Hodge Integral Identities from the Cut-and-Join Equation of Mariño-Vafa Formula

Shengmao Zhu

Abstract: In this paper, we calculate the Laplace transform of cut-and-join equation of Mariño-Vafa formula and obtain a polynomial identity of Hodge integral. Subsequently, we present how to obtain some Hodge integral identities from this polynomial identity. Lastly, we will give a recursion formula for Hodge integral $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$ where $b_L = (b_1, ..., b_l)$.

Keywords: Hodge integral, Cut-and-join equation, Mariño-Vafa formula, Recursion formula.

Contents

1. Introduction

In a recent paper [7], B. Eynard, M. Mulase and B. Safnuk stated the Laplace transform of the cut-and-join equation of the partition function of Hurwitz numbers and obtained a polynomial identity of linear Hodge integrals. As an application, they proved the Bouchard-Mariño conjecture on Hurwitz numbers [2]. Then, M. Mulase and N. Zhang [14] showed that this polynomial identity can also be used to derive the DVV equation and λ_g-integral with the same method in [9].

In 2003, C.-C. Liu, K. Liu and J. Zhou [11] proved the celebrated Mariño-Vafa conjecture [13]. One of the main step in their proof is to show the generating function of Hodge integral with triple λ-classes $\mathcal{C}(\lambda, p; \tau)$ satisfies the cut-and-join equation. Combining the cut-and-join equation of the combinatorial side [16], they finished the proof of Mariño-Vafa formula. Moreover, the famous ELSV formula [6] for Hurwitz numbers is the large framing limit of Mariño-Vafa formula [12].

With the above motivations, we calculate the Laplace transform of the cut-and-join equation for Mariño-Vafa formula in the first part of this paper. The main result is

Theorem 1.1. *For $g \geq 1$ and $l \geq 1$, we have the following equation:*

$$
\begin{aligned}
(1) \quad & -(\tau^2 + \tau)^{l-2} \sum_{b_L \geq 0} \left((l-1)(2\tau+1)\langle \tau_{b_L} \Gamma_g(\tau) \rangle_g + (\tau^2 + \tau)\langle \tau_{b_L} \frac{d}{d\tau} \Gamma_g(\tau) \rangle_g \right) \\
& \cdot \hat{\Psi}_{b_L}(t_L; \tau) - (\tau^2 + \tau)^{l-1} \sum_{b_L \geq 0} \langle \tau_{b_L} \Gamma_g(\tau) \rangle_g \\
& \cdot \sum_{i=1}^{l} \left(\frac{\partial}{\partial \tau} \hat{\Psi}_{b_i}(t_i; \tau) + \frac{1}{t_i \tau + 1} \hat{\Psi}_{b_i+1}(t_i; \tau) \right) \hat{\Psi}_{b_{L \setminus \{i\}}}(t_{L \setminus \{i\}}; \tau) \\
& = -\frac{(\tau^2 + \tau)^{l-2}}{\tau + 1} \sum_{1 \leq i < j \leq l} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \Gamma_g(\tau) \rangle_g \hat{\Psi}_{b_{L \setminus \{i,j\}}}(t_{L \setminus \{i,j\}}; \tau)
\end{aligned}
$$

$$\cdot \frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau) - (t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i-t_j}$$

$$+\frac{(\tau^2+\tau)^{l-1}}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\a_2\geq 0\\b_{L\setminus\{i\}}\geq 0}}\Bigg((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}$$

$$-\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\coprod\mathcal{J}=L\setminus\{i\}\\2g_1-1+|\mathcal{I}|>0\\2g_2-1+|\mathcal{J}|>0}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Gamma_{g_2}(\tau)\rangle_{g_2}\Bigg)$$

$$\cdot\prod_{n=1}^{2}\hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)$$

where $L=\{1,2,..,l\}$ is an index set, and for any subset $I\subset L$, we denote

$$t_I=(t_i)_{i\in I},\ b_I=(b_i)_{i\in I},\ \tau_{b_I}=\prod_{i\in I}\tau_{b_i},\ \hat{\Psi}_{b_I}(t_I,\tau)=\prod_{i\in I}\hat{\Psi}_{b_i}(t_i,\tau)$$

and $\Gamma_g(\tau)=\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)$, $\hat{\Psi}_n(t;\tau)=\left(\frac{(t^2-t)(t\tau+1)}{\tau+1}\frac{d}{dt}\right)^n\left(\frac{t-1}{\tau+1}\right)$ for $n\geq 0$.

We remark that Theorem 1.1 is equivalent to the symmetrized cut-and-join equation of Mariño-Vafa formula obtained by L. Chen [3].

Furthermore, we use the result of Theorem 1.1 to obtain some Hodge integral identities. In fact, $\hat{\Psi}_b(t,\tau)$ can be written as

$$\hat{\Psi}_b(t;\tau)=\sum_{k=0}^{b}\frac{\tau^k}{(\tau+1)^{b+1}}\Psi_b^k(t)$$

and $\Psi_b^k(t)$, $0\leq k\leq b$ could be calculated by definition.

Firstly, we consider the expansion of τ at ∞. Taking the highest level of formula (1), we get

Corollary 1.2.

$$\sum_{b_L\geq 0}\langle\tau_{b_L}\Lambda_g^\vee(1)\rangle\left((2g-2+l)\Psi_{b_L}^{b_L}(t_L)+\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\Psi_{b_i}^{b_i}(t_i)\Psi_{b_{L\setminus\{i\}}}^{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})\right)$$

$$= \sum_{\substack{1 \le i < j \le l \\ b_{L\setminus\{i,j\}} \ge 0}} \sum_{a \ge 0} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \Lambda_g^\vee(1) \rangle_g \Psi_{b_{L\setminus\{i,j\}}}^{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$

$$\cdot \frac{(t_j - 1)t_i^2 \Psi_{a+1}^{a+1}(t_i) - (t_i - 1)t_j^2 \Psi_{a+1}^{a+1}(t_j)}{t_i - t_j}$$

$$+ \frac{1}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \ge 0 \\ a_2 \ge 0 \\ b_{L\setminus\{i\}} \ge 0}} \left(\langle \tau_{a_1} \tau_{a_2} \tau_{b_{L\setminus\{i\}}} \Lambda_{g-1}^\vee(1) \rangle_{g-1} + \sum_{\substack{g_1 + g_2 = g \\ \mathcal{I} \cup \mathcal{J} = L\setminus\{i\} \\ 2g_1 - 1 + |\mathcal{I}| > 0 \\ 2g_2 - 1 + |\mathcal{J}| > 0}} \langle \tau_{a_1} \tau_{b_\mathcal{I}} \Lambda_{g_1}^\vee(1) \rangle_{g_1} \right.$$

$$\left. \cdot \langle \tau_{a_2} \tau_{b_\mathcal{J}} \Lambda_{g_2}^\vee(1) \rangle_{g_2} \right) \Psi_{a_1+1}^{a_1+1}(t_i) \Psi_{a_2+1}^{a_2+1}(t_i) \Psi_{b_{L\setminus\{i\}}}^{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}}).$$

We show in Section 2 that $\Psi_b^b(t)$ is equal to $\hat{\xi}_b(t)$ which is defined by $\hat{\xi}_b(t) = ((t^3 - t^2)\frac{d}{dt})^b(t - 1)$ in formula (1.2) in [7]. Hence, the main Theorem 1.1 in [7] is the special case of formula (1) in this paper. Similarly, taking the sub-highest level of Theorem 1.1, we also get another Hodge integral identity in Section 3.

Next, we consider the expansion of τ at $\tau = 0$. In this case, the lowest level of formula (1) is

Corollary 1.3.

$$\sum_{b_L \ge 0} \langle \tau_{b_L} \lambda_g \rangle_g \Psi_{b_L}^0(t_L) = \frac{1}{l - 1} \sum_{\substack{1 \le i < j \le l \\ b_{L\setminus\{i,j\}} \ge 0}} \sum_{a \ge 0} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_g \rangle_g \Psi_{b_{L\setminus\{i,j\}}}^0(t_{L\setminus\{i,j\}})$$

$$\cdot \frac{(t_j - 1)t_i^2 \Psi_{a+1}^0(t_i) - (t_i - 1)t_j^2 \Psi_{a+1}^0(t_j)}{t_i - t_j}.$$

We can rederive the λ_g-integral [8] from Corollary 1.3.

Furthermore, we also pick up the sub-lowest level of formula (1) and get the following identity:

Corollary 1.4.

$$\sum_{b_L \ge 0} \langle \tau_{b_L} \lambda_g \rangle_g l \left(-(|b_L| + 1)\Psi_{b_L}^0(t_L) + \sum_{j=1} \Psi_{b_j}^1(t_j) \Psi_{b_{L\setminus\{j\}}}^0(t_{L\setminus\{j\}}) \right)$$

$$+ \sum_{b_L \geq 0} \langle \tau_{b_L} \lambda_g \rangle_g \sum_{i=1}^{l} (t_i^2 - t_i) \frac{\partial}{\partial t_i} \Psi_{b_i}^0(t_i) \Psi_{b_{L \setminus \{i\}}}^0(t_{L \setminus \{i\}})$$

$$+ \sum_{b_L \geq 0} \langle \tau_{b_L} \sum_{d=g-1}^{3g-3} P_d(\lambda) \rangle_g l \Psi_{b_L}^0(t_L)$$

$$= \sum_{\substack{1 \leq i < j \leq l}} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \lambda_g \rangle_g$$

$$\cdot \left(\frac{(t_j - 1)t_i \left(\Psi_{a+1}^1(t_i) + (t_i - |b_{L \setminus \{i,j\}}| - a - 3)\Psi_{a+1}^0(t_i) \right)}{t_i - t_j} \right.$$

$$\left. - \frac{(t_i - 1)t_j \left(\Psi_{a+1}^1(t_j) + (t_j - |b_{L \setminus \{i,j\}}| - a - 3)\Psi_{a+1}^0(t_j) \right)}{t_i - t_j} \right) \Psi_{b_{L \setminus \{i,j\}}}^0(t_{L \setminus \{i,j\}})$$

$$+ \sum_{\substack{1 \leq i < j \leq l}} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \lambda_g \rangle_g \sum_{r \neq i,j} \Psi_{b_r}^1(t_r) \Psi_{b_{L \setminus \{i,j\}}}^0(t_{L \setminus \{i,j\}})$$

$$\cdot \frac{(t_j - 1)t_i \Psi_{a+1}^0(t_i) - (t_i - 1)t_j \Psi_{a+1}^0(t_j)}{t_i - t_j} + \sum_{\substack{1 \leq i < j \leq l}} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \sum_{d=g-1}^{3g-3} P_d(\lambda) \rangle_g$$

$$\cdot \Psi_{b_{L \setminus \{i,j\}}}^0(t_{L \setminus \{i,j\}}) \frac{(t_j - 1)t_i \Psi_{a+1}^0(t_i) - (t_i - 1)t_j \Psi_{a+1}^0(t_j)}{t_i - t_j}$$

$$+ \frac{1}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \geq 0 \\ a_2 \geq 0 \\ b_{L \setminus \{i\}} \geq 0}} \sum_{\substack{g_1 + g_2 = g \\ \mathcal{I} \cup \mathcal{J} = L \setminus \{i\} \\ 2g_1 - 1 + |\mathcal{I}| > 0 \\ 2g_2 - 1 + |\mathcal{J}| > 0}} \langle \tau_{a_1} \tau_{b_{\mathcal{I}}} \lambda_{g_1} \rangle_{g_1} \langle \tau_{a_2} \tau_{b_{\mathcal{J}}} \lambda_{g_2} \rangle_{g_2}$$

$$\cdot \Psi_{a_1+1}^0(t_i) \Psi_{a_2+1}^0(t_i) \Psi_{b_{L \setminus \{i\}}}^0(t_{L \setminus \{i\}})$$

where $\sum_{d=g-1}^{3g-3} P_d(\lambda) = \Lambda_g^\vee(1) a_1(\lambda)$ and $a_1(\lambda) = \sum_{m=1}^{g} m \lambda_{g-m} \lambda_g - (-1)^g \Lambda_g^\vee(-1) \lambda_{g-1}$.

