Pure and Applied Mathematics Quarterly
Volume 8, Number 4
979—992, 2012

Weakly δ-Koszul Modules

Jia-Feng Lü, Qing-Zhou Xu and Guo-Jun Wang*

Abstract: In order to study the finite generation property of the Yoneda-Ext algebras of positively graded algebras, Green and Marcos introduced δ-Koszul objects in 2005 [2]. Motivated by ([7]-[11]), we are interested in the δ-Koszulity of a finitely generated graded module over a δ-Koszul algebra and introduce the notion of weakly δ-Koszul module in this paper. The following are proved to be equivalent and are the main results of this paper:

- M is a weakly δ-Koszul module;
- the associated graded module $\mathbf{G}(M)$ is a δ-Koszul module;
- M admits a chain of graded submodules: $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M$, such that all M_i/M_{i-1} are δ-Koszul modules.

Keywords: δ-Koszul algebras, δ-Koszul modules, weakly δ-Koszul modules.

1. Motivations and Definitions

It is well known that it is very difficult to decide whether the Yoneda-Ext algebra of a positively graded algebra is finitely generated or not in general. In 2005, Green and Marcos introduced the so-called *δ-Koszul algebra (module)* (see [2]), which is an attempt to discuss such "finite generation property" for the

---

Received March 30, 2008
2010 Mathematics Subject Classification. 16E40, 16E45, 16S37, 16W50.
The first author was supported by National Natural Science Foundation of China (No. 11001245), Zhejiang Province Department of Education Fund (No. Y201016432) and Zhejiang Innovation Project (No. T200905). The third author was supported by Zhejiang Province Department of Education Fund (No. Y201224566).
* Corresponding author.

Yoneda-Ext algebras of a positively graded algebra. It should be noted that δ-Koszul modules are a class of "bi-pure" finitely generated graded modules: firstly, δ-Koszul modules are graded pure modules, i.e., they are generated in a single degree; secondly, in their minimal graded projective resolutions, each term is a graded pure module. It is obvious that Koszul algebras (Priddy, 1970, [12]), d-Koszul algebras (Berger, 2001, [1]) and piecewise-Koszul algebras (Lü, He and Lu, 2007, [6]) are special δ-Koszul algebras. This paper is motivated by [11], [7] and [8], where the authors studied the Koszulity, d-Koszulity and piecewise-Koszulity of finitely generated graded modules respectively. Now one can ask a natural question: Can we study the δ-Koszulity of finitely generated graded modules? The answer is positive and the main aim of this paper is to generalize the results from weakly Koszul modules, weakly d-Koszul modules and weakly piecewise-Koszul modules to the δ-Koszul case, and we introduce the notion of *weakly δ-Koszul module.*

We use the same notations as in [7]. For example, $\mathbb{K}$ is a fixed base field and $A = \bigoplus_{i\geq 0} A_i$ is a positively graded $\mathbb{K}$-algebra such that (a) A_0 is a semisimple Artin algebra, (b) A is generated in degrees zero and one; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$, and (c) A_1 is a finitely generated $\mathbb{K}$-module. The graded Jacobson radical of A, which we denote by J, is $\bigoplus_{i\geq 1} A_i$. We are interested in the category $Gr(A)$ of graded A-modules, and its full subcategory $gr(A)$ of finitely generated modules. The morphisms in these categories, denoted by $\mathrm{Hom}_{Gr(A)}(M, N)$ and $\mathrm{Hom}_{gr(A)}(M, N)$, are the graded A-module maps of degree zero. We denote $Gr_s(A)$ and $gr_s(A)$ the full subcategories of $Gr(A)$ and $gr(A)$ respectively, whose objects are generated in degree s. An object in $Gr_s(A)$ or $gr_s(A)$ is called a graded *pure* A-module.

Now let us recall some definitions which will be used later.

