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Curved A_∞ -Algebras and Chern Classes

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Abstract: We describe two constructions giving rise to curved A_∞ -algebras. The first consists of deforming A_∞ -algebras, while the second involves transferring curved dg structures that are deformations of (ordinary) dg structures along chain contractions. As an application of the second construction, given a vector bundle on a polyhedron X , we exhibit a curved A_∞ -structure on the complex of matrix-valued cochains of sufficiently fine triangulations of X . We use this structure as a motivation to develop a homotopy associative version of Chern-Weil theory.

Keywords: A_∞ -algebra, Chern-Weil theory.

1. INTRODUCTION

Developing discrete analogues of all fundamental differential geometric structures is desirable for several reasons. Such discretization techniques provide efficient tools for explicit numerical computations needed in applications. They also lead to simple and transparent proofs of combinatorial analogues of many classical results in differential geometry (see e.g. R. Forman’s work on discrete Morse theory [8]) or to new results obtained by passing to the “continuum limit”, as is the case with constructive quantum gauge field theory based on lattice approximations (see the survey [19] and the references therein.)

In this paper, we propose notions of connection and curvature on a piecewise smooth complex vector bundle over a polyhedron based on the notion of a

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curved A_∞ -algebra and derive explicit combinatorial formulas for the real Chern classes of the bundle in terms of the curvature. Our starting point is the recent observation of D. Sullivan ([31, Appendix], see also [16]) and [6]) that the complex of simplicial cochains on a polyhedron possesses a canonical A_∞ -structure.

We recall that A_∞ -algebras [30] are a generalization of differential graded (dg) algebras in which the associativity condition is replaced by an infinite sequence of identities involving higher “multiplications”. Curved A_∞ -algebras [17] (or weak A_∞ -algebras in the terminology of [21]) are a natural generalization of both A_∞ -algebras and curved dg algebras [29]. The latter are graded associative algebras endowed with a degree 1 derivation whose square equals the commutator with a fixed closed element of degree 2.

Let E be a complex vector bundle over a finite polyhedron X . For each piecewise smooth idempotent P representing the K -theory class of E we construct a canonical curved A_∞ -structure on the space of matrix-valued simplicial cochains on a sufficiently fine subdivision of X . This structure is obtained in Section 3 by transferring a curved dg algebra structure on the space of piecewise smooth matrix-valued differential forms on X using homological perturbation theory; its elements are given by convergent infinite series.

Various algebraic generalizations of the classical Chern-Weil theory have appeared during the last few decades. We mention, among others, the Connes-Karoubi-Chern character in cyclic homology ([20],[4]) in the context of associative algebras and Huebschmann’s work [13] in the context of Lie-Rinehart algebras. We develop a homotopy associative extension of the Chern-Weil construction for curved dg algebras from [29] in which the input data is a curved A_∞ -algebra equipped with certain additional structure. As an application, in Section 4 we establish our main result which can be stated as follows.

Theorem. There exist explicit formulae for simplicial cochains representing the real Chern classes of the bundle E in terms of the curved A_∞ -structure on matrix-valued cochains discussed above.

We point out that the problem of discretizing the vector bundle itself is not treated in this paper. We conjecture that it is possible to construct a curved A_∞ -algebra out of a purely combinatorial data representing the bundle and then obtain the characteristic classes from this algebra. We also hope that the notions

of connection and curvature presented in this work might be useful in developing an alternative approach to lattice gauge theory and to the problem of constructing continuous quantum gauge fields.

We note that formulas for simplicial representatives of the integral Chern classes of a principal bundle have been obtained in [28] by completely different methods, using a particular model of the classifying space of the general linear group.

Let us describe the contents of the paper in more detail.

In Section 2, after recalling some basic definitions we describe a general procedure of deforming A_∞ -algebras to curved ones using infinite series along a “connection”, i.e. an arbitrary element of degree 1. We use the opportunity to derive analogous results for L_∞ -algebras, the homotopy associative generalization of Lie dg algebras, building on an observation from [15].

In Section 3, we begin with a brief review of homological perturbation theory and then show how to use it to transfer curved dg structures that are deformations of ordinary dg algebras in the above sense along contractions. We also generalize the sum over rooted planar tree formulas of Kontsevich and Soibelman [23] to this setting. As an application, we obtain the curved A_∞ -structure on matrix-valued simplicial cochains discussed above.

In Section 4, we develop an algebraic homotopy associative Chern-Weil theory. We introduce the notion of a Chern-Weil triple, a curved A_∞ -algebra together with a map into a chain complex satisfying certain symmetry properties. We define the Chern character of a Chern-Weil triple, show that it is natural with respect to an appropriate class of morphisms and finally prove the theorem stated above.

2. CURVED A_∞ - AND L_∞ -STRUCTURES. DEFORMATIONS

2.1. Basic definitions. Throughout this paper, we will work with non-negatively graded vector spaces over a fixed field \mathbf{k} that is either the real or the complex numbers. By a chain complex we always mean a graded vector space equipped with a differential of degree 1. We write $V^{\otimes k}$ for the k -th (*graded*) tensor power of a graded vector space $V = \bigoplus_p V_p$ so that $V^{\otimes 0} = \mathbf{k}$. When applying graded maps

to graded objects, we always use the *Koszul sign rule*, by which we mean the appearance of a sign $(-1)^{\deg(a)\deg(b)}$ when switching two adjacent graded symbols a and b .

Definition 2.1. A **curved A_∞ -structure** on a graded vector space V is given by a family of linear maps

$$m_k : V^{\otimes k} \rightarrow V, \quad k = 0, 1, 2, \dots$$

satisfying for every $n \geq 0$ the identity

$$(2.1) \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+t+1} (\mathbf{1}_V^{\otimes r} \otimes m_s \otimes \mathbf{1}_V^{\otimes t}) = 0.$$

The map m_k is assumed to have degree $2 - k$.

It appears that the term curved A_∞ -algebra V has been used for first time in [27]. In the special case when the map $m_0 : \mathbf{k} \rightarrow V_2$ is 0 we obtain the usual notion of A_∞ -algebra (see [30],[17] and the surveys [12] and [21]). We will (by a slight abuse of notation) denote the degree 2 element of V that is the image of $1 \in \mathbf{k}$ under m_0 also by m_0 and we will refer to it as the **curvature** of the curved A_∞ -algebra V . This terminology is motivated by the following example. When $m_k = 0$ for all $k > 2$ we obtain the definition of **curved differential graded (dg) algebra** [29]. In this case, the first four identities in Eq. (2.1) read

$$(2.2) \quad m_1 m_0 = 0,$$

$$(2.3) \quad m_1 m_1 = m_2(m_0 \otimes \mathbf{1}) - m_2(\mathbf{1} \otimes m_0),$$

$$(2.4) \quad m_1 m_2 = m_2(m_1 \otimes \mathbf{1}) - m_2(\mathbf{1} \otimes m_1),$$

$$(2.5) \quad m_2(m_2 \otimes \mathbf{1}) = m_2(\mathbf{1} \otimes m_2).$$

Eq. (2.2) can be interpreted as an abstract Bianchi identity, Eq. (2.3) says that square of the “covariant derivative” m_1 equals to a commutator with the curvature, (2.4) expresses the fact that m_1 is a (graded) derivation and the last identity amounts to the associativity of the algebra.

In this paper, we will be mostly interested in examples of curved dg algebras arising from vector bundles.

Example 2.2. Let E be a real or complex vector bundle with connection ∇ over a smooth manifold M and let $\text{End}(E)$ denote the bundle of endomorphisms of E . The graded space $\Omega^\bullet(M, \text{End}(E))$ of differential forms on M with values in $\text{End}(E)$ becomes a curved dg algebra by taking m_0 to be the curvature of ∇ , m_1 to be the induced connection on $\text{End}(E)$ and m_2 to be the wedge product combined with the composition on $\text{End}(E)$.

Example 2.3. Let A be (an ordinary) dg algebra with differential d and let γ be a fixed degree 1 element in A . It is easy to check that we can *deform* A to a curved dg algebra as follows. Set

$$m_0 = d\gamma + \gamma \cdot \gamma,$$

$$m_1 = d + [\gamma, \cdot]$$

and take m_2 to be the multiplication “ \cdot ” in A (here $[\cdot, \cdot]$ denotes the (graded) commutator with respect to the multiplication in A).

Clearly when the bundle $\text{End}(E)$ is trivial, the curved dg algebra $\Omega^\bullet(M, \text{End}(E))$ from Example 2.2 is a particular example of such a deformation. In general, a related example of a curved dg algebra that is a deformation of a dg algebra may be obtained as follows.

Example 2.4. Endow the bundle E with an inner product and embed it into a trivial bundle \tilde{E} of rank l in order to obtain a matrix idempotent P over the algebra of smooth functions on M (i.e., P is a section of $\text{End}(\tilde{E})$ with $P^2 = P$, that projects onto $E \subseteq \tilde{E}$). Now we can deform the dg algebra of matrix-valued forms $\Omega^\bullet(M, \mathbf{M}_l(\mathbf{k})) \cong \Omega^\bullet(M) \otimes \mathbf{M}_l(\mathbf{k})$ by the degree 1 element $\gamma = PdP$ as above.

Next we describe a well-known alternative description of curved A_∞ -algebras. Let $T(V) = \mathbf{k} \oplus V \oplus V^{\otimes 2} \oplus \dots$ be the *full* tensor coalgebra of the graded vector space V . If $V = \bigoplus_p V_p$ its **suspension** sV is defined by $(sV)_p = V_{p-1}$.

Proposition 2.5. *There is a natural bijection between curved A_∞ -structures on V and square zero coderivations of degree 1 on $T(sV)$.*

The reader is referred to [17, Proposition 1.2] for the proof and the relevant definitions. This proposition allows us to make the following

Definition 2.6. A homomorphism between two curved A_∞ -algebras V and V' is a map $F : T(sV) \rightarrow T(sV')$ of dg coalgebras.

More explicitly, such a homomorphism F is given by a family of linear maps $F_k : V^{\otimes k} \rightarrow V'$ of degree $1 - k$, $k = 0, 1, 2, \dots$ satisfying

$$(2.6) \quad F_1 m_0^V = m_0^{V'},$$

and for every $n > 0$ the identity (cf. [27])

$$(2.7) \quad \begin{aligned} & \sum_{r+s+t=n} (-1)^{r+st} F_{r+t+1} (\mathbf{1}^{\otimes r} \otimes m_s^V \otimes \mathbf{1}^{\otimes t}) \\ &= \sum_{\substack{1 \leq q \leq n \\ i_1 + \dots + i_q = n}} (-1)^w m_q^{V'} (F_{i_1} \otimes \dots \otimes F_{i_q}), \end{aligned}$$

where $w = \sum_{\ell=1}^{q-1} (q - \ell)(i_\ell - 1)$. Note that F_0 necessarily is 0. The morphism F is called **strict** if $F_k = 0$ for every $k > 1$.

Let $F = \{F_k\} : V \rightarrow V'$ and $G = \{G_k\} : V' \rightarrow V''$ be morphisms between curved A_∞ -algebras. Then the p -th component of their composition is given by the formula (cf. [27])

$$(2.8) \quad (GF)_p = \sum_{\substack{1 \leq q \leq p \\ i_1 + \dots + i_q = p}} (-1)^w G_q (F_{i_1} \otimes \dots \otimes F_{i_q}).$$

We proceed to define curved L_∞ -algebras following the sign conventions of [24]. Given graded variables x_1, x_2, \dots, x_n and a permutation $\sigma \in \mathbf{S}_n$, let $\epsilon(\sigma; x_1, \dots, x_n)$ be the Koszul sign obtained when passing from x_1, \dots, x_n to $x_{\sigma(1)}, \dots, x_{\sigma(n)}$. Set

$$(2.9) \quad \chi(\sigma) = \text{sgn}(\sigma) \epsilon(\sigma; x_1, \dots, x_n).$$

We say that $\sigma \in \mathbf{S}_n$ is a $(k, n - k)$ -**unshuffle**, $0 \leq k \leq n$, if $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k + 1) < \dots < \sigma(n)$.

