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$L^{n/2}$ Pinching Theorem for Submanifolds with Parallel Mean Curvature in $\mathbb{H}^{n+p}(-1)$*

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Abstract: Let H and S be the mean curvature and the squared length of the second fundamental form of submanifold M respectively. We prove that if M is an $n(\geq 3)$-dimensional complete submanifold with parallel mean curvature in $\mathbb{H}^{n+p}(-1)$, and if $\int_M (S-nH^2)^{n/2}dM < C(n,H)$, where $H>1$ and $C(n,H)$ is an explicit positive constant depending on n and H, then $S \equiv nH^2$, i.e., M is the totally umbilical sphere $S^n(1/\sqrt{H^2-1})$. Consequently, we show that if M is an $n(\geq 3)$-dimensional complete submanifold with parallel mean curvature in S^{n+p}, and if $\int_M (S-nH^2)^{n/2}dM < C'(n)$, where $C'(n)$ is an explicit positive constant depending only on n, then M is the totally umbilical sphere $S^n(1/\sqrt{1+H^2})$.
Keywords: Complete submanifolds, gap theorem, mean curvature vector, second fundamental form, Sobolev inequality.

1. Introduction

An important problem in global differential geometry is the study of relations between geometrical invariants and structures of Riemannian manifolds or submanifolds. After the pioneering rigidity theorem for minimal submanifolds in a

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sphere due to Simons [23], Lawson [9] and Chern-do Carmo-Kobayashi [5] obtained a famous rigidity theorem for oriented compact minimal submanifolds in S^{n+p} with $S \leq n/(2-1/p)$. It was partially extended to compact submanifolds with parallel mean curvature in a sphere by Okumura [18, 19], Yau [35] and others. In 1990, the first named author [29] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

Theorem A. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere S^{n+p}. If $S \leq C_1(n,p,H)$, then M is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, a Clifford hypersurface in an $(n+1)$-sphere, or the Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Here the constant $C_1(n,p,H)$ is defined by*

$$C_1(n,p,H) = \begin{cases} B(n,H), & for\ p=1,\ or\ p=2\ and\ H\neq 0,\\ \frac{n}{2-\frac{1}{p}}, & for\ p\geq 2\ and\ H=0,\\ \min\left\{B(n,H), \frac{n+nH^2}{2-\frac{1}{p-1}}+nH^2\right\}, & for\ p\geq 3\ and\ H\neq 0,\end{cases}$$

$$B(n,H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}.$$

Afterwards, the above pinching constant $C_1(n,p,H)$ was improved, by Li-Li [10] for $H=0$ and by Xu [30] for $H\neq 0$, to

$$C_2(n,p,H) = \begin{cases} B(n,H), & \text{for } p=1, \text{ or } p=2 \text{ and } H\neq 0,\\ \min\left\{B(n,H), \frac{1}{3}(2n+5nH^2)\right\}, & \text{otherwise.}\end{cases}$$

Further discussions in this direction have been carried out by Shiohama, Xu and other geometers (see [25, 26, 32], etc.). However, all these results have pointwise condition for S. It seems to be very interesting to study rigidity for minimal submanifolds under $L^{n/2}$-pinching condition for S. The $L^{n/2}$-pinching theorem for minimal hypersurfaces in a sphere was initiated by Shen [21]. Later, the $L^{n/2}$-pinching problem for submanifolds with parallel mean curvature, including minimal submanifolds, was investigated by Wang [28], Lin-Xia [12], Xu [29, 31], Bérard [1], Shiohama-Xu [24], Shen-Zhu [22], Ni [17], Xu-Gu [33, 34] and others. In [29], the first named author proved the following rigidity theorem.

Theorem B (also see [31]**).** *Let M be an n-dimensional closed submanifold* *with parallel mean curvature in* S^{n+p}*. Then there exists an explicit positive constant* $C_3(n)$ *depending only on* n *such that if* $\int_M |\mathring{A}|^n dM < C_3(n)$*, then* M *must be a totally umbilical sphere. Here* $|\mathring{A}|^2$ *is the squared length of the trace free second fundamental form of* M*, i.e.,* $|\mathring{A}|^2 = S - nH^2$.

In the case where M is a compact submanifold with parallel mean curvture in spaces forms, Shiohama and Xu [24] obtained the following theorem.

Theorem C. *Let* M *be an* n*-dimensional compact submanifold with parallel mean curvature in an* $(n+p)$*-dimensional simply connected space form* $\mathbb{F}^{n+p}(c)$ *with* $c \geq 0$*. Then there exists an explicit positive constant* $C_4(n)$ *depending only on* n *such that if* M *is not totally umbilical, then*

$$\int_M (S - nH^2)^{n/2} dM \geq C_4(n)(\sum_{i=0}^{n} \beta_i).$$

Here β_i *is the* i*-th Betti number of* M *with respect to an arbitrary fixed coefficient* J.