The above identity contains Hodge integral of type $\langle \tau_{b_L} \sum_{d=g-1}^{3g-3} P_d(\lambda) \rangle_g$. By some direct calculations,

$$P_{g-1}(\lambda) = \lambda_{g-1}, P_g(\lambda) = g\lambda_g, \ P_{g+1}(\lambda) = -\lambda_g \lambda_1,$$

$$\cdots, \ P_{3g-3}(\lambda) = (-1)^{g+1} \lambda_g \lambda_{g-1} \lambda_{g-2}.$$

As an application of Corollary 1.4, we get the following Hodge integral recursion.

Theorem 1.5. *If $\sum_{i=1}^{l} b_i = 2g - 4 + l$, there exists a constant $C(g, l, b_1, .., b_l)$ related to $g, l, b_1, .., b_l$, such that*

$$\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g = \frac{1}{l} \sum_{1 \leq i < j \leq l} \langle \tau_{b_i + b_j - 1} \tau_{b_{L \setminus \{i,j\}}} \lambda_g \lambda_1 \rangle_g \frac{(b_i + b_j)!}{b_i! b_j!} + C(g, l, b_1, .., b_l)$$

where $C(g, l, b_1, .., b_l)$ is a very verbose combinatoric constant.

The initial value $\langle \tau_{2g-3} \lambda_g \lambda_1 \rangle_g = \frac{1}{12}[g(2g-3)b_g + b_1 b_{g-1}]$ has been obtained by Y. Li [10]. Combining the recursion formula in Theorem 1.5, we can get all the Hodge integral $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$. Moreover, the Hodge integrals $\langle \tau_{b_L} P_d(\lambda) \rangle_g$ appeared in Corollary 1.4 could also be calculated via the same method. But the computation will be more complicated.

2. The Laplace transform of cut-and-join equation of Mariño-Vafa formula

2.1. The Laplace transform of the generating function $\mathcal{C}_{g,\mu}(\tau)$.
At first, we introduce some notations followed by [11]. Let

$$\Lambda_g^\vee(t) = t^g - \lambda_1 t^{g-1} + \cdots + (-1)^g \lambda_g$$

be the Chern polynomial of $\mathbb{E}^\vee$, the dual of the Hodge bundle. For a partition μ given by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0,$$

we will use the standard notation $|\mu| = \sum_{i=1}^{l(\mu)} \mu_i$.

Let $\Gamma_g(\tau) = \Lambda_g^\vee(1) \Lambda_g^\vee(-\tau - 1) \Lambda_g^\vee(\tau)$ and we define,

$$\mathcal{C}_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|Aut(\mu)|} [\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i - 1}(\mu_i \tau + a)}{(\mu_i - 1)!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Gamma_g(\tau)}{\prod_{i=1}^{l(\mu)}(1 - \mu_i \psi_i)}$$

where the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,l}$ is the moduli space of stable curves satisfying the stability condition $2g - 2 + l > 0$. For the unstable cases $(g, l) = (0, 1)$ and $(0, 2)$, we define

$$\int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{1 - \mu\psi} = \frac{1}{\mu^2},$$

$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{(1 - \mu_1\psi_1)(1 - \mu_2\psi_2)} = \frac{1}{\mu_1 + \mu_2}.$$

It follows that

$$\mathcal{C}_{0,d}(\tau) = -\sqrt{-1}^{d+1} \frac{1}{\tau} \frac{\prod_{a=0}^{d-1}(d\tau + a)}{d!} d^{-2},$$

$$\mathcal{C}_{0,(\mu_1,\mu_2)}(\tau) = \frac{\sqrt{-1}^{\mu_1 + \mu_2}}{|Aut(\mu_1, \mu_2)|} \frac{\tau + 1}{\tau} \prod_{i=1}^{2} \frac{\prod_{a=0}^{\mu_i-1}(\mu_i\tau + a)}{\mu_i!} \frac{1}{\mu_1 + \mu_2}.$$

The Mariño-Vafa formula proved in [11] gives a direct combinatorial formula associated to representation of symmetric groups for the generation function of $\mathcal{C}_{g,\mu}(\tau)$. But we don't go further to this formula here.

Through a direct calculation, we get

$$(2) \qquad \mathcal{C}_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|Aut(\mu)|} [\tau(\tau + 1)]^{l(\mu)-1}$$

$$\cdot \sum_{\substack{b_i \geq 0 \\ i=1,...,l(\mu)}} \langle \prod_{i=1}^{l(\mu)} \tau_{b_i} \Gamma_g(\tau) \rangle_g \prod_{i=1}^{l(\mu)} \frac{1}{\tau} \frac{\prod_{a=0}^{\mu_i-1}(\mu_i\tau + a)}{\mu_i!} \mu_i^{b_i}.$$

Let $\mathcal{C}_g(\mu; \tau) = |Aut(\mu)|\mathcal{C}_{g,\mu}(\tau)$, we define the Laplace transform of $\mathcal{C}_{g,\mu}(\tau)$ as

$$\mathcal{C}_{g,l}(w_1, .., w_l) = \sum_{\mu \in \mathbb{N}^l} \frac{1}{\sqrt{-1}^{l+|\mu|}} \mathcal{C}_g(\mu; \tau) e^{-(\mu_1 w_1 + \cdots + \mu_l w_l)}$$

where we have let $l = l(\mu)$.

In the following of this subsection, we will show that the Laplace transform of $\mathcal{C}_{g,\mu}(\tau)$ has a nice expression in the new variable t. We need the following Lagrange inversion theorem,

Theorem 2.1. *Let $\phi(\lambda)$ be an invertible formal power series in the indeterminate λ. Then the functional equation $y = x\phi(y)$ has a unique formal power series solution $y = y(x)$. Moreover, if f is a formal power series, then*

$$(3) \qquad f(y(x)) = f(0) + \sum_{n \geq 1} \frac{x^n}{n} [\lambda^{n-1}] \frac{df(\lambda)}{d\lambda} \phi(\lambda)^n$$

and

$$(4) \qquad \frac{f(y(x))}{y} \frac{x dy(x)}{dx} = \sum_{n \geq 0} x^n [\lambda^n] f(\lambda) \phi(\lambda)^n.$$

We consider the framed Lambert curve $x = y(1-y)^\tau$. Put $\phi(y) = \frac{1}{(1-y)^\tau}$ and $f(y) = y$ in formula (3), we have $y = \sum_{k \geq 1} \frac{\prod_{a=0}^{k-2}(k\tau+a)}{k!} x^k$. We now introduce the new variables t, w through $t = \frac{1}{1-(\tau+1)y}$ and $x = e^{-w}$. It is easy to see that the derivative of variables w, x, y, t satisfy the following relations:

$$-\frac{d}{dw} = x\frac{d}{dx} = \frac{(1-y)y}{1-(\tau+1)y}\frac{d}{dy} = \left(\frac{t(t-1)(t\tau+1)}{\tau+1}\right)\frac{d}{dt}.$$

One can get that

$$t = 1 - \frac{(\tau+1)y}{\tau} + \frac{(\tau+1)^2}{\tau}x\frac{dy}{dx} = 1 + \frac{1+\tau}{\tau}\sum_{k \geq 1} \frac{\prod_{a=0}^{k-1}(k\tau+a)}{k!} x^k.$$

We now define the polynomials

$$\hat{\Psi}_n(t;\tau) = \left(\left(\frac{t(t-1)(t\tau+1)}{\tau+1}\right)\frac{d}{dt}\right)^n \frac{t-1}{\tau+1}$$

for $n \geq 0$. Finally, the Laplace transform of $\mathcal{C}_{g,\mu}(\tau)$ can be written in variable t,

$$(5) \qquad \hat{\mathcal{C}}_{g,l}(t_1,..,t_l) = \sum_{\mu \in \mathbb{N}^l} \frac{1}{\sqrt{-1}^{l+|\mu|}} \mathcal{C}_g(\mu;\tau) e^{-(\mu_1 w_1 + \cdots + \mu_l w_l)}$$

$$= -(\tau(\tau+1))^{l-1} \sum_{\substack{b_i \geq 0 \\ i=1,..,l}} \langle \prod_{i=1}^{l} \tau_{b_i} \Gamma_g(\tau) \rangle_g \prod_{i=1}^{l} \hat{\Psi}_{b_i}(t_i;\tau).$$

2.2. **The Laplace transform of the cut-and-join equation.** Let us introduce the formal variables $p = (p_1,..,p_n,..)$, and define $p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$ for a partition μ. We also introduce the generating functions

$$\mathcal{C}_{g,l}(\lambda, p; \tau) = \sum_{\mu,l(\mu)=l} \frac{\mathcal{C}_g(\mu;\tau)}{|Aut(\mu)|} p_\mu \lambda^{2g-2+l},$$

$$\mathcal{C}(\lambda, p; \tau) = \sum_{\substack{g \geq 0 \\ l \geq 1}} \mathcal{C}_{g,l}(\lambda, p; \tau).$$

In [11], C. Liu, K. Liu and J. Zhou proved that $\mathcal{C}(\lambda, p; \tau)$ satisfies the following cut-and-join equation

$$\frac{\partial \mathcal{C}}{\partial \tau} = \frac{\sqrt{-1}}{2} \lambda \sum_{i,j \geq 1} \left((i+j) p_i p_j \frac{\partial \mathcal{C}}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2 \mathcal{C}}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial \mathcal{C}}{\partial p_i} \frac{\partial \mathcal{C}}{\partial p_j} \right).$$

For every choice of $g \geq 1$ and a partition μ, the coefficient of $p_\mu \lambda^{2g-2+l(\mu)}$ in the above formula is

(6) $\quad \dfrac{\partial \mathcal{C}_g(\mu; \tau)}{\partial \tau} = \sqrt{-1} \displaystyle\sum_{i<j} (\mu_i + \mu_j) \mathcal{C}_g((\mu(\hat{i}, \hat{j}), \mu_i + \mu_j); \tau)$

$$+ \frac{\sqrt{-1}}{2} \sum_{i=1}^{l} \sum_{\alpha+\beta=\mu_i} \alpha\beta \left(\mathcal{C}_{g-1}(\mu(\hat{i}); \tau)) + \sum_{\substack{g_1+g_2=g \\ \nu_1 \coprod \nu_2 = \mu(\alpha,\hat{i})}} \mathcal{C}_{g_1}(\nu_1; \tau) \mathcal{C}_{g_2}(\nu_2; \tau) \right).$$

Let us first calculate the Laplace transform of the cut-and-join equation when $l = 1$.