Definition 2.7. A curved L_∞ -structure on a graded vector space V is given by a family of linear maps $l_k : V^{\otimes k} \rightarrow V$ of degree $2 - k$, $k = 0, 1, 2, \dots$ that are *graded antisymmetric* in the sense that

$$l_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \dots, x_k)$$

for all $\sigma \in \mathbf{S}_n$ and $x_1, \dots, x_k \in V$, and moreover satisfy the following generalized Jacobi identities for every $n \geq 0$

$$(2.10) \quad \sum_{k=0}^n (-1)^{k(n-k)} \sum_{\sigma} \chi(\sigma) l_{n-k+1}(l_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}), x_{\sigma(k+1)}, \dots, x_{\sigma(n)}) = 0,$$

where the summation is taken over all $(k, n - k)$ unshuffles σ .

In the case when the curvature $l_0 : k \rightarrow V_2$ is 0, we obtain the definition of an L_∞ -algebra (cf. [25]). When the maps l_k are equal to 0 for all $k > 2$ we obtain the notion of **curved dg Lie algebra**.

Example 2.8. Let L be (an ordinary) dg Lie algebra with differential d and let γ be a fixed element of degree 1 in L . Then it is straightforward to check that we can deform L to a curved dg Lie algebra as follows. Set

$$l_0 = d\gamma + \frac{1}{2}[\gamma, \gamma],$$

$$l_1 = d + [\gamma, \cdot]$$

and take l_2 to be commutator $[\cdot, \cdot]$ in L .

Example 2.9. Let G be Lie group with Lie algebra \mathfrak{g} and let P be a smooth principal G -bundle. The space $\Omega^\bullet(P, \mathfrak{g})$ of \mathfrak{g} -valued differential forms on the total space of P has a natural structure of a dg Lie algebra. One can use a connection on P , i.e. an invariant 1-form in $\Omega^\bullet(P, \mathfrak{g})$, to deform this structure to a curved dg Lie algebra as above. A related and perhaps more interesting example can be obtained by considering the subalgebra of basic (with respect to the action of G) forms on P instead of $\Omega^\bullet(P, \mathfrak{g})$.

As in the A_∞ case, one has an alternative description of curved L_∞ -structures in terms of coderivations. Let $S^k(V)$ denote the k -th (graded) symmetric power of the graded vector space V and let $S(V) = \mathbf{k} \oplus V \oplus S^2(V) \oplus \dots$ be the full symmetric coalgebra of V .

Proposition 2.10. *There is a natural bijection between curved L_∞ -structures on V and square zero coderivations of degree 1 on $S(sV)$.*

It was proved in [25] that there is a bijection between (ordinary) L_∞ -structures on V and square zero coderivations of degree 1 on the em reduced symmetric

algebra on sV . The proof in the curved case is completely analogous; one simply has to replace the reduced symmetric algebra with the full symmetric algebra in order to incorporate the curvature l_0 . As above, this proposition allows us to define homomorphisms of curved L_∞ -algebras.

2.2. Curved A_∞ - and L_∞ -structures as deformations. In this subsection we describe a method of perturbing A_∞ - and L_∞ -algebras to curved ones, generalizing the examples already given in the previous subsection. We first treat the (slightly easier) case of L_∞ -algebras.

Let L be an L_∞ -algebra with maps $\ell_k : L^{\otimes k} \rightarrow L$ and let γ be an element in L of degree 1. For $x_1, \dots, x_k \in L$ consider the (formal) infinite series:

$$(2.11) \quad \ell_0^\gamma = \sum_{k=1}^{\infty} \frac{1}{k!} \ell_k(\gamma, \gamma, \dots, \gamma),$$

$$\ell_n^\gamma(x_1, \dots, x_n) = \ell_n(x_1, \dots, x_n) + \sum_{k=1}^{\infty} \frac{1}{k!} \ell_{k+n}(\gamma, \dots, \gamma, x_1, \dots, x_n), \quad n \geq 1.$$

We have the following generalization of Proposition 4.4 and Lemma 4.5 in [15].

Proposition 2.11. *Let L be an L_∞ -algebra which is a Banach space and assume that all maps ℓ_k are continuous (with respect to the induced norm on the tensor powers of L). Suppose also that the series (2.11) are point-wise norm convergent. Then these series define a curved L_∞ -structure on L .*

Proof. Let us introduce the abbreviation $\ell_k(\gamma^{\wedge(k-i)}, x_1, \dots, x_i)$ for $\ell_k(\gamma, \dots, \gamma, x_1, \dots, x_i)$ (γ appears $k-i$ times). A direct computation, via rearranging the terms in the sums, shows that

$$\begin{aligned} & \sum_{k=0}^n (-1)^{k(n-k)} \sum_{\sigma} \chi(\sigma) \ell_{n-k+1}^\gamma \left(\ell_k^\gamma(x_{\sigma(1)}, \dots, x_{\sigma(k)}), x_{\sigma(k+1)}, \dots, x_{\sigma(n)} \right) \\ &= \sum_{k=0}^n \sum_{\sigma} (-1)^{k(n-k)} \chi(\sigma) \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{s=0}^p \frac{p!}{s!(p-s)!} \\ & \quad \times \ell_{n-k+p+1}^\gamma \left(\gamma^{\wedge s}, \ell_{k+p-s}^\gamma(\gamma^{\wedge(p-s)}, x_{\sigma(1)}, \dots, x_{\sigma(k)}), x_{\sigma(k+1)}, \dots, x_{\sigma(n)} \right) \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{k=0}^{p+n} (-1)^{k(p+n-k)} \sum_{\varrho} \chi(\varrho) \end{aligned}$$

$$\times \ell_{p+n-k+1} \left(\ell_k(y_{\varrho(1)}, \dots, y_{\varrho(k)}), y_{\varrho(k+1)}, \dots, y_{\varrho(p+n)} \right) = 0,$$

where $y_1 = \gamma, \dots, y_p = \gamma, y_{p+1} = x_1, \dots, y_{p+n} = x_n$ and the sum \sum_{ϱ} is over all $(k, p + n - k)$ -unshuffles ϱ . □

Remark 2.12. Note that we can similarly deform a curved L_∞ -algebra by allowing k to run from 0 to ∞ in the first of Eqs. (2.11). Proposition 2.11 with its proof remains valid in this more general setting.

The process of deformation just described is functorial in the following sense. Let L and M be two curved L_∞ -algebras, let $\gamma \in L^1$ and $F = \{F_k\} : L \rightarrow M$ be a morphism of curved L_∞ -algebras. Suppose that the series

$$\gamma' = \sum_{k=1}^{\infty} \frac{1}{k!} F_k(\gamma, \dots, \gamma)$$

is norm convergent. Assume further that we can deform L and M by γ and γ' respectively, as in Proposition 2.11, to curved L_∞ -algebras that we denote by L^γ and $M^{\gamma'}$, respectively. Now for every $n \geq 1$ consider the series

$$(2.12) \quad F_n^\gamma(x_1, \dots, x_n) = F_n(x_1, \dots, x_n) + \sum_{k=1}^{\infty} \frac{1}{k!} F_{n+k}(\gamma, \dots, \gamma, x_1, \dots, x_n).$$

By a computation similar to the one in the proof of Proposition 2.11, one can show that the following holds.

Proposition 2.13. *Assume that all maps F_k representing the morphism $F : L \rightarrow M$ are continuous and that the series (2.12) are point-wise norm convergent. Then these series define a morphism of curved L_∞ -algebras*

$$(2.13) \quad F^\gamma : L^\gamma \rightarrow M^{\gamma'}.$$

Curved A_∞ -algebras can be obtained as deformations of ordinary A_∞ -algebras in an analogous fashion. More generally, let A be a curved A_∞ -algebra with maps $m_k : A^{\otimes k} \rightarrow A$ ($k = 0, 1, \dots$) and let γ be in A_1 . For $x_1, \dots, x_k \in A$ consider the infinite series

$$(2.14) \quad m_0^\gamma = \sum_{k=0}^{\infty} m_k(\gamma, \gamma, \dots, \gamma),$$

$$m_n^\gamma(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \sum_{\sigma} \chi(\sigma) m_{n+k}(y_{\sigma(1)}, \dots, y_{\sigma(k)}, y_{\sigma(k+1)}, \dots, y_{\sigma(k+n)}),$$

where $y_1 = \gamma, \dots, y_k = \gamma, y_{k+1} = x_1, \dots, y_{k+n} = x_n$, the summation is taken over all permutations $\sigma \in \mathbf{S}_{k+n}$ that are $(k, k+n)$ -unshuffles, and $\chi(\sigma)$ is the sign appearing in (2.10).

Proposition 2.14. *Let A be a curved A_∞ -algebra that is a Banach space. Assume that the maps $\{m_k\}_{k=0}^\infty$ are continuous and that the series (2.14) are point-wise norm convergent. Then these series define a curved A_∞ -structure on A .*

Proof. We check that Eq. (2.1) holds and obtain

$$\begin{aligned}
 & \sum_{r+s+t=n} (-1)^{r+st} m_{r+t+1}^\gamma \left(x_1, \dots, x_r, m_s^\gamma(x_{r+1}, \dots, x_{r+s}), x_{r+s+1}, \dots, x_n \right) \\
 &= \sum_{r+s+t=n} \sum_{p=0}^{r+t+1} \sum_{q=0}^s (-1)^{r+st} \chi_{p,q,r,s} \\
 & \quad \times m_{p+r+t+1} \left(\overset{\check{\gamma}}{\cdot}, x_1, \overset{\check{\gamma}}{\cdot}, x_r, \overset{\check{\gamma}}{\cdot}, m_{q+s} \left(\overset{\check{\gamma}}{\cdot}, x_{r+1}, \overset{\check{\gamma}}{\cdot}, x_{r+s}, \overset{\check{\gamma}}{\cdot} \right), \right. \\
 (2.15) \quad & \left. \overset{\check{\gamma}}{\cdot}, x_{r+s+1}, \overset{\check{\gamma}}{\cdot}, x_n, \overset{\check{\gamma}}{\cdot} \right),
 \end{aligned}$$

where $\overset{\check{\gamma}}{\cdot}$ stands for insertions of γ 's, we have inserted p γ 's outside m_s and q γ 's inside, and $\chi_{p,q,r,s}$ is the resulting sign factor. Then the right hand side of Eq. (2.15) reads

$$\begin{aligned}
 & \sum_{k=0}^\infty \sum_{\sigma} \chi(\sigma) \sum_{r+s+t=k+n} (-1)^{r+st} \\
 & \quad \times m_{r+t+1} \left(y_{\sigma(1)}, \dots, y_{\sigma(r)}, m_s(y_{\sigma(r+1)}, \dots, y_{\sigma(r+s)}), y_{\sigma(r+s+1)}, \dots, y_{\sigma(k+n)} \right) = 0,
 \end{aligned}$$

where the same notation as in Eq. (2.14) is used. □

As in the L_∞ case, we have the following functoriality result. Let A and B be two curved A_∞ -algebras, let $\gamma \in A^1$ and $F = \{F_k\} : A \rightarrow B$ be a morphism of curved A_∞ -algebras. Assume that the series

$$\gamma' = \sum_{k=1}^\infty F_k(\gamma, \dots, \gamma)$$

is norm convergent. Suppose further that we can deform A and B by γ and γ' respectively, as in Proposition 2.14, to curved A_∞ -algebras that we denote by A^γ

and $B^{\gamma'}$. Now for every $n \geq 1$ consider the series

$$(2.16) \quad F_n^\gamma(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \sum_{\sigma} \chi(\sigma) F_{n+k}(y_{\sigma(1)}, \dots, y_{\sigma(k+n)}),$$

where again $y_1 = \gamma, \dots, y_k = \gamma, y_{k+1} = x_1, \dots, y_{k+n} = x_n$ and the summation is taken over all permutations $\sigma \in \mathbf{S}_{k+n}$ that are $(k, k+n)$ -unshuffles.

