F_n it then follows that $\delta^{-1}\nabla_{y_k} v \in F_n$ for $k = 1, 2, ..., n-1$ if $v \in F_n$. Hence for

$$\alpha = \min_{i=1,..,N} \alpha_i$$

it holds that $\alpha\delta^{-1}\nabla_{y_{k,i}} v_1 r_i \in F_n$ and $\{\varepsilon_n^{-1}\nabla_{y_k} v_1(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_{n-1}}) \cdot r(\frac{x}{\varepsilon_n})\}$ is thus bounded in $H^{-1}(\Omega)$ for $k = 1, 2, ..., n-1$ by Theorem 15. Observing that $\{u^\varepsilon\}$ is assumed to be bounded in $H^1_0(\Omega)$ this means that

$$\left| \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k} \int_\Omega u^\varepsilon(x)\, \varepsilon_n^{-1}\nabla_{y_k} v_1 \left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_{n-1}}\right) \cdot r\left(\frac{x}{\varepsilon_n}\right) dx \right|$$

$$\leq \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k} \|u^\varepsilon\|_{H^1_0(\Omega)} \left\| \varepsilon_n^{-1}\nabla_{y_k} v_1 \left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_{n-1}}\right) \cdot r\left(\frac{x}{\varepsilon_n}\right) \right\|_{H^{-1}(\Omega)}$$

$$\leq \alpha' \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k}$$

for some $\alpha' > 0$. Hence, all the terms in the sum (28) vanish as $\varepsilon \to 0$ as a result of the separatedness of the scales. Then (27) is all what remains after passing to the limit in (26) and integrating by parts we obtain

$$\int_\Omega \int_{Y^n} u_n(x, y^n)\, v_1\left(x, y^{n-1}\right) \nabla_{y_n} \cdot r(y_n)\, dy^n dx$$

$$= \int_\Omega \int_{Y^n} u_n(x, y^n)\, v_1\left(x, y^{n-1}\right) v_2(y_n)\, dy^n dx.$$

The proof is complete. ■

Remark 19 *The class F_n may seem restricted but is in fact quite rich and does not mean any complication while e.g. applying multiscale convergence to homogenization problems. A discussion of this class of functions is found in Remark 2.13 and in connection with Lemma 3.11 in [2].*

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