Proposition 2.2. *When $g \geq 1$ and $l = 1$, the Laplace transform of the cut-and-join equation is:*

$$-\sum_{b \geq 0} \langle \tau_b \frac{d}{d\tau} \Gamma_g(\tau) \rangle_g \hat{\Psi}_b(t; \tau) - \sum_{b \geq 0} \langle \tau_b \Gamma_g(\tau) \rangle_g \left(\frac{d}{d\tau} \hat{\Psi}_b(t; \tau) + \frac{1}{t\tau+1} \hat{\Psi}_{b+1}(t; \tau) \right)$$

$$= \frac{1}{2} \sum_{a_1, a_2 \geq 0} \left((\tau(\tau+1)) \langle \tau_{a_1} \tau_{a_2} \Gamma_{g-1}(\tau) \rangle_{g-1} - \sum_{\substack{g_1+g_2=g \\ g_1>0 \\ g_2>0}} \langle \tau_{a_1} \Gamma_{g_1}(\tau) \rangle_{g_1} \langle \tau_{a_2} \Gamma_{g_2}(\tau) \rangle_{g_2} \right)$$

$$\cdot \hat{\Psi}_{a_1+1}(t; \tau) \hat{\Psi}_{a_2+1}(t; \tau).$$

Proof. When $l = 1$, the formula (6) is reduced to

(7) $\quad \dfrac{\partial \mathcal{C}_g(\mu; \tau)}{\partial \tau} = \dfrac{\sqrt{-1}}{2} \displaystyle\sum_{\alpha+\beta=\mu} \alpha\beta \left(\mathcal{C}_{g-1}((\alpha, \beta); \tau) + \sum_{g_1+g_2=g} \mathcal{C}_{g_1}(\alpha; \tau) \mathcal{C}_{g_2}(\beta; \tau) \right).$

It then follows that the Laplace transformation of LHS of formula (7) is

(8) $\quad \dfrac{\partial}{\partial \tau} \hat{\mathcal{C}}_{g,1}(t; \tau) = \dfrac{\partial t}{\partial \tau} \dfrac{\partial}{\partial t} \hat{\mathcal{C}}_{g,1}(t; \tau) + \dfrac{\partial}{\partial \tau} \hat{\mathcal{C}}_{g,1}(t; \tau),$

and the stable part of RHS of (7) becomes

$$\frac{1}{2}(\tau(\tau+1)) \sum_{a_1,a_2 \geq 0} \langle \tau_{a_1} \tau_{a_2} \Gamma_{g-1}(\tau) \rangle_{g-1} \hat{\Psi}_{a_1+1}(t;\tau) \hat{\Psi}_{a_2+1}(t;\tau)$$

$$-\frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_1>0 \\ g_2>0}} \langle \tau_{a_1} \Gamma_{g_1}(\tau) \rangle_{g_1} \langle \tau_{a_2} \Gamma_{g_2}(\tau) \rangle_{g_2} \hat{\Psi}_{a_1+1}(t;\tau) \hat{\Psi}_{a_2+1}(t;\tau),$$

In the following, we calculate the Laplace transform of the unstable term $\mathcal{C}_0(\alpha;\tau)\mathcal{C}_g(\beta;\tau) + \mathcal{C}_g(\alpha;\tau)\mathcal{C}_0(\beta;\tau)$. Since $\mathcal{C}_{0,d}(\tau) = -\sqrt{-1}^{d+1} \frac{1}{\tau} \frac{\prod_{a=0}^{d-1}(d\tau+a)}{d!} \frac{1}{d^2}$, it follows that its Laplace transform $\hat{\mathcal{C}}_{0,1}(w;\tau) = \sum_{d\geq 1} \frac{1}{\sqrt{-1}^{d+1}} \mathcal{C}_{0,d}(\tau) e^{-dw}$ satisfies:

$$\left(-\frac{d}{dw}\right)^2 \hat{\mathcal{C}}_{0,1}(w;\tau) = -\frac{t-1}{\tau+1} = -\frac{y}{1-(\tau+1)y}.$$

Hence $\frac{d}{dw}\hat{\mathcal{C}}_{0,1}(w;\tau) = -\ln(1-y)$ followed by $-\frac{d}{dw} = \frac{(1-y)y}{1-(1+\tau)y}\frac{d}{dy}$. Moreover, recall that the framed Lambert curve is $y(1-y)^\tau = x$, where y depends on τ. Taking the derivative of τ, we have the identity

$$-\ln(1-y) = \frac{\partial y}{\partial \tau} \frac{1-(\tau+1)y}{y(1-y)}.$$

Therefore, the Laplace transform of the unstable part of RHS of (6) is

$$(9) \qquad \frac{\partial y}{\partial \tau} \frac{1-(\tau+1)y}{y(1-y)} \frac{(t^2-t)(t\tau+1)}{\tau+1} \frac{\partial}{\partial t} \hat{\mathcal{C}}_{g,1}(t;\tau) = \frac{\partial y}{\partial \tau} t^2(\tau+1) \frac{\partial}{\partial t} \hat{\mathcal{C}}_{g,1}(t;\tau),$$

where we have used $t = \frac{1}{1-(\tau+1)y}$. We also have

$$(10) \qquad \frac{\partial t}{\partial \tau} = t^2 y + \frac{\partial y}{\partial \tau} t^2(\tau+1) = \frac{t^2-t}{\tau+1} + \frac{\partial y}{\partial \tau} t^2(\tau+1).$$

Hence, moving the unstable part to left hand side of formula (7), then by formulas (8),(9) and (10), we obtain

$$\left(\frac{\partial}{\partial \tau} + \frac{t^2-t}{(\tau+1)} \frac{\partial}{\partial t}\right) \hat{C}_{g,1}(t;\tau) = \text{stable part}$$

which is the Proposition 2.2. $\qquad \square$

To finish the computation for $l \geq 2$, we introduce two lemmas firstly.

Lemma 2.3. *When* $(g, l) = (0, 2)$, *we have the following Laplace transformation formula:*

$$\hat{\mathcal{C}}_{0,2}(w_1, w_2; \tau) = -\sum_{\alpha, \beta \geq 1} \frac{\tau + 1}{\tau} \frac{1}{\alpha + \beta} \frac{\prod_{a=0}^{\alpha-1}(\alpha\tau + a)}{\alpha!} \frac{\prod_{a=0}^{\beta-1}(\beta\tau + a)}{\beta!} e^{-\alpha w_1} e^{-\beta w_2}$$

$$= -\ln\left(\frac{y_1 - y_2}{x_1 - x_2}\right) - \tau \left(\ln(1 - y_1) + \ln(1 - y_2)\right).$$

Proof. By definition,

$$\hat{\mathcal{C}}_{0,2}(w_1, w_2; \tau) = \sum_{\alpha, \beta \geq 1} \frac{1}{\sqrt{-1}^{2+\alpha+\beta}} |Aut(\alpha, \beta)| \mathcal{C}_{0,(\alpha,\beta)}(\tau)$$

$$= -\sum_{\alpha, \beta \geq 1} \frac{\tau + 1}{\tau} \frac{1}{\alpha + \beta} \frac{\prod_{a=0}^{\alpha-1}(\alpha\tau + a)}{\alpha!} \frac{\prod_{a=0}^{\beta-1}(\beta\tau + a)}{\beta!} e^{-\alpha w_1} e^{-\beta w_2},$$

it follows that

$$\left(-\frac{d}{dw_1} - \frac{d}{dw_2}\right) \hat{\mathcal{C}}_{0,2}(w_1, w_2; \tau) = -\tau(\tau + 1)\hat{\Psi}_0(t_1; \tau)\hat{\Psi}_0(t_2; \tau)$$

$$= -\tau(\tau + 1)\frac{y_1 y_2}{(1 - (\tau + 1)y_1)(1 - (\tau + 1)y_2)}.$$

Then, we obtain

$$\hat{\mathcal{C}}_{0,2}(w_1, w_2; \tau) = -\ln\left(\frac{y_1 - y_2}{x_1 - x_2}\right) - \tau \left(\ln(1 - y_1) + \ln(1 - y_2)\right)$$

which followed by the relation

$$-\frac{d}{dw} = x\frac{d}{dx} = \frac{(1 - y)y}{1 - (\tau + 1)y}\frac{d}{dy}.$$

$\square$

Lemma 2.4. *We have the following identity:*

$$\sum_{\alpha, \beta \geq 1} \frac{1}{\tau} \frac{\prod_{a=0}^{(\alpha+\beta)-1}((\alpha + \beta)\tau + a)}{(\alpha + \beta)!} (\alpha + \beta)^{a+1} e^{-\alpha w_i} e^{-\beta w_j}$$

$$= \frac{x_i}{x_i - x_j}\hat{\Psi}_{a+1}(t_i; \tau) - \frac{x_j}{x_i - x_j}\hat{\Psi}_{a+1}(t_j; \tau) - \hat{\Psi}_{a+1}(t_i; \tau) - \hat{\Psi}_{a+1}(t_j; \tau).$$

Proof. This calculation is same as formula (3.15) showed in paper [7].

Let $\mu = \alpha + \beta$ and $\nu = \beta$, then

$$\sum_{\alpha,\beta \geq 1} \frac{1}{\tau} \frac{\prod_{a=0}^{(\alpha+\beta)-1}((\alpha+\beta)\tau + a)}{(\alpha+\beta)!}(\alpha+\beta)^{a+1}e^{-\alpha w_i}e^{-\beta w_j}$$

$$= \sum_{\alpha,\beta \geq 0} \frac{1}{\tau} \frac{\prod_{a=0}^{(\alpha+\beta)-1}((\alpha+\beta)\tau + a)}{(\alpha+\beta)!}(\alpha+\beta)^{a+1}e^{-\alpha w_i}e^{-\beta w_j}$$

$$- \sum_{\alpha \geq 1} \frac{1}{\tau} \frac{\prod_{a=0}^{\alpha-1}(\alpha\tau + a)}{\alpha!}\alpha^{a+1}e^{-\alpha w_i} - \sum_{\beta \geq 1} \frac{1}{\tau} \frac{\prod_{a=0}^{\beta-1}(\beta\tau + a)}{\beta!}\beta^{a+1}e^{-\beta w_j}$$

$$= \sum_{\mu \geq 0}\sum_{\nu=0}^{\mu} \frac{1}{\tau} \frac{\prod_{a=0}^{\mu-1}(\mu\tau + a)}{\mu!}\mu^{a+1}e^{-(\mu-\nu)w_i}e^{-\nu w_j} - \hat{\Psi}_{a+1}(t_i;\tau) - \hat{\Psi}_{a+1}(t_j;\tau)$$

$$= \sum_{\mu \geq 0} \frac{1}{\tau} \frac{\prod_{a=0}^{\mu-1}(\mu\tau + a)}{\mu!}\mu^{a+1}\frac{x_i^{\mu+1} - x_j^{\mu+1}}{x_i - x_j} - \hat{\Psi}_{a+1}(t_i;\tau) - \hat{\Psi}_{a+1}(t_j;\tau)$$

$$= \frac{x_i}{x_i - x_j}\hat{\Psi}_{a+1}(t_i;\tau) - \frac{x_j}{x_i - x_j}\hat{\Psi}_{a+1}(t_j;\tau) - \hat{\Psi}_{a+1}(t_i;\tau) - \hat{\Psi}_{a+1}(t_j;\tau).$$

$$\square$$

Now we present our main result in this paper.