A general gap theorem for complete submanifolds with parallel mean curvature in $\mathbb{R}^{n+p}$ was proved by Xu and Gu [33], as stated:

Theorem D. *Let* $M^n (n \geq 3)$ *be a complete submanifold with parallel mean curvature in* $\mathbb{R}^{n+p}$*. Denote by* H *and* S *the mean curvature and the squared length of the second fundamental form of* M *respectively. If* $\int_M (S - nH^2)^{n/2} dM < C_5(n)$*, where* $C_5(n)$ *is an explicit positive constant depending only on* n*, then* $S \equiv nH^2$*, i.e.,* M *is a totally umbilical submanifold. In particular, if* $H = 0$*, then* $M = \mathbb{R}^n$*; if* $H \neq 0$*, then* $M = S^n(1/H)$.

In the present paper, we mainly study the $L^{n/2}$-pinching problem for n-dimensional complete submanifolds with parallel mean curvature in the standard hyperbolic space $\mathbb{H}^{n+p}(-1)$ with constant curvature -1, and obtain the following gap theorem.

Theorem 1(Main Theorem). *Let* $M^n (n \geq 3)$ *be an* n*-dimensional complete submanifold with parallel mean curvature in an* $(n+p)$*-dimensional hyperbolic space* $\mathbb{H}^{n+p}(-1)$*. Denote by* H *and* S *the mean curvature and the squared length* *of the second fundamental form of M respectively. If $\int_M (S - nH^2)^{n/2} dM < C(n, H)$, where $H > 1$ and $C(n, H)$ is an explicit positive constant depending on n and H, then $S \equiv nH^2$, i.e., M is the totally umbilical submanifold $S^n(1/\sqrt{H^2 - 1})$.*

Consequently, we have the following corollary.

Corollary. *Let $M^n (n \geq 3)$ be a complete submanifold with parallel mean curvature in the $(n + p)$-dimensional unit sphere S^{n+p}. Denote by H and S the mean curvature and the squared length of the second fundamental form of M respectively. If $\int_M (S - nH^2)^{n/2} dM < C'(n)$, where $C'(n)$ is an explicit positive constant depending only on n, which is defined in (12), then $S \equiv nH^2$, i.e., M is the totally umbilical sphere $S^n(1/\sqrt{1 + H^2})$.*

2. Notation and Lemmas

Let M^n be an n-dimensional Riemannian submanifold immersed in the $(n+p)$-dimensional standard hyperbolic space $\mathbb{H}^{n+p}(-1)$. We shall make use of the following convention on the rang of indices:

$$1 \leq A, B, C, \ldots \leq n + p;\ 1 \leq i, j, k, \ldots \leq n;\ n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p.$$

Choose a local orthonormal frame field $\{e_A\}$ in $\mathbb{H}^{n+p}(-1)$ such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of $\mathbb{H}^{n+p}(-1)$ respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j,\ h^\alpha_{ij} = h^\alpha_{ji}, \tag{1}$$

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{2}$$

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}), \tag{3}$$

$$A = \sum_{\alpha,i,j} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha, \tag{4}$$

$$\xi = \frac{1}{n}\sum_{\alpha,i} h_{ii}^{\alpha} e_{\alpha}, \tag{5}$$

where R_{ijkl}, $R_{\alpha\beta kl}$, A and ξ are the curvature tensor, the normal curvature tensor, the second fundamental form and the mean curvature vector of M respectively. The trace free second fundamental form of M is denoted by

$$\mathring{A} = \sum_{i,j,\alpha} \mathring{h}_{ij}^{\alpha} \omega_i \otimes \omega_j \otimes e_{\alpha}, \quad \text{where} \quad \mathring{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - \frac{1}{n}\sum_k h_{kk}^{\alpha}\delta_{ij}.$$

We define $S = |A|^2,\ H = |\xi|,\ H_{\alpha} = (h_{ij}^{\alpha})_{n\times n}$, then $|\mathring{A}|^2 = S - nH^2$.

Definition 1. *M is called a submanifold with parallel mean curvature if ξ is parallel in the normal bundle of M. In particular, M is called a minimal submanifold if $\xi = 0$.*

When $\xi \neq 0$, we choose e_{n+1} such that $e_{n+1}//\xi$, $\mathrm{tr}H_{n+1} = nH$ and $\mathrm{tr}H_{\beta} = 0$, $n+2 \leq \beta \leq n+p$. Set

$$S_H = \sum_{i,j} (h_{ij}^{n+1})^2, \quad S_I = \sum_{i,j,\beta\neq n+1} (h_{ij}^{\beta})^2.$$

The following lemmas will be used in the proof of our main results.

Lemma 1. *If M^n is a submanifold with parallel mean curvature in a space form of constant curvature, then either $H = 0$, or H is constant and $H_{n+1}H_{\alpha} = H_{\alpha}H_{n+1}$, for all α.*

Lemma 2 ([29])**.** *Let M^n be a submanifold with parallel mean curvature in $\mathbb{H}^{n+p}(-1)$. If $H \neq 0$, then*

$$\begin{aligned} \frac{1}{2}\Delta S_H \geq & \sum_{i,j,k} (h_{ijk}^{n+1})^2 + (S_H - nH^2)\Bigg[2nH^2 - n \\ & -S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S - nH^2}\Bigg], \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{1}{2}\Delta S_I \geq & \sum_{i,j,k,\beta\neq n+1} (h_{ijk}^{\beta})^2 + nH \sum_{\beta\neq n+1} tr(H_{n+1}H_{\beta}^2) \\ & - \sum_{\beta\neq n+1} [tr(H_{n+1}H_{\beta})]^2 - nS_I - \mu(p-1)S_I^2, \qquad \text{for } p \neq 1. \end{aligned} \tag{7}$$

