How Dennis and I Intersected

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Abstract: This paper is a slightly modified version of the talk I gave at the Dennisfest. After some reminiscences about how things were in the good old days, including how Dennis' work and mine were somewhat transverse ;-) I will review and update my ancient, unpublished (now arXived) paper with Mike Schlessinger on the deformation theory of rational homotopy types. Then, I will return briefly to another transverse intersection of my work with Dennis'.

CONTENTS

1. Introduction

Dennis refers to me at times as Mr. A_∞. Perhaps I should call him Mr. E_∞ since his work has been so much more diverse than mine. Indeed, E_∞ implies a lot more structure than A_∞, but we have overlapped in small but significant ways.

One of the advantages of being of a certain age is that I am free to reminisce about how things were in the "good old days" - or at least how I remember them.

At one time, our interaction might be pictured

$$\chi$$

That intersection is primarily Dennis' minimalist approach to rational homotopy theory which, for some of us, is more accessible and hands on than Quillen's 'maximalist', that is, categorical. On the other hand, Quillen's version does highlight certain functors which were hidden for a while in Dennis'.

I mourn the tragic loss of Dan even as I rejoice in Dennis' presence.

One of the key aspects of his models of rational homotopy types, though overlooked at the time, was that his models were of the form of a free graded commutative algebra on the suspension of an L_∞-coalgebra. The L_∞ structure was in essence the differential on the algebra.

Perhaps this structure or the problem of commutative cochains is what led to his interest in ∞ -structures. Commutativity over the integers destroys associativity, but only up to homotopy, in fact strong homotopy, i.e. an A_∞-structure. (In my talk, I said: "Maybe Dennis would care to set the record straight now, continuing his fine imitation of Gel'fand.", referring to his ability to take over the 'podium' of the seminar as called for.)

Some people refer to Sullivan models, meaning the minimal ones, but sometimes there are advantages to being not so efficient, as we will see.

2. Manifolds?

That graphic above (the large χ) is a deliberate double entendre. Not only are our interests transverse, but I have never used manifold structures.

What never?

Well, hardly ever.

Of course the "Characteristic Classes" book with Milnor starts off with a treatment of manifolds, but that part Milnor wrote years after the original lectures, for which I was just the scribe - a very fortunate scribe.

In "Differential homological algebra and homogeneous spaces," with Dale Husemoller and John Moore [13], the emphasis was on the first topic, though motivated by the example of the second.

One place where I really was interested in manifolds was "Manifolds of the homotopy type of (non-Lie) groups" [28], which I owe in part to Dennis in his 1970 MIT notes [30] for the idea of breaking apart a space into its p-primary components and to Zabrodsky [33] for the idea of mixing together p-primary components from different spaces.

3. A point of intersection - Poincaré duality spaces

One paper I wrote was devoted to rational Poincaré duality spaces [27], motivated in part by a duality at the level of Massey products.

My favorite example is

$$Sp(5)/SU(5).$$

Its cohomology algebra is isomorphic to that of

$$S^6 \times S^{25} \,\#\, S^{10} \times S^{21},$$

where $\#$ denotes connected sum. But that space has no non-trivial Massey products, while $Sp(5)/SU(5)$ has two non-trivial 3-fold Massey products $< \ , \ , \ >$ (as follows from computations by Borel [2]). If we denote the algebra generators by their dimensions, then $< 6, 6, 10 >= 21$ and $< 6, 10, 10 >= 25$. The duality also exhibits the fundamental slide relation [18]:

$$< 6, 6, 10 > 10 = 6 < 6, 10, 10 > .$$

There is no homotopy type with this cohomology algebra and only one non-trivial Massey product.

Recently Larry Taylor [31] has found a much better way to exhibit the duality at the level of Massey products with clever control of the indeterminacy. The chief idea is to convert the Massey triple product $< x, y, z >$ to a linear operation $< x, ?, z >$, defined for all y such that $xy = 0 = yz$ by fixing x and z. The usual slide relation

$$w < x, y, z >= \pm < w, x, y > z$$

then gives an explanation for the pairs of Massey product in the above example.