Theorem 2.5. *For $g \geq 1$ and $l \geq 1$, we have the following equation:*

$$(11) \quad -(\tau^2 + \tau)^{l-2}\sum_{b_L \geq 0}\left((l-1)(2\tau+1)\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g + (\tau^2+\tau)\langle\tau_{b_L}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\right)$$

$$\cdot \hat{\Psi}_{b_L}(t_L;\tau) - (\tau^2+\tau)^{l-1}\sum_{b_L \geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g$$

$$\cdot \sum_{i=1}^{l}\left(\frac{\partial}{\partial\tau}\hat{\Psi}_{b_i}(t_i;\tau) + \frac{1}{t_i\tau+1}\hat{\Psi}_{b_i+1}(t_i;\tau)\right)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)$$

$$= -\frac{(\tau^2+\tau)^{l-2}}{\tau+1}\sum_{1 \leq i < j \leq l}\sum_{\substack{a \geq 0 \\ b_{L\setminus\{i,j\}} \geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)$$

$$\cdot \frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau) - (t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i - t_j}$$

$$+ \frac{(\tau^2+\tau)^{l-1}}{2}\sum_{i=1}^{l}\sum_{\substack{a_1 \geq 0 \\ a_2 \geq 0 \\ b_{L\setminus\{i\}} \geq 0}}\left((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\right.$$

$$\left.- \sum_{\substack{g_1+g_2=g \\ \mathcal{I} \amalg \mathcal{J}=L\setminus\{i\} \\ 2g_1-1+|\mathcal{I}|>0 \\ 2g_2-1+|\mathcal{J}|>0}} \langle \tau_{a_1}\tau_{b_\mathcal{I}}\Gamma_{g_1}(\tau)\rangle_{g_1} \langle \tau_{a_2}\tau_{b_\mathcal{J}}\Gamma_{g_2}(\tau)\rangle_{g_2} \right)$$

$$\cdot \prod_{n=1}^{2} \hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau).$$

Proof. By definition, the Laplace transform of LHS of equation (6) is

(12)
$$\sum_{\mu\in\mathbb{N}^l} \frac{1}{\sqrt{-1}^{l(\mu)+|\mu|}} \frac{\partial}{\partial\tau} \mathcal{C}_g(\mu;\tau)e^{-(\mu_1 w_1+\cdots+\mu_l w_l)}$$

$$= \frac{\partial}{\partial\tau}\left(\sum_{\mu\in\mathbb{N}} \frac{1}{\sqrt{-1}^{l(\mu)+|\mu|}} \mathcal{C}_g(\mu;\tau)e^{-(\mu_1 w_1+\cdots+\mu_l w_l)} \right)$$

$$= \frac{\partial}{\partial\tau}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)$$

$$= \sum_{i=1}^{l} \frac{\partial t_i}{\partial\tau} \frac{\partial}{\partial t_i}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau) + \frac{\partial}{\partial\tau}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau).$$

The Laplace transform of stable geometry in the cut-term of RHS of (6) is

(13)
$$\frac{\sqrt{-1}}{2} \sum_{\mu\in\mathbb{N}^l} \frac{1}{\sqrt{-1}^{l(\mu)+|\mu|}} \sum_{i=1}^{l} \sum_{\alpha+\beta=\mu_i} \alpha\beta(\mathcal{C}_{g-1}(\mu(\alpha,\hat{i});\tau)$$

$$+ \sum_{\substack{g_1+g_2=g \\ \nu_1 \amalg \nu_2=\mu(\alpha,\hat{i}) \\ 2g_1-2+|\nu_1|>0 \\ 2g_2-2+|\nu_2|>0}} \mathcal{C}_{g_1}(\nu_1;\tau)\mathcal{C}_{g_2}(\nu;\tau))e^{-(\mu_1 w_1+\cdots+\mu_l w_l)}$$

$$= \frac{(\tau^2+\tau)^{l-1}}{2} \sum_{i=1}^{l} \sum_{\substack{a_1\geq 0 \\ a_2\geq 0 \\ b_{L\setminus\{i\}}\geq 0}} \left((\tau^2+\tau)\langle \tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Gamma_{g-1}(\tau)\rangle_{g-1} \right.$$

$$\left. -\sum_{\substack{g_1+g_2=g \\ \mathcal{I}\coprod\mathcal{J}=\{1,..,\hat{i},.l\} \\ 2g_1-1+|\mathcal{I}|>0 \\ 2g_2-1+|\mathcal{J}|>0}} \langle \tau_{a_1}\tau_{b_{\mathcal{I}}}\Gamma_{g_1}(\tau)\rangle_{g_1} \langle \tau_{a_2}\tau_{b_{\mathcal{J}}}\Gamma_{g_2}(\tau)\rangle_{g_2} \right)$$

$$\cdot \prod_{n=1}^{2} \hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau).$$

The unstable geometry in the cut-term of RHS has two terms for $l \geq 2$:

$$U_1 = \frac{\sqrt{-1}}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\beta\left(\mathcal{C}_0(\alpha;\tau)\cdot\mathcal{C}_g((\mu(\hat{i}),\beta);\tau) + \mathcal{C}_g((\mu(\hat{i}),\alpha);\tau)\cdot\mathcal{C}_0(\beta;\tau)\right),$$

$$U_2 = \frac{\sqrt{-1}}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\beta\sum_{j\neq i}\left(\mathcal{C}_0((\mu_j,\alpha);\tau)\cdot\mathcal{C}_g((\mu(\hat{i},\hat{j}),\beta);\tau)\right.$$

$$\left. +\mathcal{C}_g((\mu(\hat{i},\hat{j}),\alpha);\tau)\cdot\mathcal{C}_0((\mu_j,\beta);\tau)\right).$$

As calculated in the proof of Proposition 2.2, the Laplace transform of U_1 is

$$\text{(14)} \qquad \sum_{i=1}^{l}\frac{\partial y_i}{\partial\tau}t_i^2(\tau+1)\frac{\partial}{\partial t_i}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau).$$

Moving the formula (14) to left hand side of (6), it will cancel the first term of formula (12), i.e.

$$\text{(15)} \qquad \text{(12)} - \text{(14)} = \left(\frac{\partial}{\partial\tau} + \sum_{i=1}^{l}\frac{t_i^2-t_i}{\tau+1}\frac{\partial}{\partial t_i}\right)\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)$$

$$= -(\tau^2+\tau)^{l-2}\sum_{b_L\geq 0}((l-1)(2\tau+1)\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g$$

$$+(\tau^2+\tau)\langle\tau_{b_L}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\right)\hat{\Psi}_{b_L}(t_L;\tau) - (\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g$$

$$\cdot\sum_{i=1}^{l}\left(\frac{\partial}{\partial\tau}\hat{\Psi}_{b_i}(t_i;\tau) + \frac{1}{t_i\tau+1}\hat{\Psi}_{b_i+1}(t_i;\tau)\right)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau).$$

Furthermore, the Laplace transform of U_2 equals to

$$\text{(16)} \qquad \sum_{\mu\in\mathbb{N}^l}\frac{1}{\sqrt{-1}^{l+|\mu|}}U_2 e^{-(\mu_1 w_1+\cdots+\mu_l w_l)}$$

$$= -\sum_{i=1}^{l} \sum_{j \neq i} \sum_{\substack{\mu_j \geq 1 \\ \alpha \geq 1}} \frac{1}{\sqrt{-1}^{2+\mu_j+\alpha}} \alpha \mathcal{C}_0((\mu_j, \alpha); \tau) e^{-\mu_j w_j} e^{-\alpha w_i}$$

$$\times \sum_{\substack{\beta \geq 0 \\ \mu_k \geq 0 \\ k \neq i,j}} \frac{1}{\sqrt{-1}^{l+|\mu|-\alpha-1}} \beta \mathcal{C}_g((\mu(\hat{i},\hat{j}), \beta); \tau) e^{-\beta w_i} e^{-\sum_{k \neq i,j} \mu_k w_k}$$

$$= \sum_{i=1}^{l} \sum_{j \neq i} \frac{\partial}{\partial w_i} \hat{\mathcal{C}}_{0,2}(w_i, w_j; \tau)(\tau^2 + \tau)^{l-2}$$

$$\cdot \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \Gamma_g(\tau) \rangle_g \hat{\Psi}_{a+1}(t_i; \tau) \hat{\Psi}_{b_{L \setminus \{i,j\}}}(t_{L \setminus \{i,j\}}; \tau)$$

$$= \sum_{i=1}^{l} \sum_{j \neq i} \left(-\frac{y_i}{y_i - y_j} \left(1 + \frac{\tau y_j}{1 - (\tau + 1)y_i} \right) + \frac{x_i}{x_i - x_j} \right)$$

$$\cdot (\tau^2 + \tau)^{l-2} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \Gamma_g(\tau) \rangle_g \hat{\Psi}_{a+1}(t_i; \tau) \hat{\Psi}_{b_{L \setminus \{i,j\}}}(t_{L \setminus \{i,j\}}; \tau)$$

$$= (\tau^2 + \tau)^{l-2} \sum_{1 \leq i < j \leq l} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \Gamma_g(\tau) \rangle_g \hat{\Psi}_{b_{L \setminus \{i,j\}}}(t_{L \setminus \{i,j\}}; \tau)$$

$$\cdot \left(\frac{x_i}{x_i - x_j} \hat{\Psi}_{a+1}(t_i; \tau) - \frac{x_j}{x_i - x_j} \hat{\Psi}_{a+1}(t_j; \tau) \right.$$

$$- \frac{y_i}{y_i - y_j} \left(1 + \frac{\tau y_j}{1 - (\tau + 1)y_i} \right) \hat{\Psi}_{a+1}(t_i; \tau)$$

$$\left. + \frac{y_j}{y_i - y_j} \left(1 + \frac{\tau y_i}{1 - (\tau + 1)y_j} \right) \hat{\Psi}_{a+1}(t_j; \tau) \right)$$

where we have used the Lemma 2.4.

Finally, we calculate the Laplace transform of the join term in RHS of (6)

$$(17) \quad \sum_{\mu \in \mathbb{N}^l} \sum_{i < j} \frac{\sqrt{-1}}{\sqrt{-1}^{l(\mu)+|\mu|}} (\mu_i + \mu_j) \mathcal{C}_g(\mu(\hat{i}, \hat{j}); \tau) e^{-\mu_i w_i} e^{-\mu_j w_j} e^{-\sum_{k \neq i,j} \mu_k w_k}$$

$$= -(\tau^2 + \tau)^{l-2} \sum_{\mu \in \mathbb{N}^l} \sum_{i < j} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \Gamma_g(\tau) \rangle_g \prod_{k \neq i,j} \frac{1}{\tau} \frac{\prod_{a=0}^{\mu_k - 1}(\mu_k \tau + a)}{\mu_k!}$$

$$\cdot \mu_k^{b_k} e^{-\sum_{k\neq i,j}\mu_k w_k} \frac{1}{\tau} \frac{\prod_{a=0}^{\mu_i+\mu_j-1}((\mu_i+\mu_j)\tau+a)}{(\mu_i+\mu_j)!}(\mu_i+\mu_j)^{a+1}e^{-\mu_i w_i}e^{-\mu_j w_j}$$

$$= -(\tau^2+\tau)^{l-2}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)$$

$$\cdot\left(\frac{x_i}{x_i-x_j}\hat{\Psi}_{a+1}(t_i;\tau)-\frac{x_j}{x_i-x_j}\hat{\Psi}_{a+1}(t_j;\tau)-\hat{\Psi}_{a+1}(t_i;\tau)-\hat{\Psi}_{a+1}(t_j;\tau)\right).$$

Combining (16) and (17), we get

$$(18) \qquad (16)+(17)$$

$$= -\frac{(\tau^2+\tau)^{l-2}}{\tau+1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)$$

$$\cdot\frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau)-(t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i-t_j}.$$

Collecting the remainder equations (15),(16) and (18), we obtain the identity (11) of Theorem 2.5. □

3. Some Hodge integral identities

In this section, we will show how to use the formula (1) to obtain some Hodge integral identities.