Here

$$\mu(m) = \begin{cases} 1, & \textit{for } m = 1, \\ \frac{3}{2}, & \textit{for } m \geq 2. \end{cases} \tag{8}$$

By using the same argument as in [31], we have the following

Lemma 3. *Let M^n be a submanifold with parallel mean curvature in $\mathbb{H}^{n+p}(-1)$. Set $f_\varepsilon = (S_H - nH^2 + n\varepsilon^2)^{1/2}$, $g_\varepsilon = [S_I + n(p-1)\varepsilon^2]^{1/2}$. If $H \neq 0$, then*

$$\sum_{i,j,k} (h_{ijk}^{n+1})^2 \geq \frac{n+2}{n} |\nabla f_\varepsilon|^2, \tag{9}$$

$$\sum_{i,j,k,\beta \neq n+1} (h_{ijk}^{\beta})^2 \geq \frac{n+2}{n} |\nabla g_\varepsilon|^2, \quad \textit{for } p \neq 1. \tag{10}$$

From [8] and [31], we have

Lemma 4. *Let $M^n (n \geq 3)$ be a compact submanifold with or without boundary with parallel mean curvature in $\mathbb{H}^{n+p}(-1)$. Then for all $t \in \mathbb{R}^+$, and $f \in C^1(M)$, $f \geq 0$ (if the boundary $\partial M \neq \emptyset$, $f|_{\partial M} = 0$), f satisfies*

$$\|\nabla f\|_2^2 \geq \frac{(n-2)^2}{4(n-1)^2(1+t)} \left[\frac{1}{D^2(n)} \|f\|_{2n/(n-2)}^2 - H^2 (1 + \frac{1}{t}) \|f\|_2^2 \right], \tag{11}$$

where $D(n) = 2^n (1+n)^{(n+1)/n} (n-1)^{-1} \sigma_n^{-1/n}$ and σ_n =volume of the unit ball in $\mathbb{R}^n$.

3. Proof of Main Theorem

We first define our pinching constants as follows.

$$C(n,H) = \min\{\alpha^{n/2}(n,H),\ [\frac{2}{3}\beta(n,H)]^{n/2}\},$$
$$C'(n) = \min\{\alpha'^{n/2}(n),\ [\frac{2}{3}\beta'(n)]^{n/2}\},$$

$$\hat{C}(n,p,H)=\begin{cases}\alpha(n,H), & \text{for } p=1,\\ \min\{\alpha(n,H),\ \beta(n,H)\}, & \text{for } p=2,\\ \min\{\alpha(n,H),\ \frac{2}{3}\beta(n,H)\}, & \text{for } p\geq 3,\end{cases} \tag{12}$$

where

$$\begin{aligned}
\alpha(n,H) &= 2na(n,H)D^{-2}(n)[(a(n,H)b(n,H))^{1/2}\\
&\quad +(1+a(n,H))^{1/2}(2+b(n,H))^{1/2}]^{-2},\\
\beta(n,H) &= n(n^2-n+2)(n-2)^2(H^2-1)D^{-2}(n)[n^4(n-1)^2(H^2-1)\\
&\quad +(n^2-n+2)(n-2)^2H^2]^{-1},\\
a(n,H) &= (n^2-n+2)(n-2)^2H^2[n^4(n-1)^2(H^2-1)]^{-1},\\
b(n,H) &= (n-2)^2H^2[(2n-2)(H^2-1)]^{-1},\\
\alpha'(n) &= na'(n)D^{-2}(n)[(a'(n)b'(n))^{1/2}+(1+a'(n))^{1/2}(2+b'(n))^{1/2}]^{-2},\\
\beta'(n) &= n(n^2-n+2)(n-2)^2D^{-2}(n)[n^4(n-1)^2+2(n^2-n+2)(n-2)^2]^{-1},\\
a'(n) &= 2(n^2-n+2)(n-2)^2[n^4(n-1)^2]^{-1},\ b'(n)=(n-2)^2(n-1)^{-1}.
\end{aligned}$$

To prove Theorem 1, we give the following key lemma.

Lemma 5. *Let $M^n(n\geq 3)$ be a complete submanifold with parallel mean curvature in $\mathbb{H}^{n+p}(-1)$. Suppose that $H>1$ and $\|S-nH^2\|_{n/2}<\alpha(n,H)$. Then M is a pseudo-umbilical submanifold. In particular, if $p=1$, then M is a totally umbilical sphere in $\mathbb{H}^{n+1}(-1)$. Here $\|S-nH^2\|_K=[\int_M(S-nH^2)^K dM]^{1/K}$.*

Proof. Putting $f_\varepsilon=(S_H-nH^2+n\varepsilon^2)^{1/2}$, $f=(S_H-nH^2)^{1/2}$, we have $\Delta f_\varepsilon^2=\Delta f^2$. By Lemmas 2 and 3, we obtain

$$\frac{1}{2}\Delta f_\varepsilon^2\geq \frac{n+2}{n}|\nabla f_\varepsilon|^2+f^2\left[2nH^2-n-S-\frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right]. \tag{13}$$