Definition 1.1. ([2]) Let A be a positively graded algebra. A is called a *δ-Koszul algebra* provided the following two conditions:

(1) The trivial A-module A_0 admits a minimal graded projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0,$$

such that each P_n is generated in a single degree, say $\delta(n)$ for all $n \geq 0$, where δ is a strictly increasing set function from $\{0, 1, 2, \cdots\}$ to $\{0, 1, 2, \cdots\}$;

(2) The Yoneda-Ext algebra, $E(A) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_A(A_0, A_0)$, is finitely generated as a graded algebra.

Let A be a δ-Koszul algebra and $M \in gr(A)$. We call M a *δ-Koszul module* if there is a minimal graded projective resolution

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

such that each Q_n is generated in degree $\delta(n) + t$, where $t \geq 0$ a fixed integer.

Let $\mathcal{K}^\delta(A)$ denote the category of δ-Koszul modules.

The first assertion of the following proposition is a motivation for the definition of weakly δ-Koszul modules.

Proposition 1.2. *The following statements are true for a δ-Koszul algebra A.*

(1) *Let $M \in gr_s(A)$ and*

$$\cdots \longrightarrow Q_n \xrightarrow{f_n} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0,$$

a minimal graded projective resolution of M. Then M is a δ-Koszul module if and only if $\ker f_n \subseteq J^{\delta(n+1)-\delta(n)}Q_n$ and $J \ker f_n = \ker f_n \cap J^{\delta(n+1)-\delta(n)+1}Q_n$ for all $n \geq 0$.

(2) *Let $M \in gr(A)$. Then M is a δ-Koszul module if and only if $\mathcal{E}(M) = \bigoplus_{i\geq 0} \mathrm{Ext}^i_A(M, A_0)$ is generated in degree 0 as a graded $E(A)$-module.*

Proof. The proof of the first assertion is similar to that of (Proposition 2.1, [7]) and the second is immediate from (Proposition 3.5, [5]). □

Definition 1.3. Let A be a δ-Koszul algebra and $M \in gr(A)$. Let

$$\cdots \longrightarrow Q_n \xrightarrow{f_n} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then M is called a *weakly δ-Koszul module* if for all $n,\ k \geq 0$, we have $J^k \ker f_n = \ker f_n \cap J^{\delta(n+1)-\delta(n)+k}Q_n$. Let $\mathcal{WK}^\delta(A)$ denote the category of weakly δ-Koszul modules.

Example 1.4. (1) Weakly Koszul modules, introduced by Martínez-Villa and Zacharia in 2003 ([11]), are a special class of weakly δ-Koszul modules in the sense of $\delta(i) = i$ for all $i \geq 0$.

(2) Weakly d-Koszul modules, introduced by Lü, He and Lu in 2007 ([7]), are a special class of weakly δ-Koszul modules with respect to

$$\delta(n) = \begin{cases} \frac{nd}{2}, & \text{if } n \text{ is even;} \\ \frac{(n-1)d}{2} + 1, & \text{if } n \text{ is odd,} \end{cases}$$

where $d \geq 2$ is a given integer.

(3) Weakly piecewise-Koszul modules, introduced by Lü in 2009 ([8]), are another special class of weakly δ-Koszul modules in the sense of

$$\delta(n) = \begin{cases} \frac{nd}{p}, & \text{if } n \equiv 0(\text{mod}p); \\ \frac{(n-1)d}{p} + 1, & \text{if } n \equiv 1(\text{mod}p); \\ \cdots & \cdots \\ \frac{(n-p+1)d}{p} + p-1, & \text{if } n \equiv p-1(\text{mod}p), \end{cases}$$

where $d \geq p \geq 2$ are fixed integers.