Proposition 2.15. *Assume that the maps $\{F_k\}_{k=1}^\infty$ representing the morphism $F = A \rightarrow B$ are continuous and that the series (2.16) are point-wise norm convergent. Then these series define a morphism of curved A_∞ -algebras*

$$(2.17) \quad F^\gamma : A^\gamma \rightarrow B^{\gamma'}.$$

2.3. Examples. Let X be a locally finite polyhedron. We briefly recall, following [16], how the complex $C^\bullet(X)$ of \mathbf{k} -valued oriented simplicial cochains on X can be endowed with an A_∞ -structure (which is in fact a C_∞ -structure). For alternative constructions of such a structure the reader is referred to the Appendix of [31] and to [6].

We recall the definition of the complex of the piece-wise smooth differential forms on X introduced by H. Whitney in [32] (see also [7, Chapter 2] and [3]). A piece-wise smooth differential n -form on X is a collection of forms $\omega = \{\omega_\sigma\}$, one for each simplex σ in X , such that ω_σ is a smooth n -form on σ and for every inclusion of simplices $i : \sigma_1 \hookrightarrow \sigma_2$ we have $i^* \omega_{\sigma_2} = \omega_{\sigma_1}$. We denote the space of all such collections of forms by $\Omega^\bullet(X)$. Since the exterior differential commutes with pull-backs we obtain a well-defined differential on $\Omega^\bullet(X)$.

The following theorem is due to J. Dupont [7, Theorem 2.16].

Theorem 2.16. *There exists a chain contraction from $\Omega^\bullet(X)$ to $C^\bullet(X)$. Namely, there exist chain maps $R : \Omega^\bullet(X) \rightarrow C^\bullet(X)$ and $W : C^\bullet(X) \rightarrow \Omega^\bullet(X)$, and a linear map $H : \Omega^\bullet(X) \rightarrow \Omega^{\bullet-1}(X)$ satisfying the relations*

$$(2.18) \quad \begin{aligned} RW &= \mathbf{1}_{C^\bullet(X)} \\ WR - \mathbf{1}_{\Omega^\bullet(X)} &= dH + Hd. \end{aligned}$$

The map R is given by integration over simplices. The map W can be described as a generalized linear interpolation; its image consist of piece-wise linear forms

(see [32]). We shall refer to R (respectively W) as the de Rham (respectively Whitney) map.

Note that the complex $\Omega^\bullet(X)$ is a dg algebra with respect to the exterior product of forms. Thus, we can transfer this particular A_∞ -structure to an A_∞ -structure on the complex $C^\bullet(X)$ along the contraction given from Theorem 2.16, using the sum over rooted planar tree formulas of Kontsevich and Soibelman [23]. In Sect. 3 we will review these formulas in some detail.

Using a piece-wise smooth metric on X one can define an L_2 -norm on forms in $\Omega^\bullet(X)$ and, using the Whitney embedding W , also on cochains in $C^\bullet(X)$. We shall denote this norm by $\|\cdot\|$ in what follows. Let $\{m_k\}_{k=1}^\infty$ be the maps defining the A_∞ -structure on $C^\bullet(X)$ just described.

Lemma 2.17. *For all $k > 1$ and all $x_1, \dots, x_k \in C^\bullet(X)$ one has*

$$\|m_k(x_1, \dots, x_k)\| \leq c_{k-1} E^k \|x_1\| \cdots \|x_k\|,$$

where $c_k = \frac{(2k)!}{(k+1)!k!}$ is the k -th Catalan number and E is a constant depending only on X and the norm $\|\cdot\|$.

Proof. Observe that the expression for m_k given in [23] in this case reduces to a sum over all completely binary trees with k tails (see Sect. 3 for the relevant definitions) and the number of these trees is exactly c_{k-1} . Now the estimate follows since the exterior product and the homotopy H are bounded maps in the L_2 -norm. \square

Observe that we can extend all maps involved in Theorem 2.16 entry-wise to matrix-valued forms and matrix-valued cochains. We can apply Theorem 2.16 and Lemma 2.17 to each matrix entry and conclude that these two statements remain valid in the matrix setting. Thus we obtain an A_∞ -structure on the complex of matrix-valued cochains $C^\bullet(X, \mathbf{M}_l(\mathbf{k}))$. We shall deform this to a curved A_∞ -structure using a degree 1 element as in Proposition 2.14.

Let E be a (continuous) complex vector bundle over X . As in Example 2.4, there is a continuous matrix idempotent representing E . Observe that $\Omega^0(X)$, the algebra of piece-wise smooth functions on X , is a dense (by the Stone-Weierstrass theorem) subalgebra of the Banach algebra $C(X)$ of continuous functions on X , which is closed under holomorphic functional calculus. It follows (see e.g. [4,

Chapter III, Appendix C]) that the inclusion $\Omega^0(X) \hookrightarrow C(X)$ induces isomorphism of K -theory groups: $K_0(\Omega^0(X)) \cong K_0(C(X))$. In other words, there exists a piece-wise smooth idempotent $P \in \Omega^0(X, \mathbf{M}_l(\mathbf{k}))$ representing the K -theory class of E .

Proposition 2.18. *For every fine enough subdivision \tilde{X} of X , the series (2.14) with $\gamma = R(PdP)$ (where R is the de Rham map for the complex \tilde{X}) define a curved A_∞ -structure on $C^\bullet(\tilde{X}, \mathbf{M}_l(\mathbf{k}))$.*

Proof. Since the analytical hypotheses of Proposition 2.14 are not satisfied in this setting, we shall show that the series (2.14) are locally convergent in an appropriate sense. To this end, observe that the maps $\{m_k\}$ are quasi-local in the sense that the value of the cochain $m_k(x_1, \dots, x_k)$ at a simplex Δ depends only on the values of x_1, \dots, x_k restricted to $\mathbf{St}(\Delta)$, the star of Δ . Thus Lemma 2.17 gives

$$\|m_k(x_1, \dots, x_k)|_\Delta\| \leq c_{k-1} E^k \|x_1|_{\mathbf{St}\Delta}\| \cdots \|x_k|_{\mathbf{St}\Delta}\|.$$

Using the elementary estimate $c_k < 4^k$ we obtain

$$\|m_{p+k}(\gamma, \dots, \gamma, x_1, \dots, x_k)\| \leq (4E)^{p+k} \|\gamma\|^p \|x_1\| \cdots \|x_k\|.$$

Thus we see that if we choose a fine enough subdivision \tilde{X} of X so that the norm of γ restricted to the star of each simplex in \tilde{X} is less than $1/(4E)$, the series (2.14) will locally (i.e. at each simplex) be dominated by convergent geometric series and hence will be convergent. \square

Finally, we briefly sketch how one can obtain an example of a curved L_∞ -structure based on Example 2.9. Let G be Lie group with Lie algebra \mathfrak{g} , P a smooth principal G -bundle, and X a triangulation of the total space of P . Using Theorem 2.16 and the main result of [11] one can transfer the dg Lie algebra structure on $\Omega^\bullet(P, \mathfrak{g})$ to an L_∞ -structure on the complex of \mathfrak{g} -valued cochains $C^\bullet(X, \mathfrak{g})$. Let α be connection 1-form on P . Then one has the following analogue of Proposition 2.18.

Proposition 2.19. *For every fine enough subdivision \tilde{X} of X , the series (2.11) with $\gamma = R(\alpha)$ (where R is the de Rham map for the complex \tilde{X}) define a curved L_∞ -structure on $C^\bullet(\tilde{X}, \mathfrak{g})$.* \square

It is clear however that this curved L_∞ -structure on X does not reflect the G action on P ; perhaps a more relevant example of a curved L_∞ -structure can be obtained by considering a G -equivariant triangulation of P and G -invariant cochains.

3. TRANSFERRING CURVED DG AND LIE DG STRUCTURES THAT ARE DEFORMATIONS

3.1. Review of homological perturbation theory. In this section we recall the classical homological perturbation lemma ([18]) and show how to use it to transfer curved dg and Lie dg algebra structures that are deformations of ordinary ones. Given a chain complex (C, d) , we say that a map $\delta : C^\bullet \rightarrow C^{\bullet+1}$ is a **perturbation** of d if $(d + \delta)^2 = 0$.

Definition 3.1. Assume that we are given two chain complexes (C_1, d_1) and (C_2, d_2) and chain maps $p : C_1 \rightarrow C_2$ and $i : C_2 \rightarrow C_1$. We say that the pair (p, i) is a **chain contraction** of (C_1, d_1) to (C_2, d_2) if $pi = \mathbf{1}_{C_1}$ and there exists a homotopy between ip and the identity map on C_1 , i.e. a map $H : C_1^\bullet \rightarrow C_1^{\bullet-1}$ such that $ip - \mathbf{1}_{C_1} = d_1H + Hd_1$. We say that the contraction (p, i) is **special** if the following so called annihilation conditions hold.

$$Hi = 0, \quad pH = 0, \quad H^2 = 0.$$

It is shown in [15] that the chain contraction described in Theorem 2.16 is special.

The first part of the following statement is known as the main homological perturbation lemma. The second part is proved in [14, Section 2].

Theorem 3.2. (a) Let (C_1, d_1) be a chain complex and δ_1 be a perturbation of d_1 . Suppose that we are given a special chain contraction (p, i, H) between (C_1, d_1) and a chain complex (C_2, d_2) . Assume further that $\mathbf{1}_{C_1} - \delta_1H$ is invertible and set $\Sigma = (\mathbf{1}_{C_1} - \delta_1H)^{-1}\delta_1$. Then $\delta_2 = p\Sigma i$ is a perturbation of d_2 and the formulas

$$(3.1) \quad \tilde{i} = i + H\Sigma i, \quad \tilde{p} = p + p\Sigma H, \quad \tilde{H} = H + H\Sigma H$$

define a special chain contraction between $(C_1, d_1 + \delta_1)$ and $(C_2, d_2 + \delta_2)$.

(b) Assume, in addition to the hypotheses in part (a), that (C_1, d_1) is dg (co)algebra, δ_1 is a (co)derivation, and p and i are (co)algebra homomorphisms. Then δ_2 is a coderivation and the maps \tilde{p} and \tilde{i} are (co)algebra homomorphisms.

3.2. Transferring curved dg and Lie dg algebra structures that are deformations of ordinary ones.

Proposition 3.3. *Let A be dg algebra equipped with a Banach norm such that the multiplication map is continuous. Let γ be a degree 1 element of A and put $\widehat{\gamma}(x) = [\gamma, x]$ for $x \in A$. Let A^γ denote the curved dg algebra obtained from A via γ as in Example 2.2. Let (p, i, H) be a special contraction from A to a chain complex B which is a Banach space, such that H is continuous and p and i are continuous with norms not exceeding 1. Assume further that $\|\widehat{\gamma}\| \|H\| < 1$. Then there exists a curved A_∞ -structure on B and morphisms of curved A_∞ -algebras $P : A^\gamma \rightarrow B$ and $I : B \rightarrow A^\gamma$.*

Proof. According to Proposition 2.5, the curved dg algebra structure on A obtained via γ is equivalent to a coderivation D_A on $T(sA)$ such that $D_A^2 = 0$. Let $T(d_A)$ denote the usual extension of the differential d_A on A to $T(sA)$. It follows from the proof of Proposition 2.5 that we can write D_A as a perturbation of $T(d_A)$: $D_A = T(d_A) + \delta_A$. Since $T(d_A)$ and D_A are coderivations, so is δ_A .