We choose a cut-off function $\phi_R\in C^\infty(M)$ such that

$$\phi_R(x)=\begin{cases}1, & \text{if x}\in B_R(q),\\ 0, & \text{if x}\in M\setminus B_{2R}(q),\\ \phi_R(x)\in[0,1],\ \text{and } |\nabla\phi_R|\leq\frac{1}{R}, & \text{if x}\in B_{2R}(q)\setminus B_R(q),\end{cases}$$

where $B_r(q)$ is the geodesic ball in M with radius r centered at $q\in M$. In particular, if M is compact, and if $R\geq d$, where d is the diameter of M, then $\phi_R\equiv 1$ on M. Multiplying $\phi_R^2 f_\varepsilon^{2k-2}(k\geq 1)$ on the both sides of inequality (13) and integrating by parts, we get

$$
\begin{aligned}
0 \geq & \int_M \phi_R^2 f_\varepsilon^{2k-2} f^2 \left[2nH^2 - n - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right] dM \\
& + \frac{n+2}{nk^2}\int_M \phi_R^2 |\nabla f_\varepsilon^k|^2 dM + \frac{1}{2}\int_M \nabla(\phi_R^2 f_\varepsilon^{2k-2})\nabla f_\varepsilon^2 dM \\
& - \frac{1}{2}\int_M \mathrm{div}(\phi_R^2 f_\varepsilon^{2k-2}\nabla f_\varepsilon^2) dM \\
= & \int_M \phi_R^2 f_\varepsilon^{2k-2} f^2 \left[2nH^2 - n - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right] dM \\
& + \frac{2nk-n+2}{nk^2}\int_M \phi_R^2 |\nabla f_\varepsilon^k|^2 dM + 2\int_M \phi_R f_\varepsilon^{2k-1}\nabla\phi_R \nabla f_\varepsilon dM \\
\geq & \int_M \phi_R^2 f_\varepsilon^{2k-2} f^2 \left[2nH^2 - n - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right] dM \\
& + \frac{2nk-n+2}{nk^2}\int_M \phi_R^2 |\nabla f_\varepsilon^k|^2 dM + 2\int_M \phi_R f_\varepsilon^{2k-1}\nabla\phi_R \nabla f_\varepsilon dM \\
& + \sigma\int_M \phi_R f_\varepsilon^{2k-1}\nabla\phi_R\nabla f_\varepsilon dM - \frac{\rho\sigma}{2}\int_M \phi_R^2 f_\varepsilon^{2k-2}|\nabla f_\varepsilon|^2 dM \\
& - \frac{\sigma}{2\rho}\int_M |\nabla\phi_R|^2 f_\varepsilon^{2k} dM,
\end{aligned}
\tag{14}
$$

for all $\rho, \sigma \in \mathbb{R}^+$. Taking $k = \frac{n}{2}, \sigma = \frac{4(n^2-n+2)}{n^2} - \frac{n}{R} - 2$ and $\rho = \frac{n^2}{2R\sigma}$, where $R > \frac{n^3}{2(n^2-2n+4)}$, we get

$$
\begin{aligned}
0 \geq & \int_M \phi_R^2 f_\varepsilon^{n-2} f^2 \left[2nH^2 - n - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right] dM \\
& - \frac{R\sigma^2}{n^2}\int_M f_\varepsilon^n |\nabla\phi_R|^2 dM + \left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R}\right]\bigg(\int_M \phi_R^2 |\nabla f_\varepsilon^{n/2}|^2 dM \\
& + n\int_M \phi_R f_\varepsilon^{n-1}\nabla\phi_R\nabla f_\varepsilon dM\bigg) \\
\geq & -\left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right]\int_M f_\varepsilon^n |\nabla\phi_R|^2 dM \\
& + \left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R}\right]\int_M |\nabla(\phi_R f_\varepsilon^{n/2})|^2 dM \\
& + \int_M \phi_R^2 f_\varepsilon^{n-2} f^2 \left[2nH^2 - n - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right] dM.
\end{aligned}
\tag{15}
$$

By Lemma 4, we have

$$\begin{aligned}\|\nabla(\phi_R f_\varepsilon^{n/2})\|_2^2 \geq \frac{(n-2)^2}{4(n-1)^2(1+t)}\Big[&\frac{1}{D^2(n)}\|\phi_R f_\varepsilon^{n/2}\|_{2n/(n-2)}^2\\ &-H^2(1+\frac{1}{t})\|\phi_R f_\varepsilon^{n/2}\|_2^2\Big],\end{aligned} \tag{16}$$

for all $t \in \mathbb{R}^+$. From (15) and (16), we obtain

$$\begin{aligned}0 \geq &-\Big[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\Big]\int_M f_\varepsilon^n |\nabla\phi_R|^2 dM\\ &+\frac{[4R(n^2-n+2)-n^3](n-2)^2}{4Rn^3(n-1)^2(1+t)}\Big[\frac{1}{D^2(n)}\|\phi_R f_\varepsilon^{n/2}\|_{2n/(n-2)}^2\\ &-H^2(1+\frac{1}{t})\|\phi_R f_\varepsilon^{n/2}\|_2^2\Big] + \int_M \phi_R^2 f_\varepsilon^{n-2} f^2\Big\{nH^2 - n - (S-nH^2)\\ &-\frac{1}{2}\Big[\frac{n(n-2)^2H^2}{l(n-1)} + l(S-nH^2)\Big]\Big\}dM,\end{aligned} \tag{17}$$