3.1. **Preliminary calculations.** For convenience, let us recall the formula (1) first: for $g\geq 1$ and $l\geq 1$,

$$(19) \qquad LHS := -(\tau^2+\tau)^{l-2}\sum_{b_L\geq 0}\Big((l-1)(2\tau+1)\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g$$

$$+ (\tau^2+\tau)\langle\tau_{b_L}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\Big)\hat{\Psi}_{b_L}(t_L;\tau)-(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g$$

$$\cdot\sum_{i=1}^{l}\left(\frac{\partial}{\partial\tau}\hat{\Psi}_{b_i}(t_i;\tau)+\frac{1}{t_i\tau+1}\hat{\Psi}_{b_i+1}(t_i;\tau)\right)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)$$

$$= T_1+T_2+T_3$$

where

$$T_1 := -\frac{(\tau^2+\tau)^{l-2}}{\tau+1} \sum_{1 \le i < j \le l} \sum_{\substack{a \ge 0 \\ b_{L\setminus\{i,j\}} \ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \Gamma_g(\tau) \rangle_g \hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}; \tau)$$

$$\cdot \frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau) - (t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i - t_j},$$

$$T_2 := \frac{(\tau^2+\tau)^l}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \ge 0 \\ a_2 \ge 0 \\ b_{L\setminus\{i\}} \ge 0}} \langle \tau_{a_1} \tau_{a_2} \tau_{b_{L\setminus\{i\}}} \Gamma_{g-1}(\tau) \rangle_{g-1}$$

$$\cdot \prod_{n=1}^{2} \hat{\Psi}_{a_n+1}(t_i;\tau) \hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}}; \tau),$$

$$T_3 := -\frac{(\tau^2+\tau)^{l-1}}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \ge 0 \\ a_2 \ge 0 \\ b_{L\setminus\{i\}}}} \sum_{\substack{g_1+g_2=g \\ \mathcal{I} \coprod \mathcal{J} = L\setminus\{i\} \\ 2g_1-1+|\mathcal{I}|>0 \\ 2g_2-1+|\mathcal{J}|>0}} \langle \tau_{a_1} \tau_{b_{\mathcal{I}}} \Gamma_{g_1}(\tau) \rangle_{g_1} \langle \tau_{a_2} \tau_{b_{\mathcal{J}}} \Gamma_{g_2}(\tau) \rangle_{g_2}$$

$$\cdot \prod_{n=1}^{2} \hat{\Psi}_{a_n+1}(t_i;\tau) \hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}}; \tau),$$

$$\hat{\Psi}_n(t;\tau) = \left(\frac{(t^2-t)(t\tau+1)}{\tau+1} \frac{d}{dt} \right)^n \left(\frac{t-1}{\tau+1} \right) \text{ for } n \ge 0,$$

and

$$\Gamma_g(\tau) = \Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau).$$

One can expand $\hat{\Psi}_n(t;\tau)$ and $\Gamma_g(\tau)$ as follow,

(20)
$$\hat{\Psi}_b(t;\tau) = \sum_{k=0}^{b} \frac{\tau^k}{(\tau+1)^{b+1}} \Psi_b^k(t)$$

and

(21)
$$\Gamma_g(\tau) = \sum_{m=0}^{2g} \Lambda_g^\vee(1) a_m(\lambda) \tau^m.$$

By the definition of $\Lambda_g^\vee(t) = \sum_{j=0}^{g}(-1)^{g-j}\lambda_{g-j}t^j$ and Mumford's relation $\Lambda_g^\vee(t)\Lambda_g^\vee(-t) = (-1)^g t^{2g}$, one can obtain the coefficients $a_m(\lambda)$ in (21). For example,

(22)
$$a_{2g}(\lambda) = (-1)^g, a_{2g-1}(\lambda) = (-1)^g g, \cdots$$

$$a_1(\lambda) = \sum_{m=1}^{g} m\lambda_{g-m}\lambda_g - (-1)^g\lambda_{g-1}\Lambda_g^\vee(-1),$$

$$a_0(\lambda) = (-1)^g\lambda_g\Lambda_g^\vee(-1).$$

On the other hand, by definition $\hat{\Psi}_{b+1}(t,\tau) = (t^2-t)(\frac{t\tau+1}{\tau+1})\frac{d}{dt}\hat{\Psi}_b(t,\tau)$, $\hat{\Psi}_0(t,\tau) = \frac{t-1}{\tau+1}$, it follows that $\hat{\Psi}_{b+1}(t,\tau)$ has the expansion form (20). Moreover, we have

$$(23) \qquad \Psi_{b+1}^k(t) = (t^3 - t^2)\frac{d}{dt}\Psi_b^{k-1}(t) + (t^2 - t)\frac{d}{dt}\Psi_b^k(t), \ k = 0, .., b+1.$$

and all the $\Psi_b^k(t)$ can be calculated.

As an illustration, we calculate some cases which will be used in the following discussions. By formula (23),

$$(24) \qquad\qquad \Psi_{b+1}^{b+1}(t) = (t^3 - t^2)\frac{d}{dt}\Psi_b^b(t),$$

$$(25) \qquad\qquad \Psi_{b+1}^b(t) = (t^3 - t^2)\frac{d}{dt}\Psi_b^{b-1}(t) + (t^2 - t)\frac{d}{dt}\Psi_b^b(t),$$

$$(26) \qquad\qquad \Psi_{b+1}^1(t) = (t^3 - t^2)\frac{d}{dt}\Psi_b^0(t) + (t^2 - t)\frac{d}{dt}\Psi_b^1(t),$$

$$(27) \qquad\qquad \Psi_{b+1}^0(t) = (t^2 - t)\frac{d}{dt}\Psi_b^0(t).$$

It is clear that, the recursion relation for $\Psi_b^b(t)$ in (24) is same as the definition of $\hat{\xi}_b(t)$ in [7] and they share the same initial. Hence $\Psi_b^b(t) = \hat{\xi}_b(t)$. We write $\Psi_b^b(t)$ as

$$(28) \qquad\qquad \Psi_b^b(t) = \sum_{i=b+1}^{2b+1} f^b(b,i)t^i$$

then all the $f^b(b,i)$ could be calculated from the recursion (24). For example, $f^b(b, 2b+1) = (2b-1)!!$, $f^b(b, b+1) = (-1)^b b!$ which are the coefficients used in the proof of DVV equation and λ_g integral respectively[14, 5, 9]. One can also let

$$(29) \qquad\qquad \Psi_b^{b-1}(t) = \sum_{i=b}^{2b} f^{b-1}(b,i)t^i$$

and by (25), we obtain

$$f^b(b+1,k) = (k-2)f^{b-1}(b,k-2) - (k-1)f^{b-1}(b,k-1)$$

$$+ (k-1)f^b(b, k-1) - kf^b(b, k).$$

For example,

$$f^b(b+1, 2b+2) = 2bf^{b-1}(b, 2b) + (2b+1)f^b(b, 2b+1)$$
$$= 2bf^{b-1}(b, 2b) + (2b+1)!!,$$

it follows that

$$(30) \qquad f^{b-1}(b, 2b) = \sum_{k=0}^{b-1} \frac{(2b-2)!!(2k+1)!!}{(2k)!!}.$$

Similarly,

$$f^b(b+1, b+1) = -bf^{b+1}(b, b) - (b+1)f^b(b, b+1)$$
$$= -bf^b(b, b) + (-1)^{b+1}(b+1)!,$$

hence

$$(31) \qquad f^{b-1}(b, b) = (-1)^b \frac{(b+1)!}{2}.$$

Moreover, we let

$$\Psi_b^0(t) = \sum_{i=1}^{b+1} f^0(b, i)y^i$$

and we have $f^0(b+1, i) = (i-1)f^0(b, i-1) - if^0(b, i)$ from (27). It is easy to show

$$(32) \qquad f^0(b, b+1) = b!.$$

Similarly, one can write $\Psi_b^1(t)$ as

$$(33) \qquad \Psi_b^1(t) = \sum_{j=2}^{b+2} f^1(b, j)t^j$$

where

$$f^1(b+1, k) = (k-2)f^0(b, k-2) - (k-1)f^0(b, k-1)$$
$$+ (k-1)f^1(b, k-1) - kf^1(b, k).$$

Hence,

$$(34) \qquad f^1(b+1, b+3) = (b+1)f^0(b, b+1) + (b+2)f^1(b, b+2),$$

by initial value $f^1(1,3) = 1$ and $f^0(b, b+1) = b!$, it follows

$$(35) \qquad f^1(b, b+2) = (b+1)! \sum_{k=2}^{b+1} \frac{1}{k}.$$

Now, let us substitute (20) and (21) to (19), one gets

$$
(36) \quad LHS = - \sum_{b_L \geq 0} \sum_{0 \leq k_L \leq b_L} \sum_{m=0}^{2g} \langle \tau_{b_L} \Lambda_g^\vee(1) a_m(\lambda) \rangle_g \Psi_{b_L}^{k_L}(t_L)
$$

$$
\cdot \frac{(l-2+m+|k_L|-|b_L|)\tau^{|k_L|+l+m-1} + (l-1+m+|k_L|)\tau^{|k_L|+l+m-2}}{(\tau+1)^{|b_L|+2}}
$$

$$
- \sum_{b_L \geq 0} \sum_{0 \leq k_L \leq b_L} \sum_{m=0}^{2g} \langle \tau_{b_L} \Lambda_g^\vee(1) a_m(\lambda) \rangle_g
$$

$$
\sum_{i=1}^{l} (t_i^2 - t_i) \frac{\partial}{\partial t_i} \Psi_{b_i}^{k_i}(t_i) \Psi_{b_{L\setminus\{i\}}}^{k_{L\setminus\{i\}}}(t_{L\setminus\{i\}}) \frac{\tau^{|b_L|+m+l-2}}{(\tau+1)^{|b_L|+2}},
$$

(37)

$$
T_1 = - \sum_{\substack{1 \leq i < j \leq l \\ b_{L\setminus\{i,j\}} \geq 0}} \sum_{a \geq 0} \sum_{m=0}^{2g} \sum_{\substack{0 \leq k \leq a+1 \\ 0 \leq k_{L\setminus\{i,j\}} \leq b_{L\setminus\{i,j\}}}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \Lambda_g^\vee(1) a_m(\lambda) \rangle_g
$$

$$
\cdot \Psi_{b_{L\setminus\{i,j\}}}^{k_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{(t_j-1)t_i^2 \Psi_{a+1}^k(t_i) - (t_i-1)t_j^2 \Psi_{a+1}^k(t_j)}{t_i - t_j} \frac{\tau^{|k_{L\setminus\{i,j\}}|+k+l+m-1}}{(\tau+1)^{|b_{L\setminus\{i,j\}}|+a+3}}
$$

$$
- \sum_{\substack{1 \leq i < j \leq l \\ b_{L\setminus\{i,j\}} \geq 0}} \sum_{a \geq 0} \sum_{m=0}^{2g} \sum_{\substack{0 \leq k \leq a+1 \\ 0 \leq k_{L\setminus\{i,j\}} \leq b_{L\setminus\{i,j\}}}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \Lambda_g^\vee(1) a_m(\lambda) \rangle_g
$$

$$
\cdot \Psi_{b_{L\setminus\{i,j\}}}^{k_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{(t_j-1)t_i \Psi_{a+1}^k(t_i) - (t_i-1)t_j \Psi_{a+1}^k(t_j)}{t_i - t_j} \frac{\tau^{|k_{L\setminus\{i,j\}}|+k+l+m-2}}{(\tau+1)^{|b_{L\setminus\{i,j\}}|+a+3}},
$$