Now let us define maps $T(p) : T(sA) \rightarrow T(sB)$, $T(i) : T(sB) \rightarrow T(sA)$ and $T(H) : T(sA) \rightarrow T(sA)$ by setting for every $n \geq 1$:

$$(3.2) \quad (T(p))_n = p^{\otimes n}, \quad (T(i))_n = i^{\otimes n}, \quad (T(H))_n = \sum_{k+j+1=n} \mathbf{1}^{\otimes k} \otimes H \otimes (ip)^{\otimes j}.$$

(The last formula is a particular case of the so-called tensor trick, see [10] and the references therein.) It is easy to check that (3.2) defines a special contraction from $T(sA)$ to $T(sB)$ and that $T(p)$ and $T(i)$ are coalgebra morphisms. The assumption $\|\widehat{\gamma}\| \|H\| < 1$ implies that the series

$$\sum_{k=0}^{\infty} (\delta_A T(H))^k$$

is dominated by a convergent geometric series. Thus $1 - \delta_A T(H)$ is invertible and we can apply Theorem 3.2 to the contraction (3.2). As a result, we obtain a coderivation δ_B on $T(sB)$, which is a perturbation of $T(d_B)$ and hence defines a curved A_∞ -structure on B , and also curved A_∞ -algebra morphisms between A^γ and B given by (3.1). \square

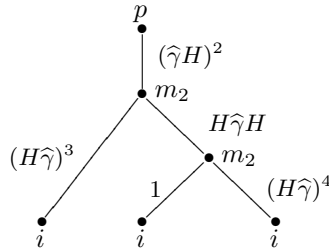
We note that the case of an ordinary dg algebra in the proposition above has already been considered in [10] where it is also shown that the formulas for the

transferred A_∞ -structure obtained from the perturbation lemma coincide with the sum of over rooted tree formulas obtained in [23]. We will now generalize these formulas to the case of a transfer of a curved dg structure.

Recall that a rooted planar tree has a natural orientation of its edges, which we use to define the **valency** of a vertex of such a tree to be the number of edges going out from the vertex. A **tail** is a vertex of valency zero. A **completely binary** tree is rooted planar tree whose root has valency 1 and all other (non-tail) vertices have valency 2. Let us denote by Υ_n the set of all completely binary rooted planar trees with n tails. Define further Υ_n^{dec} to be the set whose elements consist of a tree in Υ_n and a sequence of arbitrary non-negative integers, one for each edge of the tree. Assuming the hypotheses of Proposition 3.3, we can think of an element in Υ_n^{dec} as a tree in Υ_n decorated according to the following rules:

- (1) Assign to the root the map p and to each tail the map i .
- (2) Assign to each vertex of valency 2 the map m_2^A .
- (3) Assign to the edge e containing the root the map $(\widehat{\gamma}H)^{k_e}$, $k_e = 0, 1, 2, \dots$
- (4) Assign to each edge containing a tail e the map $(H\widehat{\gamma})^{k_e}$, $k_e = 0, 1, 2, \dots$
- (5) Assign to each interior edge e the map $H(\widehat{\gamma}H)^{k_e}$, $k_e = 0, 1, 2, \dots$

Now to each element τ in Υ_n^{dec} we can assign an operation $m_\tau : B^{\otimes n} \rightarrow B$ by composing the maps assigned to the vertices and the edges of τ in the natural order. For example, to the tree



there corresponds the map

$$p(\widehat{\gamma}H)^2 m_2^A((H\widehat{\gamma})^3 i \otimes H\widehat{\gamma}H m_2^A(i \otimes (H\widehat{\gamma})^4 i)).$$

Proposition 3.4. *The transferred curved A_∞ -structure on B obtained in Proposition 3.3 is given by*

$$(3.3) \quad m_0^B = \sum_{k=0}^{\infty} p(\widehat{\gamma}H)^k(m_0^A),$$

$$(3.4) \quad m_1^B = pd_Ai + \sum_{k=0}^{\infty} p\widehat{\gamma}(\widehat{\gamma}H)^k i,$$

$$(3.5) \quad m_n^B = \sum_{\tau \in \Upsilon_n^{dec}} (\pm)m_\tau, \quad n > 1.$$

Proof. Formulas (3.3) and (3.4) follow easily from Proposition 3.3. We shall verify (3.5) closely following [2]. We first observe that the transfer formulas from Theorem 3.2 can be rewritten as recursive formulas:

$$\begin{aligned} \delta_B &= T(p)\delta_A I & I &= T(i) + T(H)\delta_A I \\ P &= T(p) + P\delta_A T(H) & \widetilde{H} &= T(H) + \widetilde{H}\delta_A T(H). \end{aligned}$$

Using this, we can derive an inductive formula for the components of the curved A_∞ -morphism $I : B \rightarrow A^\gamma$. Proceeding as in the proof of Theorem 2.1 in [2] we find that

$$I_1 = i + H\widehat{\gamma}I_1.$$

Solving for I_1 and expanding $(1 - H\widehat{\gamma})^{-1}$ in geometric series we obtain

$$(3.6) \quad I_1 = \sum_{k=0}^{\infty} (H\widehat{\gamma})^k i.$$

Next, for every $n > 1$ we find

$$I_n = H\widehat{\gamma}I_n + \sum_{l+s=n} (-1)^{l(s-1)} Hm_2^A(I_l, I_s),$$

which implies the desired recursive formula:

$$(3.7) \quad I_n = \sum_{k=0}^{\infty} \sum_{l+s=n} (-1)^{l(s-1)} H(\widehat{\gamma}H)^k m_2^A(I_l, I_s),$$

Similarly one shows that

$$m_n^B = p\widehat{\gamma}I_n + \sum_{l+s=n} (-1)^{l(s-1)} m_2^A(I_l, I_s),$$

and substituting (3.7) in the last expression gives

$$(3.8) \quad m_n^B = \sum_{k=0}^{\infty} \sum_{l+s=n} (-1)^{l(s-1)} p(\widehat{\gamma}H)^k m_2^A(I_l, I_s).$$

Now substituting (3.6) and (3.7) into (3.8) one obtains (see [2] and [26])

$$m_n^B = \sum_{\tau \in \Upsilon_n^{dec}} (-1)^{\epsilon(\tau)} m_\tau,$$

where $\epsilon(\tau)$ is defined as follows. Let v be a vertex of τ which is not a tail and not the root. Let l_v be the number of tails t of τ such that the unique path from t to the root of τ contains the first output edge of v and let s_v be the number of tails t of τ such that the unique path from t to the root of τ contains the second output edge of v . Then define $\epsilon(\tau) = \sum_v l_v(s_v - 1)$ (the sum is over all vertices v which are neither a tail nor the root).

□

Remark 3.5. Clearly (3.7) implies a sum over rooted trees formula for the components of I similar to (3.5); the only difference is that we assign H instead of p to the roots of the trees. One can also derive a similar formula for the morphism $P : A^\gamma \rightarrow B$ as in [2]. In this case each tree in the sum has H assigned to *one* of its tails.

We shall need the additional properties of the transferred curved A_∞ -structure described in the following three lemmas in the next section.

Lemma 3.6. *Let (p_k, i_k, H_k) be a special contraction from a dg algebra A_k to a chain complex B_k , $k = 1, 2$, satisfying the hypotheses of Proposition 3.3. Let $\gamma_k \in A_k$ be degree 1 elements and let $\psi : A_1^{\gamma_1} \rightarrow A_2^{\gamma_2}$ be a strict morphism of curved dg algebras such that*

$$(3.9) \quad \psi H_1 = H_2 \psi$$

Consider B_k , $k = 1, 2$, with the transferred curved A_∞ -structure obtained in Proposition 3.3 and let $P_k = \{P_{k,l}\}_{l=1}^\infty$ and $I_k = \{I_{k,l}\}_{l=1}^\infty$ be the corresponding curved A_∞ -algebra morphisms. Then the morphism $P_2 \psi I_1 : B_1 \rightarrow B_2$ is also strict.

Proof. We compute the components of $P_2 \psi I_1$ using (2.8) and Remark 3.5 and find that all terms contain compositions either of the form $H_2 \psi H_1$ or of the form

$p_2\psi H_1$. Now the claim follows from (3.9), using the annihilation conditions stated in Definition 3.1. \square

Lemma 3.7. *Assume, in addition to the hypotheses of Proposition 3.3, that for some $l > 0$ we have*

$$A = \tilde{A} \otimes \mathbf{M}_l(\mathbf{k}), \quad B = \tilde{B} \otimes \mathbf{M}_l(\mathbf{k}),$$

where \tilde{A} is a commutative dg algebra, and that the contraction between A and B is induced by a contraction between \tilde{A} and \tilde{B} . Suppose further that $x_1, \dots, x_n \in B$ are of even degree and n is odd and greater than 1. Then the transferred curved A_∞ -structure on B satisfies

$$(3.10) \quad \sum_{\sigma\text{-cyclic}} \text{Tr}(m_n^B(x_{\sigma(1)}, \dots, x_{\sigma(n)})) = 0,$$

where the sum is over all cyclic permutations and Tr denotes the matrix trace. The same identity holds for the odd components of the morphism $I : B \rightarrow A^\gamma$ given by Proposition 3.3.

Proof. By Proposition 3.4 it suffices to show that

$$(3.11) \quad \sum_{\sigma\text{-cyclic}} \sum_{\tau \in \Upsilon_n^{\text{dec}}} (-1)^{\epsilon(\tau)} \text{Tr}(m_\tau(x_{\sigma(1)}, \dots, x_{\sigma(n)}))$$

is identically 0. Observe that for every τ we can write $m_\tau = pm_2^A(N_1, N_2)$ for some maps $N_1 : B^{\otimes k} \rightarrow A$, $N_2 : B^{\otimes n-k} \rightarrow A$ and some $k > 0$. Using the definition of $\epsilon(\tau)$ one checks that the terms in (3.11) corresponding to $pm_2^A(N_1, N_2)$ and $pm_2^A(N_2, N_1)$ appear with opposite signs. But it is easy to see that our parity assumptions imply

$$\begin{aligned} & \text{Tr}(m_2^A(N_1(x_1, \dots, x_k), N_2(x_{k+1}, \dots, x_n))) \\ &= \text{Tr}(m_2^A(N_2(x_{k+1}, \dots, x_n), N_1(x_1, \dots, x_k))), \end{aligned}$$

hence all terms in (3.11) cancel in pairs. \square

Lemma 3.8. *Assume, in addition to the hypotheses of Proposition 3.3, that A is commutative and extend the transferred curved A_∞ -structure on B to $\mathbf{M}_l(B)$. If we deform the latter structure by an element $\beta \in \mathbf{M}_l(B_1)$ to a curved A_∞ -structure $\{m_k^{B,\beta}\}$ as in Proposition 2.14, we have*

$$\sum_{\sigma\text{-cyclic}} \text{Tr}(m_n^{B,\beta}(x_{\sigma(1)}, \dots, x_{\sigma(n)})) = 0$$

for every odd $n > 1$ and all $x_1, \dots, x_n \in \mathbf{M}_l(B)$ of even degree.