4. Quillen's approach to rational homotopy theory

Quillen emphasized differential graded Lie algebras (dgLs) as well as differential graded commutative coalgebras (dgccs): for a simply connected space X, there is a dgL L_X such that $H(L_X)$ is isomorphic as graded Lie algebra to the rational homotopy groups of the loop space of X with respect to the Samelson product. Baues and Lemaire [1] gave a construction of L_X as a free graded Lie algebra on the despsuspension of the reduced homology of X. Hain [7] studied this construction from the point of view of Chen's connections. Simple connectivity in the dg algebra setting mean $A^0 = \mathbf{Q}$ and $A^1 = 0$.

Let $\mathcal{L}$ denote the free graded Lie algebra functor on the suspension of a graded vector space. For a simply connected rational Poincaré duality space Y with fundamental class $\mu \in H^N$, choose a self-dual basis $\{x_1, \ldots, x_q, x^q, \ldots, x^1\}$ for degrees $k : \ 0 < k < N$, so that $x_i x^i = \mu$. I was able to show:

Theorem 1. [27] *There is a differential graded Lie algebra model $\mathcal{L}(H(Y))$ with*

$$d(\mu) = 1/2 \ \Sigma[x_i, x^i].$$

Equivalently $Y = X \cup e^N$ where e^N is attached by ordinary Whitehead products (not iterated) with respect to some basis of the rational homotopy groups of X.

5. WORK WITH STEVE HALPERIN

Disclaimer The results here are stated in the hopes of conceptual accessibility, *not* in maximal generality; i.e. being simply connected can often be generalized to being nilpotent.

Sullivan models sometimes refer to minimal ones, but Halperin and I [8] made good use of perturbing the minimal model of a formal space to a non-minimal model of a general simply connected rational space (henceforth, for the rest of the paper, called *space*).

Let $\mathcal{H}$ denote a graded algebra such that for some space or dgca (A, d_A) we have an isomorphism $\phi : \mathcal{H} \to H((A, d_A))$. Given this, we constructed a canonically filtered *non-minimal* Sullivan model for (A, d_A) by perturbing the minimal model for the graded algebra $\mathcal{H}$. (Specification of ϕ plays a role in the classification to come.) In fact, the minimal model for $\mathcal{H}$ is a purely algebraic construct, closely related to a multiplicative resolution of $\mathcal{H}$ by free commutative algebras, sometimes called the Koszul-Tate resolution (and extended to the graded case by Jozefiak) [25, 24]. The filtration corresponds to the resolution degree.

We refer to the total degree minus resolution degree as the *weight*.

By a *perturbed differential D*, we mean a derivation that can be written as $D = d + p$ where p lowers weight by at least 2. That is, $D = d + d_2 + d_3 + \cdots$, where d_i lowers weight by i.

A *perturbation* could also be called a *deformation*. Conversely, deformations can often be studied as perturbed differentials.

If S and T are simply connected (more generally, rationally nilpotent) spaces and $f : H^*(S; \mathbf{Q}) \to H^*(T; \mathbf{Q})$ is an isomorphism of algebras, we used the filtered models to construct a sequence of classes, $O_n(f)$, to be called *obstructions*.

Theorem 2. [8] *The isomorphism f can be realized by a rational homotopy equivalence if and only if the obstructions $O_n(f)$ all vanish.*

The next obvious question was "how different can rational homotopy types be if the cohomology algebras agree?".

For example, Theorem 2 is one of many ways that it can be seen that:

Theorem 3. [5, 22, 8] *For $n > 1$ and $H^i = 0$ for $i < n$ and for $i \geq 3n - 1$, then H is intrinsically formal, i.e., there is only one homotopy type with the given cohomology algebra.*

For rational Poincaré duality spaces, the hypothesis can be extended to $H^i = 0$ for $i < n$ and for $i \geq 4n - 1$.

We gave several other illustrative examples with specific computations, but not a general answer, except to remark that our procedure identifies the set of homotopy types with given rational cohomology algebra $\mathcal{H}$ in terms of the set of homology preserving deformations of the minimal model of $(\mathcal{H}, 0)$.