Proposition 1.5. *Let A be a δ-Koszul algebra and $M \in gr_s(A)$. Then $M \in \mathcal{K}^\delta(A)$ if and only if $M \in \mathcal{WK}^\delta(A)$.*

Proof. Let $\cdots \longrightarrow Q_n \xrightarrow{f_n} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0$ be a minimal graded projective resolution of M. By Proposition 1.2, $M \in \mathcal{K}^\delta(A)$ if and only if $\ker f_n \subseteq J^{\delta(n+1)-\delta(n)}Q_n$ and $J \ker f_n = \ker f_n \cap J^{\delta(n+1)-\delta(n)+1}Q_n$ for all $n \geq 0$. Note that M is a graded pure module, thus the conditions $\ker f_n \subseteq J^{\delta(n+1)-\delta(n)}Q_n$ and $J \ker f_n = \ker f_n \cap J^{\delta(n+1)-\delta(n)+1}Q_n$ for all $n \geq 0$ are equivalent to that for all $n,\ k \geq 0$, $J^k \ker f_n = \ker f_n \cap J^{\delta(n+1)-\delta(n)+k}Q_n$, i.e., $M \in \mathcal{WK}^\delta(A)$. □

2. Weakly δ-Koszul modules

In this section, we will discuss the δ-Koszulity of finitely generated graded modules over a δ-Koszul algebra.

Lemma 2.1. *Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $gr(A)$ and A a δ-Koszul algebra.*

(1) *If $K,\ M \in \mathcal{WK}^\delta(A)$ and $J^kK = K \cap J^kM$ for all $k \geq 0$, then $N \in \mathcal{WK}^\delta(A)$;*

(2) *If* $K,\ N \in \mathcal{WK}^\delta(A)$ *and* $JK = K \cap JM$, *then* $M \in \mathcal{WK}^\delta(A)$.

Proof. We will only prove (1) since (2) can be proved similarly.

For (1), consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i+1}(K) & \longrightarrow & \Omega^{i+1}(M) & \longrightarrow & \Omega^{i+1}(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_i & \longrightarrow & Q_i & \longrightarrow & L_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i}(K) & \longrightarrow & \Omega^{i}(M) & \longrightarrow & \Omega^{i}(N) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $P_i \to \Omega^i(K) \to 0$, $Q_i \to \Omega^i(M) \to 0$ and $L_i \to \Omega^i(N) \to 0$ are graded projective covers, respectively. Note that $K,\ M \in \mathcal{WK}^\delta(A)$, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i+1}(K) & \longrightarrow & \Omega^{i+1}(M) & \longrightarrow & \Omega^{i+1}(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^{\delta(i+1)-\delta(i)}P_i & \longrightarrow & J^{\delta(i+1)-\delta(i)}Q_i & \longrightarrow & J^{\delta(i+1)-\delta(i)}L_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^{\delta(i+1)-\delta(i)}\Omega^{i}(K) & \longrightarrow & J^{\delta(i+1)-\delta(i)}\Omega^{i}(M) & \longrightarrow & J^{\delta(i+1)-\delta(i)}\Omega^{i}(N) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Apply the functor $A/J^k \otimes_A -$ to the above diagram, one can get the map

$$A/J^k \otimes_A \Omega^{i+1}(N) \longrightarrow A/J^k \otimes_A J^{\delta(i+1)-\delta(i)} L_i$$

is a monomorphism, which is equivalent to

$$J^k \Omega^{i+1}(N) = \Omega^{i+1}(N) \cap J^{\delta(i+1)-\delta(i)+k} L_i$$

for all $i,\ k \geq 0$. i.e., $N \in \mathcal{WK}^\delta(A)$. □

Lemma 2.2. *Let A be a δ-Koszul algebra and $M = \bigoplus_{i\geq 0} M_i$ be a weakly δ-Koszul module with $M_0 \neq 0$. Set $K_M = \langle M_0 \rangle$. Then*

(1) $K_M \in \mathcal{K}^\delta(A)$;
(2) $K_M \cap J^k M = J^k K_M$ *for all* $k \geq 0$;
(3) $M/K_M \in \mathcal{WK}^\delta(A)$.