Proof. Lemma 3.7 easily implies that the curved A_∞ -structure on $\mathbf{M}_l(B)$ satisfies (3.10). A computation similar to the proof of Lemma 3.7 shows that the trace vanishes on the sums appearing in (2.14) for every $k > 0$, which proves the claim. \square

Example 3.9. Let X be a finite polyhedron. Given a vector bundle E on X , we can find a piece-wise smooth idempotent P representing the K -theory class of E , as already noted in Sect. 2. Thus, we obtain a curved dg structure on the complex $\Omega^\bullet(X, \mathbf{M}_l(\mathbf{k}))$ of piece-wise smooth forms on X as in Example 2.4. As in Sect. 2, we equip $\Omega^\bullet(X, \mathbf{M}_l(\mathbf{k}))$ with an L_2 -norm and use the inclusion W to transfer this norm to $C^\bullet(X, \mathbf{M}_l(\mathbf{k}))$. Assume that we have fixed fine enough subdivision \tilde{X} of X so that the condition $\|\hat{\gamma}\| \|H\| < 1$ (where H is the contraction from Theorem 2.16 and $\hat{\gamma}$ denotes commutator with $\gamma = PdP$) is satisfied locally, i.e. on each simplex. Then we can transfer the curved dg structure on $\Omega^\bullet(\tilde{X}, \mathbf{M}_l(\mathbf{k}))$ obtaining a curved A_∞ -structure on $C^\bullet(\tilde{X}, \mathbf{M}_l(\mathbf{k}))$ as in Proposition 3.3 (all series are locally convergent as in Proposition 2.18).

We remark that the analogue of Proposition 3.3 for curved Lie dg structures holds. Indeed, in the absence of curvature, this is the main result of [11] (see also [1] for an alternative approach). The generalization to the curved case is done as in the proof of Proposition 3.3. As an application, one can transfer the curved dg Lie algebra structure associated to a principal bundle P from Example 2.9 to a curved L_∞ -structure on the complex of the \mathfrak{g} -valued cochains of a fine enough triangulation of the total space of P .

We conclude this section by summarizing our results obtained so far as follows. Given a curved dg algebra (or curved Lie dg algebra) A that is obtained as a deformation of a (Lie) dg algebra via a degree 1 element γ and a special contraction from A to a complex B , there are two procedures to construct a curved A_∞ (respectively L_∞)-structure on B : One can first transfer the (Lie) dg algebra structure of A to B and then deform this transferred structure via the transferred element γ' or one can directly transfer the curved (Lie) dg structure of A to B . We shall see in the next section that one can obtain non-trivial characteristic classes from a curved A_∞ -structure obtained by the second procedure,

while for curved A_∞ -structures obtained by the first procedure that appears to be impossible.

4. A GENERALIZED CHERN-WEIL THEORY

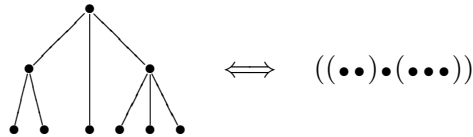
4.1. **Chern-Weil triples.** In this section, we develop an abstract algebraic version of the classical Chern-Weil algorithm for producing characteristic classes of vector bundles.

Definition 4.1. Let $(A, \{m_k\}_{k=0}^\infty)$ be a curved A_∞ -algebra, (C, d) a chain complex, and $\Phi : A \rightarrow C$ a linear map of degree zero. We call (A, Φ, C) a **Chern-Weil triple** if for every x of even degree in A , every odd $n > 1$ and all even x_1, \dots, x_n in A the following conditions hold.

- (i) $\Phi(m_1(x)) = d(\Phi(x))$
- (ii) $\sum_{\sigma \in \mathbf{S}_n} \Phi(m_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})) = 0.$

Given a Chern-Weil triple (A, Φ, C) , it is easy to check that $\Phi(m_2(m_0, m_0))$ is a cocycle in C . We shall now construct higher cocycles in C by introducing higher powers of the curvature m_0 .

In order to formulate our next result, we recall that there is a one-to-one correspondence between rooted planar trees and bracket expressions, for example:



Using this correspondence, given a curved A_∞ -algebra A , we define an element \mathcal{M}_T in A for every rooted planar tree T by assigning to each vertex of valency k the operation m_k in A . Thus in the example above we have:

$$T = \begin{array}{c} \bullet \\ \diagup \quad | \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad | \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \implies \mathcal{M}_T = m_3(m_2(m_0, m_0), m_0, m_3(m_0, m_0, m_0)).$$

We call the degree of the element \mathcal{M}_T in A (which is independent of A) the **degree** of the tree T . We say that a rooted planar tree is **completely even** if all of its vertices have even valency and denote the set of all completely even trees of degree n by $\text{C.E.T.}(n)$. Given a rooted tree, consider for each tail the number

of edges in the shortest path joining the tail and the root. We call the largest among these numbers the **depth** of the rooted tree.

Remark 4.2. It is instructive to note that every completely even tree can be presented as

$$\mathcal{T} = \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ \mathcal{T}_1 \quad \dots \quad \mathcal{T}_n \end{array} \implies \mathcal{M}_{\mathcal{T}} = m_n(\mathcal{M}_{\mathcal{T}_1}, \dots, \mathcal{M}_{\mathcal{T}_n}).$$

Hence, degree of $\mathcal{M}_{\mathcal{T}} = 2 - n + \text{degree of } \mathcal{M}_{\mathcal{T}_1} + \dots + \text{degree of } \mathcal{M}_{\mathcal{T}_n}$, in other words, degree of $\mathcal{M}_{\mathcal{T}} - 1 = 1 + (\text{degree of } \mathcal{M}_{\mathcal{T}_1} - 1) + \dots + (\text{degree of } \mathcal{M}_{\mathcal{T}_n} - 1)$. It follows then by induction on the number of vertices of \mathcal{T} that

$$\text{degree of } \mathcal{T} = 1 + \text{number of vertices of } \mathcal{T}.$$

Theorem 4.3. *Let (A, Φ, C) be a Chern-Weil triple. Then*

$$(4.1) \quad \mathcal{C}_n = \sum_{\mathcal{T} \in \text{C.E.T.}(2n)} \Phi(\mathcal{M}_{\mathcal{T}})$$

is a cocycle in C^{2n} for every $n > 0$.

Proof. According to property (i) in Definition 4.1, it suffices to show that

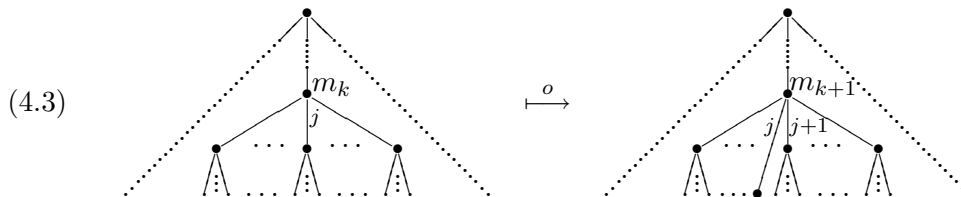
$$\sum_{\mathcal{T} \in \text{C.E.T.}(2n)} m_1(\mathcal{M}_{\mathcal{T}})$$

is in the kernel of Φ . This will be done using the following formula valid for every completely even tree \mathcal{T} (in fact, for every tree with no vertex of valency 1):

$$(4.2) \quad m_1(\mathcal{M}_{\mathcal{T}}) = \sum_{o \in B(\mathcal{T})} (-1)^{\varepsilon(\mathcal{T}, \mathcal{T}^o)} \mathcal{M}_{\mathcal{T}^o},$$

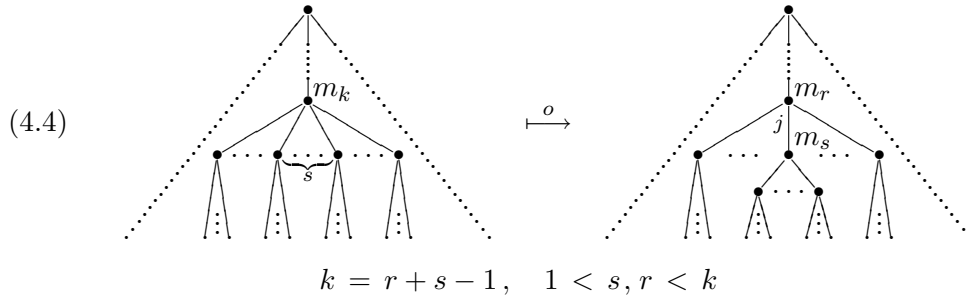
where $B(\mathcal{T})$ is a set consisting of two types of operations $\mathcal{T} \xrightarrow{o} \mathcal{T}^o$ defined as follows:

(a) *Attaching* an edge to a vertex of \mathcal{T} which is not a tail. The new vertex of the added edge then becomes a new tail for the resulting tree. This operation can schematically be represented as follows:



Note that if this operation is performed on a vertex of valency k there are $k + 1$ possibilities for attachment. We assign to the tree \mathcal{T}^o so obtained the sign $(-1)^{\varepsilon(\mathcal{T}, \mathcal{T}^o)} = (-1)^{j+1}$, where $j = 0, \dots, k$ represents the position of the attached edge moving from left to right.

(b) *Grafting* an edge at a vertex which is not a tail. We can represent this operation schematically as follows:



In detail, we choose $s > 1$ adjacent edges coming out of a vertex of valency k and attach them to a new vertex of valency s together with the subtrees attached to them, then join the vertex of valency s with the original vertex which now becomes of valency $r = k - s + 1$. In this case we have

(4.5)
$$(-1)^{\varepsilon(\mathcal{T}, \mathcal{T}^o)} = (-1)^{rs+(j+1)(s+1)},$$

where $j = 0, \dots, r - 1$ represents the position of the group of s adjacent edges.

We remark that one can treat the operations of type (a) as “generalized” operations of type (b) with $s = 0$ so that Eq. (4.5) remains true in the case of operations of type (a) as well.

We verify Eq. (4.2) using induction on the depth of the tree \mathcal{T} . If this depth is 1 the identity (2.1) gives

$$\begin{aligned}
 m_1(m_k(m_0, \dots, m_0)) &= \sum_{j=0}^k (-1)^{j+1} m_{k+1}(m_0, \dots, m_0) + \\
 + \sum_{\substack{r+s-1=k \\ s>1}} \sum_{j=0}^{r-1} &(-1)^{rs+(j+1)(s+1)} m_r(m_0, \dots, m_0, m_s(m_0, \dots, m_0), m_0, \dots, m_0) + \\
 &+ \sum_{j=0}^{k-1} (-1)^k m_k(m_0, \dots, m_0, m_1(m_0), m_0, \dots, m_0).
 \end{aligned}$$

Note that the last sum vanishes due to the identity $m_1(m_0) = 0$. Hence, we obtain formula (4.2) in the case of tree depth 1 since the sum over the operations of type (a) corresponds to the sum in the first line and the sum over the operations of type (b) corresponds to the sum in the second line above.