$$
(38) \quad T_2 = \frac{1}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \geq 0 \\ a_2 \geq 0 \\ b_{L\setminus\{i\}} \geq 0}} \sum_{m=0}^{2(g-1)} \sum_{\substack{0 \leq n_1 \leq a_1+1 \\ 0 \leq n_2 \leq a_2+1 \\ 0 \leq k_{L\setminus\{i\}} \leq b_{L\setminus\{i\}}}} \langle \tau_{a_1} \tau_{a_2} \tau_{b_{L\setminus\{i\}}} \Lambda_{g-1}^\vee(1) a_m(\lambda) \rangle_{g-1}
$$

$$
\cdot \Psi_{a_1+1}^{n_1}(t_i) \Psi_{a_2+1}^{n_2}(t_i) \Psi_{b_{L\setminus\{i\}}}^{k_{L\setminus\{i\}}}(t_{L\setminus\{i\}}) \frac{\tau^{|k_{L\setminus\{i\}}|+n_1+n_2+l+m}}{(\tau+1)^{a_1+a_2+|k_{L\setminus\{i\}}|+3}},
$$

$$(39) \qquad T_3 = -\frac{1}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \geq 0 \\ a_2 \geq 0 \\ b_{L \setminus \{i\}} \geq 0}} \sum_{\substack{g_1 + g_2 = g \\ \mathcal{I} \cup \mathcal{J} = L \setminus \{i\} \\ 2g_1 - 1 + |\mathcal{I}| > 0 \\ 2g_2 - 1 + |J| > 0}} \sum_{\substack{0 \leq n_1 \leq a_1 + 1 \\ 0 \leq n_2 \leq a_2 + 1 \\ 0 \leq m_1 \leq 2g_1 \\ 0 \leq m_2 \leq 2g_2 \\ 0 \leq k_{L \setminus \{i\}} \leq b_{L \setminus \{i\}}}} \langle \tau_{a_1} \tau_{b_{\mathcal{I}}} \Lambda^{\vee}_{g_1}(1) a_{m_1}(\lambda) \rangle_{g_1}$$

$$\cdot \langle \tau_{a_2} \tau_{b_{\mathcal{J}}} \Lambda^{\vee}_{g_2}(1) a_{m_2}(\lambda) \rangle_{g_2} \Psi^{n_1}_{a_1 + 1}(t_i) \Psi^{n_2}_{a_2 + 1}(t_i)$$

$$\cdot \Psi^{k_{L \setminus \{i\}}}_{b_{L \setminus \{i\}}}(t_{L \setminus \{i\}}) \frac{\tau^{|k_{L \setminus \{i\}}| + n_1 + n_2 + l + m_1 + m_2 - 1}}{(\tau + 1)^{a_1 + a_2 + |k_{L \setminus \{i\}}| + 4}},$$

where we have used the notation $|b_L| = \sum_{i=1}^{l} b_i$ in above formulas. We note that (36), (37), (38) and (39) are all the functions of τ. If we expand them as the formal power series of τ at different points, we will get some identities of Hodge integrals after recollecting the coefficients of τ at both sides of the identity $(36) = (37) + (38) + (39)$.

3.2. **Expansion of τ at ∞.** For $b \geq 0$, one can expand at $\tau = \infty$ by

$$\frac{\tau^a}{(\tau + 1)^b} = \sum_{h \geq 0} \binom{-b}{h} \tau^{a - b - h}.$$

For example, the term containing τ in (36) is

$$\frac{\tau^{|k_L| + l + m - 1}}{(\tau + 1)^{|b_L| + 2}} = \sum_{h \geq 0} \binom{-|b_L| - 2}{h} \tau^{l + m - 3 + |k_L| - |b_L| - h}.$$

It is easy to see that the largest degree of τ is $2g + l - 3$ in the formulas (36), (37), (38), (39). If $F(t, \tau) \in Q[t][[\tau]]$, we will use the notation $[\tau^k]F(t, \tau)$ to denote the coefficient of τ^k in $F(t, \tau)$. By $a_{2g} = (-1)^g$ in (22) and some calculations, we obtain

$$(40) \qquad [\tau^{2g + l - 3}]LHS = (-1)^{g+1} \sum_{b_L \geq 0} \langle \tau_{b_L} \Lambda^{\vee}_g(1) \rangle \left((2g - 2 + l) \Psi^{b_L}_{b_L}(t_L) \right.$$

$$\left. + \sum_{i=1}^{l} (t_i^2 - t_i) \frac{\partial}{\partial t} \Psi^{b_i}_{b_i}(t_i) \Psi^{b_{L \setminus \{i\}}}_{b_{L \setminus \{i\}}}(t_{L \setminus \{i\}}) \right)$$

$$(41) \qquad [\tau^{2g + l - 3}]T_1 = (-1)^{g+1} \sum_{1 \leq i < j \leq l} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \Lambda^{\vee}_g(1) \rangle_g$$

$$\cdot \Psi^{b_{L\setminus\{i,j\}}}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{(t_j-1)t_i^2\Psi^{a+1}_{a+1}(t_i)-(t_i-1)t_j^2\Psi^{a+1}_{a+1}(t_j)}{t_i-t_j}$$

$$(42) \qquad [\tau^{2g+l-3}]T_2 = (-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0 \\ a_2\geq 0 \\ b_{L\setminus\{i\}}\geq 0}} \langle \tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda^{\vee}_{g-1}(1)\rangle_{g-1}$$

$$\cdot \Psi^{a_1+1}_{a_1+1}(t_i)\Psi^{a_2+1}_{a_2+1}(t_i)\Psi^{b_{L\setminus\{i\}}}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

$$(43) \qquad [\tau^{2g+l-3}]T_3 = (-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0 \\ a_2\geq 0 \\ b_{L\setminus\{i\}}\geq 0}}\sum_{\substack{g_1+g_2=g \\ \mathcal{I}\cup\mathcal{J}=L\setminus\{i\} \\ 2g_1-1+|\mathcal{I}|>0 \\ 2g_2-1+|\mathcal{J}|>0}} \langle \tau_{a_1}\tau_{b_{\mathcal{I}}}\Lambda^{\vee}_{g_1}(1)\rangle_{g_1}$$

$$\cdot \langle \tau_{a_2}\tau_{b_{\mathcal{J}}}\Lambda^{\vee}_{g_2}(1)\rangle_{g_2} \prod_{n=1}^{2}\Psi^{a_n+1}_{a_n+1}(t_i)\Psi^{b_{L\setminus\{i\}}}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

Since $\Psi^b_b(t)=\hat{\xi}_b(t)$, we obtain the following corollary from the identity $(40)=(41)+(42)+(43)$,

Corollary 3.1.

$$(44) \quad \sum_{b_L\geq 0}\langle\tau_{b_L}\Lambda^{\vee}_g(1)\rangle\left((2g-2+l)\hat{\xi}_{b_L}(t_L)+\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t}\hat{\xi}_{b_i}(t_i)\hat{\xi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})\right)$$

$$=\sum_{\substack{1\leq i<j\leq l}}\sum_{\substack{a\geq 0 \\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda^{\vee}_g(1)\rangle_g\hat{\xi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus i,j})$$

$$\cdot\frac{(t_j-1)t_i^2\hat{\xi}_{a+1}(t_i)-(t_i-1)t_j^2\hat{\xi}_{a+1}(t_j)}{t_i-t_j}$$

$$+\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0 \\ a_2\geq 0 \\ b_{L\setminus\{i\}}\geq 0}}\left(\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda^{\vee}_{g-1}(1)\rangle_{g-1}+\sum_{\substack{g_1+g_2=g \\ \mathcal{I}\cup\mathcal{J}=L\setminus\{i\} \\ 2g_1-1+|\mathcal{I}|>0 \\ 2g_2-1+|\mathcal{J}|>0}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Lambda^{\vee}_{g_1}(1)\rangle_{g_1}\right.$$

$$\left.\cdot\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Lambda^{\vee}_{g_2}(1)\rangle_{g_2}\right)\hat{\xi}_{a_1+1}(t_i)\hat{\xi}_{a_2+1}(t_i)\hat{\xi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

which is Theorem 1.1, one of main results, in the paper [7]. Formula (44) has been studied by many people to derive DVV equation [5, 14], λ_g-conjecture [9, 14] and λ_{g-1} Hodge integral recursion [18].

Obviously, we can obtain more such corollaries with the same procedure. For example, by $a_{2g} = (-1)^g$, $a_{2g-1}(\lambda) = (-1)^g g$ in (22), we have

$$(45) \quad [\tau^{2g+l-4}]LHS = (-1)^{g+1}(2g-3+l)\sum_{b_L \geq 0}\langle \tau_{b_L}\Lambda_g^{\vee}(1)\rangle_g$$

$$\cdot \left[(g-1-|b_L|)\Psi_{b_L}^{b_L}(t_L) + \sum_{i=1}^{l}\Psi_{b_i}^{b_i-1}(t_i)\Psi_{b_{L\backslash\{i\}}}^{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})\right] + (-1)^{g+1}$$

$$\cdot \sum_{b_L \geq 0}\langle \tau_{b_L}\Lambda_g^{\vee}(1)\rangle_g\left[\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\left((g-|b_L|-2)\Psi_{b_i}^{b_i}(t_i) + \Psi_{b_i}^{b_i-1}(t_i)\right)\right.$$

$$\left.\left(\Psi_{b_{L\backslash\{i\}}}^{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}}) + \sum_{j\neq i}\Psi_{b_j}^{b_j-1}(t_j)\Psi_{b_{L\backslash\{i,j\}}}^{b_{L\backslash\{i,j\}}}(t_{L\backslash\{i,j\}})\right)\right]$$

$$(46) \quad [\tau^{2g+l-4}]T_1 = (-1)^{g+1}\sum_{\substack{1\leq i<j\leq l}}\sum_{\substack{a\geq 0\\b_{L\backslash\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\backslash\{i,j\}}}\Lambda_g^{\vee}(1)\rangle_g\Psi_{b_{L\backslash\{i,j\}}}^{b_{L\backslash\{i,j\}}}(t_{L\backslash\{i,j\}})$$

$$\cdot \left(\frac{(t_j-1)t_i^2\left[\Psi_{a+1}^a(t_i) + (g-a-|b_{L\backslash\{i,j\}}|-3+\frac{1}{t_i})\Psi_{a+1}^{a+1}(t_i)\right]}{t_i-t_j}\right.$$

$$\left.-\frac{(t_i-1)t_j^2\left[\Psi_{a+1}^a(t_j) + (g-a-|b_{L\backslash\{i,j\}}|-3+\frac{1}{t_j})\Psi_{a+1}^{a+1}(t_j)\right]}{t_i-t_j}\right)$$

$$(47) \quad [\tau^{2g+l-4}]T_2 = (-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\a_2\geq 0\\b_{L\backslash\{i\}}\geq 0}}\langle \tau_{a_1}\tau_{a_2}\tau_{b_{L\backslash\{i\}}}\Lambda_{g-1}^{\vee}(1)\rangle_{g-1}$$

$$\cdot \left[(g-a_1-a_2-|b_{L\backslash\{i,j\}}|-4)\Psi_{a_1+1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)\right.$$

$$\left.+\Psi_{a_1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i) + \Psi_{a_1+1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2}(t_i)\right]\Psi_{b_{L\backslash\{i\}}}^{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})$$

$$+ (-1)^{g+1} \frac{1}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \geq 0 \\ a_2 \geq 0 \\ b_{L\setminus\{i\}} \geq 0}} \langle \tau_{a_1} \tau_{a_2} \tau_{b_{L\setminus\{i\}}} \Lambda_{g-1}^{\vee}(1) \rangle_{g-1}$$