for all $l \in \mathbb{R}^+$. As $\varepsilon \to 0$, (17) becomes

$$\begin{aligned}0 \geq &-\Big[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\Big]\int_M f^n |\nabla\phi_R|^2 dM\\ &+\frac{[4R(n^2-n+2)-n^3](n-2)^2}{4Rn^3(n-1)^2(1+t)}\Big[\frac{1}{D^2(n)}\|\phi_R f^{n/2}\|_{2n/(n-2)}^2\\ &-H^2(1+\frac{1}{t})\|\phi_R f^{n/2}\|_2^2\Big] + \Big[nH^2 - n - \frac{n(n-2)^2H^2}{2l(n-1)}\Big]\|\phi_R f^{n/2}\|_2^2\\ &-(1+\frac{l}{2})\|\phi_R^2 f^n\|_{n/(n-2)}\|S-nH^2\|_{n/2}\\ = &-\Big[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\Big]\int_M f^n |\nabla\phi_R|^2 dM\\ &+\Big\{\frac{[4R(n^2-n+2)-n^3](n-2)^2}{4Rn^3(n-1)^2D^2(n)(1+t)}\\ &-(1+\frac{l}{2})\|S-nH^2\|_{n/2}\Big\}\|\phi_R^2 f^n\|_{n/(n-2)}\\ &+\Big\{nH^2 - n - \frac{n(n-2)^2H^2}{2l(n-1)}\\ &-\frac{[4R(n^2-n+2)-n^3](n-2)^2H^2}{4Rn^3(n-1)^2 t}\Big\}\|\phi_R f^{n/2}\|_2^2,\end{aligned} \tag{18}$$

where $l \in \mathbb{R}^+$. Since $\int_M f^n dM < \alpha^{n/2}(n, H)$, as $R \to \infty$, we get

$$0 \leq \lim_{R\to\infty} \left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right] \int_M f^n |\nabla \phi_R|^2 dM$$

$$\leq \lim_{R\to\infty} \left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right] \frac{1}{R^2} \int_M f^n dM = 0. \tag{19}$$

From (18) and (19), we obtain

$$0 \geq \left[\frac{(n^2-n+2)(n-2)^2}{n^3(n-1)^2 D^2(n)(1+t)} - (1+\frac{l}{2})\|S - nH^2\|_{n/2}\right] \lim_{R\to\infty} \|\phi_R^2 f^n\|_{n/(n-2)}$$

$$+\left[nH^2 - n - \frac{n(n-2)^2 H^2}{2l(n-1)} - \frac{(n^2-n+2)(n-2)^2 H^2}{n^3(n-1)^2 t}\right] \lim_{R\to\infty} \|\phi_R f^{n/2}\|_2^2 \tag{20}$$

for all t, $l \in \mathbb{R}^+$. We take

$$t = t(l) = \frac{(n^2-n+2)(n-2)^2}{n^4(n-1)^2}\left[\frac{H^2-1}{H^2} - \frac{(n-2)^2}{2l(n-1)}\right]^{-1}, \ \ l > \frac{(n-2)^2 H^2}{2(n-1)(H^2-1)}.$$

This together with (20) yields

$$\left[\frac{(n^2-n+2)(n-2)^2}{n^3(n-1)^2 D^2(n)(1+t(l))} - (1+\frac{l}{2})\|S - nH^2\|_{n/2}\right] \lim_{R\to\infty} \|\phi_R^2 f^n\|_{n/(n-2)} \leq 0,$$

where $l \in \mathbb{R}^+$ satisfying

$$l > \frac{(n-2)^2 H^2}{2(n-1)(H^2-1)}.$$

By a computation, we have

$$\max_{l > \frac{(n-2)^2 H^2}{2(n-1)(H^2-1)}} \frac{1}{(2+l)(1+t(l))} = \frac{n^3(n-1)^2 D^2(n)\alpha(n,H)}{2(n^2-n+2)(n-2)^2}.$$

So,

$$\left[\alpha(n,H) - \|S - nH^2\|_{n/2}\right] \lim_{R\to\infty} \|\phi_R^2 f^n\|_{n/(n-2)} \leq 0.$$

From the assumption

$$\|S - nH^2\|_{n/2} < \alpha(n,H),$$

we conclude that $f \equiv 0$, i.e., $S_H \equiv nH^2$. Therefore, M is a pseudo-umbilical submanifold. If $p = 1$, then $S - nH^2 = f^2 \equiv 0$, i.e., M is the totally umbilical sphere $S^n(1/\sqrt{H^2-1})$ in $\mathbb{H}^{n+1}(-1)$.

Now we are in a position to give the proof of Theorem 1.

Proof of Theorem 1. If $p = 1$, the assertion follows from Lemma 5.