6. Work with Mike Schlessinger

Sometime after my work with Halperin, Mike Schlessinger and I [25, 24] made contact mathematically - geographics were not a problem; our offices at UNC were across the hall from each other. As I recall, it happened like this:

Something led us to compare notes on the Koszul-Tate resolution of

$$\mathbf{Q}[x_1, x_2]/x_i x_j = 0,$$

which Mike regarded as pure algebra and I regarded as the cohomology of $S^2 \vee S^2$. Similarly, Mike spoke the language of classical deformation theory while I used that of homological perturbation theory. Fortunately, we each learned to speak the other's language. (In contrast, it took decades for Murray Gerstenhaber and me to realize we had common techniques, though our breakthrough papers [6, 26] had each appeared in 1963 - it was a very good year.)

It was from Mike that I learned the yoga: deformation problems are controlled by dg Lie algebras. Later this yoga was extended to control by L_∞-algebras so as to include the appropriate L_∞-uniqueness.

In order to classify homotopy types, we first chose to use a particular *non-minimal* Sullivan model of the formal space with the given cohomology algebra $\mathcal{H}$, using the adjointness between dgcas and dg Lie *co*algebras.

Quillen's approach to rational homotopy theory is at the chain rather than cochain level. In [23], he constructs a functor from simply connected rational spaces to dg Lie algebras and then applies a generalized Chevalley-Eilenberg construction $\mathcal{C}$ to obtain a dgc coalgebra model. (The functor $\mathcal{C}$ also appears in work of J. C. Moore [20] the following year, I believe independently.) The functor $\mathcal{A}(\ ) = Hom\,(\mathcal{C}(\ ), \mathbf{Q})$ from dg Lie algebras to dga's fits more readily into a traditional cohomological exposition, but the usual subtleties of the Hom functor necessitate the detour into the more natural differential graded coalgebras. We will still need to have recourse to $\hat{\mathcal{C}}$, the completion of $\mathcal{C}$ with respect to a suitable filtration.

Apply the cofree graded Lie coalgebra $\mathcal{L}^c$ to the shifted augmentation ideal of $\mathcal{H}$, with the differential $d_{\mathcal{L}^c}$ defined by extending the multiplication in $\mathcal{H}$ as a coderivation (after shifting). Then form the free graded commutative algebra $\mathcal{A}(s\mathcal{L}^c\mathcal{H})$ with differential d determined by $d_{\mathcal{L}^c}$ and the cobracket. This construction defines a rational dgca A which is manifestly formal. For a general rational space/dgca A, the argument in my work with Halperin can be used to perturb $d_{\mathcal{A}(s\mathcal{L}^c s\mathcal{H})}$ to give a model for (A, d_A).

Although perturbations are always of total degree 1, they sit naturally in a sub-dg Lie algebra of $Der(\mathcal{A}(s\mathcal{L}^c\mathcal{H}))$. We will also have need for perturbations of the differential of $\mathcal{L}^c s\mathcal{H}$, which sit naturally in a sub-dg Lie algebra of $Coder(\mathcal{L}^c s\mathcal{H})$. These we formalize with a definition.

Definition 4. *Denote by $Pert\mathcal{A}(s\mathcal{L}^c s\mathcal{H})$ the dg Lie algebra of weight decreasing derivations of $\mathcal{A}(s\mathcal{L}^c s\mathcal{H})$. Similarly denote by $Pert(\mathcal{L}^c s\mathcal{H})$ the dg Lie algebra of weight decreasing coderivations of $\mathcal{L}^c\mathcal{H}$, where a perturbation of $\mathcal{L}^c s\mathcal{H}$ is a Lie coderivation p of the same degree as d such that p increases cobracket length by at least 2 and $(d + p)^2 = 0$.*

Theorem 5. *Let $\mathcal{H}$ be a simply connected graded commutative algebra of finite type. The set of augmented homotopy types of dgca's*

$$(A, \phi : \mathcal{H} \approx H(A))$$

is in 1–1 correspondence with the path components of $\hat{\mathcal{C}}(Pert\mathcal{A}(s\mathcal{L}^c s\mathcal{H}))$.

A graded coalgebra map from the rationals $\mathbf{Q}$ as a coalgebra serves as a point and a path is an appropriate kind of homotopy between points, using an 'interval' dgc coalgebra. (There is indeed some subtlety there [24].)

The easy part of the theorem is the observation as in [8] that two perturbed models have the same augmented homotopy type provided they are related by an automorphism (of the underlying gca) of the form: Id plus "terms which decrease weight".