If \mathcal{T} has depth greater than 1 we can write, as in Remark 4.2, $\mathcal{M}_{\mathcal{T}} = m_k(\mathcal{M}_{\mathcal{T}_1}, \dots, \mathcal{M}_{\mathcal{T}_k})$ for some trees $\mathcal{T}_1, \dots, \mathcal{T}_k$ whose depths are smaller than the depth of \mathcal{T} . Using (2.1) again we obtain

$$\begin{aligned}
 m_1(\mathcal{M}_{\mathcal{T}}) &= \sum_{j=0}^k (-1)^{j+1} m_{k+1}(\mathcal{M}_{\mathcal{T}_1}, \dots, \mathcal{M}_{\mathcal{T}_j}, m_0, \mathcal{M}_{\mathcal{T}_{j+1}}, \dots, \mathcal{M}_{\mathcal{T}_k}) + \\
 &+ \sum_{\substack{r+s-1=k \\ s>1}} \sum_{j=0}^{r-1} (-1)^{rs+(j+1)(s+1)} m_r(\mathcal{M}_{\mathcal{T}_1}, \dots, m_s(\mathcal{M}_{\mathcal{T}_j}, \dots, \mathcal{M}_{\mathcal{T}_{j+s}}), \dots, \mathcal{M}_{\mathcal{T}_r}) + \\
 &\qquad\qquad\qquad + \sum_{j=0}^{k-1} (-1)^k m_k(\mathcal{M}_{\mathcal{T}_1}, \dots, m_1(\mathcal{M}_{\mathcal{T}_j}), \dots, \mathcal{M}_{\mathcal{T}_k}).
 \end{aligned}$$

Applying the inductive hypothesis to $m_1(\mathcal{M}_{\mathcal{T}_j})$ in the last sum, we obtain (4.2), as we did above for the case of depth 1.

It remains to show that

$$(4.6) \quad \sum_{\mathcal{T} \in \text{C.E.T.}(2n)} m_1(\mathcal{M}_{\mathcal{T}}) = \sum_{\mathcal{T} \in \text{C.E.T.}(2n)} \sum_{o \in B(\mathcal{T})} (-1)^{\varepsilon(\mathcal{T}, \mathcal{T}^o)} \mathcal{M}_{\mathcal{T}^o}$$

is in the kernel of Φ .

Observe that all trees \mathcal{T}^o in (4.6) contain exactly one vertex of odd valency. Indeed, this is the vertex where an operation either of type (a) or type (b) has been performed. More precisely, in the type (b) case, the operation is applied to a vertex of even valency k and we obtain two new vertices of valencies r and s , respectively, so that exactly one of these two numbers is odd due to the relation $r + s = k + 1$. When s is even we will say that the operation is performed “below” the vertex of odd valency in \mathcal{T}^o , otherwise we will say that the operation is performed “above” the vertex. If the operation performed is of type (a) we will treat it as a “generalized” operation of type (b) with $s = 0$ and will say that is performed “below” the vertex.

We separate the sum in (4.6) into two subsums: (1) terms $\mathcal{M}_{\mathcal{T}^o}$ for which the vertex of odd valency is not the root of \mathcal{T}^o and (2) terms $\mathcal{M}_{\mathcal{T}^o}$ for which this vertex is the root of \mathcal{T}^o .

We claim that the subsum (1) is identically zero. Indeed, to each tree \mathcal{T}^o whose odd vertex is not the root there correspond even number terms in the subsum (1) which appear with alternating signs. To show this, observe that such a tree \mathcal{T}^o can be obtained from completely even trees \mathcal{T} by performing one operation “above” the odd vertex and an odd number (which equals to the valency of this vertex) of operations performed “below”. Now using (4.5) we see that the term that corresponds to the operation performed “above” appears with positive sign and the remaining odd number of terms appear with alternating signs, $(-1)^{rs+(j+1)(s+1)} = (-1)^{j+1}$ ($j = 0, \dots, r - 1$), beginning with minus. All these terms cancel and the claim is proved.

Now consider the subsum (2). To each tree \mathcal{T}^o whose odd vertex is the root there correspond odd terms in this subsum. Indeed, such a tree \mathcal{T}^o can be obtained from \mathcal{T} by performing odd number of operations “below” only. Using (4.5) again we conclude that the signs of these terms alternate beginning with minus sign. Thus, after cancelation only one term corresponding to every \mathcal{T}^o remains. As a result, we obtain

$$(4.7) \quad \sum_{\mathcal{T} \in \text{C.E.T.}(2n)} m_1(\mathcal{M}_{\mathcal{T}}) = - \sum_{\substack{k - \text{odd} \\ 1 < k \leq n}} \sum_{\substack{\mathcal{T}_j \in \text{C.E.T.}(2n_j) \\ n_1 + \dots + n_k = n}} m_k(\mathcal{M}_{\mathcal{T}_1}, \dots, \mathcal{M}_{\mathcal{T}_k}).$$

Now Property (ii) in Definition 4.1 implies that the subsum (2) is in the kernel of Φ and the theorem is proved. □

Remark 4.4. The action of m_1 on the trees considered above essentially coincides with that of the differential in Kontsevich’s graph cohomology complex [22]. Namely, the result of the application of m_1 is a sum (with appropriate signs) of trees that are obtained from the original tree by expanding a vertex to an edge.

Definition 4.5. We call the cohomology classes defined by $\frac{1}{n!c_{n-1}} \cdot \mathcal{C}_n, n = 1, 2, \dots$ the components of the **Chern character** of the Chern-Weil triple (A, Φ, C) and denote them by $\mathbf{Ch}^\bullet(A, \Phi, C)$.

In the definition above we divide by c_{n-1} , the number of all *binary* trees appearing in the sum (4.1), in order to obtain the usual definition of Chern character when A is associative.

4.2. **Pre-Chern elements.** Equation (4.1) suggests that we introduce

$$\mathfrak{C}_n := \sum_{\mathcal{T} \in \text{C.E.T.}(2n)} \mathcal{M}_{\mathcal{T}}$$

(Eq. (4.1) then reads $\mathfrak{C}_n = \Phi(\mathfrak{C}_n)$), which we call **pre-Chern elements**. From the recursion in Remark 4.2 we obtain

$$(4.8) \quad \mathfrak{C}_n = \sum_{\substack{k \text{ - even} \\ 2 \leq k \leq n}} \sum_{n_1 + \dots + n_k = n} m_k(\mathfrak{C}_{n_1}, \dots, \mathfrak{C}_{n_k}).$$

On the other hand, Eq. (4.7) implies that

$$(4.9) \quad m_1(\mathfrak{C}_n) = - \sum_{\substack{k \text{ - odd} \\ 1 < k \leq n}} \sum_{n_1 + \dots + n_k = n} m_k(\mathfrak{C}_{n_1}, \dots, \mathfrak{C}_{n_k}), \quad \text{or}$$

$$(4.10) \quad \sum_{\substack{k \text{ - odd} \\ 1 \leq k \leq n}} \sum_{n_1 + \dots + n_k = n} m_k(\mathfrak{C}_{n_1}, \dots, \mathfrak{C}_{n_k}) = 0.$$

Thus, we see that Eq. (4.9), which can be easily verified using Eq. (4.8) and Eq. (2.1), is essentially what is needed to prove Theorem 4.3.

It is instructive to give an independent proof of Eq. (4.10). To this end it is convenient to introduce the formal sum

$$\mathfrak{C} = \sum_{n=1}^{\infty} \mathfrak{C}_n,$$

which is nothing but the formal generating series of all \mathfrak{C}_n : the degree $2n$ part of \mathfrak{C} is \mathfrak{C}_n . Then Eq. (4.8) becomes

$$(4.11) \quad \mathfrak{C} = \sum_{k=0}^{\infty} m_{2k}(\mathfrak{C}^{\otimes 2k}),$$

while Eq. (4.10) reads

$$(4.12) \quad \sum_{k=1}^{\infty} m_{2k-1}(\mathfrak{C}^{\otimes (2k-1)}) = 0.$$

Now, to prove (4.12) we use the A_∞ -identity (2.1) to find that

$$(4.13) \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+t+1} \left(\mathfrak{e}^{\otimes r} \otimes m_s(\mathfrak{e}^{\otimes s}) \otimes \mathfrak{e}^{\otimes t} \right) = 0.$$

Setting $\mathfrak{e}' := \sum_{k=1}^{\infty} m_{2k-1} \left(\mathfrak{e}^{\otimes (2k-1)} \right)$ and summing (4.13) over *even* n , we obtain

$$\begin{aligned} 0 &= \sum_{N=0}^{\infty} \sum_{r+s+t=2N} (-1)^{r+st} m_{r+t+1} \left(\mathfrak{e}^{\otimes r} \otimes m_s(\mathfrak{e}^{\otimes s}) \otimes \mathfrak{e}^{\otimes t} \right) \\ &= \sum_{\substack{s=0,2,\dots(\text{even}) \\ r+t+1=1,3,\dots(\text{odd})}} (-1)^r m_{r+t+1} \left(\mathfrak{e}^{\otimes r} \otimes m_s(\mathfrak{e}^{\otimes s}) \otimes \mathfrak{e}^{\otimes t} \right) \\ &\quad + \sum_{\substack{s=1,3,\dots(\text{odd}) \\ r+t+1=2,4,\dots(\text{even})}} \underbrace{(-1)^{r+t}}_{(-1)} m_{r+t+1} \left(\mathfrak{e}^{\otimes r} \otimes m_s(\mathfrak{e}^{\otimes s}) \otimes \mathfrak{e}^{\otimes t} \right) \\ &= \sum_{k=1}^{\infty} \left(\sum_{r=0}^{2k-2} (-1)^r \right) m_{2k-1} \left(\mathfrak{e}^{\otimes (2k-1)} \right) \\ &\quad - \sum_{k=1}^{\infty} \sum_{r=0}^{2k-1} m_{2k} \left(\mathfrak{e}^{\otimes r} \otimes \mathfrak{e}' \otimes \mathfrak{e}^{\otimes (2k-r-1)} \right) \\ &= \mathfrak{e}' - \sum_{k=1}^{\infty} \sum_{r=0}^{2k-1} m_{2k} \left(\mathfrak{e}^{\otimes r} \otimes \mathfrak{e}' \otimes \mathfrak{e}^{\otimes (2k-r-1)} \right) \end{aligned}$$

and hence

$$(4.14) \quad \mathfrak{e}' = \sum_{k=1}^{\infty} \sum_{r=0}^{2k-1} m_{2k} \left(\mathfrak{e}^{\otimes r} \otimes \mathfrak{e}' \otimes \mathfrak{e}^{\otimes (2k-r-1)} \right).$$

The homogeneous part of degree $2n - 1$ in Eq. (4.14) gives

$$(4.15) \quad \mathfrak{e}'_n = \sum_{k=1}^{\infty} \sum_{\substack{n_1 + \dots + n_{2k} = n+k \\ n_1, \dots, n_{2k} \geq 1}} \sum_{r=0}^{2k-1} m_{2k} \left(\mathfrak{e}_{n_1}, \dots, \mathfrak{e}_{n_r}, \mathfrak{e}'_{n_{r+1}}, \mathfrak{e}_{n_{r+2}}, \dots, \mathfrak{e}_{n_{2k}} \right)$$

($\mathfrak{e}' = \sum_{n=1}^{\infty} \mathfrak{e}'_n$, $\deg \mathfrak{e}'_n = 2n - 1$). Now Eq. (4.15) easily implies by induction on n that $\mathfrak{e}'_n = 0$ for all n , since $\mathfrak{e}'_1 = m_0$ and $\mathfrak{e}'_1 = m_1(m_0) = 0$.

Next we discuss the independence of our Chern character on the choice of “connection”.