Proof. Let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow K_M \longrightarrow 0$$

be a minimal graded projective resolution of K_M. We want to show that each P_i is generated in degree $\delta(i)$. Note that M is a weakly δ-Koszul module, by definition, M admits a minimal graded projective resolution

$$\cdots \longrightarrow Q_n \xrightarrow{f_n} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0$$

with $J^k \ker f_n = \ker f_n \cap J^{\delta(n+1)-\delta(n)+k} Q_n$ for all $n,\ k \geq 0$. Up to an isomorphism, $Q_i = A \otimes_{A_0} S^{\Omega^i(M)}$, where $S^{\Omega^i(M)} = S_{\delta(i)} \oplus S_{\delta(i+1)} \oplus \cdots \subseteq \Omega^i(M)$ is a minimal graded sub-A_0-module of $\Omega^i(M)$ such that $A \cdot S^{\Omega^i(M)} = \Omega^i(M)$. Set $P_i = (Q_i)_{\delta(i)} = A \otimes_{A_0} S_{\delta(i)}$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{i+1}(K_M) & \longrightarrow & (Q_i)_{\delta(i)} & \longrightarrow & \Omega^i(K_M) & \longrightarrow & 0 \\
 & & \downarrow & & \subseteq\downarrow & & \subseteq\downarrow & & \\
0 & \longrightarrow & \Omega^{i+1}(M) & \longrightarrow & Q_i & \longrightarrow & \Omega^i(M) & \longrightarrow & 0,
\end{array}$$

Now we claim that $\langle \Omega^{i+1}(M)_{\delta(i+1)} \rangle = \Omega^{i+1}(K_M)$. Indeed, note that M is a weakly δ-Koszul module, we have $\Omega^{i+1}(M) \subseteq J^{\delta(i+1)-\delta(i)} Q_i$. Thus, we have the following exact sequence

$$0 \longrightarrow \Omega^{i+1}(M) \longrightarrow J^{\delta(i+1)-\delta(i)}Q_i \longrightarrow J^{\delta(i+1)-\delta(i)}\Omega^i(M) \longrightarrow 0.$$

Note that $\Omega^{i+1}(M)\cap J^k(J^{\delta(i+1)-\delta(i)}Q_i) = \Omega^{i+1}(M)\cap J^{k+\delta(i+1)-\delta(i)}Q_i = J^k\Omega^{i+1}(M)$, which implies that $\langle\Omega^{i+1}(M)_{\delta(i+1)}\rangle = \Omega^{i+1}(M)\cap\langle(J^{\delta(i+1)-\delta(i)}Q_i)_{\delta(i+1)}\rangle = \Omega^{i+1}(M)\cap \langle(J^{\delta(i+1)-\delta(i)}(Q_i)_{\delta(i)})_{\delta(i+1)}\rangle$. Obviously $(Q_i)_{\delta(i)}$ is generated in degree $\delta(i)$, we have

$$\Omega^{i+1}(M) \cap \langle(J^{\delta(i+1)-\delta(i)}(Q_i)_{\delta(i)})_{\delta(i+1)}\rangle = \Omega^{i+1}(M) \cap (Q_i)_{\delta(i)} = \Omega^{i+1}(K_M).$$

Hence $\Omega^{i+1}(K_M) = \langle\Omega^{i+1}(M)_{\delta(i+1)}\rangle$ which is generated in degree $\delta(i+1)$ for all $i \geq 0$. Thus assertion (1) holds.

The assertion (2) is standard and the third assertion is immediate from the exact sequence

$$0 \longrightarrow K_M \longrightarrow M \longrightarrow M/K_M \longrightarrow 0$$

and Lemma 2.1. □

Theorem 2.3. *Let A be a δ-Koszul algebra and $M \in gr(A)$. Let $\{S_{d_1}, S_{d_2}, \cdots, S_{d_m}\}$ denote the set of the minimal homogeneous generating spaces of M and S_{d_i} consists of homogeneous elements of degree d_i. Consider the following natural filtration of M:*

$$\mathcal{F}(M): \quad 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M,$$

where $M_1 = \langle S_{d_1}\rangle$, $M_2 = \langle S_{d_1}, S_{d_2}\rangle$, $\cdots$, $M_m = \langle S_{d_1}, S_{d_2}, \cdots, S_{d_m}\rangle$. Then $M \in \mathcal{WK}^\delta(A)$ if and only if all $M_i/M_{i-1} \in \mathcal{K}^\delta(A)$ for all $1 \leq i \leq m$.