The hard part is to show that a homotopy gives rise to such an equivalence.

Now $\mathcal{A}(s\mathcal{L}^c s\mathcal{H})$ is rather large, so we are interested in comparing perturbations of $\mathcal{L}^c s\mathcal{H}$ with the corresponding induced perturbations of $\mathcal{A}(s\mathcal{L}^c s\mathcal{H})$, for the freeness of the latter implies that a perturbation of $\mathcal{L}^c(s\mathcal{H})$ generates one in $\mathcal{A}(\mathcal{L}^c(s\mathcal{H}))$ with respect to the induced filtration.

Theorem 6. *For simply connected $\mathcal{H}$ of finite type, the natural dg Lie algebra map $Pert(\mathcal{L}^c s\mathcal{H})) \to Pert\mathcal{A}(s\mathcal{L}^c s\mathcal{H})$ is a homology isomorphism.*

6.1. **Classifying space versus moduli space.** Theorem 5 indicates $\hat{\mathcal{C}}(Pert\mathcal{A}(s\mathcal{L}^c(\mathcal{H})))$ is rather like a model of a classifying space, an idea we will reinforce later. More to the point right now, the set of path components can be regarded as a topological space, but then we need to make a further identification by the action of $Aut\ \mathcal{H}$, the group of automorphisms of $\mathcal{H}$, though the quotient can fail to be even Hausdorff.

On the other hand, since perturbations satisfy the defining equation (equivalent to the Maurer-Cartan equation), we can change point of view to that of algebraic geometry: varieties and moduli spaces.

Definition 7. *For any dg Lie algebra L, we define the* incorporated *dg Lie algebra $(L[d], ad\ d)$ by adjoining a single new generator also called d with the obvious relations. The* variety $V_L \subset L[d]$ *is defined as*

$$\{p \in L^1 \mid (d + p)^2 = 0\}.$$

If L is complete with respect to the L^0 filtration, the gauge action *is*

$$p \mapsto (exp\ b)(d + p) - d \quad for\ b \in L^0.$$

Neither the variety V_L nor the gauge action is uniquely determined by the problem, but the quotient by the gauge action is (up to appropriate isomorphism).

Implicit in the use of quasi-isomorphisms, even for strict dg Lie algebras, is the fact that L_∞-morphisms respect the deformation and moduli space functors.

For the classification problem, we want V_L for $L = Pert(\mathcal{L}^c s\mathcal{H})$. A simple example is $\mathcal{H} = H(S^2 \vee S^2 \vee S^5)$ [8] §6.6. Let the generators be x_2, y_2, z_5. For dimensional reasons V_L is $\mathbf{Q}^2$. Finally, $Aut\, \mathcal{H} = GL(2) \times GL(1)$ acts on $H^1(L)$ so as to give two orbits: $(0,0)$ and the rest. The space of rational homotopy types is

$$\odot \quad \bullet$$

meaning the non-Hausdorff two-point space with one open point and one closed. We can also represent this as

$$\bullet \to \bullet$$

meaning one orbit is a limit point of the other.

6.2. L_∞-structure on $H(L)$.

Since $Pert(\mathcal{L}^c(\mathcal{H}))$ is still rather large for effective computation, it is helpful to have at hand the L_∞-structure induced on $H(L)$ for any dg Lie algebra L. Even though $H(L)$ inherits a strict dgLie structure (with $d = 0$), there is in general a highly non-trivial L_∞-structure such that L and $H(L)$ are equivalent as L_∞ algebras. This transfer of structure began in the case of dg associative algebras with the work of Kadeishvili [14]. The definitive treatment in the Lie case (which is more subtle) is due to Huebschmann [11, 12].

As we have recalled, the structure of a dg Lie algebra is captured by the Chevalley-Eilenberg complex $\mathcal{C}(sL)$. We can construct a minimal model of $\mathcal{C}(sL)$ which is a perturbation of $\mathcal{C}(sH(L))$, i.e.

$$d = 0 + [\ ,\ ]_{H(L)} + \text{terms of bracket length} > 2.$$

These higher order terms correspond precisely to the higher order brackets of an L_∞-algebra structure on $H(L)$.