Proposition 4.6. *Let A be a finite dimensional curved A_∞ -algebra. For $\gamma \in A_1$ let A^γ denote the deformed algebra described in Proposition 2.14. Assume that the series (2.14) are uniformly convergent with respect to γ on compact subsets of A_1 . Suppose further that for every $\gamma \in A_1$ we have a Chern-Weil triple (A^γ, Φ, C) . Then the first two components of the Chern characters of these Chern-Weil triples do not depend on γ .*

Proof. We adapt the argument given in [29, Section 5.2]. We denote by d_R the DeRham differential on $\Omega^\bullet(A_1, A)$, the complex of smooth differential forms on A_1 with values in A . Since A is finite dimensional we can define for every $\gamma \in A_1$ the tautological 1-form $d_R\gamma \in \Omega^1(A_1, A_1)$. Then it is not hard to check, differentiating term-by-term with respect to γ , that we have

$$(4.16) \quad d_R m_0^\gamma = m_1^\gamma d_R \gamma,$$

and the claim for the first component of the Chern character easily follows. A simple computation using (4.16) and (2.1) shows that $d_R \mathfrak{C}_2^\gamma = m_1^\gamma \omega$ for some $\omega \in \Omega^1(A_1, A)$ which completes the proof. \square

4.3. Morphisms of Chern-Weil triples.

Definition 4.7. Let (A, Φ, C) and (B, Ψ, D) be two Chern-Weil triples. Let $F : A \rightarrow B$ be a morphism of A_∞ -algebras and let $f : C \rightarrow D$ be a chain map. We say that the pair (F, f) is a **morphism of Chern-Weil triples** if for every even x in A , every odd $n > 1$ and all even $x_1, \dots, x_n \in A$ the following conditions hold.

- (i) $\Psi(F_1(x)) = f(\Phi(x))$
- (ii) $\sum_{\sigma \in \mathbf{S}_n} \Psi(F_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})) = 0$.

Clearly Chern-Weil triples with this notion of morphism do *not* form a category. It turns out however that the Chern character we have defined is natural with respect to certain class of morphisms. This will easily follow from our next lemma.

Let us introduce as in Sect. 4.2 the pre-Chern elements \mathfrak{C}_n^A and \mathfrak{C}_n^B , and their formal sums \mathfrak{C}_A and \mathfrak{C}_B .

Lemma 4.8. *Let $F = \{F_k\} : (A, \{m_k^A\}) \rightarrow (B, \{m_k^B\})$ be a morphism of two curved A_∞ -algebras such that F_k vanishes for all even k . Then*

$$(4.17) \quad \mathfrak{C}_B = \sum_{k=1}^{\infty} F_{2k-1} \left(\mathfrak{C}_A^{\otimes(2k-1)} \right),$$

or equivalently

$$(4.18) \quad \mathfrak{C}_n^B = \sum_{k=1}^{\infty} \sum_{\substack{n_1 + \dots + n_{2k-1} \\ = n + k - 1}} F_{2k-1} \left(\mathfrak{C}_{n_1}^A, \dots, \mathfrak{C}_{n_{2k-1}}^A \right).$$

(Note that the sum over k in (4.18) is in fact finite.)

Proof. Consider first a general morphism $F = \{F_k\} : (A, \{m_k^A\}) \rightarrow (B, \{m_k^B\})$ of two curved A_∞ -algebras. We can transfer \mathfrak{C}_A to the following elements in B

$$(4.19) \quad \mathfrak{G}_{A,B}^\pm := \sum_{k=1}^{\infty} (\pm 1)^{k-1} F_k \left(\mathfrak{C}_A^{\otimes k} \right).$$

Thus $\mathfrak{G}_{A,B}^\pm$ are the formal generating series for $\mathfrak{G}_n^{A,B}$ which are defined by

$$\mathfrak{G}_{A,B}^\pm := \sum_{n=2}^{\infty} (\pm 1)^n \mathfrak{G}_n^{A,B}, \quad \deg \mathfrak{G}_n^{A,B} = n$$

and Eq. (4.19) reads

$$(4.20) \quad \mathfrak{G}_n^{A,B} = \sum_{k=1}^{\infty} \sum_{\substack{2n_1 + \dots + 2n_k \\ = n + k - 1}} F_k \left(\mathfrak{C}_{n_1}^A, \dots, \mathfrak{C}_{n_k}^A \right).$$

It is convenient to split $\mathfrak{G}_{A,B}^\pm$ into even and odd part, $\mathfrak{G}_{A,B}^0$ and $\mathfrak{G}_{A,B}^1$, respectively, so that

$$\begin{aligned} \mathfrak{G}_{A,B}^\pm &= \mathfrak{G}_{A,B}^0 \pm \mathfrak{G}_{A,B}^1, & \mathfrak{G}_{A,B}^{0/1} &= \frac{1}{2} (\mathfrak{G}_{A,B}^+ \pm \mathfrak{G}_{A,B}^-), \\ \mathfrak{G}_{A,B}^0 &:= \sum_{k=1}^{\infty} F_{2k-1} \left(\mathfrak{C}_A^{\otimes(2k-1)} \right), & \mathfrak{G}_{A,B}^1 &:= \sum_{k=1}^{\infty} F_{2k} \left(\mathfrak{C}_A^{\otimes 2k} \right). \end{aligned}$$

We now proceed as in Sect. 4.2 and start with the identity for A_∞ -morphism (2.7):

$$\sum_{r+s+t=n} (-1)^{r+st} F_{r+t+1} \left(\mathfrak{C}_A^{\otimes r} \otimes m_s^A(\mathfrak{C}_A^{\otimes s}) \otimes \mathfrak{C}_A^{\otimes t} \right)$$

$$(4.21) \quad = \sum_{\substack{1 \leq q \leq n \\ i_1 + \dots + i_q = n}} (-1)^w m_q^B \left(F_{i_1}(\mathbf{e}_A^{\otimes i_1}), \dots, F_{i_q}(\mathbf{e}_A^{\otimes i_q}) \right).$$

Using Eq. (2.6) and summing (4.21) over n , we obtain

$$(4.22) \quad \sum_{r,s,t \geq 0} (-1)^{r+st} F_{r+t+1} \left(\mathbf{e}_A^{\otimes r} \otimes m_s^A(\mathbf{e}_A^{\otimes s}) \otimes \mathbf{e}_A^{\otimes t} \right) \\ = \sum_{\substack{q \geq 0 \\ i_1, \dots, i_q \geq 1}} m_q^B \left((\pm 1)^{i_1-1} F_{i_1}(\mathbf{e}_A^{\otimes i_1}), \dots, (-1)^{i_{q-1}-1} F_{i_{q-1}}(\mathbf{e}_A^{\otimes i_{q-1}}), F_{i_q}(\mathbf{e}_A^{\otimes i_q}) \right)$$

The right hand side of Eq. (4.22) contains alternating sign factors $(-1)^{(q-\ell)(i_\ell-1)}$ and simply gives

$$(4.23) \quad \sum_{q=0}^{\infty} m_q^B(\mathfrak{G}_{A,B}^{\pm}, \dots, \mathfrak{G}_{A,B}^-, \mathfrak{G}_{A,B}^+),$$

where the signs in the arguments alternate, so that sign in the first argument is + when q is odd and - when q is even. The left hand side of Eq. (4.22) can be transformed as follows

$$\sum_{r,s,t \geq 0} (-1)^{r+st} F_{r+t+1} \left(\mathbf{e}_A^{\otimes r} \otimes m_s^A(\mathbf{e}_A^{\otimes s}) \otimes \mathbf{e}_A^{\otimes t} \right) \\ = \sum_{r,t \geq 0} (-1)^r F_{r+t+1} \left(\mathbf{e}_A^{\otimes r} \otimes \left(\sum_{s=0,2,\dots \text{ (even)}} m_s^A(\mathbf{e}_A^{\otimes s}) \right) \otimes \mathbf{e}_A^{\otimes t} \right) \\ + \sum_{r,t \geq 0} (-1)^{r+t} F_{r+t+1} \left(\mathbf{e}_A^{\otimes r} \otimes \left(\sum_{s=1,3,\dots \text{ (odd)}} m_s^A(\mathbf{e}_A^{\otimes s}) \right) \otimes \mathbf{e}_A^{\otimes t} \right) \\ = \sum_{r,t \geq 0} (-1)^r F_{r+t+1} \left(\mathbf{e}_A^{\otimes r} \otimes \mathbf{e}_A \otimes \mathbf{e}_A^{\otimes t} \right) \\ = \sum_{\ell=1}^{\infty} \left(\sum_{r=0}^{\ell-1} (-1)^r \right) F_{\ell}(\mathbf{e}_A^{\otimes \ell}) = \sum_{k=1}^{\infty} F_{2k-1}(\mathbf{e}_A^{\otimes (2k-1)}) = \mathfrak{G}_{A,B}^0.$$

Thus Eq. (4.22) gives us

$$(4.24) \quad \mathfrak{G}_{A,B}^0 = \sum_{q=0}^{\infty} m_q^B(\mathfrak{G}_{A,B}^{\pm}, \dots, \mathfrak{G}_{A,B}^-, \mathfrak{G}_{A,B}^+),$$

$$(4.25) \iff \begin{cases} \mathfrak{G}_{2n}^{A,B} = \sum_{q=0}^{\infty} \sum_{\substack{n_1 + \dots + n_q \\ = 2n + q - 2 \\ n_1, \dots, n_q \geq 2}} (-1)^{\sum_{\ell=1}^q (q-\ell)n_\ell} m_q^B(\mathfrak{G}_{n_1}^{A,B}, \dots, \mathfrak{G}_{n_q}^{A,B}) \\ 0 = \sum_{q=0}^{\infty} \sum_{\substack{n_1 + \dots + n_q \\ = 2n + q - 1 \\ n_1, \dots, n_q \geq 2}} (-1)^{\sum_{\ell=1}^q (q-\ell)n_\ell} m_q^B(\mathfrak{G}_{n_1}^{A,B}, \dots, \mathfrak{G}_{n_q}^{A,B}). \end{cases}$$

Now let us assume that the A_∞ -morphism $F = \{F_k\}$ is such that $F_k = 0$ for all even k . It follows that

$$\mathfrak{G}_{A,B}^+ = \mathfrak{G}_{A,B}^- = \mathfrak{G}_{A,B}^0, \quad \mathfrak{G}_{A,B}^1 = 0.$$

Hence Eq. (4.24) reads

$$\mathfrak{G}_{A,B}^0 = \sum_{q=0}^{\infty} m_q^B \left((\mathfrak{G}_{A,B}^0)^{\otimes q} \right),$$

which is equivalent to

$$\mathfrak{G}_{A,B}^0 = \sum_{k=0}^{\infty} m_{2k}^B \left((\mathfrak{G}_{A,B}^0)^{\otimes 2k} \right), \quad \sum_{k=1}^{\infty} m_{2k-1}^B \left((\mathfrak{G}_{A,B}^0)^{\otimes (2k-1)} \right) = 0$$

(since $\mathfrak{G}_{A,B}^0$ is even). But these equations recursively determine \mathfrak{C}_B and we conclude that

$$\mathfrak{C}_B = \mathfrak{G}_{A,B}^0, \quad \text{i.e.,} \quad \mathfrak{C}_k^B = \mathfrak{G}_{2k}^{A,B} \quad k = 1, 2, \dots$$

The last identity together with Eq. (4.19) implies Eq. (4.17) and the lemma is proved. \square

Proposition 4.9. *Let (F, f) be a morphism of Chern-Weil triples from (A, Φ, C) to (B, Ψ, D) such that F_k vanishes for all even k . Then*

$$f(\mathfrak{C}_n^A) = \mathfrak{C}_n^B.$$

Proof. In view of (2.6) it suffices to show that for every n the expression

$$\sum_{\mathcal{T} \in \text{C.E.T.}(2n)} F_1(\mathcal{M}_{\mathcal{T}}^A) - \mathcal{M}_{\mathcal{T}}^B = F_1(\mathfrak{C}_n^A) - \mathfrak{C}_n^B$$

is in the kernel of Ψ . But Eq. (4.18) gives us

$$\mathfrak{e}_n^B - F_1(\mathfrak{e}_n^A) = \sum_{k=2}^{\infty} \sum_{\substack{n_1 + \dots + n_{2k-1} \\ = n + k - 1}} F_{2k-1}(\mathfrak{e}_{n_1}^A, \dots, \mathfrak{e}_{n_{2k-1}}^A),$$

and we conclude that the right hand side is indeed in the kernel of Ψ according to condition (ii) in Definition (4.7). \square

In what follows we will apply the above proposition only in the (trivial) case when F is a strict morphism of curved A_∞ -algebras, i.e. when F_k vanishes for every $k > 1$.