Proof. Let $M \in \mathcal{WK}^\delta(A)$. If $m = 1$, then the theorem is trivial. If $m \geq 2$, then by Lemma 2.2, $M_1 \in \mathcal{K}^\delta(A)$ and $M_1 \cap J^kM = J^kM_1$ for all $k \geq 0$. Set $W = M/M_1$. By Lemma 2.1, $W \in \mathcal{WK}^\delta(A)$. Consider the exact sequence

$$0 \longrightarrow K_W \longrightarrow W \longrightarrow W/K_W \longrightarrow 0,$$

where $K_W = \langle W_{d_2}\rangle = M_2/M_1$. By Lemma 2.2 again, we get that $K_W \in \mathcal{K}^\delta(A)$ and $K_W \cap J^kM = J^kK_W$ for all $k \geq 0$. Repeat the above argument, we obtain $M_i/M_{i-1} \in \mathcal{K}^\delta(A)$ for all $1 \leq i \leq m$.

Conversely, consider the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_2/M_1 \longrightarrow 0.$$

Evidently, $JM_1 = M_1 \cap JM_2$. Since M_1 and M_2/M_1 are δ-Koszul modules, by Lemma 2.1, we get that M_2 is a weakly δ-Koszul module. Apply Lemma 2.1 several times, we get that M is a weakly δ-Koszul since $JM_i = M_i \cap JM_{i+1}$ for all $1 \leq i \leq m$. □

3. Some applications

In this section, we will give some applications of Theorem 2.3.

3.1. **On the finitistic dimension conjecture.** As an application of Theorem 2.3, we first prove that the finitistic dimension conjecture is true in $\mathcal{WK}^\delta(A)$. The *finitistic dimension conjecture*, is one of the important and interesting conjectures in the representation theory of Artin algebras. Now we will recall the contents of the original conjecture.

Let Λ be an arbitrary Artin R-algebra, where R is a commutative Artin ring with identity. Let $\text{mod}(\Lambda)$ be the category of finitely generated Λ-modules and

$$\textbf{Bound}(\Lambda) := \{M \in \text{mod}(\Lambda) | pd_\Lambda(M) < \infty\}.$$

The following is the well-known finitistic dimension conjecture:

Let Λ be as above. Then

$$\sup\{pd_\Lambda(M) | M \in \textbf{Bound}(\Lambda)\} < \infty.$$

It is too far to solve it completely and still remains open now. Therefore it is also interesting to find certain subcategories in which the finitistic dimension conjecture holds. Here, we will show that in $\mathcal{WK}^\delta(A)$, the finitistic dimension conjecture is true provided A is a finite dimensional δ-Koszul algebra.

Lemma 3.1. *Let A be a finite dimensional δ-Koszul algebra. Then the finitistic dimension conjecture holds in the category $\mathcal{K}^\delta(A)$.*

Proof. It is immediate from (Theorem 4.5, [4]). □

Lemma 3.2. *([14]) Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be an exact sequence in $gr(A)$. Then $pd_A(M) \leq \max\{\mathrm{pd_A(M_1)},\ \mathrm{pd_A(M_2)}\}$.* □

Theorem 3.3. *Let A be a finite dimensional δ-Koszul algebra. Then the finitistic dimension conjecture holds in the category $\mathcal{WK}^\delta(A)$.*

Proof. Observe that $M \in \mathcal{WK}^\delta(A)$, by Theorem 2.3, there exists a chain of graded submodules,

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M,$$

such that all M_i/M_{i-1} are δ-Koszul modules. Consider the following exact sequence,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_2/M_1 \longrightarrow 0,$$

by Lemma 3.2, $pd_A(M_2) \leq \max\{\mathrm{pd_A(M_1)},\ \mathrm{pd_A(M_2/M_1)}\}$. By an easy induction, we have

$$pd_A(M) \leq \max\{\mathrm{pd_A(M_1)},\ \mathrm{pd_A(M_2/M_1)},\ \mathrm{pd_A(M_3/M_2)}, \cdots,\ \mathrm{pd_A(M_m/M_{m-1})}\}.$$