$$\cdot \Psi_{a_1}^{a_1+1}(t_i) \Psi_{a_2+1}^{a_2+1}(t_i) \sum_{j \neq i} \Psi_{b_j}^{b_j-1}(t_j) \Psi_{b_{L\setminus\{i,j\}}}^{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$

$$(48) \qquad [\tau^{2g+l-4}] T_3 = (-1)^{g+1} \frac{1}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \geq 0 \\ a_2 \geq 0 \\ b_{L\setminus\{i\}} \geq 0}} \sum_{\substack{g_1+g_2=g \\ \mathcal{I} \cup \mathcal{J} = L\setminus\{i\}}}^{stable} \langle \tau_{a_1} \tau_{b_{\mathcal{I}}} \Lambda_{g_1}^{\vee}(1) \rangle_{g_1}$$

$$\cdot \langle \tau_{a_2} \tau_{b_{\mathcal{J}}} \Lambda_{g_2}^{\vee}(1) \rangle_{g_2} \Big[(g - a_1 - a_2 - |b_{L\setminus\{i,j\}}| - 4) \Psi_{a_1+1}^{a_1+1}(t_i) \Psi_{a_2+1}^{a_2+1}(t_i)$$

$$+ \Psi_{a_1+1}^{a_1}(t_i) \Psi_{a_2+1}^{a_2+1}(t_i) + \Psi_{a_1+1}^{a_1+1}(t_i) \Psi_{a_2+1}^{a_2}(t_i) \Big] \Psi_{b_{L\setminus\{i\}}}^{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

$$+ (-1)^{g+1} \frac{1}{2} \sum_{i=1}^{l} \sum_{\substack{a_1 \geq 0 \\ a_2 \geq 0 \\ b_{L\setminus\{i\}} \geq 0}} \sum_{\substack{g_1+g_2=g \\ \mathcal{I} \cup \mathcal{J} = L\setminus\{i\} \\ 2g_1-1+|\mathcal{I}|>0 \\ 2g_2-1+|\mathcal{J}|>0}} \langle \tau_{a_1} \tau_{b_{\mathcal{I}}} \Lambda_{g_1}^{\vee}(1) \rangle_{g_1}$$

$$\cdot \langle \tau_{a_2} \tau_{b_{\mathcal{J}}} \Lambda_{g_2}^{\vee}(1) \rangle_{g_2} \Psi_{a_1}^{a_1+1}(t_i) \Psi_{a_2+1}^{a_2+1}(t_i) \sum_{j \neq i} \Psi_{b_j}^{b_j-1}(t_j) \Psi_{b_{L\setminus\{i,j\}}}^{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$

Then, we also get a Hodge integral identity from the identity $(45) = (46) + (47) + (48)$ which contains the Hodge integral of type $\langle \tau_{b_L} \Lambda_g^{\vee}(1) \rangle_g$.

In order to get the Hodge integral identity with more than one λ-class, we need to compare the coefficients of τ^{2g+l-5} with the same procedure, we will obtain a Hodge integral identity which involves $\langle \tau_{b_L} \Lambda_g^{\vee}(1) \rangle_g$ and $\langle \tau_{b_L} \Lambda_g^{\vee}(1) \lambda_1 \rangle_g$, because $a_{2g-2}(\lambda) = (-1)^g \left(\frac{g(g-1)}{2} - \lambda_1 \right)$. And by this Hodge integral identity, we can calculate all the Hodge integral of the type $\langle \tau_{b_L} \lambda_k \lambda_1 \rangle_g$, $k = 1, .., g$. However, many terms will appear in the computation of the coefficients of τ^{2g+l-5} which makes the computation very complicated. So far, we have not written down the explicit formula to calculate the Hodge integral of the type $\langle \tau_{b_L} \lambda_k \lambda_1 \rangle_g$.

3.3. **Expansion of τ at 0.** Now, let us consider the expansion of $\frac{\tau^a}{(\tau+1)^b}$ at $\tau = 0$,

$$\frac{\tau^a}{(\tau+1)^b} = \sum_{i \geq 0} \binom{-b}{i} \tau^{a+i}.$$

It is easy to see that the lowest degree of τ in formula (36) is $l - 2$. By $a_0(\lambda) = (-1)^g \lambda_g \Lambda_g^\vee(-1)$ show in (22) and $\Lambda_g^\vee(1)\Lambda_g^\vee(-1) = (-1)^g$, we have

$$(49) \qquad [\tau^{l-2}]LHS = -(l-1) \sum_{b_L \geq 0} \langle \tau_{b_L} \lambda_g \rangle_g \Psi_{b_L}^0(t_L),$$

$$(50) \qquad [\tau^{l-2}]T_1 = - \sum_{\substack{1 \leq i < j \leq l}} \sum_{\substack{a \geq 0 \\ b_{L\setminus\{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_g \rangle_g \Psi_{b_{L\setminus\{i,j\}}}^0(t_{L\setminus\{i,j\}})$$
$$\times \frac{(t_j - 1)t_i^2 \Psi_{a+1}^0(t_i) - (t_i - 1)t_j^2 \Psi_{a+1}^0(t_j)}{t_i - t_j},$$

$$(51) \qquad [\tau^{l-2}]T_2 = [\tau^{l-2}]T_3 = 0.$$

Therefore, we obtain

Corollary 3.2.

$$(52) \qquad \sum_{b_L \geq 0} \langle \tau_{b_L} \lambda_g \rangle_g \Psi_{b_L}^0(t_L) = \frac{1}{l-1} \sum_{1 \leq i < j \leq l} \sum_{\substack{a \geq 0 \\ b_{L\setminus\{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_g \rangle_g$$
$$\cdot \Psi_{b_{L\setminus\{i,j\}}}^0(t_{L\setminus\{i,j\}}) \frac{(t_j - 1)t_i^2 \Psi_{a+1}^0(t_i) - (t_i - 1)t_j^2 \Psi_{a+1}^0(t_j)}{t_i - t_j}.$$

We now introduce two notations for the following expositions. Let

$$g(x_1, .., x_l) = \sum_{i_k \geq 0} a_{i_1 i_2 \cdots i_l} x_1^{i_1} \cdots x_l^{i_l} \in Q[x_1, .., x_l],$$

and denote by

$$F_d(g(x_1, \cdots, x_l)) = \sum_{\sum_{k=1}^{l} i_k = d} a_{i_1 i_2 \cdots i_l} x_1^{i_1} \cdots x_l^{i_l},$$

and

$$[x_1^{j_1} \cdots x_l^{j_l}]g(x_1, \cdots, x_l) = a_{j_1 j_2 \cdots j_l}.$$

the two operators acting on $g(x_1, .., x_l)$. Applying the above two operators on Corollary 3.2, we have

$$F_{2g-3+2l}(LHS \text{ of } (52)) = (l-1) \sum_{b_L \geq 0} \langle \tau_{b_L} \lambda_g \rangle_g b_L! y_L^{b_L+1},$$

$$F_{2g-3+2l}(RHS\ of\ (52)) = \sum_{1 \le i < j \le l} \sum_{\substack{a \ge 0 \\ b_L \ge 0}} \langle \tau_{b_{L \setminus \{i,j\}}} \lambda_g \rangle_g (a+1)!$$

$$\cdot\, b_{L \setminus \{i,j\}}! t_{L \setminus \{i,j\}}^{b_{L \setminus \{i,j\}}+1} \frac{t_j t_i^{a+4} - t_i t_j^{a+4}}{t_i - t_j}.$$

It follows that

$$[t_L^{b_L+1}] F_{2g-3+2l}(LHS\ of\ (52)) = (l-1) \langle \tau_{b_L} \lambda_g \rangle_g b_L!,$$

$$[t_L^{b_L+1}] F_{2g-3+2l}(RHS\ of\ (52)) = \sum_{1 \le i < j \le l} \langle \tau_{b_i+b_j-1} \tau_{b_{L \setminus \{i,j\}}} \lambda_g \rangle_g (b_i + b_j)! b_{L \setminus \{i,j\}}!.$$

So we get

$$\langle \tau_{b_L} \lambda_g \rangle_g = \frac{1}{l-1} \sum_{1 \le i < j \le l} \langle \tau_{b_i+b_j-1} \tau_{b_{L \setminus \{i,j\}}} \lambda_g \rangle_g \frac{(b_i + b_j)!}{b_i! b_j!}.$$

By induction on l, it follows the λ_g-integral formula:

$$\langle \tau_{b_L} \lambda_g \rangle_g = \binom{2g-3+l}{b_L} c_g.$$

In the following, we calculate the sub-lowest degree of τ. By $a_0(\lambda) = (-1)^g \lambda_g \Lambda_g^\vee(-1)$ and $\Lambda_g^\vee(1) \Lambda_g^\vee(-1) = (-1)^g$, we have

(53)

$$[\tau^{l-1}]LHS = -\sum_{b_L \ge 0} \langle \tau_{b_L} \lambda_g \rangle_g l \left(-(|b_L|+1) \Psi_{b_L}^0(t_L) + \sum_{j=1} \Psi_{b_j}^1(t_j) \Psi_{b_{L \setminus \{j\}}}^0(t_{L \setminus \{j\}}) \right)$$

$$- \sum_{b_L \ge 0} \langle \tau_{b_L} \lambda_g \rangle_g \sum_{i=1}^{l} (t_i^2 - t_i) \frac{\partial}{\partial t_i} \Psi_{b_i}^0(t_i) \Psi_{b_{L \setminus \{i\}}}^0(t_{L \setminus \{i\}})$$

$$- \sum_{b_L \ge 0} \langle \tau_{b_L} \Lambda_g^\vee(1) a_1(\lambda) \rangle_g l \Psi_{b_L}^0(t_L),$$

(54)

$$[\tau^{l-1}]T_1 = -\sum_{1 \le i < j \le l} \sum_{\substack{a \ge 0 \\ b_{L \setminus \{i,j\}} \ge 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \lambda_g \rangle_g$$

$$\cdot \left[\frac{(t_j - 1) t_i \left(\Psi_{a+1}^1(t_i) + (t_i - |b_{L \setminus \{i,j\}}| - a - 3) \Psi_{a+1}^0(t_i) \right)}{t_i - t_j} \right.$$

$$\left. -\frac{(t_i-1)t_j\left(\Psi^1_{a+1}(t_j)+(t_j-|b_{L\setminus\{i,j\}}|-a-3)\Psi^0_{a+1}(t_j)\right)}{t_i-t_j}\right]\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$

$$-\sum_{\substack{1\le i<j\le l\\ b_{L\setminus\{i,j\}}\ge 0}}\sum_{\substack{a\ge 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\sum_{r\ne i,j}\Psi^1_{b_r}(t_r)\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$

$$\cdot\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$-\sum_{\substack{1\le i<j\le l\\ b_{L\setminus\{i,j\}}\ge 0}}\sum_{\substack{a\ge 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda^\vee_g(1)a_1(\lambda)\rangle_g$$

$$\cdot\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j},$$

$$(55)\qquad\qquad [\tau^{l-1}]T_2=0,$$

$$(56)\qquad [\tau^{l-1}]T_3=-\frac{1}{2}\sum_{i=1}^l\sum_{\substack{a_1\ge 0\\ a_2\ge 0\\ b_{L\setminus\{i\}}\ge 0}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{i\}\\ 2g_1-1+|\mathcal{I}|>0\\ 2g_2-1+|\mathcal{J}|>0}}\langle\tau_{a_1}\tau_{b_\mathcal{I}}\lambda_{g_1}\rangle_{g_1}\langle\tau_{a_2}\tau_{b_\mathcal{J}}\lambda_{g_2}\rangle_{g_2}$$

$$\cdot\Psi^0_{a_1+1}(t_i)\Psi^0_{a_2+1}(t_i)\Psi^0_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}}).$$

Recall $a_1(\lambda)=\sum_{m=1}^g m\lambda_{g-m}\lambda_g-(-1)^g\Lambda^\vee_g(-1)\lambda_{g-1}$ in (22) and Mumford's relation $\Lambda^\vee_g(1)\Lambda^\vee_g(-1)=(-1)^g$. We have the form

$$(57)\qquad\qquad \Lambda^\vee_g(1)a_1(\lambda)=\sum_{d=g-1}^{3g-3}P_d(\lambda)$$

where $P_d(\lambda)$ is some combinatoric of λ-class with degree d. For example

$$(58)\qquad P_{g-1}(\lambda)=\lambda_{g-1},P_g(\lambda)=g\lambda_g,P_{g+1}(\lambda)=-\lambda_g\lambda_1,$$

$$\cdots,P_{3g-3}(\lambda)=(-1)^{g+1}\lambda_g\lambda_{g-1}\lambda_{g-2}.$$

By the above calculation, substituting (57) to the identity (53) = (54) + (56), we obtain Corollary 1.4 as showed in Section 1.