If $p \geq 2$, we see from the assumption that $\|S - nH^2\|_{n/2} < \alpha(n, H)$. It follows from Lemma 5 that $S_H = nH^2$, i.e., $H_{n+1} = HI$, where I is the unit matrix. By Lemmas 1, 2 and 3, we have

$$\frac{1}{2}\Delta g_\varepsilon^2 = \frac{1}{2}\Delta g^2 \geq \frac{n+2}{n}|\nabla g_\varepsilon|^2 + g^2[nH^2 - n - \mu(p-1)g^2], \tag{21}$$

where $g_\varepsilon = [S_I + n(p-1)\varepsilon^2]^{1/2}$, $g = S_I^{1/2}$. Multiplying $\phi_R^2 g_\varepsilon^{2k-2} (k \geq 1)$ on the both sides of the inequality above and integrating by parts, where ϕ_R is the cut-off function defined in Lemma 5, we obtain

$$\begin{aligned}
0 \geq & \int_M \phi_R^2 g_\varepsilon^{2k-2} g^2[nH^2 - n - \mu(p-1)(S - nH^2)]dM + \frac{n+2}{nk^2}\int_M \phi_R^2 |\nabla g_\varepsilon^k|^2 dM \\
& + \frac{1}{2}\int_M \nabla(\phi_R^2 g_\varepsilon^{2k-2})\nabla g_\varepsilon^2 dM - \frac{1}{2}\int_M \operatorname{div}(\phi_R^2 g_\varepsilon^{2k-2}\nabla g_\varepsilon^2)dM \\
\geq & \int_M \phi_R^2 g_\varepsilon^{2k-2} g^2[nH^2 - n - \mu(p-1)(S - nH^2)]dM \\
& + \frac{2nk - n + 2}{nk^2}\int_M \phi_R^2|\nabla g_\varepsilon^k|^2 dM + 2\int_M \phi_R g_\varepsilon^{2k-1}\nabla\phi_R \nabla g_\varepsilon dM \\
& + \sigma\int_M \phi_R g_\varepsilon^{2k-1}\nabla\phi_R\nabla g_\varepsilon dM - \frac{\rho\sigma}{2}\int_M \phi_R^2 g_\varepsilon^{2k-2}|\nabla g_\varepsilon|^2 dM \\
& - \frac{\sigma}{2\rho}\int_M |\nabla\phi_R|^2 g_\varepsilon^{2k} dM,
\end{aligned} \tag{22}$$

for all $\rho, \sigma \in \mathbb{R}^+$. Taking $k = \frac{n}{2}$, $\sigma = \frac{4(n^2-n+2)}{n^2} - \frac{n}{R} - 2$, $\rho = \frac{n^2}{2R\sigma}$, where $R > \frac{n^3}{2(n^2-2n+4)}$, we get

$$\begin{aligned}
0 \geq & -\left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right]\int_M g_\varepsilon^n |\nabla\phi_R|^2 dM \\
& + \left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R}\right]\int_M |\nabla(\phi_R g_\varepsilon^{n/2})|^2 dM \\
& + \int_M \phi_R^2 g_\varepsilon^{n-2} g^2[nH^2 - n - \mu(p-1)(S - nH^2)]dM.
\end{aligned} \tag{23}$$

By Lemma 4, we have

$$\begin{aligned}
\|\nabla(\phi_R g_\varepsilon^{n/2})\|_2^2 \geq \frac{(n-2)^2}{4(n-1)^2(1+t)}\Bigg[& \frac{1}{D^2(n)}\|\phi_R g_\varepsilon^{n/2}\|_{2n/(n-2)}^2 \\
& - H^2(1 + \frac{1}{t})\|\phi_R g_\varepsilon^{n/2}\|_2^2\Bigg],
\end{aligned} \tag{24}$$

for all $t \in \mathbb{R}^+$. From (23) and (24), we obtain

$$\begin{aligned}0 \geq & -\left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right]\int_M g_\varepsilon^n |\nabla \phi_R|^2 dM \\ & + \frac{[4R(n^2-n+2)-n^3](n-2)^2}{4Rn^3(n-1)^2(1+t)}\left[\frac{1}{D^2(n)}\|\phi_R g_\varepsilon^{n/2}\|^2_{2n/(n-2)}\right. \\ & \left. -H^2(1+\frac{1}{t})\|\phi_R g_\varepsilon^{n/2}\|_2^2\right] \\ & + \int_M \phi_R^2 g_\varepsilon^{n-2} g^2 [nH^2 - n - \mu(p-1)(S-nH^2)]dM. \qquad (25)\end{aligned}$$

As $\varepsilon \to 0$, we have

$$\begin{aligned}0 \geq & -\left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right]\int_M g^n |\nabla \phi_R|^2 dM \\ & + \frac{[4R(n^2-n+2)-n^3](n-2)^2}{4Rn^3(n-1)^2(1+t)}\left[\frac{1}{D^2(n)}\|\phi_R g^{n/2}\|^2_{2n/(n-2)}\right. \\ & \left. -H^2(1+\frac{1}{t})\|\phi_R g^{n/2}\|_2^2\right] \\ & + \left[(nH^2-n)\|\phi_R g^{n/2}\|_2^2 - \mu(p-1)\|S-nH^2\|_{n/2}\|\phi_R^2 g^n\|_{n/(n-2)}\right] \\ \geq & -\left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right]\int_M g^n |\nabla \phi_R|^2 dM \\ & + \left\{\frac{[4R(n^2-n+2)-n^3](n-2)^2}{4Rn^3(n-1)^2 D^2(n)(1+t)} - \mu(p-1)\|S-nH^2\|_{n/2}\right\}\|\phi_R^2 g^n\|_{n/(n-2)} \\ & + \left\{nH^2 - n - \frac{[4R(n^2-n+2)-n^3](n-2)^2H^2}{4Rn^3(n-1)^2 t}\right\}\|\phi_R g^{n/2}\|_2^2, \qquad (26)\end{aligned}$$