Therefore, $\sup\{pd_A(M) |\ M \in \mathcal{WK}^\delta(A)\} \leq \sup\{\max\{\mathrm{pd_A(M_1)}, \mathrm{pd_A(M_2/M_1)}, \mathrm{pd_A(M_3/M_2)}, \cdots, \mathrm{pd_A(M_m/M_{m-1})}\} | \mathrm{M_i/M_{i-1}} \in \mathcal{K}^\delta(\mathrm{A})\}$, which is finite by Lemma 3.1. □

3.2. **On the finite generation of $\mathcal{E}(M)$.** As another application of Theorem 2.3, we will prove that the Koszul dual $\mathcal{E}(M)$ of a weakly δ-Koszul module M is finitely 0-generated graded $E(A)$-module.

Lemma 3.4. *Let A be a δ-Koszul algebra and $M \in \mathcal{WK}^\delta(A)$. Then $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module.*

Proof. Assume that the generators of M lie in the degrees $d_1, d_2, \cdots, d_m$ with $d_1 < d_2 < \cdots < d_m$. We will do it by induction. If $m = 1$, then M is pure. By Theorem 2.3, M is a δ-Koszul module, which implies that $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module. Assume that the statement holds for less than m. Note that M is a weakly δ-Koszul module, by Theorem 2.3, M admits a chain of submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M,$$

such that all M_i/M_{i-1} are δ-Koszul modules. Consider the following exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0.$$

We have the following exact sequence

$$0 \longrightarrow \Omega^n(M_1) \longrightarrow \Omega^n(M) \longrightarrow \Omega^n(M/M_1) \longrightarrow 0,$$

which induces the following exact sequence

$$0 \to \mathrm{Hom}_A(\Omega^n(M/M_1), A_0) \to \mathrm{Hom}_A(\Omega^n(M), A_0) \to \mathrm{Hom}_A(\Omega^n(M_1), A_0) \to 0.$$

for all $n \geq 0$. Thus we have the following exact sequence

$$0 \longrightarrow \mathrm{Ext}_A^n(M/M_1, A_0) \longrightarrow \mathrm{Ext}_A^n(M, A_0) \longrightarrow \mathrm{Ext}_A^n(M_1, A_0) \longrightarrow 0$$

for all $n \geq 0$. Now apply the exact functor "$\bigoplus$" to the exact sequence above, we get the following exact sequence

$$0 \to \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(M/M_0, A_0) \to \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(M, A_0) \to \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(M_0, A_0) \to 0.$$

That is, we have the following exact sequence

$$0 \longrightarrow \mathcal{E}(M/M_1) \longrightarrow \mathcal{E}(M) \longrightarrow \mathcal{E}(M_1) \longrightarrow 0.$$

It is easy to see that $\mathcal{E}(M_1)$ is a finitely generated graded $E(A)$-module and the number of the generating spaces of M/M_1 is less than m. By induction hypothesis, we have that $\mathcal{E}(M/M_1)$ is a finitely generated graded $E(A)$-module. Therefore, $\mathcal{E}(M)$ is a finitely generated graded $E(A)$-module and we finish the proof. □

Lemma 3.5. *If M is a weakly δ-Koszul module, then $\mathcal{E}(M)$ is generated in degree 0 as a graded $E(A)$-module.*

Proof. It is no harm to assume that M is a weakly δ-Koszul module with homogeneous generators of degrees d_1 and d_2 ($d_1 < d_2$) and without generality, we may assume that $d_1 = 1$ and $d_2 = 2$. We have a chain of graded submodules of M

$$0 \subset M_1 \subset M_2 = M,$$

such that M_1 and M_2/M_1 are δ-Koszul modules and $J^k M_2 \cap M_1 = J^k M_1$ for $k \geq 0$. Similarly, we can get the following exact sequence

$$0 \longrightarrow \mathcal{E}(M/M_1) \longrightarrow \mathcal{E}(M) \longrightarrow \mathcal{E}(M_1) \longrightarrow 0.$$