Since $\langle\tau_{b_L}\lambda_g\rangle_g=\binom{2g-3+l}{b_L}c_g$, thus all the Hodge integrals of type

$$(59)\qquad\qquad \langle\tau_{b_L}\sum_{d=g-1}^{3g-3}P_d(\lambda)\rangle_g$$

can be calculated by Corollary 1.4.

3.4. **A recursion formula for Hodge integral** $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$. Although Corollary 1.4 says that all the Hodge integral (59) is computable, it is not easy to write down their explicit formula. In this subsection, we will show how to obtain an recursion formula for $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$.

We have known that $\langle \tau_{b_L} \lambda_g \rangle_g = \binom{2g-3+l}{b_L} c_g$, and $\langle \tau_{b_L} \lambda_{g-1} \rangle_g$ can also be calculated easily from a recursion formula showed in [18]. Thus, we have

Theorem 3.3. *If* $\sum_{i=1}^{l} b_i = 2g - 4 + l$, *there exists a constant* $C(g, l, b_1, .., b_l)$ *related to* $g, l, b_1, .., b_l$, *such that*

$$(60) \quad \langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g = \frac{1}{l} \sum_{1 \leq i < j \leq l} \langle \tau_{b_i + b_j - 1} \tau_{b_{L \setminus \{i,j\}}} \lambda_g \lambda_1 \rangle_g \frac{(b_i + b_j)!}{b_i! b_j!} + C(g, l, b_1, ..b_l)$$

where $C(g, l, b_1, .., b_l)$ *is a combinatoric constant related to the Hodge integrals* $\langle \tau_{b_L} \lambda_g \rangle_g = \binom{2g-3+l}{b_L} c_g$, *and* $\langle \tau_{b_L} \lambda_{g-1} \rangle_g$.

Proof. Taking all the terms with degree $2g + 2l - 4$ in formula Corollary 1.4, we have

$$(61) \qquad F_{2g+2l-4}(LHS) = -l \sum_{b_L \geq 0} \langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g b_L! t_L^{b_L+1} + G_1(t_1, .., t_l)$$

$$(62) \qquad F_{2g+2l-4}(RHS) = - \sum_{\substack{1 \leq i < j \leq l \\}} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \lambda_g \lambda_1 \rangle_g$$

$$\cdot b_{L \setminus \{i,j\}}! (a+1)! \sum_{m=0}^{a+1} t_{L \setminus \{i,j\}}^{b_{L \setminus \{i,j\}}+1} t_i^{a+2-m} t_j^{m+1} + G_2(t_1, .., t_l)$$

where $G_i(t_1, .., t_l) = \sum_{|d_L| = 2g+2l-4} C_i(g, l, d_1, .., d_l) t_L^{d_L} \in \mathbb{Q}[t_1, ..t_l]$, $C_i(g, l, d_1, .., d_l)$ *is a combinatoric constant related to* $g, l, d_1, .., d_l$. *It follows that*

$$(63) \qquad \sum_{b_L \geq 0} \langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g b_L! t_L^{b_L+1} = \frac{1}{l} \sum_{1 \leq i < j \leq l} \sum_{\substack{a \geq 0 \\ b_{L \setminus \{i,j\}} \geq 0}} \langle \tau_a \tau_{b_{L \setminus \{i,j\}}} \lambda_g \lambda_1 \rangle_g$$

$$\cdot b_{L \setminus \{i,j\}}! (a+1)! \sum_{m=0}^{a+1} t_{L \setminus \{i,j\}}^{b_{L \setminus \{i,j\}}+1} t_i^{a+2-m} t_j^{m+1} + G(t_1, .., t_l)$$

where $G(t_1, .., t_l) = \sum_{|d_L|=2g+2l-4} \hat{C}(g, l, b_1, .., b_l) t_L^{d_L} = \frac{1}{l}(G_1(t_1, ..t_l) - G_2(t_1, .., t_l))$.
Then taking the coefficients of $t_L^{b_L+1}$ in (63), we get

$$(64) \qquad \langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g = \frac{1}{l} \sum_{1 \le i < j \le l} \langle \tau_{b_i+b_j-1} \tau_{b_{L \setminus \{i,j\}}} \lambda_g \lambda_1 \rangle_g \frac{(b_i + b_j)!}{b_i! b_j!}$$

$$+ \frac{\hat{C}(g, l, b_1 + 1, .., b_l + 1)}{b_L!}$$

Let $C(g, l, b_1, .., b_l) = \frac{\hat{C}(g,l,b_1+1,..,b_l+1)}{b_L!}$, Theorem 3.4 is proved.   □

The initial value $\langle \tau_{2g-3} \lambda_g \lambda_1 \rangle_g = \frac{1}{12}[g(2g-3)b_g + b_1 b_{g-1}]$ has been obtained by
Y. Li in [10]. Therefore, by the above recursion formula (64), one can calculate
all the Hodge integral $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$. This Hodge integral can also be computed
using Mumford's Chern character formula for λ_1 and the λ_g theorem [15].

4. Conclusion

With the help of the Laplace transform in this paper or the symmetrization
method in [3], we can convert the cut-and-join equation of Mariño-Vafa formula
to a polynomial identity which can be used to prove the BKMP conjecture [1] for
$\mathbb{C}^3$ [4, 17]. In this paper, we derived some corollaries about the Hodge integral
identities from this polynomial identity. In fact, we have found an algorithm to
calculate the Hodge integrals appearing in Mariño-Vafa formula by the following
two steps.

Step1: expanding τ at special points, then collect the corresponding level of τ
in Theorem 1.1.

Step2: taking certain degrees of t_L to obtain the corresponding Hodge integral
identities.

In this paper, we only calculate four cases by using Step1 and calculate a new
Hodge integral recursion for $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$ via Step2 from the result of last case after
doing Step1. We have not considered other cases here, because the computation
will be more complicated. We hope to implement this algorithm on a computer.

We note that, the recursion formulas for $\langle \tau_{b_L} \lambda_{g-1} \rangle_g$ [18] and $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$ in
Theorem 1.5 have the similar recursion structure. For a given g, Hodge integral
with l points will reduce to 1 points after only l times recursions. So it is a very

effective recursion relation. In fact, many Hodge integral recursions have such structure if they are calculated by above algorithm. We formulate the following combinatoric problem.

Let $Q_l(b_L) = Q_l(b_1, b_2, .., b_l)$ be a symmetric function on $(b_1, b_2, .., b_l) \in (\mathbb{Z}^+)^l$ which is defined by the following recursion relation. For given g, k,

$$Q_1(b) = \begin{cases} \text{constant}, & b = 3g - 2 - k, \\ 0, & \text{others.} \end{cases}$$

$$Q_l(b_L) = \begin{cases} \displaystyle\sum_{1 \leq i < j \leq l} Q_{l-1}(b_i + b_j - 1, b_{L \setminus \{i,j\}}) A_l(b_L) + B_l(b_L) & \sum_{i=1}^l b_i = 3g - 3 + l - k, \\ 0 & \text{others.} \end{cases}$$

where $A_l(b_L) = A_l(b_1, .., b_l)$ and $B_l(b_L) = B_l(b_1, .., b_l)$ are some fixed functions defined on $(b_1, b_2, .., b_l) \in (\mathbb{Z}^+)^l$.

It is interesting to study the properties of $Q_l(b_L)$ defined above. We hope that such combinatoric structure of Hodge integral will be studied further in future.

References

[1] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Remodeling the B-model*, Commun. Math. Phys. **287**, 117-178 (2009).

[2] V. Bouchard and M. Mariño, *Hurwitz numbers, matrix models and enumerative geometry*, Proc. Symposia Pure Math. **78**, 263–283 (2008).

[3] L. Chen, *Symmetrized cut-join equation of the Mariño-Vafa formula*, Pacific Journal of Math. **235**, No.2, 201-212 (2008).

[4] L. Chen, *Bouchard-Klemm-Mariño-Pasquetti Conjecture for* $\mathbb{C}^3$, arXiv:0910.3739v1.

[5] L. Chen, Y. Li and K. Liu, *Localization, Hurwitz numbers and the Witten conjecture*, Asian J. Math. **12** No.4, 511-518 (2008).

[6] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** No. 2, 297-327 (2001).

[7] B. Eynard, M. Mulase and B. Safnuk, *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, math.AG/0907.5224v1

[8] C. Faber and R. Pandharipande, *Hodge integrals, partition matrices, and the* λ_g *conjecture*, Ann. Math.(2) **157** No.1, 97-124 (2003).

[9] I.P. Goulden, D.M. Jackson and R. Vakil, *A short prooof of the* λ_g*-Conjecture without Gromov-Witten theory: Hurwitz theory and the moduli of curves*, math.AG/0604297.

[10] Y. Li, *Some results of the Mariño-Vafa Formula*, Math. Res. Lett. **13**, 10001-10018 (2006).

[11] C.-C. Liu, K. Liu and J. Zhou, *A proof of a conjecture of Mariño-Vafa on Hodge integrals*, J. Differential Geom. **65** (2003).

[12] C.-C. Liu, K. Liu and J. Zhou, *Mariño-Vafa formula and Hodge integral identities*, J. Algebric Geom. **15**, 379-398 (2006).

[13] M. Mariño and C. Vafa, *Framed knots at large N*, Orbifolds in mathematics and physics (Madison, WI, 2001), 185-204, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.

[14] M. Mulase and N. Zhang, *Polynomial recursion formula for linear Hodge integrals*, Communications in Number Theory and Physics **4** No.2, 267-294 (2010).

[15] H. Xu, *Private discussion*, 2010.

[16] J. Zhou, *Hodge integrals, Hurwitz numbers, and symmetric groups*, math.AG/0308024.

[17] J. Zhou, *Local Mirror Symmetry for One-Legged Topological Vertex*, arXiv:0910.4320.

[18] S. Zhu, *Hodge integral with one λ-class*. Sci. China Math. **55** No.1 (2012), 119-130.

Shengmao Zhu

Department of Mathematics and Center of Mathematical Sciences,

Zhejiang University,

Hangzhou, Zhejiang 310027, China

E-mail: zhushengmao@gmail.com