4.4. Examples. Let M be a smooth manifold and let $\Omega^\bullet(M, \text{End}(E))$ be the curved dg algebra from Example 2.2. Then obviously $(\Omega^\bullet(M, \text{End}(E)), \text{Tr}, \Omega^\bullet(M))$ (where Tr denotes the matrix trace) is a Chern-Weil triple. The Chern character of this triple coincides with the Chern character of E .

Let E be a vector bundle over a finite polyhedron X and let $\Omega^\bullet(X, \mathbf{M}_l(\mathbb{C}))$ be the curved dg algebra from Example 3.9 obtained from a piece-wise smooth idempotent P representing E . We shall now explain how one can extend the classical Chern-Weil theory to the piece-wise smooth setting. Indeed, one can apply the general construction of [20, Chapter 1] to the dg algebra $\Omega^\bullet(X)$ to obtain a Chern character

$$\text{Ch} : K_0(\Omega^0(X)) \rightarrow H_{dR}^\bullet(X)$$

whose k -th component $\text{Ch}_k(P)$ is given by the usual formula $(1/k!) \text{Tr}(P(dP)^k)$. Then one can define (see [20]) Chern classes $c_k(P)$ that are related to the Chern character by the formula

$$\text{Ch}_k(P) = \frac{1}{k!} Q_k(c_1(P), \dots, c_k(P)),$$

where Q_k denotes the k -th elementary symmetric function.

Clearly Ch is natural (with respect to piece-wise smooth maps) and coincides with the Chern character given by Chern-Weil theory when P represents a smooth bundle over a smooth compact manifold. It follows that $c_k(P)$ satisfy the axioms for Chern classes stated in [9, Chapter 1, §4] and therefore coincide with the standard Chern classes defined for bundles over arbitrary compact space X . In other

words, the Chern character of the Chern-Weil triple $(\Omega^\bullet(X, \mathbf{M}_l(\mathbb{C})), \text{Tr}, \Omega^\bullet(X))$ is the Chern character of E .

It is not hard to see that $(C^\bullet(X, \mathbf{M}_l(\mathbb{C})), \text{Tr}, C^\bullet(X))$, where $C^\bullet(X, \mathbf{M}_l(\mathbb{C}))$ is the curved A_∞ -algebra from Example 3.9, is a Chern-Weil triple. Indeed, condition (i) in Definition 4.1 is easily checked using Proposition 3.4 and the fact that the trace vanishes on commutators. Condition (ii) directly follows from Lemma 3.7 since we can write the sum over all permutations as a sum of subsums over cyclic permutations.

Similarly, the curved A_∞ -morphism from $C^\bullet(X, \mathbf{M}_l(\mathbb{C}))$ to $\Omega^\bullet(X, \mathbf{M}_l(\mathbb{C}))$ given by Proposition 3.3 induces a morphism of Chern-Weil triples. This morphism however does not satisfy the assumptions of Proposition 4.9. Thus, we choose a different route to show that the Chern character of the Chern-Weil triple defined by the curved A_∞ -algebra $C^\bullet(X, \mathbf{M}_l(\mathbb{C}))$ coincide with the Chern character of the bundle E .

Lemma 4.10. *Let X be locally finite polyhedron and let $C^\bullet(X)$ be the A_∞ -algebra defined after Theorem 2.16. There exists a strict quasi-isomorphism of A_∞ -algebras from $C^\bullet(X)$ to $C^\bullet(\tilde{X})$ for every sufficiently fine subdivision \tilde{X} of X .*

Proof. If \tilde{X} is a fine enough subdivision of X then there exists a simplicial map $\varphi : \tilde{X} \rightarrow X$ which can be obtained by applying the simplicial approximation theorem to the identity map on X . The map φ clearly induces a dg algebra quasi-isomorphism

$$\varphi^* : \Omega^\bullet(X) \rightarrow \Omega^\bullet(\tilde{X}).$$

It follows directly from the definition of the homotopy H given in [7, Chapter 2] that it commutes with simplicial maps. In particular, we have

$$\varphi^* H_X = H_{\tilde{X}} \varphi^*$$

Let $\tilde{W}_X : C^\bullet(X) \rightarrow \Omega^\bullet(X)$ and $\tilde{R}_{\tilde{X}} : \Omega^\bullet(\tilde{X}) \rightarrow C^\bullet(\tilde{X})$ be the A_∞ -morphisms given by Proposition 3.3. Then Lemma 3.6 implies that the composition $\tilde{R}_{\tilde{X}} \varphi^* \tilde{W}_X$ is a strict morphism of A_∞ -algebras. \square

Theorem 4.11. *The images under the De Rham isomorphism of the Chern character of the Chern-Weil triple $(C^\bullet(X, \mathbf{M}_l(\mathbb{C})), \text{Tr}, C^\bullet(X))$ coincide with the Chern character of E .*

Proof. Given a polyhedron Y and $\gamma \in \Omega^1(Y, \mathbf{M}_l(\mathbb{C}))$, denote the curved dg algebra obtained by deforming the dg algebra $\Omega^\bullet(Y, \mathbf{M}_l(\mathbb{C}))$ along γ by $\Omega^\bullet(Y)^\gamma$. Proceeding as in the proof of Lemma 4.10, we see that there exists a sequence of subdivisions $\{X_j\}_{j=1}^\infty$ of X satisfying the following conditions:

(1) For every j there exists a simplicial map $\varphi_j : X_j \rightarrow X$ that induces a morphism of curved dg algebras

$$\varphi_j^* : \Omega^\bullet(X)^\gamma \rightarrow \Omega^\bullet(X_j)^{\varphi_j^*(\gamma)}$$

and isomorphisms

$$\begin{aligned} \varphi_{\Omega,j}^* &: H_{dR}^\bullet(X) \rightarrow H_{dR}^\bullet(X_j), \\ \varphi_{C,j}^* &: H_{\text{simp}}^\bullet(X) \rightarrow H_{\text{simp}}^\bullet(X_j) \end{aligned}$$

on De Rham and simplicial cohomology respectively, that commute with the De Rham isomorphisms.

(2) The fullness of X_j is bounded from below and the mesh of X_j tends to 0 as $j \rightarrow \infty$ (see [5] for the relevant definitions).

Let $C^\bullet(X)^\gamma$ (respectively $C^\bullet(X_j)^{\varphi_j^*(\gamma)}$) denote the curved A_∞ -algebras obtained by transferring the curved dg structure on $\Omega^\bullet(X)^\gamma$ (respectively $\Omega^\bullet(X_j)^{\varphi_j^*(\gamma)}$) as in Example 3.9. Then Lemma 3.6 implies that there exists a strict morphism of curved A_∞ -algebras between $C^\bullet(X)^\gamma$ and $C^\bullet(X_j)^{\varphi_j^*(\gamma)}$ which clearly coincides with $\varphi_{C,j}^*$ at the cohomology level.

Let us denote the Chern characters of the Chern-Weil triples defined by the curved dg algebras $\Omega^\bullet(X)^\gamma$ and $\Omega^\bullet(X_j)^{\varphi_j^*(\gamma)}$ by $\mathbf{Ch}_\Omega^\bullet(X)$ and $\mathbf{Ch}_\Omega^\bullet(X_j)$ respectively, and the Chern characters of the Chern-Weil triples defined by the curved A_∞ -algebras $C^\bullet(X)^\gamma$ and $C^\bullet(X_j)^{\varphi_j^*(\gamma)}$ by $\mathbf{Ch}_C^\bullet(X)$ and $\mathbf{Ch}_C^\bullet(X_j)$ respectively. Then by naturality of the Chern character under strict A_∞ -morphisms we have

$$(4.26) \quad \overline{W}_X((\varphi_{C,j}^*)^{-1}(\mathbf{Ch}_C^\bullet(X_j))) = \overline{W}_X(\mathbf{Ch}_C^\bullet(X))$$

for every j , where \overline{W}_X denotes the De Rham isomorphism induced by the Whitney map W_X .

Now using condition (2) above one can show exactly as in [33, Theorems 5.4 and 5.12] that for all $p > 2$ and all forms $\omega_1, \dots, \omega_p$ in $\Omega^\bullet(X, \mathbf{M}_l(\mathbb{C}))$ one has

$$\lim_{j \rightarrow \infty} \|W_{X_j}(m_2^j(R_{X_j}(r_j(\omega_1)), R_{X_j}(r_j(\omega_2)))) - \omega_1 \wedge \omega_2\|_j = 0,$$

$$\lim_{j \rightarrow \infty} \|W_{X_j}(m_p^j(R_{X_j}(r_j(\omega_1)), \dots, R_{X_j}(r_j(\omega_p))))\|_j = 0,$$

where $\{m_p^j\}$ is the curved A_∞ -structure on $C^\bullet(X_j)^{\varphi_j^*(\gamma)}$, the map

$$r_j : \Omega^\bullet(X, \mathbf{M}_l(\mathbb{C})) \rightarrow \Omega^\bullet(X_j, \mathbf{M}_l(\mathbb{C}))$$

is the natural restriction of forms, and $\|\cdot\|_j$ is the L_2 -norm on $\Omega^\bullet(X_j, \mathbf{M}_l(\mathbb{C}))$. Using this, it is not hard to show that

$$(4.27) \quad \lim_{j \rightarrow \infty} \|\overline{W}_{X_j}(\mathbf{Ch}_C^\bullet(X_j)) - \mathbf{Ch}_\Omega^\bullet(X_j)\|_j = 0.$$

One can easily verify that the maps φ_j^* are uniformly bounded from below. It follows that $(\varphi_{\Omega,j}^*)^{-1}$ are uniformly bounded from above, hence Eq. (4.27) implies that

$$(4.28) \quad \lim_{j \rightarrow \infty} \|(\varphi_{\Omega,j}^*)^{-1}(\overline{W}_{X_j}(\mathbf{Ch}_C^\bullet(X_j)) - \mathbf{Ch}_\Omega^\bullet(X))\|_j = 0.$$

Now since

$$(\varphi_{\Omega,j}^*)^{-1}\overline{W}_{X_j} = \overline{W}_X(\varphi_{C,j}^*)^{-1},$$

comparing Eqs. (4.26) and (4.28) we conclude that $\overline{W}_X(\mathbf{Ch}_C^\bullet(X))$ coincides with $\mathbf{Ch}_\Omega^\bullet(X)$, the Chern character of E . \square

We note that Lemma 3.8 implies that the curved A_∞ -algebra $C^\bullet(X, \mathbf{M}_l(\mathbf{k}))^\gamma$ from Proposition 2.18 defines a Chern-Weil triple $(C^\bullet(X, \mathbf{M}_l(\mathbf{k}))^\gamma, \text{Tr}, C^\bullet(X))$. However, since $C^\bullet(X, \mathbf{M}_l(\mathbf{k}))^\gamma$ is a deformation of an ordinary A_∞ -algebra, Proposition 4.6 shows that one cannot obtain nontrivial Chern character from this Chern-Weil triple.

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