By Proposition 1.2, we have that $\mathcal{E}(M_1)$ and $\mathcal{E}(M/M_1)$ are generated in degree 0 since M_1 and M/M_1 are δ-Koszul modules. Therefore $\mathcal{E}(M)$ is generated in degree 0 as a graded $E(A)$-module. □

Theorem 3.6. *Let M be a weakly δ-Koszul module. Then $\mathcal{E}(M)$ is finitely 0-generated as a graded $E(A)$-module.*

Proof. By Lemmas 3.4 and 3.5. □

4. On the associated graded module $\mathbf{G}(M)$

In this section, we will discuss the δ-Koszulity of the associated graded module of a weakly δ-Koszul module.

Let A be a positively graded $\mathbb{K}$-algebra and $M \in gr(A)$, we can get another graded module, denoted by $\mathbf{G}(A)$, called the *associated graded module of M* as follows:

$$\mathbf{G}(M) = M/JM \oplus JM/J^2M \oplus J^2M/J^3M \oplus \cdots .$$

Similarly, we can define $\mathbf{G}(A)$ for a positively graded algebra.

Proposition 4.1. *Let A be a positively graded $\mathbb{K}$-algebra and $M \in gr(A)$. Then*

(1) $\mathbf{G}(A) \cong A$ *as a graded $\mathbb{K}$-algebra.*
(2) $\mathbf{G}(M)$ *is a finitely generated graded A-module.*
(3) *If M is pure, then $\mathbf{G}(M) \cong M[l]$ as graded A-modules for some integer l.*

Proof. By the definition, $\mathbf{G}(A)_i = J_i/J_{i+1} = A_i$ for all $i \geq 0$ since the positively graded $\mathbb{K}$-algebra $A = A_0 \oplus A_1 \oplus \cdots$ is generated in degrees 0 and 1. Now the first assertion is clear. For the second assertion, by (1), we only need to prove that $\mathbf{G}(M)$ is a graded $\mathbf{G}(A)$-module. We define the module action as follows:

$$\mu : \mathbf{G}(A) \otimes \mathbf{G}(M) \longrightarrow \mathbf{G}(M)$$

via

$$\mu((a + J^iA) \otimes (m + J^jM)) = a \cdot m + J^{i+j-1}M$$

for all $a + J^iA \in \mathbf{G}(A)$ and $m + J^jM \in \mathbf{G}(M)$. It is easy to check that μ is well-defined and under μ, $\mathbf{G}(M)$ is a graded $\mathbf{G}(A)$-module. The proof of the third assertion is similar to that of (1) and we omit it. □

Lemma 4.2. *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a split exact sequence in* $gr(A)$*, where* A *is a* δ*-Koszul algebra. Then* M *is a* δ*-Koszul module if and only if* K *and* N *are both* δ*-Koszul modules.*

Lemma 4.3. *([11]) Let* A *be a positively graded* $\mathbb{K}$*-algebra and let* $M = M_{d_1} \oplus M_{d_2} \oplus M_{d_3} \oplus \cdots$ *be a finitely generated* A*-module with* $M_{d_1} \neq 0$*. Let* $K = \langle M_{d_1} \rangle$ *be the graded submodule of* M *generated by* M_{d_1}*. Then we have a split exact sequence in* $gr(\mathbf{G}(A))(= gr(A))$

$$0 \longrightarrow \mathbf{G}(K) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(M/K) \longrightarrow 0.$$

Theorem 4.4. *Let* A *be a* δ*-Koszul algebra and* $M \in gr(A)$*. Then* $M \in \mathcal{WK}^{\delta}(A)$ *if and only if* $\mathbf{G}(M) \in \mathcal{K}^{\delta}(A)$*.*

Proof. Since M is finitely generated, assume that M is generated by a minimal set of homogeneous elements lying in degrees $d_1 < d_2 < \cdots < d_m$. Set $K = \langle M_{d_1} \rangle$. By Lemma 4.3, we get a split exact sequence

$$0 \longrightarrow \mathbf{G}(K) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(M/K) \longrightarrow 0.$$