In the case in which the dg Lie algebra is $Pert(\mathcal{L}^c(\mathcal{H}))$, some of these higher order terms can be related to Massey products and attaching maps (see the next section). Although Massey products suffer from having restricted domains and considerable indeterminacy, when defined, they represent the higher order brackets on $H(Pert(\mathcal{L}^c(\mathcal{H})))$, which are better defined.

7. Examples and computations

There are some easy consequences for the classification from conditions on $H(Pert(\mathcal{L}^c(\mathcal{H})))$ which follow the route of classical deformation theory: results that depend on vanishing of $H(Pert(\mathcal{L}^c(\mathcal{H})))$ in low degrees. Let's turn now to computations of manageable size.

$Pert(\mathcal{L}^c(\mathcal{H}))$ can be identified with a subspace of $Hom\,(\mathcal{L}^c(\mathcal{H}), \mathcal{H})$ and hence each perturbation as a sum of elements of $Hom\,(\bar{\mathcal{H}}^{\otimes k+2}, \mathcal{H})$ which lower the internal $\mathcal{H}$-degree by k. This permits an interpretation in terms of attaching maps, for example, in case the formal space is a wedge of spheres

$$X = \bigvee S^{n_i}.$$

The dual of the algebra of rational homotopy groups $\pi_*(\Omega X) \otimes \mathbf{Q}$ is then isomorphic to $\mathcal{L}^c(H(X))$.

Recall that attaching a cell by an ordinary Whitehead product $[S^p, S^q]$ means the cell carries the product cohomology class. Massey (and Uehara) [32, 19] introduced Massey products in order to detect cells attached by iterated Whitehead products such as $[S^p, [S^q, S^r]]$.

The Massey product interpretation is particularly helpful when the perturbation consists of only one term θ_k, if we identify such a perturbation θ_k with a homomorphism $\theta_k : H^{\otimes k+2} \to H$,

Adopting the language of deformation theory, we refer to a candidate perturbation θ as an *infinitesimal deformation*. If $(d + \theta)^2 \neq 0$, we say θ is *obstructed*.

One of the simplest examples of this phenomenon is $X = S^3 \vee S^3 \vee S^8 \vee S^{13}$. We use a dual notation and write:

$$\theta = [x_1, [x_1, x_2]]\partial x_8 + [x_2, [x_1, x_8]]\partial x_{13},$$

then $[\theta,\ \theta]$ is not cohomologous to 0.

In terms of cells, this means we cannot attach simultaneously both e^8 to realize $\langle x_1, x_1, x_2 \rangle$ **and** attach e^{13} to realize $\langle x_2, x_1, x_8 \rangle$.

The first example of "continuous moduli", i.e., of a one–parameter family of homotopy types, was first mentioned to us by John Morgan (cf. [21, 5]). Let $H = H(S^3 \vee S^3 \vee S^{12})$, so that the attaching map α is in $\pi_{11}(S^3 \vee S^3) \otimes \mathbf{Q}$ which is of dimension 6, while $Aut\,H = GL(2) \times GL(1)$ is of dimension 5. Alternatively, the

space of 5–fold Massey products $H^{\otimes 5} \to H$ is of dimension 6 and so distinguishes at least a 1- parameter family.

8. Extension to fibrations

The construction of a rational homotopy model for a classifying space for fibrations with given fibre was sketched briefly by Sullivan [29]. The classification of rational homotopy types by homotopy classes of dg coalgebra maps of $\mathbf{Q} \to \hat{C}(\mathcal{L})$ generalizes naturally to classification of rational homotopy fibrations by maps of the base dgcc to $\hat{C}(\mathcal{L})$ where $\mathcal{L}$ is the dg Lie coalgebra corresponding to self homotopy equivalences of the fibre F.