where $t \in \mathbb{R}^+$. Since $\int_M g^n dM < C(n,H)$, as $R \to \infty$, we get

$$\begin{aligned}0 & \leq \lim_{R\to\infty}\left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right]\int_M g^n |\nabla \phi_R|^2 dM \\ & \leq \lim_{R\to\infty}\left[\frac{4(n^2-n+2)}{n^3} - \frac{1}{R} + \frac{R\sigma^2}{n^2}\right]\frac{1}{R^2}\int_M g^n dM = 0. \qquad (27)\end{aligned}$$

Hence, as $R \to \infty$, (26) becomes

$$\begin{aligned}0 \geq & \left[\frac{(n^2-n+2)(n-2)^2}{n^3(n-1)^2 D^2(n)(1+t)} - \mu(p-1)\|S-nH^2\|_{n/2}\right]\lim_{R\to\infty}\|\phi_R^2 g^n\|_{n/(n-2)} \\ & + \left[nH^2 - n - \frac{(n^2-n+2)(n-2)^2H^2}{n^3(n-1)^2 t}\right]\lim_{R\to\infty}\|\phi_R g^{n/2}\|_2^2, \qquad (28)\end{aligned}$$

for all $t \in \mathbb{R}^+$. By taking $t = (n^2 - n + 2)(n-2)^2 H^2 [n^4(n-1)^2(H^2-1)]^{-1}$, we have

$$\left\{\frac{n(n^2-n+2)(n-2)^2(H^2-1)}{D^2(n)[n^4(n-1)^2(H^2-1)+(n^2-n+2)(n-2)^2H^2]}\right. \\ \left. -\mu(p-1)\|S-nH^2\|_{n/2}\right\} \lim_{R\to\infty} \|\phi_R^2 g^n\|_{n/(n-2)} \leq 0,$$

which implies

$$\left[\beta(n,H) - \mu(p-1)\|S-nH^2\|_{n/2}\right] \lim_{R\to\infty} \|\phi_R^2 g^n\|_{n/(n-2)} \leq 0. \tag{29}$$

It is easy to see from the assumption that

$$\|S-nH^2\|_{n/2} < \beta(n,H)/\mu(p-1).$$

This together with (29) gives $g \equiv 0$. So,

$$S - nH^2 = f^2 + g^2 \equiv 0.$$

Therefore, M is the totally umbilical sphere $S^n(1/\sqrt{H^2-1})$ in $\mathbb{H}^{n+p}(-1)$. This completes the proof of Theorem 1.

Remark 1. We see from the proof above that the pinching constant in Theorem 1 can be replaced by the constant $\hat{C}^{\frac{n}{2}}(n,p,H)$ defined in (12), which is not less than $C(n,H)$.

Proof of Corollary. We consider the composition of isometric immersions

$$j \circ i \circ \varphi : M^n \to S^{n+p} \to \mathbb{R}^{n+p+1} \to \mathbb{H}^{n+p+2}(-1),$$

where $\varphi : M^n \to S^{n+p}$ is the isometric immersion, i is the standard isometric embedding of S^{n+p} into $\mathbb{R}^{n+p+1}$, and $j : \mathbb{R}^{n+p+1} \to \mathbb{H}^{n+p+2}(-1)$ is the standard totally umbilical immersion. Denote by $\widetilde{H}$ and $\widetilde{S}$ the mean curvature and the squared length of the second fundamental form of the isometric immersion $j \circ i \circ \varphi$ respectively. Then $j \circ i \circ \varphi(M)$ is a complete submanifold in $\mathbb{H}^{n+p+2}(-1)$ with parallel mean curvature vector having norm $\widetilde{H}$. By the Gauss equation, we have

$$n(n-1) + n^2H^2 - S = -n(n-1) + n^2\widetilde{H}^2 - \widetilde{S}.$$

Substituting $\widetilde{H}^2 = 2 + H^2$ into the above, we get

$$S - nH^2 = \widetilde{S} - n\widetilde{H}^2. \tag{30}$$

Since

$$1 \leq \frac{\widetilde{H}^2}{\widetilde{H}^2 - 1} = \frac{H^2+2}{H^2+1} \leq 2,$$

we obtain

$$\alpha(n, \widetilde{H}) \geq \alpha'(n), \qquad \beta(n, \widetilde{H}) \geq \beta'(n). \tag{31}$$

From (30), (31) and Theorem 1, we complete the proof of Corollary.

4. Final Remarks

Let M^n be an n-dimensional compact submanifold with parallel mean curvature H in an $(n+p)$-dimensional hyperbolic space $\mathbb{H}^{n+p}(-1)$. Note that $\int_M (S - nH^2)^{n/2} dM$ is a conformal invariant of submanifolds and $\mathbb{H}^{n+p}(-1)$ is conformally equivalent to $\mathbb{R}^{n+p}$. It follows from a theorem due to Shiohama and Xu [24] that $\int_M (S - nH^2)^{n/2} dM \geq C_6(n)(\sum_{i=1}^{n-1} \beta_i)$, where $C_6(n)$ is an explicit positive constant depending only on n and β_i is the i-th Betti number of M with respect to an arbitrary fixed coefficient J.