($\Rightarrow$) Let $M \in \mathcal{WK}^{\delta}(A)$. If $m = 1$, M is a pure weakly δ-Koszul module, by Proposition 1.2, we obtain that M is a δ-Koszul module and by Proposition 4.1, we have $M[l] \cong \mathbf{G}(M)$ as graded A-modules. Hence $\mathbf{G}(M)$ is a δ-Koszul module. Now we assume that the statement holds for less than m. By Theorem 2.3, K is a δ-Koszul module, of course, K is a weakly δ-Koszul module. Consider the exact sequence

$$0 \longrightarrow \mathbf{G}(K) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(M/K) \longrightarrow 0,$$

by Lemma 2.1, we have M/K is a weakly δ-Koszul module. Since the number of generators of M/K is less than m, by the induction hypothesis, we have $\mathbf{G}(M/K)$ is a δ-Koszul module. Since $\mathbf{G}(K)$ is obvious a δ-Koszul module, by Proposition 4.2, we get that $\mathbf{G}(M)$ is a δ-Koszul module.

($\Leftarrow$) Conversely, assume that $\mathbf{G}(M)$ is a δ-Koszul module, by Proposition 4.2, we obtain that $\mathbf{G}(K)$ and $\mathbf{G}(M/K)$ are both δ-Koszul modules. By induction hypothesis, K and M/K are weakly δ-Koszul modules. By Lemma 2.1, we have that M is a weakly δ-Koszul module. $\square$

References

1. R. Berger, *Koszulity for nonquadratic algebras*, J. Alg., **239** (2001), 705-734.
2. E. L. Green, E. N. Marcos, *δ-Koszul algebras*, Comm. Alg., **33**(6) (2005), 1753-1764.
3. E. L. Green, R. Martínez-Villa, *Koszul and Yoneda algebras*, Representation theory of algebras (Cocoyoc, 1994), CMS Conference Proceedings, American Mathematical Society, Providence, RI, **18** (1996), 247-297.
4. E. L. Green, R. Martínez-Villa, I. Reiten, ϕ. Solberg, D. Zacharia, *On modules with linear presentations*, J. Alg., **205**(2) (1998), 578-604.
5. E. L. Green, E. N.Marcos, R. Martínez-Villa, P. Zhang, *D-Koszul algebras*, J. Pure Appl. Alg., **193** (2004) 141-162.
6. J.-F. Lü, J.-W. He, D.-M. Lu, *Piecewise-Koszul algebras*, Sci. China, **50** (2007), 1785-1794.
7. J.-F. Lü, J.-W. He, D.-M. Lu, *On modules with d-Koszul towers*, Chinese J. Contemp. Math., **28**(2) (2007), 191-200.
8. J.-F. Lü, *On modules with piecewise-Koszul towers*, Houston J. Math., **35**(1) (2009), 185-207.
9. J.-F. Lü, *On modules with d-Koszul-type submodules*, Acta Math. Sin., **25**(6) (2009), 1015-1030.
10. J.-F. Lü, G.-J. Wang, *Weakly d-Koszul Modules*, Vietnam J. Math., **34**(3) (2006), 341-351.
11. R. Martínez-Villa, D. Zacharia, *Approximations with modules having linear resolutions*, J. Alg., **266** (2003), 671-697.
12. S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc., **152** (1970), 39-60.
13. G.-J. Wang, F. Li, *On minimal Horse-shoe Lemma*, Taiwanese J. Math., **12**(2) (2008), 373-387.
14. C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Avanced Mathematics, **38**, Cambridge Univ. Press, 1995.

Jia-Feng Lü
Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang, 321004, P.R. China
E-mail: jiafenglv@zjnu.edu.cn

Qing-Zhou Xu
Department of Mathematics
Xuchang University
Henan 461000, P.R. China

E-mail: xuqz88@163.com

Guo-Jun Wang
Department of Mathematics
Zhejiang University
Hangzhou 310027, P.R. China
E-mail: wgj@zju.edu.cn