8.1. **Algebraic model of a fibration.** For simplicity of exposition, let's assume enough conditions so we can deal with the algebraic model of a fibration as a twisted tensor product. Consider topological fibrations, i.e., maps of spaces

$$\mathcal{F} \to \mathcal{E} \xrightarrow{p} \mathcal{B}$$

such that $p^{-1}(*) = \mathcal{F}$ and p satisfies the homotopy lifting property. We have not only the corresponding maps of generalized differentil form cgda's

$$A^*(\mathcal{B}) \to A^*(\mathcal{E}) \to A^*(\mathcal{F})$$

but $A^*(\mathcal{E})$ is an $A^*(\mathcal{B})$-algebra and, under reasonable assumptions, there is an $A^*(\mathcal{B})$-derivation D on $A^*(\mathcal{B}) \otimes A^*(\mathcal{F})$ and an equivalence of cdgas between

$$A^*(\mathcal{E}) \text{ and } (A^*(\mathcal{B}) \otimes A^*(\mathcal{F}), D).$$

By a sleight of hand, let us rename these dgcas F, E and B. Both the algebra structure and the differential may be twisted, but if we *assume that F is free as a cga*, e.g. a model for F, then it follows that E is strongly equivalent to

$$B \xrightarrow{i} B \otimes F \xrightarrow{p} F$$

with the $\otimes$–algebra structure. The differential in $E = B \otimes F$ then has the form

$$d_\otimes + \tau,$$

where

$$d_\otimes = d_B \otimes 1 + 1 \otimes d_F.$$

Let $\bar{B}$ denote the augmentation ideal of B. The *"twisting term"* τ lies in the sub-dg Lie algebra of $Der(B \otimes F)$ consisting of those derivations of $B \otimes F$ which vanish on B and reduce to 0 on F via the augmentation.

8.2. **Twisting cochains as perturbations.** Assuming B is connected, τ does not increase the F–degree so we regard τ as a perturbation of $d_\otimes$ on $B \otimes F$ with respect to the filtration by F degree. The twisting term must satisfy the integrability conditions (aka the Maurer-Cartan equation):

$$(1) \qquad (d+\tau)^2 = 0 \;\; \text{or} \;\; [d,\tau] + \frac{1}{2}[\tau,\tau] = 0.$$

To obtain strong equivalence classes of fibrations, we pass to the quotient by the action of automorphisms θ of $B \otimes F$ which are the identity on B and reduce to the identity on F via augmentation. Assuming B is connected, then $\theta - 1$ must take F to $\bar{B} \otimes F$ and therefore lowers the degree in F, so that $\phi = \log \theta = \log(1+(\theta-1))$ exists; thus $\theta = exp\,(\phi)$ for some ϕ in $Der^0(F, B \otimes F)$.

If we set $\mathcal{L}(B,F) = Der\,(F, \bar{B} \otimes F)$, then for B connected, we may repeat much of what we did for the case $B = \mathbf{Q}$. Dualizing with impunity, we consider

$$\mathcal{A}(s\mathcal{L}(B,F)).$$

Theorem 8. *For connected B and reasonable F, free of finite type, the set of strong fibre homtopy equivalence classes of fibrations*

$$B \to E \to F$$

is in bijection with the set of homotopy classes of maps $\mathcal{A}(s\mathcal{L}(B,F)) \to B$.

Thus $\mathcal{A}(s\mathcal{L}(B,F))$ is very much like a model for the classifying space. The problem is that it has terms of negative degree, so, presto chango, we truncate it appropriately.

Mike, of course, went further and provided the moduli space specification.

Theorem 9. [24] *For fixed $\mathcal{H}$, the set of homotopy types of pairs $(X, \phi : \mathcal{H} \simeq H(X))$ can be represented as the quotient V/G of a (perhaps infinite dimensional) conical rational algebraic variety V modulo a pro-unipotent (algebraic) group action G.*

9. A STRING OF INTERSECTIONS

Dennis and I have each dealt with homotopy theory arising from strings. For Dennis, there is his work with Chas [3] which gave rise to a major industry of *string topology*, whereas in my work with Kajiura [15], [17],[16] and in work of Hoefel [10], [9], there is translation of the physicists *string field theory* into homotopical algebra. Earlier, I had planned to compare and contrast the various ∞-algebra structures that arise in these different approaches, but I find that there are now established a plethora of such structures and considerable further work in progress, e.g. the minimal model of BV_∞-structures of Drummond-Cole and Vallette [4].

Of course, Dennis and his progeny work in the context of manifolds or at least Poincaré Duality, while I impose no such constraint. A more important difference is that string topology is developed as a homology or even a chain level structure, whereas string field theory is contravariant, in terms of functions or forms on a space of strings.

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