Using our main theorem and the inequality above, we obtain the following rigidity theorem for compact submanifolds with parallel mean curvature in a hyperbolic space.

Theorem 2. *Let M be an $n(\geq 3)$-dimensional compact submanifold with parallel mean curvature in an $(n+p)$-dimensional hyperbolic space $\mathbb{H}^{n+p}(-1)$. Assume that $H > 1$. Then there exists an explicit positive constant $C_7(n, H)$ depending on n and H, and an explicit positive constant $C_8(n)$ depending only on n such that*
(i) if

$$\int_M (S - nH^2)^{n/2} dM < C_7(n, H) + C_8(n)(\sum_{i=1}^{n-1} \beta_i),$$

then $S \equiv nH^2$, i.e., M is the totally umbilical sphere $S^n(1/\sqrt{H^2-1})$;
(ii) if M is not totally umbilical, then

$$\int_M (S - nH^2)^{n/2} dM \geq C_7(n, H) + C_8(n)(\sum_{i=1}^{n-1} \beta_i).$$

Here β_i is the i-th Betti number of M with respect to an arbitrary fixed coefficient J.

When $n = 2$, we have the following rigidity theorem.

Theorem 3. *Let M be a 2-dimensional oriented compact surface with parallel mean curvature in a $(2+p)$-dimensional hyperbolic space $\mathbb{H}^{2+p}(-1)$. Assume that $H > 1$. Then*
(i) if

$$\int_M (S - 2H^2)dM < \frac{(H^2-1)\pi}{144H^2} + 2\pi\beta_1,$$

then $S \equiv 2H^2$, i.e., M is the totally umbilical sphere $S^2(1/\sqrt{H^2-1})$;
(ii) if M is not totally umbilical, then

$$\int_M (S - 2H^2)dM \geq \frac{(H^2-1)\pi}{144H^2} + 2\pi\beta_1.$$

Here β_1 is defined as in Theorem 2.

Proof. (i) Following [24], we have $C_6(n) = 4\pi$ for $n = 2$. This means $\int_M (S - 2H^2)dM \geq 4\pi\beta_1$. This together with the assumption implies that

$$\int_M (S - 2H^2)dM < \frac{(H^2-1)\pi}{72H^2}. \tag{32}$$

By Theorem 4 of [35], M is either a minimal surface in the totally umbilical sphere $S^{1+p}(1/\sqrt{H^2-1}) \subset \mathbb{H}^{2+p}(-1)$ for $p \geq 2$, or a surface with constant mean curvature H_0 in a three dimensional space form $F^3(H^2 - H_0^2 - 1)$, where $F^3(H^2 - H_0^2 - 1)$ is a totally umbilical submanifold with constant mean curvature $\sqrt{H^2 - H_0^2}$ in $\mathbb{H}^{2+p}(-1)$. By the Gauss equation, we have

$$2K_M = -2 + 4H^2 - S.$$

Using the Gauss-Bonnet theorem, we obtain

$$\int_M (S - 2H^2)dM = 2(H^2-1)\mathrm{Area}(M) + 8\pi(g-1), \tag{33}$$

where g is the genus of M. On the other hand, it's seen from the isoperimetric inequality due to Hoffman-Spruck-Otsuki [8, 20] that

$$\int_M HdM \geq \frac{1}{12}\sqrt{\pi Area(M)},$$

which implies

$$\text{Area}(M) \geq \frac{\pi}{144H^2}. \tag{34}$$

Combining (32), (33) and (34), we have that $g = 0$, and

$$\text{Area}(M) < \frac{4\pi}{H^2 - 1} + \frac{\pi}{144H^2} \leq \frac{8\pi}{H^2 - 1}. \tag{35}$$

When M is a minimal surface in $S^{1+p}(1/\sqrt{H^2 - 1})$, it follows from (35) and a theorem due to Calabi [2] that M is a great sphere in $S^{1+p}(1/\sqrt{H^2 - 1})$.

When M is a surface with constant mean curvature H_0 in $F^3(H^2 - H_0^2 - 1)$, it is a topological sphere with constant mean curvature H_0 in $F^3(H^2 - H_0^2 - 1)$. By a theorem due to Chern [6], M is a totally umbilical surface in $F^3(H^2 - H_0^2 - 1)$.

Therefore, we conclude that M is the totally umbilical sphere $S^2(1/\sqrt{H^2 - 1})$ in $\mathbb{H}^{2+p}(-1)$.

(ii) The assertion (ii) follows from (i). This completes the proof of Theorem 3.

Remark 2. Motivated by our $L^{n/2}$ pinching theorem, and Liu, Xu, Ye and Zhao's convergence theorems [13, 14, 15, 16] on mean curvature flow, one can investigate the convergence for the mean curvature flow with pinched integral curvature in hyperbolic spaces.

Remark 3. There are close relations between rigidity results for self-shrinkers of the mean curvature flow [3, 7] and ones for submanifolds with parallel mean curvature (including minimal submanifolds). Our method can be used in the study of rigidity of self-shrinkers of the mean curvature flow of arbitrary codimension.

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